The Annals of Probability
2000, Vol. 28, No. 3, 1384–1391

SUM THE ODDS TO ONE AND STOP

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The objective of this paper is to present two theorems which are directly applicable to optimal stopping problems involving independent indicator functions. The proofs are elementary. One implication of the results is a convenient solution algorithm to obtain the optimal stopping rule and the value. We will apply it to several examples of sequences of independent indicators, including sequences of random length. Another interesting implication of the results is that the well-known asymptotic value $1/e$ for the classical best-choice problem is in fact a typical lower bound in a much more general class of problems.

1. Introduction. Suppose we toss a fair die a fixed number n of times. It is trivial to announce the first 6 when it appears, but what about the last 6? This should not be hard to figure out: the last l tosses contain exactly one 6 with (binomial) probability $l6^{-1}(5/6)^{l-1}$. This is maximal for $l = 6$ (or 5). Thus, intuitively, it should be optimal to announce the first 6 (if any) in the last six tosses as being the last one. And it is easy to make this precise. Note also that the answer "6" coincides with the number of times we have to add up the success probability to reach the target 1. (Is this a coincidence?)

For games allowing different numbers of dice in each toss, or biased dice, the above approach would become harder and for a random number of dice or tosses not evident at all.

Many stopping problems are of a similar kind. One often wants to stop on the very last success. For instance, investors are typically interested in stopping on the last success in a given period, where a success is a price increase in a long position and a decrease in a short position. Similarly, venture capital investors often try to put all reserved capital in the last technological innovation in the targeted field. In secretary problems, we want to select the best candidate (which means stopping on the last record value) and so on.

In this note we address the general problem of deriving the optimal rule and reward for sequences of independent but otherwise arbitrary trials.

The problem. Let $I_1, I_2, \dots, I_n$ be indicators of independent events $A_1, A_2, \dots, A_n$ defined on some probability space $(\Omega, \mathscr{A}, P)$. We observe $I_1, I_2, \dots$ sequentially and may stop at any of these, but may not recall on preceding I_k's. If $I_k = 1$, we say that k is a "success time." Let $\mathscr{T}$ denote the class of all rules t such that $\{t = k\} \in \sigma(I_1, I_2, \dots, I_k)$, the sigma field generated by $I_1, I_2, \dots, I_k$. We want to find an optimal rule, that is, a stopping

Received November 1998.
AMS 1991 *subject classification*. 60G40.
Key words and phrases. Optimal stopping, stopping islands, monotone case, generating function, arithmetic–geometric mean, dice problems, best choice, speculation, investment, $1/e$-law.

rule $\tau_n \in \mathscr{T}$ maximizing $P(I_t = 1; I_{t+1} = I_{t+2} = \cdots I_n = 0)$ over all $t \in \mathscr{T}$ and its value.

Problems of this kind can of course be framed as problems of stopping Markov chains. This is done, for instance, in a well-known solution of the classical best-choice problem [see Dynkin and Juschkewitsch (1969) 81–103]. The attractive part of the theory of stopping Markov chains is that it links the optimality principle, excessive functions and the theory of martingales in a convincing way [see, e.g., Shiryayev (1978) and Billingsley (1995)]. However, this general framework need not be efficient for more specific problems, because it fails to exploit simpler structures of problems such as, in our problem, the independence. Showing that this case allows for an elegant unification is the scope of this paper.

2. Optimal stopping rule. The following theorem is the main result of this paper.

THEOREM 1 (Odds-theorem). *Let $I_1, I_2, \dots, I_n$ be a sequence of independent indicator functions with $p_j = E(I_j)$. Let $q_j = 1 - p_j$ and $r_j = p_j/q_j$. Then an optimal rule τ_n for stopping on the last success exists and is to stop on the first index* (*if any*) *k with $I_k = 1$ and $k \geq s$, where*

$$s = \sup\left\{1, \sup\left\{1 \leq k \leq n\colon \sum_{j=k}^{n} r_j \geq 1\right\}\right\},$$

with $\sup\{\varnothing\} := -\infty$.

The optimal reward (win probability) is given by $V(n) = \prod_{j=s}^{n} q_j \Sigma_{j=s}^{n} r_j$.

PROOF. We first assume that $p_j < 1$ for all $j = 1, 2, \dots, n$. Let $g_j(t)$ and $G_k(t)$ denote the probability generating functions of I_j and $S_k := I_{k+1} + I_{k+2} + \cdots + I_n$, respectively. Then, from independence of the I_k's,

$$(1) \qquad g_t(t) = q_j + p_j t; \qquad G_k(t) = \prod_{j=k+1}^{n} (q_j + p_j t) = \prod_{j=k+1}^{n} q_j(1 + r_j t).$$

Deriving $\log G_k(t)$ yields $G'_k(t)/G_k(t) = \Sigma_{j=k+1}^{n}(r_j/(1 + r_j t))$, so that

$$(2) \qquad P(S_k = 1) = G'_k(0) = \prod_{j=k+1}^{n} q_j \sum_{j=k+1}^{n} r_j.$$

Here we used $p_j < 1$ for all $j = 1, 2, \dots, n$ which implies $G_k(0) > 0$. Now look at the class $\mathscr{C} \subset \mathscr{T}$ of rules which stop at the *first* success (if any) *after* a fixed waiting time k. The first essential observation is if we apply a stopping rule $t \in \mathscr{C}$, then we stop on the last success iff $S_t = 1$. The optimal value in $\mathscr{C}$ is $p^* = P(S_{k^*} = 1)$, where k^* maximizes (2). Secondly, by definition, S_k is independent of $I_1, I_2, \dots I_k$. We extend now the class $\mathscr{C}$ to $\mathscr{C}' \subset \mathscr{T}$ allowing random waiting times W with $\{W = k\} \in \sigma(I_1, I_2, \dots I_k)$ after which we would stop on any success. The optimal rule within $\mathscr{C}'$ must then have the same value p^*, because $P(S_W = 1|I_1, I_2, \ldots I_k, W = k) = P(S_k = 1)$ for all k.

But now, $P(S_k = 1)$ is *always* a unimodal function of $k = 0, 1, \ldots, n-1$. To see this, suppose first that $r_2 + r_3 + \cdots + r_n \geq 1$. Then from (2) and $q_j > 0$, $j = 1, 2, \ldots, n$, after straightforward simplifications,

$$P(S_k = 1) < P(S_{k+1} = 1) \Longleftrightarrow q_{k+1} r_{k+1} < \sum_{j=k+2}^{n} r_j (1 - q_{k+1}) \Longleftrightarrow 1 < \sum_{j=k+2}^{n} r_j. \tag{3}$$

The sum on the rhs of (3) is monotone decreasing in k. As soon as inequalities in (3) change the direction for increasing k, they change for good. Thus $P(S_{k^*} = 1) \geq P(S_{k^*+1} = 1) \geq \cdots \geq P(S_{n-1} = 1)$, and so $P(S_{k^*} = 1) \geq P(S_W = 1|W \geq k^*)$. This means that τ_n has a single stopping island [or in the terminology of Chow, Robbins and Siegmund (1971), that we are in the monotone case], that is τ_n stops at the *first* success (if any) after k^*, that is, from $s = k^* + 1$ onwards.

The same is true if $r_2 + r_3 + \cdots + r_n < 1$, because then $k^* = 0$; that is, it is better to stop on the very first success than to wait for further occasions. Therefore, τ_n lies also in $\mathscr{C}$, and so its value equals

$$V(n) = p^* = \prod_{m=s}^{n} q_m \sum_{j=s}^{n} r_j. \tag{4}$$

Finally, assume that $p_j = 1$ for some $1 \leq j \leq n$. If $p_n = 1$ then it is clearly optimal to stop on n. This coincides with the answer $s = n$ as defined by the theorem, since $r_n = \infty$. If $p_n < 1$, let j^* be the last index $1 \leq j < n$ such that $p_j = 1$. Then $p_{j^*+1} < 1, \ldots, p_n < 1$, so that we are in the first case if $s \geq j^* + 1$. If not, that is, if $s \leq j^*$, then we would lose with probability 1 if we stopped on $j < j^*$. Thus we must stop on j^*. This confirms the statement $s = j^*$ of the theorem, and the proof is complete. □

2.1. *Odds-algorithm.* The following algorithm based on the Odds-theorem is convenient, in particular if we want to answer a whole sequence of questions (see, e.g., Example 1 in Section 3).

A1. List, in reversed order, the $q_j = 1 - p_j$, $r_j = p_j/q_j$ and compute recursively $Q_k := q_n q_{n-1} \cdots q_k$ and $R_k := r_n + r_{n-1} + \cdots + r_k$. Stop, when R_k hits or exceeds 1 or $k = 1$ is reached, whatever happens earlier (at index s, say).

A2. $V(n) := Q_s R_s$.

The stopped index s in A1 is the time from which onwards it is optimal to stop on the first success. The output of A2 is the value (optimal win probability). For convenience, we call R_s the sum of odds "stopped at 1" instead of "stopped at the first index s with $R_s \geq 1$."

Recall the question we asked in the Introduction: was it a coincidence that the sum of success probabilities stopped at the target value 1 gave also the right answer? Yes, it was. The coincidence is due to $R_s = 1$; that is, the stopped sum hits 1. This is the only case where we have two optimal threshold indices, and then these are adjacent. Indeed, we have the Corollary.

COROLLARY 1.

$$V(n) = R_{s-1}Q_{s-1} \iff R_s = 1. \tag{5}$$

PROOF. From the Odds-theorem we have $V(n) = Q_s R_s$. Hence $V(n) = Q_{s-1}R_{s-1}$ is equivalent to $q_{s-1}R_{s-1} = R_s$. The latter holds iff $q_{s-1}R_s + p_{s-1} = R_s$, that is, iff $R_s = 1$. $\square$

Note that the n and the sequence (p_k) can be chosen such that the difference between the thresholds becomes arbitrarily large.

In general it is thus the odds we have to sum up to the target number 1, not the probabilities. But this is almost as nice.

2.2. *Generalization.* We comment briefly on a generalization. Suppose we would like to stop on the lth last success, where $1 \le l \le n$. Two questions arise: first, is an optimal strategy for $l > 1$ of the same form as for $l = 1$, that is, of a threshold-index form? Second, it is now optimal to add up the odds to l to find this index?

The answer to the first question is "yes." The optimal strategy is indeed of the same form; that is, it is optimal to stop on the first success from some stopped index $s_l = k^*(l) + 1$ onwards. But just adding up the odds gives, at best, only an approximate answer.

To see this, it suffices to look at the case $l = 2$. From the generating function (1) we can easily write down $P(S_k = 2) = G''_k(0)/2$. Similarly to (3), it is straightforward to verify that the differences $G''_k(0) - G''_{k+1}(0)$ change at most once the sign, as k increases, so that $P(S_k = 2)$ is unimodal in k. Thus the proof of the Odds-theorem holds for $l = 2$ as well, and the optimal rule is of the same form. [This holds similarly for all l, because we can show that $P(S_k = l)$ is unimodal in k for all l.] However, the optimal $s := s_2$ is more complicated, namely,

$$s_2 = \sup\Big\{1, \sup\Big\{1 \le k \le n-1\colon R_k - R_k^{(2)}/(R_k) \ge 2\Big\}, \tag{6}$$

where $R^{(2)}$ denotes the corresponding sum of the squares of the odds. The amount of necessary computations increases with l.

Asymptotics are often much easier. If, for instance $R_k^{(2)}$ in (6) becomes small compared to R_k then "stop at 2" gives an optimal or almost optimal answer.

The case $l = 1$ is very fortunate, because then the algorithm for a precise answer is very simple. It is also the most important or most natural case (see below).

3. Applications.

This section gives a selection of problems where the Odds-algorithm can be applied directly.

1. (Betting) A casino offers a game consisting of a sequence of dice tosses of length n. In the first toss, n dice are thrown, in the second, $n-1$ dice, ..., in the last one, one die. We receive a reward β_n if 6 appears in the sequence of tosses *and* if we announce correctly the last time a 6 appears. What is, for each n, a fair entrée E_n?

We apply A1 and A2: The q_j and r_j in reversed order read 5/6, $(5/6)^2, (5/6)^3, \ldots$ and $1/5, 11/25, 91/125 \cdots$. Thus $R_n = 1/5$, $R_{n-1} = 16/25$, $R_{n-2} = 171/125 > 1$, and we stop. This gives $E_1 = \beta_1/6$, $E_2 = 10\beta_2/27$, $E_3 = \beta_3(5/6)^6(171/125) = 0.458 \cdots \times \beta_3$, and it is optimal to stop on the first occasion within the last three tosses. For $n \geq 3$, the optimal reward equals $0.458 \cdots \times \beta_n$, and we may take advantage if the casino offers different β_n for $n \geq 3$.

2. (Secretary problem) In the classical case we want to maximize the probability of stopping on rank 1 in a random permution of n candidates (all $n!$ being equally likely), that is, on the last record. Using the Odds-theorem, let $I_k = 1$ if the kth observation is a record (= 0 otherwise). It is well known that the I_k are independent with $E(I_k) = 1/k$. Hence $r_k = 1/(k-1)$ and so the Odds-algorithm yields $R_s = 1/(n-1) + 1/(n-2) + \cdots + 1/(s-1)$, stopped at 1. Also, clearly $s/n \to 1/e$ and $V(n) = ((s-1)/n)R_s \to 1/e$ as $n \to \infty$. This re-proves both parts of the well-known result.

It is interesting that this solution has been overlooked, as far as the author is aware, until 1984, although the independence of relative ranks figures in equivalent forms everywhere. Bruss (1984a) gave this solution then, but failed as well to see the more general picture of the Odds-theorem.

3. (Group interviews) Here is an example of the more general picture. Hsiau and Yang (2000) studied a new version of the secretary problem. An interviewer sees sequentially n groups of $l_1, l_2, \cdots l_n$ candidates and can recall on all candidates within the present group before having to decide whether to appoint the best in this group, or to go on. The objective is to appoint the best candidate of all. The authors solved this problem by dynamic programming. They found that the optimal rule is to stop at the first group after some index $r^* - 1$ in which the best is the best of all candidates seen so far. They obtained $r^* = \min\{k\colon l_k/b_{k-1} + \cdots + l_n/b_{n-1} \leq 1\}$, where $b_k := l_1 + l_2 + \cdots + l_k$.

We get this directly from the Odds-theorem. Let $H_j = \min\{1, I_{b_{j-1}+1} + \cdots + I_{b_j}\}$. The interviewer clearly gets the best iff he or she stops on the last group with $H = 1$. As functions of independent indicators, the H_j's are also independent. Since $E(H_j) = l_j/b_j$, $r_j = l_j/b_{j-1}$ and $s = r^*$, this is the result.

4. Random number of indicators. The Odds-theorem can cope with a random number of indicators, provided the independence of indicators is maintained.

Let $(N(t))_{t\geq 0}$ be a stochastic counting process on $\mathbb{R}^+$ with independent increments and occurrence times $T_1, T_2, \ldots$ At each occurrence time T_k we observe I_k, indicating a success or failure of a certain trial, which may depend on time, that is, where $E(I_k)$ may depend on T_k. The Odds-algorithm to find the optimal rule and value translates now into a similar integral version. Here is an example which may have a certain appeal for investment problems.

4.1. *Trials on Poisson arrivals.* Let $(N(t))$ be an inhomogeneous Poisson process with intensity rate $\lambda(t)$. Let $h(t)$ be the success parameter function for an experiment occurring at time t, where a success at time t is independent of preceding outcomes. We suppose that $\varphi(t) = \lambda(t)h(t)$ has at most finitely many discontinuities on a given horizon $[0, T]$. The objective is to stop on the last success on $[0, T]$.

Solution. Let $[0, t_1]\cup]t_1, t_2]\cup\cdots\cup]t_{m-1}, t_m]$ be an arbitrary partition of $[0, T]$ with $t_m = T$. Let p_k denote the probability of at least one success in $]t_{k-1}, t_k]$. By our assumptions, $p_k = \lambda(t_k)h(t_k)(t_k - t_{k-1}) + o(t_k - t_{k-1})$, with at most a fixed number of exceptions for k. When the caliber of the partition tends to 0 we obtain a well-defined limiting intensity φ for successes at time t, which is $\varphi(t) = \lambda(t)h(t)$. Clearly, the limiting "odds-intensity" is here the same. Therefore it suffices to define

$$\tau = \sup\left\{0, \sup\left\{0 \le t \le T\colon \int_t^T \lambda(u)h(u)\,du \ge 1\right\}\right\}, \tag{7}$$

and it is optimal to stop on the first success (if any) which occurs after time τ.

The answer (7) is intuitive. The function $h(t)$ can of course be interpreted as a thinning or time transformation of the intensity $\lambda(t)$ which maintains the Poisson process property. Stopping the new Poisson process (in which each occurrence is a success) with maximal probability on its last occurrence time means stopping as soon as the expected number of future occurrences becomes less or equal to one. Hence the optimal win probability equals $1/e$. Conversely, the Odds-theorem proves that if n is large and the p_k are (uniformly) small, then it is justified to use (7) for an integral version as a quick approximation for the discrete time solution.

4.2. *Improving special cases.* It is evident from Section 3 and the preceding example that Theorem 1 can be applied to many problems treated in the literature of optimal stopping, and that, if it can be applied, it is elegant. Tamaki (1999) for instance now uses it to solve the secretary problem with uncertain availability of candidates.

Counterexamples, due to the lack of independence, are of course easy to find. We give one example. Section 4.1 may remind the reader of the so-called $1/e$-law for best choice problems with an unknown number of options [Bruss (1984b)]. But this law does *not* follow from the Odds-theorem. Notice that, given a distribution of the number N of arrivals, the number of arrivals in subintervals depends in general on the number of preceding arrivals. Hence the success indicators for record values on subintervals are in general not independent. This dependence disappears if $N = \infty$, but then the result follows already from the work of Gianini and Samuels (1976) and, in extended form, from Rocha (1993).

For Poisson arrivals, which satisfy the hypotheses of the $1/e$-law, the result in Section 4.1 is stronger, however. The $1/e$-law thus maintains its general interest, but Theorem 1 strengthens it in special cases.

5. A typical lower bound for $V(n)$. Many modifications of the secretary problem show the asymptotic value $1/e$ or a value near $1/e$ [see, e.g., Pfeifer (1989), Samuels (1993), Samuel-Cahn (1995) and others]. The following theorem shows that this has little to do with the approximately logarithmic rate of records in such rank-based problems, the true reason being simpler.

THEOREM 2. *If $R_s = \Sigma_{j=s}^n r_j = R$, then*

(i)

$$V(n) = \sum_{j=s}^{n} r_j \prod_{j=s}^{n} q_j > Re^{-R}. \tag{8}$$

(ii) *If $R_{s(n)} \to 1$ with $R_{s(n)}^{(2)} \to 0$ as $n \to \infty$ then $V(n) \to 1/e$.*

PROOF. (i) $R = \Sigma_{j=s}^n r_j = \Sigma_{j=s}^n (1/q_j - 1)$, and hence $\Sigma_{j=s}^n 1/q_j = R + (n-s) + 1$. Put $m = n - s + 1$. By the inequality for geometric mean and arithmetic mean, we have then

$$\sqrt[m]{\prod_{j=s}^{n} \frac{1}{q_j}} = \sqrt[m]{1/\prod_{j=s}^{n} q_j} \le \frac{R+m}{m} = 1 + \frac{R}{m}$$

and thus $\prod_{j=s}^n q_j \ge (1 + R/m)^{-m}$. Since $(1 + R/m)^{-m} \downarrow e^{-R}$ as $m \to \infty$, it follows that

$$V(n) = R \prod_{j=s}^{n} q_j \ge \frac{R}{(1 + R/m)^m} > Re^{-R}. \tag{9}$$

(ii) Recall that $R_{s(n)}^{(2)} = \Sigma_{k=s(n)}^n r_k^2$. We have

$$Q_{s(n)}^{-1} = \prod_{k=s(n)}^{n} \frac{1}{q_k} = \prod_{k=s(n)}^{n} (1 + r_k).$$

Taking the logarithm and using $\log(1 + r_k) \ge r_k - r_k^2$, we obtain

$$-\log(Q_{s(n)}) = \sum_{k=s(n)}^{n} \log(1 + r_k) \ge R_{s(n)} - R_{s(n)}^{(2)}. \tag{10}$$

Hence, from (10), $Q_{s(n)} \le \exp(-R_{s(n)} + R_{s(n)}^{(2)}) \to 1/e$. Together with (8) this yields $Q_{s(n)} \to 1/e$ and $V(n) = Q_{s(n)} R_{s(n)} \to 1/e$. □

Instigated by the present paper, another proof of part (i) was given by Grey (1999). For secretary problems based on relative ranks, the corresponding result follows also from the work of Hill and Krengel (1992). [See also Hill and Kennedy (1992).]

In general, only trivial *upper* bounds are possible, of course. If $p_n > 1/2$, for instance, $s = n$ and $V(n) = q_n r_n = p_n$. But Theorem 2 tells us that $R_s e^{-R_s}$ is a lower bound for the success probability under optimal play, whatever the independent indicators are. The bound is sharp, as we have seen, and this is nice since R_s is often close to 1, not only in asymptotic cases. That is why $1/e$ may be called, without much abuse, a typical lower bound.

Acknowledgments. The author thanks Professors G. Alsmeyer, T. S. Ferguson, D. O. Siegmund and P. Whittle, as well as the referee, for their interest and their comments.

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