European Option Pricing Based on Heston-Dupire

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1. Theory

1.1 Heston-Dupire Model

The Heston-Dupire immediate local volatility model is a model that combines the local volatility obtained from the Dupire formula in its unparameterized form with the Heston stochastic volatility model.

$$dS_t = rS_t dt + L\left(t, S_t\right)\sqrt{V_t} S_t dW_{1t}$$

$$dV_t = a\left(b - V_t\right) d_t + \sigma_v \sqrt{V_t} dW_{2t}$$

1.2 Leverage function

1.2.1 Derivation of leverage function

Let the price of the European call option be

$$C(t, K) = \frac{B_{t_0}}{B_t} E\left[\left(S_t - K\right)^+\right]$$

Differentiating the the above equation, and using Fubini's theorem, we get,

$$dC(t, K) = -\frac{r}{M_t} E\left[\left(S_t - K\right)^+\right] dt + \frac{1}{M_t} E\left[d\left(S_t - K\right)^+\right]$$

This function is not differentiable at x = c and cannot be solved directly by Itoˆ 's lemma. However, this problem can be solved by the following Tanaka-Meyer formula.

$$g\left(X_t\right) = g\left(X_{t_0}\right) + \int_{t_0}^{t} 1_{X_u > b} d\tilde{B}_u + \int_{t_0}^{t} 1_{X_u > b} dV_u + \frac{1}{2}\int_{t_0}^{t} g''\left(X_u\right)\left(d\tilde{B}_u\right)^2$$

To further simplify the calculation, the well-known conclusion from Feng (2010) can be used. If S_t, V_t obey the stochastic local volatility model in 1.1, then the following equation holds for the price of a European call option.

$$-\frac{\partial C(t, K)}{\partial K} = \frac{1}{B_t} E\left[1_{S_t > K}\right], \frac{\partial C^2(t, K)}{\partial K^2} = \frac{\psi_S}{B_t}$$

Then we get the leverage function

$$L^2(t, K) = \frac{\frac{\partial C(t,K)}{\partial t} + rK\frac{\partial C(t,K)}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C(t,K)}{\partial K^2} E\left[V_t \mid S_t = K\right]} = \frac{\sigma_{LV}^2(t, K)}{E\left[V_t \mid S_t = K\right]}$$

1.2.2 Calculation of leverage function

Numerator

$$
\begin{aligned}
E\left[V_{t_i} \mid S_{t_i}=s_{i,j}\right] & \approx \frac{E\left[V_{t_i} \mathbb{1}_{S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]}\right]}{\mathbb{Q}\left[S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]\right]} \\
& \approx \frac{\frac{1}{M} \sum_{j=1}^M v_{i,j} \mathbb{1}_{s_{i,j} \in\left(b_{i,k}, b_{i,k+1}\right]}}{\mathbb{Q}\left[S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]\right]} \\
& \approx \frac{l}{M} \sum_{j \in \oint_{i,k}} v_{i,j}
\end{aligned}
$$

Denominator

$$
\sigma_{LV}^2(T, K)=\frac{\sigma_I^2+2 T \sigma_I\left(\frac{\partial \sigma_I}{\partial T}+r K \frac{\partial \sigma_I}{\partial K}\right)}{\left(1+d_1 K \sqrt{T} \frac{\partial \sigma_l}{\partial K}\right)^2+K^2 \sigma_I T\left(\frac{\partial^2 \sigma_I}{\partial K^2}-d_1 \sqrt{T}\left(\frac{\partial \sigma_l}{\partial K}\right)^2\right)}
$$

To get σ_I We first use the SVI (Stochastic Volatility Inspired) function to fit in the K-direction , followed by linear interpolation in the T-direction. And then we obtain the implied volatility surface by combining the two directions.

$$
\sigma_I^{SVI}=\sqrt{\frac{\alpha_n+\beta_n\left[\rho_n\left(x-m_n\right)+\sqrt{\left(x-m_n\right)^2+\sigma_n}\right]}{T_n}}
$$

$$
\min J_n(\boldsymbol{C})=\frac{1}{2} \sum_{i=1}^M w_i\left(\frac{\sigma_I^{SVI}\left(T_n, K_i ; \boldsymbol{C}\right)-\sigma_I^{\text{Market }}\left(T_n, K_i\right)}{\sigma_I^{\text{Market }}\left(T_n, K_i\right)}\right)^2
$$

$$
\sigma_I^{SVI}(T, x)=\frac{T_{n+1}-T}{T_{n+1}-T_n} \sigma_I^{SVI}\left(T_n, x\right)+\frac{T-T_n}{T_{n+1}-T_n} \sigma_I^{SVI}\left(T_{n+1}, x\right)
$$

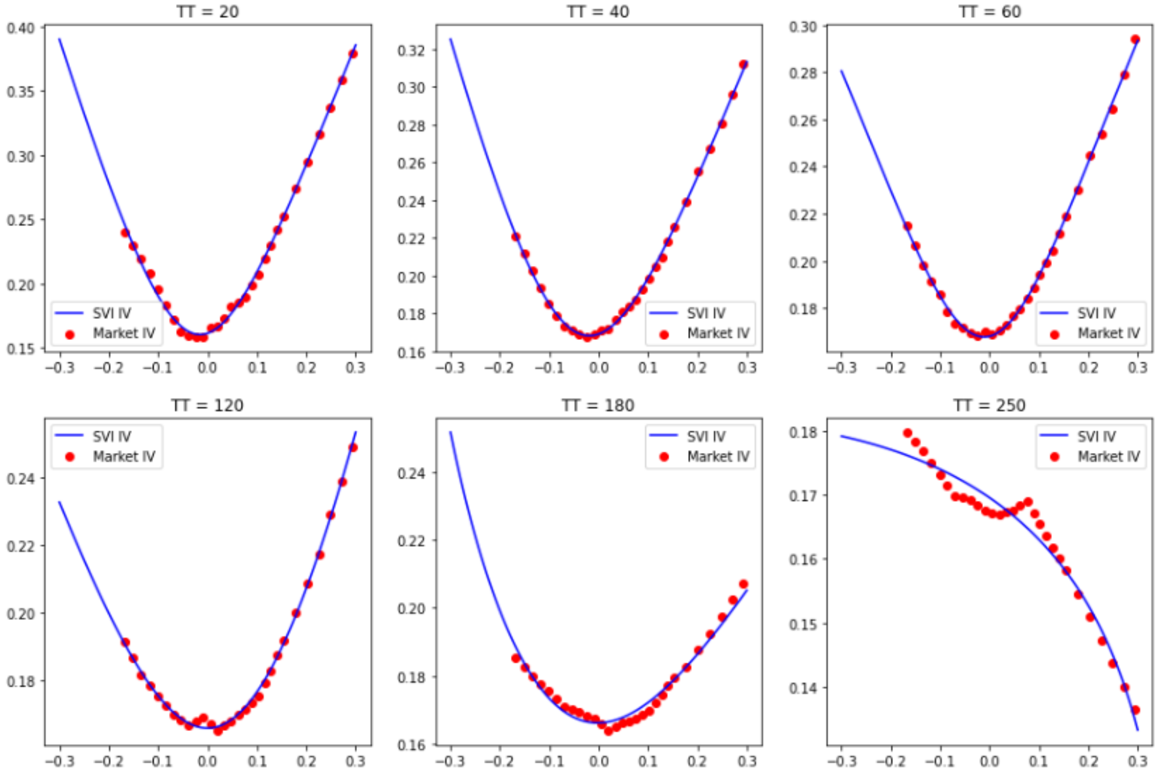
To complete the calculation of the rest of leverage function, we take the numerical derivatives by difference.

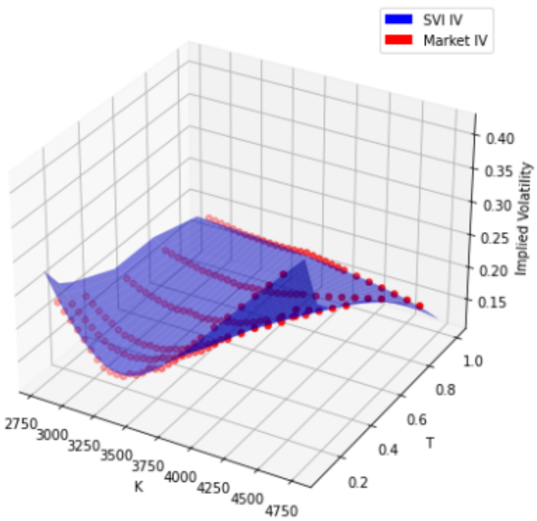
$$
\begin{aligned}
\sigma_I & =\sigma_I^{SVI}(T, x)=\lambda_n \sigma_I^{SVI}\left(T_n, x\right)+\lambda_{n+1} \sigma_I^{SVI}\left(T_{n+1}, x\right) \\
\frac{\partial \sigma_I}{\partial T} & =\frac{1}{T_{n+1}-T_n}\left[\sigma_I^{SVI}\left(T_{n+1}, x\right)-\sigma_I^{SVI}\left(T_n, x\right)\right] \\
\frac{\partial \sigma_I}{\partial x} & =\tau_n f_n^{(1)}+\tau_{n+1} f_{n+1}^{(1)} \\
\frac{\partial^2 \sigma_I}{\partial x^2} & =\tau_n f_n^{(2)}+\tau_{n+1} f_{n+1}^{(2)}
\end{aligned}
$$

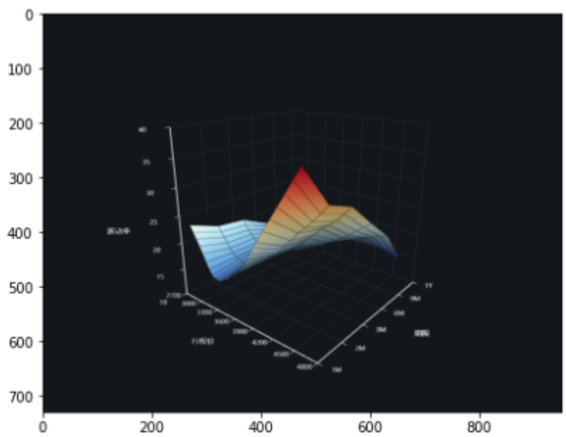
2. Practice

2.1 Fitting volatility based on SVI

We select the CSI 300 put option data on December 29, 2023, and fit the implied volatility surface using the SVI method.







2.2 Model result

- strike = np.arange(2900, 3500, 50)
- sigma, vov, mr, rho, texp, spot = 0.3, 1, 0.5, -0.9, 20, 3431.1099

2.2.1 Heston model

- Lewis AL (2000) Option valuation under stochastic volatility: with Mathematica code. Finance Press

```
array([2614.7557097 , 2605.5777012 , 2596.47637983, 2587.45031286,
       2578.49811439, 2569.61844314, 2560.81000035, 2552.07152782,
       2543.40180606, 2534.79965257, 2526.2639202 , 2517.79349561])
```

- Conditional MC for Heston model based on QE discretization scheme by Andersen (2008)

```
array([2628.65334821,  2619.48966579,  2610.40231646,  2601.38987433,
       2592.45096014,  2583.58423907,  2574.78841859,  2566.06224659,
       2557.40450947,  2548.81403046,  2540.28966797,  2531.83031408])
```

- Milstein for Heston

```
[2588.3889053   2579.35324418 2570.39391599 2561.50654925 2552.6974709
 2543.95942172 2535.28801316 2526.68473049 2518.15106307 2509.68248496
 2501.27981974 2492.95075096]
```

2.2.2 Heston-Dupire

- Var calculation is based on QE method, and we use Euler method to obtain ST

$$dX_t = \left(r - \frac{1}{2}L^2\left(t, e^{X_t}\right)V_t\right)dt + L\left(t, e^{X_t}\right), \sqrt{V_t}dW_{1t}$$

```
Heston-Dupire:  [574.08437613 558.31437139 542.96737848 528.08087414 513.67470773
 499.80181511 486.51445111 473.83205918 461.79139787 450.4427644
 439.77928339 429.82440945]
```

- Milstein for Heston-Dupire

$$v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dtZ_v + \frac{1}{4}\sigma^2 dt(Z_v^2 - 1)$$

$$X_{t+dt} = X_t + \left(r - \frac{1}{2}v_t {L_t}^2\right)dt + \sqrt{v_t}L_t dtZ_s$$

```
[543.0012140365923,  527.5087182695403,  512.4369276030873,  497.7983599724606,  483.640501
0650162,  469.9989355729829,  456.9234524661498,  444.458064480143,  432.6333222533259,  42
1.4505778410579,  410.9189585735541,  401.04508534373315]
```