European Option Pricing Based on Heston-Dupire

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1. Theory

1.1 Heston-Dupire Model

The Heston-Dupire immediate local volatility model is a model that combines the local volatility obtained from the Dupire formula in its unparameterized form with the Heston stochastic volatility model.

$$dS_t = rS_t dt + L(t, S_t)\sqrt{V_t}S_t dW_{1t}$$

$$dV_t = a(b - V_t)d_t + \sigma_v\sqrt{V_t}dW_{2t}$$

1.2 Leverage function

1.2.1 Derivation of leverage function

Let the price of the European call option be

$$C(t, K) = \frac{B_{t_0}}{B_t}E\left[(S_t - K)^+\right]$$

Differentiating the the above equation, and using Fubini's theorem, we get,

$$dC(t, K) = -\frac{r}{M_t}E\left[(S_t - K)^+\right]dt + \frac{1}{M_t}E\left[d(S_t - K)^+\right]$$

This function is not differentiable at x = c and cannot be solved directly by Itoˆ 's lemma. However, this problem can be solved by the following Tanaka-Meyer formula.

$$g(X_t) = g(X_{t_0}) + \int_{t_0}^{t} 1_{X_u > b} d\tilde{B}_u + \int_{t_0}^{t} 1_{X_u > b} dV_u + \frac{1}{2}\int_{t_0}^{t} g''(X_u)\left(d\tilde{B}_u\right)^2$$

To further simplify the calculation, the well-known conclusion from Feng (2010) can be used. If S_t, V_t obey the stochastic local volatility model in 1.1, then the following equation holds for the price of a European call option.

$$-\frac{\partial C(t, K)}{\partial K} = \frac{1}{B_t}E\left[1_{S_t > K}\right], \frac{\partial C^2(t, K)}{\partial K^2} = \frac{\psi_S}{B_t}$$

$$L^2(t, K) = \frac{\frac{\partial C(t,K)}{\partial t} + rK\frac{\partial C(t,K)}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C(t,K)}{\partial K^2}E\left[V_t \mid S_t = K\right]} = \frac{\sigma_{LV}^2(t, K)}{E\left[V_t \mid S_t = K\right]}$$

1.2.2 Calculation of leverage function

Numerator

$$
\begin{aligned}
E\left[V_{t_i} \mid S_{t_i}=s_{i,j}\right] & \approx \frac{E\left[V_{t_i} \mathbb{1}_{S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]}\right]}{\mathbb{Q}\left[S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]\right]} \\
& \approx \frac{\frac{1}{M} \sum_{j=1}^M v_{i,j} \mathbb{1}_{s_{i,j} \in\left(b_{i,k}, b_{i,k+1}\right]}}{\mathbb{Q}\left[S_{t_i} \in\left(b_{i,k}, b_{i,k+1}\right]\right]} \\
& \approx \frac{l}{M} \sum_{j \in \mathscr{I}_{i,k}} v_{i,j}
\end{aligned}
$$

Denominator

$$
\sigma_{LV}^2(T, K)=\frac{\sigma_I^2+2 T \sigma_I\left(\frac{\partial \sigma_I}{\partial T}+r K \frac{\partial \sigma_I}{\partial K}\right)}{\left(1+d_1 K \sqrt{T} \frac{\partial \sigma_I}{\partial K}\right)^2+K^2 \sigma_I T\left(\frac{\partial^2 \sigma_I}{\partial K^2}-d_1 \sqrt{T}\left(\frac{\partial \sigma_I}{\partial K}\right)^2\right)}
$$

To get σ_I We first use the SVI (Stochastic Volatility Inspired) function to fit in the K-direction , followed by linear interpolation in the T-direction. And then we obtain the implied volatility surface by combining the two directions.

$$
\sigma_I^{SVI}=\sqrt{\frac{\alpha_n+\beta_n\left[\rho_n\left(x-m_n\right)+\sqrt{\left(x-m_n\right)^2+\sigma_n}\right]}{T_n}}
$$

$$
\min J_n(\boldsymbol{C})=\frac{1}{2} \sum_{i=1}^M w_i\left(\frac{\sigma_I^{SVI}\left(T_n, K_i ; \boldsymbol{C}\right)-\sigma_I^{\text{Market }}\left(T_n, K_i\right)}{\sigma_I^{\text{Market }}\left(T_n, K_i\right)}\right)^2
$$

$$
\sigma_I^{SVI}(T, x)=\frac{T_{n+1}-T}{T_{n+1}-T_n} \sigma_I^{SVI}\left(T_n, x\right)+\frac{T-T_n}{T_{n+1}-T_n} \sigma_I^{SVI}\left(T_{n+1}, x\right)
$$

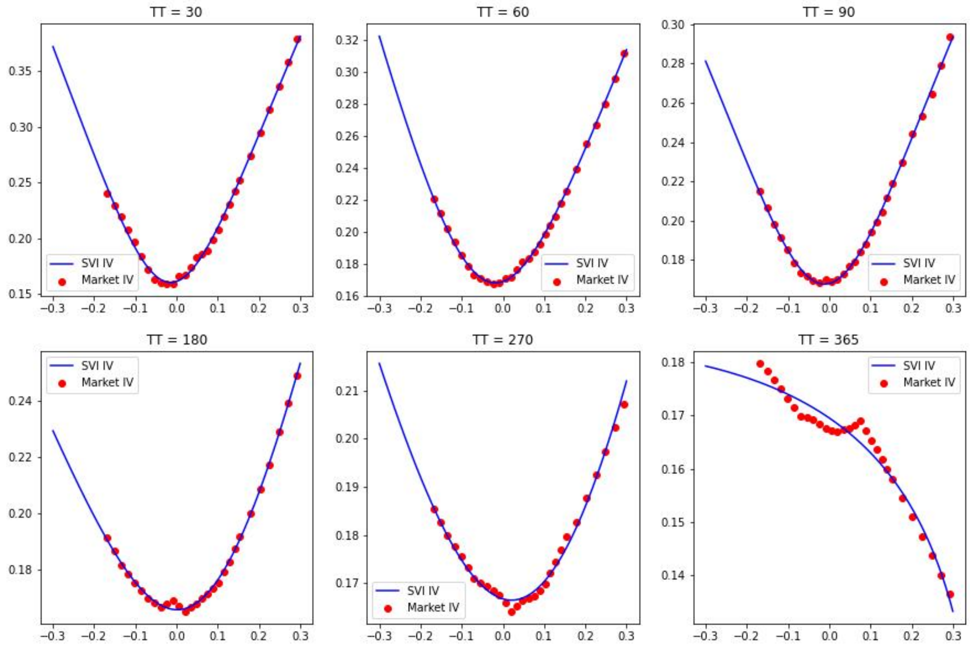
To complete the calculation of the rest of leverage function, we take the numerical derivatives by difference.

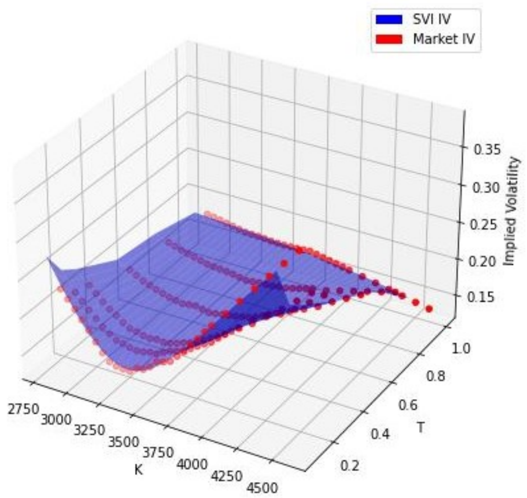
$$
\begin{aligned}
\sigma_I & =\sigma_I^{SVI}(T, x)=\lambda_n \sigma_I^{SVI}\left(T_n, x\right)+\lambda_{n+1} \sigma_I^{SVI}\left(T_{n+1}, x\right) \\
\frac{\partial \sigma_I}{\partial T} & =\frac{1}{T_{n+1}-T_n}\left[\sigma_I^{SVI}\left(T_{n+1}, x\right)-\sigma_I^{SVI}\left(T_n, x\right)\right] \\
\frac{\partial \sigma_I}{\partial x} & =\tau_n f_n^{(1)}+\tau_{n+1} f_{n+1}^{(1)} \\
\frac{\partial^2 \sigma_I}{\partial x^2} & =\tau_n f_n^{(2)}+\tau_{n+1} f_{n+1}^{(2)}
\end{aligned}
$$

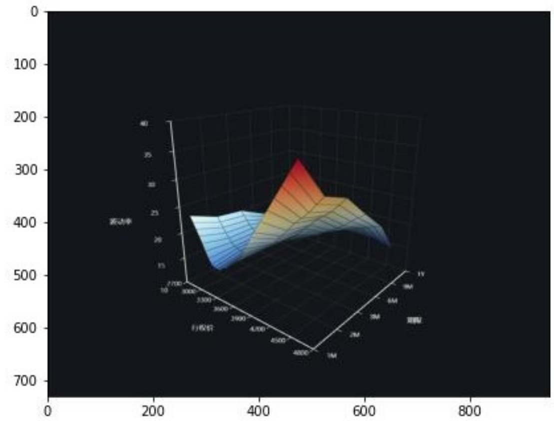
2. Practice

2.1 Fitting volatility based on SVI

We select the CSI 300 put option data on December 29, 2023, and fit the implied volatility surface using the SVI method.







2.2 Model result

- strike = np.arange(2900, 3500, 50)
- sigma, vov, mr, rho, texp, spot = 0.3, 1, 0.5, -0.9, 20, 3431.1099

2.2.1 Heston model

- Lewis AL (2000) Option valuation under stochastic volatility: with Mathematica code. Finance Press

```
array([3593.50002999, 3592.3524918 , 3591.21330625, 3590.08229183,
       3588.95927373, 3587.84408354, 3586.73655888, 3585.63654316,
       3584.54388521, 3583.45843908, 3582.38006374, 3581.30862288])
```

- Conditional MC for Heston model based on QE discretization scheme by Andersen (2008)

```
array([3332.26872614, 3331.18496626, 3330.1090163 , 3329.04070706,
       3327.97987561, 3326.92636495, 3325.88002375, 3324.84070599,
       3323.80827077, 3322.78258201, 3321.76350822, 3320.75092232])
```

2.2.2 Heston-Dupire

- Var calculation is based on QE method, and we use Euler method to obtain ST

$$dX_t = \left(r - \frac{1}{2}L^2\left(t, e^{X_t}\right)V_t\right)dt + L\left(t, e^{X_t}\right), \sqrt{V_t}dW_{1t}$$

```
Heston-Dupire:  [2440.88090474 2440.01291144 2439.14690736 2438.28335157 2437.42280354
 2436.56579268 2435.71165923 2434.86170604 2434.01561377 2433.17237949
 2432.33350623 2431.49757753]
```