Sets

Set ($\{A, B, C\}$ or $\{Jonas, Banana, 43\}$) • A collection of items of any kind. Duplicates don't matter. Order doesn't matter. Contained in brackets. Example: $\{1, 2, 2, b\} = \{1, 1, 2, b, b\} = \{b, 1, 2\}$.

Empty set ($\emptyset$ or $\{\}$) • The set that contains no elements. There exists only 1 empty set. Properties:$\#\emptyset = 0$. For any set A, $\emptyset$ is a subset of A.

Number sets ($\mathbb{N}$, $\mathbb{N}^+$ and $\mathbb{R}$ for example) • These are infinite sets containing numbers.

- Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, 5...\infty\}$ and $\mathbb{N}^+ = \{1, 2, 3, 4, 5...\infty\}$
- Integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Rational numbers: $\mathbb{Q} = \{..., \frac{-2}{2}, -2, ..., \frac{-1}{3}, \frac{-1}{2}, -1, 0, 1, \frac{1}{2}, \frac{1}{3}, ..., 2, \frac{2}{2}, ...\}$
- Real numbers: $\mathbb{R} = \{\text{all non imaginary numbers}\}$

Element of ($\in$ or $\notin$) • True if an element exists in a set at least once. Example: $3 \in \{4, 4, 3, 1\}$ but $3 \notin \{4, 4, 1\}$. Also $\{1, 2\} \in \{\{3, 4\}, \{2, 1\}\}$.

Cardinality (#) • The cardinality of a set is the number of elements it contains. Example: $\#\{1, 2, 3\} = 3$ and $\#\{\{1, 2, 3\}\} = 1$. $\#\emptyset = 0$.

Inclusion and proper inclusion ($\subseteq$ and $\subset$) • If $A \subseteq B$ it means that if $x \in A$ then $x \in B$ for all $x \in A$. One way to think of this is that A is contained in B. It may be that $A = B$. A is called subset and B superset. Proper or strict inclusion ($\subset$) means that $A \subset B$ iff $A \subseteq B$ and $A \neq B$. Same as $\subseteq$, except that the two sets can not be equal. Example: Let $A = \{a, b, c, 1, 2, 3\}$. $\{1, b, c\} \subset A$ and $\{1, 2, 3, a, b, c\} \subseteq A$ but $\{1, 2, 3, a, b, c\} \not\subset A$ and $\{a, b, c, d\} \not\subseteq A$. Properties: It is always true that $A \subseteq A$. It is always false that $A \subset A$. If $A \subset B$ then $A \subseteq B$.

Set builder notation / set comprehension • Used to create sets. Many different notations. The ":" is read *such that*. " | " may be used instead of ":". Flavor 1: $\{n \in \mathbb{N}^+ : n \text{ is prime}\} = \{2, 3, 5, 7...\}$. Flavor 2: $\{n : n \in \mathbb{N}^+, n \text{ is even }, n > 10\} = \{12, 14, 16, 18...\}$.

Union ($\cup$) • All elements that are in A or B or both. Example: If $A = \{1, 2, 3\}$ and $B = \{2, 4, 3, 5\}$ then $A \cup B = \{1, 2, 3, 4, 5\}$.

Intersection ($\cap$) • All elements that are both in A and B. Example: If $A = \{1, 2, 3\}$ and $B = \{4, 3, 5\}$ then $A \cap B = \{3\}$.

Difference ($\setminus$) • All elements that are in A and not in B. Example: If $A = \{1, 2, 3\}$ and $B = \{4, 3, 5\}$ then $A \setminus B = \{1, 2\}$.

Complement ($-A$ or $_{-U}A$ or A^- or A' or A^c) • If we have a finite local universe set called U (which contains everything we're potentially interested in) then $A^c = U\ A$ is the complement to A. This is everything in U that is not in A and has no real meaning if U is not defined and finite. Example: $U = \mathbb{N}$, $A = \{1, 2, 3\}$ then $A^c = \mathbb{N} \setminus \{1, 2, 3\} = \{0, 4, 5, 6, 7, 8...\}$.

Disjointness • A and B are disjoint if they share no elements: $A \cap B = \emptyset$.

Family of sets ($\{A_i : i \in I\}$) • A is a set of sets, i is index and I is the index set. Example: If $I = \{Carnivore, Herbivore, Omnivore\}$ then we can define a family of sets $\{A_i : i \in I\}$ such that $A_{Carnivore} = \{Cat, T-Rex, Lion, Shark\}$, $A_{Herbivore} = \{Nora, Cow, Stegosaurus\}$, $A_{Omnivore} = \{Pig, Human, Rat, Oviraptor\}$.

Generalized union and intersection ($\cap$ or $\cup\{A_i : i \in I\}$) • A is a set of sets. The generalized union: $\cup A = \{x : x \in s \text{ for at least one } s \in A\}$ is the union of all the sets in A. The generalized intersection: $\cap A = \{x : x \in s \text{ for all } s \in A\}$ is the intersection of all sets in A. Example: Using the example from *Family of sets*, the generalized union $\cup\{A_i : i \in \{Carnivore, Herbivore, Omnivore\}\} = \{Cat, T-Rex, Lion, Shark, Nora, Cow, Stegosaurus, Pig, Human, Rat, Oviraptor\}$. The generalized intersection $\cup\{A_i : i \in \{Carnivore, Herbivore, Omnivore\}\} = \emptyset$ as they share no elements.

Power set ($P(A)$ or 2^A) • The power set of A is the set of all of its possible subsets, or $\{s : s \subseteq A\}$. Example: if $A = \{a, b, c\}$, then $P(A) = \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Properties: $\emptyset \in P(A), \quad A \in P(A), \quad \#(P(A)) = 2^{\#A}$.

Relations

Ordered pairs ((a, b)) • A pair of items where order and duplicates matter. Properties: $(a, b) \neq (b, a)$ and $(a, a) \neq (a)$ contradictory to sets.

n-tuples ($(a_1, a_2, ..., a_n)$) • A collection of n items where order and duplicates matters, like ordered pairs.

Cartesian product ($A \times B$ and $A_1 \times A_2 \times ... A_n$) • The cartesian product is the set of all n-tuples or using math: $A \times B = \{(a, b) : a \in A and b \in B\}$ or for the more general case $A_1 \times ... A_n = \{(a_1, ..., a_n) : a_1 \in A, ..., a_n \in A_n\}$. Example: $\{a, b, c\} \times \{3, 4\} = \{(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)\}$. With three sets it looks like this: $\{a, b)\} \times \{3, 4\} \times \{\alpha, \beta\} = \{(a, 3, \alpha), (a, 3, \beta), (a, 4, \alpha), (a, 4, \beta), (b, 3, \alpha), (b, 3, \beta), (b, 4, \alpha), (b, 4, \beta)\}$. Properties: $\#(A \times B) = \#A \times \#B$.

Relations ($R \subseteq A \times B$) • A relation is a way of mapping values in A to values in B. The notation $R \subseteq A \times B$ reveals that the relation R over $A \times B$ is a subset of the cartesian product $A \times B$. Example: Let's specify the relation *less than* over the natural numbers, $< \subseteq \mathbb{N}^+ \times \mathbb{N}^+$. $<$ is then the set of all of the tuples (a, b) where $a < b$. $<= \{(1, 2), (1, 3), ..., (1, 654), ..., (45, 7889)...\}$. This means that, for example, $(2, 1) \notin <$.

Source and target • For any relation $R \subseteq A \times B$, A is the source and B is the target.

Domain ($dom(R)$) • For a relation $R \subseteq A \times B$ the domain is the set of all $a \in A$ that is present in a tuple in the relation. Example: if we have the set $H = \{Barney, Marshall, Ted, Lily, Robin, Ranjit\}$ and define the relation $\heartsuit \subseteq H \times H$ to mean who screwed whom in HIMYM, it would look like this: $\heartsuit = \{(Barney, Robin), (Robin, Barney), (Marshall, Lily), (Lily, Marshall), (Ted, Robin), (Robin, Ted)\}$. Then $dom(\heartsuit) = \{Barney, Marshall, Ted, Lily, Robin\}$. Notice that $Ranjit \notin dom(\heartsuit)$ as $Ranjit \not\heartsuit h$ for any $h \in H$. Properties: For any relation $R \subseteq A \times B$ it is the case that $dom(R) \subseteq A$.

Range ($range(R)$) • The range of a relation is much like the domain, except that for a relation $R \subseteq A \times B$ the range is the set of all $b \in B$ that is present in a tuple in the relation. Example: if we have the same set H and relation $\heartsuit$ defined in the previous entry, then the $range(\heartsuit) = \{Barney, Marshall, Ted, Lily, Robin\}$. In this case it happens to be equivalent to the domain. Properties: For any relation $R \subseteq A \times B$ it is the case that $range(R) \subseteq B$.

Converse or Inverse ($R^{-1} = (b, a) : aRb)$) • Given a relation R, we define R^{-1} to be the set of all ordered pairs (b,a) such that (a,b) ∈ R. In other words, the converse is simply the same set, but with its tuple's elements flipped. Example: If $R = \{(a, a), (a, b), (b, b), (b, c), (c, a), (c, c)\}$ then $R^{-1} = \{(a, a), (b, a), (b, b), (c, b), (a, c), (c, c)\}$. Properties: If $R \subseteq A \times B$ then $R^{-1} \subseteq B \times A$.
$(R^{-1})^{-1} = R$.
$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Complement ($\bar{R}$ or $_{-A \times B}R$ or $A \times B \setminus R$) • All of the tuples in the cartesian product of $A \times B$ that is not in the relation R. Example: If $R \subseteq A \times B$ and $R = \{(a, a), (a, b), (b, b), (b, c), (c, a), (c, c)\}$ and $A \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ then the complement $\bar{R} = \{(a, c), (b, a), (c, b)\}$. Properties: $\bar{R} \subseteq A \times B$, $\bar{\bar{R}} = R$, $\overline{R \cup S} = \bar{R} \cap \bar{S}$, $\overline{R \cap S} = \bar{R} \cup \bar{S}$.

Composition ($S \circ R$) • If $R \subseteq A \times B$ and $S \subseteq B \times C$ then their composition $R \circ S$ is a relation over $A \times C$. Mathematically the composition is defined like this: $S \circ R = \{(a, c) : aRb$ and bSc for some $b \in B\}$. This means that if there is a b that connects a and c it is included in the composition. Example: If $R = \{(a, x), (a, y), (b, y), (b, z)\}$ and $S = \{(x, 1), (y, 1), (z, 2)\}$ then their composition $S \circ R = \{(a, 1), (b, 1), (b, 2)\}$.

Image ($R(a)$) • Let R be any relation over $A \times B$, and let $a \in A$. We define the image of a under R, written $R(a)$, to be the set of all $b \in B$ such that $(a, b) \in R$. Example: Let $A = \{John, Mary, Peter\}$ and $B = \{1, 2, 3\}$. Let R be the relation $\{(John, 1), (Mary, 2), (Mary, 3)\}$. The image of John is $R(John) = 1$ and $R(Mary) = \{2, 3\}$. Properties: The image can be used with a parameter that is a set. If $X \subseteq A$, then $R(X) = \{b \in B : aRb$ for some $a \in X\}$.

Reflexivity • Let $R \subseteq A \times A$ be a (binary) relation. We say that it is reflexive iff $(a, a) \in R$ for all $a \in A$. In other words, for every $ainA$, aRa. Example: The relation $\geq = \{(1, 1), (2, 1), (3, 1), ..., (5, 5), (6, 5)...\}$ is reflexive because for any element a we can find the tuple (a, a), like $(2, 2)$ for example, as $2 \geq 2$.

Irreflexivity • Let $R \subseteq A \times A$ be a (binary) relation. We say that it is irreflexive iff $(a, a) \notin R$ for all $a \in A$.Example: The relation $<= \{(1, 2), (1, 3), (1, 4), ..., (5, 6), (5, 7)...\}$ is irreflexive because for any element a the tuple (a, a) does not exist, like (2,2) for example, as $2 \not< 2$. Properties: A relation R over a set may be neither reflexive nor irreflexive. For example, n is a prime divisor of n for some natural numbers(e.g. 2 is a prime divisor of itself) but not for some others(e.g. 4 is not a prime divisor of 4)

Transitivity • Let $R \subseteq A \times A$ be a (binary) relation. It is transitive iff for all $(a, b, c \in A)$: whenever aRb and bRc then aRc. Example: The relation $<= \{(1, 2), (1, 3), (1, 4), ..., (5, 6), (5, 7)...\}$ is transitive because for any element $a, b, c \in A$, if we can find the tuples (a, b) and (b, c) then (a, c) is also in $<$. In other words, if $a < b$ and $b < c$ then $a < c$.

Symmetry • Let $R \subseteq A \times A$ be a (binary) relation. It is symmetrical iff for all $a, b \in A$: if aRb then bRa. Example: The relation $\circledcirc$, which we here define to mean "married to", is symmetrical because if $a \circledcirc b$ then $b \circledcirc a$ (in most legal systems).

Asymmetry • Let $R \subseteq A \times A$ be a (binary) relation. It is asymmetrical iff for all $a, b \in A$: if aRb then **not** bRa. In other words, it is not symmetrical for any $a, b \in A$. Example: The relation $<$ is asymmetric because if $a < b$ then $b \not< a$. Properties: If a relation is asymmetric then it is also antisymmetric.

Antisymmetry • Let $R \subseteq A \times A$ be a (binary) relation. It is antisymmetrical if *whenever* aRb and bRa then $a = b$. This is equivalent to: *whenever* $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$. The keyword here being *whenever*. Example: The relation $\leq$ is antisymmetric because if $a \leq b$ and $b \leq a$, a must be equal to b. Note: The requirements that $a = b$ for antisymmetry applies only if we can find both (a, b) and (b, a) in the relation. If, for example, only (a, b) exists in the relation, antisymmetry still holds. Consider the relation $R = \{(a, b), (b, c), (b, d), (a, c), (a, d)\}$. It is antisymmetric, as we can not find a (b, a) for (a, b) (so a does not have to equal b), no (c, b) for (b, c) and so on.

Equivalence relation • Let $\approx \subseteq A \times A$ be a (binary) relation. It is an equivalence relation iff it is *Reflexive*, *Symetric* and *Transitive*. Example: The relation $=$ is an equivalence relation.

Partitions • A partition of A is a set of non-empty subsets $B \subseteq A$, that together contain all the elements of A but are pairwise disjoint (they do not contain the same elements). More mathematically explained, the non-empty subsets B_i have to exhaust A, that is $\cup\{B_i\}_{i \in I} = A$. They also have to be pairwise disjoint: for all $i, j \in I$ if $B_i \neq B_j$ then $B_i \cap B_j = \emptyset$, or even more mathematically $\forall i, j \in I, (B_i \neq B_j \implies B_i \cap B_j = \emptyset)$. If all of these are true then $\{B_i : i \in I\}$ is the partition of A. Example: If we have a set of names, $A = \{Jonas, Mary, Alex, Kim, Julia\}$. We now want to partition this set and divide the names based on female, male and unisex names. $B_1 = \{Jonas\}, B_2 = \{Mary, Julia\}, B_3 = \{Kim, Alex\}$ and the partition of A is {{Jonas}, {Mary, Julia}, {Kim, Alex}}. We see that neither is empty, none of them contain the same elements (pairwise disjointness) and they exhaust A, that is $B_1 \cup B_2 \cup B_3 = A$.

Equivalence class ($[a]$ or $[a]_\approx$) • Given an equivalence relation $\approx \subseteq A \times A$, for any $a \in A$, the equivalence class of a is defined as a set like this: $[a]_\approx = \{b \in A : a \approx b\}$. In other words, the equivalence class of $[a]_\approx$ is the set containing all the elements of A that relate to a. Example: Let $\approx$ be an equivalence relation. $\approx = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$. $[a] = \{a\}$ and $[b] = [c] = \{b, c\}$. The set of all equivalence classes for $\approx$ is $\{\{a\}, \{b, c\}\}$

Quotient set ($A/\approx$) • Given a set A and an equivalence relation $\approx$, the quotient set $A/\approx$ is the set of all equivalence classes of A. More mathematically: $A/\approx = \{[a]_\approx : a \in A\}$. Properties: Every partition on a set A is the quotient set of an equivalence relation over A. Every quotient set is a partition. Compare with the two above entries.

Similarity relations • Let $R \subseteq A \times A$ be a (binary) relation. It is a similarity relation if it is both *Reflexive* and *Symmetric*. Example: The relation "close to" ($\approx$) is defined like: $r \approx s$ for $r, s \in \mathbb{R}$

iff $|r - s| < 0.5$. It is reflexive as $|a - a| = 0 < 0.5$. It is also symmetric as if $|a - b| < 0.5$ then obviously $|b - a| < 0.5$, thanks to the absolute sign. It is however not transitive. It is true that $1.0 \approx 1.4$ as $|1.0 - 1.4| = 0.4 < 0.5$ and it is also true that $1.4 \approx 1.8$ as $|1.4 - 1.8| = 0.4 < 0.5$. Still, $1.0 \not\approx 1.8$ as $|1.0 - 1.8| = 0.8 > 0.5$.

Order relation (partial order) • A binary relation $R \subseteq A \times A$ is an *inclusive* (or *non-strict*) *partial order* iff it is *Reflexive*, *Antisymmetric* and *Transitive*. Example: The relation $\leq$ is an order relation, while $<$ is **not** as it fails to fulfill the *antisymmetry* criteria.

Partially ordered set or Poset ((A, R)) • A pair (A, R) where A is a set and $R \subseteq A \times A$ is a *partial order* on A is called a partial ordered set or poset.

Strict (partial) order • A binary relation $R \subseteq A \times A$ is a *strict partial order* iff it is *Irreflexive* and *Transitive*. Properties: Also implies *Asymmetry*. *Irreflexivity*: $a \not R\, a$. *Transitivity*: if aRb and bRc then aRc. *Asymmetry*: aRb then $a \not R\, b$.

Total (or linear) order • A binary relation $R \subseteq A \times A$ is an *inclusive* (or *non-strict*) *partial order* iff it is *Reflexive*, *Antisymmetric* and *Transitive*. It also has to be *Total (complete)*, that is for any $a, b \in A$ either aRb or bRa. Example: The relation $\leq$ is a total (or linear) order.

Transitive closure (R^*) • If a relation R is not transitive (does not satisfy aRb, bRc then aRc) it can be expanded and made transitive. The transitive closure of R is the smallest relation that includes R and is transitive. We write it as R^*. Example: Let $R = \{(a, a), (a, b), (b, c)\}$. To make it a transitive closure, it has to include R and be transitive, like so: $R^* = \{(a, a), (a, b), (b, c), (a, c)\}$

Functions

Functions ($f : A \longrightarrow B$) • A relation $R \subseteq A \times B$ is a function iff its domain is equal to its source - that is, $f(x)$ exists for all $x \in A$ - and all elements in the domain map to exactly 1 element in the codomain. These criteria can also be written as $dom(f) = A$ and $\#(f(a)) = 1$ for all $a \in A$. Example: Let A and B be sets such that $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$. The relation $R_1 = \{(a, 1), (b, 2), (c, 3)\}$ is **not** a function as it is missing a pair (d, x), where $x \in B$. Neither is $R_2 = \{(a, 1), (b, 2), (c, 3), (d, 4), (d, 5)\}$ as d can't be mapped to 2 values in a function. $R_3 = \{(a, 1), (b, 2), (c, 2), (d, 2)\}$ is a function, because all elements of A are mapped from. Note that not all the elements in B must be mapped to.

Domain, range, codomain • These properties are really best explained through this image:

	$R \subseteq A \times B$	$f : A \longrightarrow B$
actual values, left	domain	domain
actual values, right	range	range
A	source	domain
B	target	codomain

$dom(f) = A$

Set of functions ($\langle A \longrightarrow B\rangle$ or B^A) • This is the set of all possible functions from A to B. That means that the following say the same thing: $f : A \longrightarrow B$, $f \in \langle A \longrightarrow B\rangle$, $f \in B^A$. What we mean here is the set of possible functions we can create by mapping A to B in different ways. Example: Let $A = \{a, b\}$ and $B = \{0, 1\}$. According to the properties of the set of functions (see *Properties*), we have $\#(B)^{\#(A)}$ number of functions, i.e. 4. So $\langle A \longrightarrow B\rangle$ contains the following 4 sets: $f_1 = \{(a, 0), (b, 0)\}$, $f_2 = \{(a, 0), (b, 1)\}$, $f_3 = \{(a, 1), (b, 0)\}$ och $f_4 = \{(a, 1), (b, 1)\}$. Properties: The notation B^A does make sense, as the number of functions from A to B is $\#(B)^{\#(A)}$.

Mapping ($x \longmapsto$ (*something with* x)) • The mapping defines how the function works, how it transforms the input values to an output. In other words, it states how the domain maps to the range. It is best understood with an example. Example:

$$sqrt : \mathbb{R} \longrightarrow \mathbb{R} \qquad x \longmapsto x^2$$

The right hand side notation shows how the function takes an x as input and outputs $x * x$.

Multiple argument functions ($A \times A \longrightarrow B$ and $(x, y) \longmapsto (x \textit{ something } y)$) • Functions of multiple arguments are functions of cartesian products. A function with n arguments looks like $A_1 \times A_2 \times A_3 \times ... \times A_n \longrightarrow B$. Example: A function *add* that adds two numbers might be defined like this:

$$add : A \times A \longrightarrow B \qquad (x, y) \longmapsto x + y$$

Restriction ($f_x : X \longrightarrow B$ or $f|x$) • If $f : A \longrightarrow B$ is a function and $X \subseteq A$, then fs restriction to the set X is $f_x : X \longrightarrow B$ but the mapping stays the same: $a \longmapsto f(a)$. In other words, the restriction is the same function with the same rules, but with a smaller domain X. Example: If $A = \{a, b, c\}$ and $X \subseteq A$, $X = \{a, b\}$, then the restriction $f_x : X \longrightarrow B$ is best illustrated with the image below:

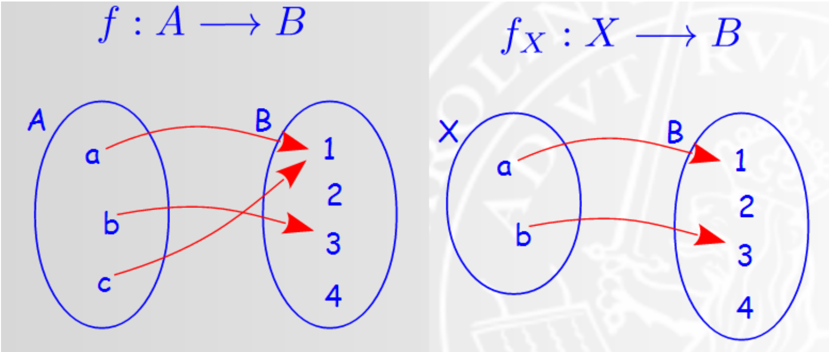


Image ($f(a)$ or $f(\{a, b\})$) • If $f : A \longrightarrow B$ and $X \subseteq A$ then we define the image of X under f as: $f(X) = \{f(x) : x \in X\}$. X can be a large subset or just a set containing 1 element. Example: Let $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 9, 16\}$, $sqrt = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$. We can now define the image of some sets:

- $f(2) = 4$
- $f(\{2, 4\}) = \{4, 16\}$
- $f(A) = \{1, 4, 9, 16\}$

Endofunction ($f : A \longrightarrow A$) • An endofunction is one where domain and codomain are the same set. Example: $f : \mathbb{N} \longrightarrow \mathbb{N}$.

Closure ($f[X]$) • Let $f : A \longrightarrow A$ be an endofunction over A and $X \subseteq A$. The closure of X under f, written $f[X]$, is defined as the smallest $Y \subseteq A$ such that $X \subseteq Y$, i.e. Y contains X, and $f(Y) \subseteq Y$, i.e. the image of Y under f is contained in Y. This might be difficult to grip, so it is best to explain how to construct Y and then use it on an example.

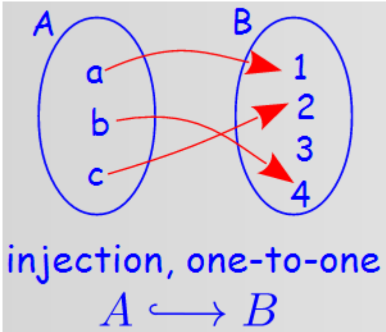
Construction of closure

1. Find $Y_0 = X$
2. Union Ys and image of Y under f together for $n \in \mathbb{N}$ like so: $Y_{n+1} = Y_n \cup f(Y_n)$
3. The closure of X under f is now the union of all Ys: $f[X] = U\{Y_i\}_{i \in \mathbb{N}}$

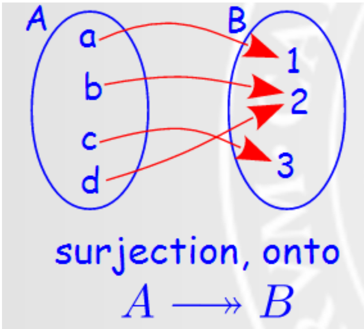
Example: Let $A = \{1, 2, 3, 4\}$ and $f = \{(1, 1), (2, 3), (3, 1), (4, 2)\}$. To find $f[\{3, 4\}]$ we first find $Y_0 = \{3, 4\}$. $Y_1 = Y_0 \cup f(Y_0) = \{3, 4\} \cup \{1, 2\} = \{1, 2, 3, 4\}$. $Y_2 = Y_1 \cup f(Y_1) = \{1, 2, 3, 4\} \cup \{1, 2, 3\} = \{1, 2, 3, 4\} = Y_1$. If we are observant, we can see that $Y_s, s \geq 2$ will be exactly the same as Y_1, so we do not have to continue. $f[\{3, 4\}] = \cup\{Y_i\}_{i \in \mathbb{N}} = Y_0 \cup Y_1 \cup Y_2 ... = \{3, 4\} \cup \{1, 2, 3, 4\} \cup \{1, 2, 3, 4\} = Y_1$. So, finally, $f[\{3, 4\}] = \{1, 2, 3, 4\}$.

Composition ($g \circ f$) • Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ be two functions. Their composition $g \circ f : A \longrightarrow C$ is then: $g(f(a))$. Example: Say we have three sets: $A = \{a, b\}$, $B = \{x, y, z\}$ and $C = \{1, 2\}$. Two functions are defined as follow: $f : A \longrightarrow B = \{(a, x), (a, y)\}$ and $g : B \longrightarrow C = \{(x, 1), (y, 1), (z, 2)\}$. The composition is then $g \circ f : A \longrightarrow C = \{(a, 1), (b, 1)\}$. For example, to find $(g \circ f)(b) = g(f(b))$, we can first find $f(b) = y$ and then $g(y) = 1$. And so $(g \circ f)(b) = 1$. Properties: Composition is associative, so $f \circ (g \circ q) = (f \circ g) \circ q$ and therefor the parentheses are often omitted.

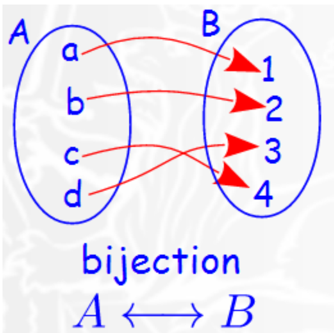
Injection ($f : A \hookrightarrow B$) • A function $f : A \hookrightarrow B$ is injective (and thus called an injection) iff $a \neq b \implies f(a) \neq f(b)$, i.e. if a and b are different, they can not map to the same element in B. In other words, no element in B is mapped to more than once, but does not have to be mapped to at all. This is illustrated well by the image below. Properties: If there is an injection $f : A \hookrightarrow B$ then $\#(A) \leq \#(B)$



Surjection ($f : A \twoheadrightarrow B$) • A function $f : A \twoheadrightarrow B$ is surjective (and thus called a surjection) iff $f(A) = B$, i.e. every element in B is mapped to **at least** once. See image below for illustration:



Bijection ($f : A \longleftrightarrow A$) • A function $f : A \longleftrightarrow B$ is bijective (and thus called a bijection) iff it is both *injective* and *surjective*, i.e. every element in B is mapped to exactly once. See image below for illustration. Properties: If there is a bijection $f : A \longleftrightarrow B$ then $\#(A) = \#(B)$.



Inverse or converse (f^{-1}) • Let $f : A \longrightarrow B$. The inverse of f is $f^{-1} \subseteq B \times A$ and is defined as: $f^{-1} = \{(f(a), a) : a \in A\}$. This can be envisioned as simply flipping all ordered pairs in the set f. Note that the domain and codomain change place. Example: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4\}$ and $f : A \longrightarrow B = \{(1, 1), (2, 3), (3, 1), (4, 2), (5, 3)\}$. The inverse $f^{-1} : B \longrightarrow A = \{(1, 1), (3, 2), (3, 1), (2, 4), (3, 5)\}$. This inverse if not a function for two reasons. First, $f^{-1}(x)$ does not exist for all $x \in B$ (which is $f^{-1}s$ domain), as $f^{-1}(4)$ is missing. Second, the value 3 is mapped to two different values (2 and 1), which violates one of the function requirements. Properties: In general, the inverse of a function is not a function. However, only in the case that a function is bijective, $f : A \longleftrightarrow B$, then the inverse if also a function: $f^{-1} : B \longleftrightarrow A$.

Equinumerosity ($A \sim B$) • Let A and B be sets. If $\#(A) = \#(B)$ then they are called equinumerous, which is written $A \sim B$.

Pigeonhole principle • The pigeonhole principle states that if you have n pigeons and m pigeonholes, and $n > m$, and try to distribute all of the pigeons into the pigeonholes, then there is always at least 1 hole containing more than 1 pigeon, which might seem obvious. More exact, if you have n items and m containers, $n > m$, then there is at least one container with $\lceil \frac{n}{m} \rceil$ items. What this means for functions is that if $\#(A) > \#(B)$ then every function $f : A \longrightarrow B$ is **not** injective. This is because the functions will map at least two distinct $a_1, a_2 \in A$ to the same $b \in B$., Example: Say we have 8 hats but only 3 boxes to put them in. This means that, however you distribute them, at least one box will contain at least $\lceil \frac{8}{3} \rceil = \lceil 2.67 \rceil = 3$ hats.

Sequences $(a_1, a_2, a_3 ...)$ • A sequence can consist of numbers, instructions or other mathematical objects. An infinite sequence is best described by a function $f : \mathbb{N}^+ \longrightarrow A$ for some appropriately chosen set A, where $f(i) = a_i$ and $a_i \in A$, for each $i \in \mathbb{N}^+$. The ith term in the sequence is then the value of $f(i)$. For finite sequences of length n, we can define it in the same way, but let $f(n + j) = f(n)$ for all $j \in \mathbb{N}^+$. This means that the function's value is constant for all $f(n)$ upwards. More intuitively, we can define it as the function $f : \{i \in \mathbb{N}^+ : i \leq n\} \longrightarrow A$, so that the domain has a finite (n) number of items. Example: Consider the set $A = \{1, 2, 3\}$ and the finite sequence of length 10, described by the function $s : \{i \in \mathbb{N}^+ : i \leq 10\} \longrightarrow A$. Let the sequence be $s = 1_1, 2_2, 3_3, 2_4, 1_5, 2_6, 3_7, 2_8, 1_9, 2_{10}$ where the subscripts represent the index i of the elements. We can find $f(2) = 2$, $f(4) = 2$, while $f(12)$ is not defined.

Identity function (i_A or ι_A) • The identity function on a set A is the function $f : A \longrightarrow A$ such that $a \mapsto a$ or alternatively, $f(a) = a$ for all $a \in A$. Properties: If $p : A \longrightarrow B$ then $p \circ i_A = f = i_B \circ p$.

Constant function (C_b) • The constant function is a function $C_b : A \longrightarrow B$ where every value in A maps to the same $b \in B$. The function therefor has a constant value b. This can also be phrased: C_b is a constant function iff there is some $b \in B$ such that $b = f(A)$ for all $a \in A$.

Projection functions () • Let $f : A \times B \longrightarrow C$ be a function of two arguments and $a \in A$. We define the *right projection of f from a* as the one-argument function $f_a : B \longrightarrow C$, defined as $b \mapsto (a, b)$ for all $b \in B$, where a is constant. In other words, we freeze a at some value, and examine what happens when b vary. Reversely, the *left projection of f from b* is the one-argument function $f_b : A \longrightarrow C$ defined as $a \mapsto (a, b)$ for all $a \in A$, where b is constant. Example: Let

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \qquad (x, y) \longmapsto x^y$$

The *right projection of f*, when $x = 3$ is then: $f_{x=3} = \{(0, 1), (1, 3), (2, 9), (3, 27), ...\}$ and so on. The *left projection of f*, $y = 2$ is then: $f_{y=2} = \{(0, 0), (1, 1), (2, 4), (3, 9), ...\}$.

Characteristic functions () • Let U be a local universe, and $A \subseteq U$. As characteristic function f_A is defined as:

$$f : U \longrightarrow \{0, 1\} \qquad a \longmapsto \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{otherwise} \end{cases}$$

This function tells you whether an element is in A or not by returning *true* (1) or *false* (0). Example: Let U be the universe containing all colors and $A = \{red, green, blue\}$ and $f_A : U \longrightarrow \{0, 1\}$. Now we can see if $magenta \in A$: $f_A(magenta) = 0 = false$. Nope ☹.

Infinity

Cantor-Schröder-Bernstein theorem • For any two sets A and B if there are two injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, then there also exists the bijection $h : A \longleftrightarrow B$. Remember that for the injections f and g, the following is true: $\#(A) \leq \#(B)$ and $\#(B) \leq \#(A)$ respectively. This implies (for the bijection h) that $\#(A) = \#(B)$. Note: For a proof of this (which is way too long to include here) see page 232 in *Book of Proof*

Infinite sets • We will define an infinite set in the following way: The set A is *infinite* if it is equinumerous (has the same cardinality as) to a proper subset of itself. If we have a proper subset $S \subset A$, we know that $S \neq A$. But if $\#(S) = \#(A)$, also written as $S \sim A$, then it must be infinite. Example: A common way to show that the set $\mathbb{N}$ is infinite is to first define a proper subset $A \subset \mathbb{N}$ of $\mathbb{N}$. For example $A = \{2n : n \in \mathbb{N}\}$ which is all the even numbers in $\mathbb{N}$. Next step is to construct a bijection $f : \mathbb{N} \longrightarrow A$, with the mapping $f : n \mapsto 2n$, so that every number in $\mathbb{N}$ maps to its double in A. Recall that if there is a bijection between $\mathbb{N}$ and A, then $\#(\mathbb{N}) = \#(A)$.

Equinumerous infinite sets • The property of equinumerosity can be defined as follows: Two sets A and B are equinumerous if there exists a bijection $f : A \longleftrightarrow B$. That is, we can match up all elements in A with B in a 1-to-1 manner. This can be used to show that two infinite sets have the same size. Example: If we want to show that $\mathbb{N}$ and $\mathbb{Z}$ are of equal size (despite what our instinct tells us) we create the bijection $f : \mathbb{Z} \longleftrightarrow \mathbb{N}$. Formally, the bijection is defined as follows:

$$f : \mathbb{Z} \longrightarrow \mathbb{N} \qquad f : x \longmapsto \begin{cases} 2x - 1, & \text{if } x > 0 \\ -2x, & \text{otherwise} \end{cases}$$

This will match up all numbers in $\mathbb{Z}$ with all numbers in $\mathbb{N}$ evenly, see table below:

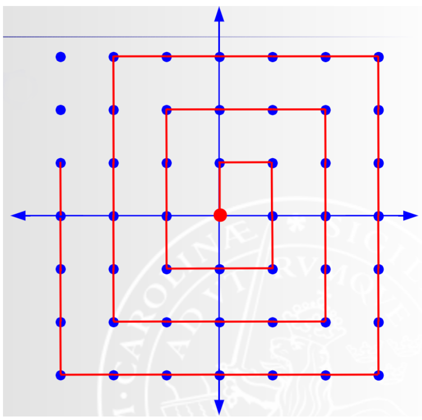
$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	11	12	13	...
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6	...

As we were able to find a bijection from $\mathbb{N}$ to $\mathbb{Z}$, we know that $\#(\mathbb{N}) = \#(\mathbb{Z})$ even though they are both infinite. Note that we could have defined the bijection like $f : \mathbb{N} \longleftrightarrow \mathbb{Z}$, but the mapping would be done in a slightly different way.

Denumerable (countable) sets • An arbitrary, infinite set A is said to be countable (or countably infinite) if it is equinumerous to $\mathbb{N}$. This is equal to saying that if the bijection $f : \mathbb{N} \longleftrightarrow A$ exists, then A is countably infinite. As shown previously $\#(\mathbb{N}) = \#(\mathbb{Z})$, and so $\mathbb{Z}$ is also countable. The cardinality of $\mathbb{N}$, and therefore all countable sets, is called $\aleph_0$, pronounced *aleph naught*. It is the smallest infinite number. As we will show later, there is no bijection $f : \mathbb{N} \longleftrightarrow \mathbb{R}$, and so $\mathbb{R}$ is not countable.

List theorem • A theorem (which I call *list theorem*) states that: a set A is countable iff its elements can be arranged in an infinite list $a_1, a_2, a_3....$ Example: We can list the elements in both $\mathbb{N}$ and $\mathbb{Z}$ as follows: $0, 1, 2, 3, 4, 5...$ and $0, 1, -1, 2, -2, 3, 3...$ respectively. Therefore they are countable. We will show later that it is also possible to arrange the elements of $\mathbb{Q}$ in a list.

Products of $\mathbb{Z}$ • So hopefully you have accepted that $\#(\mathbb{N}) = \#(\mathbb{Z})$. However, the cardinality of $\mathbb{Z}^2$ surely must be more than $\aleph_0$? Actually, it is not. Using the *list theorem*, it is quite easy to show that the elements of $\mathbb{Z}^2$ can be arranged as an infinite list. Consider the image below.



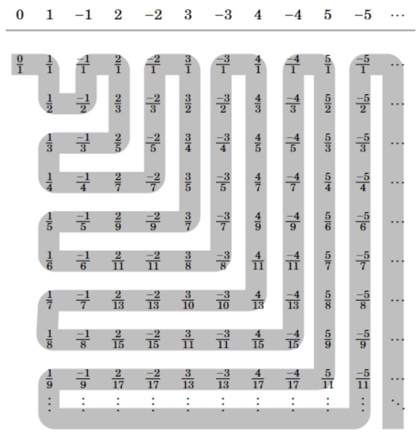
$\mathbb{Z}^2$ representation

Unfortunately the actual products are not shown in the graph, but you can imagine it as a coordinate system, where the product of $x * y$ is at position (x, y). Origo represents 0, the product $-3 * -3 = 9$ is found at position $(-3, -3)$ in the lower left corner etc. To prove that this is countable, we draw a spiral that encompasses all the numbers in the product like the image shows. Enumerating the elements from the center and out, the infinite list will be: $0, 0, 1, 0, -1, 0, 1, 0, -1, -2, 0, 2, 4, 2, 0, -2, -4, -2, 0....$ This proves that $\mathbb{Z}^2$ is in fact countable and that $\#(\mathbb{Z}^2) = \aleph_0$.

Counting $\mathbb{Q}$ • We can show that $\mathbb{Q}$ is countably infinite by creating a list of all of its elements. If we order the elements in $\mathbb{Q}$ like in the table below (left image), where the numerator is shown in the top row and the denominator increases as we go down the list, it is possible to draw a snakelike trail that will pass through every single element.

0	1	−1	2	−2	3	−3	4	−4	5	−5	⋯
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	⋯
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	⋯
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	⋯
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	⋯
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	⋯
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	⋯
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$\Rightarrow$



The list of all rational numbers will then be: $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{1}, \frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{-1}{3}, \frac{1}{3}...$, and so we have proven that $\mathbb{Q}$ is countable.

Beyond $\aleph_0$ • Cantor showed that there are infinities beyond $\aleph_0$ and that $\#(\mathbb{R}) > \#(\mathbb{N})$. To show this, one has to show that the bijection $f : \mathbb{N} \longleftrightarrow \mathbb{R}$ does not exist. To do this, we start enumerating the natural numbers $n \in \mathbb{N}$ vertically in the left hand column, and their arbitrarily mapped value $f(n) \in \mathbb{R}$ in the right. The first few entries is shown in the image below:

n	$f(n)$
1	0 . 4 0 0 0 0 0 0 0 0 0 0 0 0 0...
2	8 . 5 0 0 6 0 7 0 8 6 6 6 9 0 0...
3	7 . 5 0 5 0 0 9 4 0 0 4 4 1 0 1...
4	5 . 5 0 7 0 4 0 0 8 0 4 8 0 5 0...
5	6 . 9 0 0 2 6 0 0 0 0 0 0 5 0 6...
6	6 . 8 2 8 0 9 5 8 2 0 5 0 0 2 0...
7	6 . 5 0 5 0 5 5 5 0 6 5 5 8 0 8...
8	8 . 7 2 0 8 0 6 4 0 0 0 0 4 4 8...
9	0 . 5 5 0 0 0 0 8 8 8 8 0 0 7 7...
10	0 . 5 0 0 2 0 7 2 2 0 7 8 0 5 1...
11	2 . 9 0 0 0 0 8 8 0 0 0 0 9 0 0...
12	6 . 5 0 2 8 0 0 0 8 0 0 9 6 7 1...
13	8 . 8 9 0 0 8 0 2 4 0 0 8 0 5 0...
14	8 . 5 0 0 0 8 7 4 2 0 8 0 2 2 6...
⋮	⋮

The shaded band covers the 1st decimal of $f(1)$, the 2nd decimal of $f(2)$ and so on. For each $n \in \mathbb{N}$, the band covers the nth decimal place of $f(n)$. With this diagonal, we can create a unique number, that is unequal to any number $f(n)$ for any $n \in \mathbb{N}$. All we do is change the nth number of the diagonal so that it does not match the nth decimal in $f(n)$. We call this number $b \in \mathbb{R}$ and, for example, change its entries according to this rule: if the entry equals 0, change it to 1, else set it to 0. By these rules, our b now equals $b = 01010001001000....$ We now let $d \in \mathbb{R}$ be the number that is less than 0 and has the decimals in b, so $d = 0.01010001001000....$ This number d is unlike any other number $f(n)$ for all $n \in \mathbb{N}$ because it differs from all the numbers in the right hand column by at least one digit in its decimals. If we compare d to the 10000th entry, that is $f(10000)$, we know that they are not equal, because we changed the 10000th decimal in d to be specifically unlike the 10000th decimal in $f(10000)$.

What this means is that the function f is not surjective, because we found a number in $\mathbb{R}$ that was not mapped to, and can therefore not be bijective either. We know that we can map every $n \in \mathbb{N}$ to the same integer in $\mathbb{R}$ because $\mathbb{R}$ contains all the integers. This means that $\#(\mathbb{N}) \leq \#(\mathbb{R})$. But we have just showed that there is no bijection, so $\#(\mathbb{N}) \neq \#(\mathbb{R})$, therefore $\#(\mathbb{N}) < \#(\mathbb{R})$. Accordingly, there is a greater infinity than $\aleph_0$, namely $\aleph_1 = \#(\mathbb{R})$. Properties: It can be shown that $\aleph_1 = 2^{\aleph_0}$.

Finite sequences/strings (A^*) • Let A be a finite set of n symbols, $A = \{a_1, a_2, ..., a_n\}$. A^* is defined as the set of all finite sequences of the symbols in A. Example: Let $A = \{0, 1\}$. The finite sequence of A is then: $A^* = \{0, 00, 000..., 1, 11, 111..., 01, 001, 10, ...\}$. Properties: The empty sequence, denoted ϵ, is an element of all finite sequences: $\epsilon \in A^*$ for any A.

Infinite sequences/strings ($A^{\mathbb{N}}$) • Let A be a finite set of n symbols, $A = \{a_1, a_2, ..., a_n\}$. An infinite sequence in A is a function $f : \mathbb{N} \longrightarrow A$ that maps every element in the sequence to a number in $\mathbb{N}$. The set of all infinite sequences that can be constructed with the symbols in A is denoted $A^{\mathbb{N}}$.

Power set cardinality • As we showed before, $\aleph_0 < \aleph_1 \Leftrightarrow \aleph_0 < 2^{\aleph_0}$. Remember the cardinality of power sets: $\#\mathcal{P}(A) = 2^{\#(A)}$. Combining this we get the theorem that states $\#(A) < \#(\mathcal{P}(A))$ for any set A. This will not be proven, see page 229 of *Book of Proof*.