
Übungsblatt Nr. 4

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Aufgabe 1: Newtonsche Reibung

a) $F_{ges} = F_g + \kappa v^2$, also

$$m\ddot{z} = ma + \kappa \dot{z}^2$$

b) für die Endgeschwindigkeit gilt $a = 0 \text{ m/s}^2$, also

$$\begin{aligned} F_{ges} &= 0 \frac{\text{kg m}}{\text{s}^2} \\ F_g + \kappa v_\infty^2 &= 0 \frac{\text{kg m}}{\text{s}^2} \\ \kappa v_\infty^2 &= -F_g \\ v^2 &= -\frac{F_g}{\kappa} \\ |v_\infty| &= \sqrt{-\frac{F_g}{\kappa}} \quad | \quad \text{da die Fallschirmspringerin nach unten fällt} \\ v_\infty &= -\sqrt{-\frac{F_g}{\kappa}} \\ v_\infty &= -\sqrt{-\frac{6.5 \cdot 10^1 \text{ kg} \cdot (-1.0 \cdot 10^1 \text{ m/s})}{2.6 \cdot 10^{-1} \text{ kg/m}}} \\ v_\infty &= -5.0 \cdot 10^1 \frac{\text{m}}{\text{s}} \end{aligned}$$

Die Endgeschwindigkeit beträgt demnach $-5.0 \cdot 10^1 \text{ m/s}$

c) $u := \sqrt{-\frac{\kappa}{F_g}} v, \xi := \sqrt{-\frac{m}{4\kappa a}}, \tau := u - 1, \vartheta := u + 1$, daraus folgt:

$$v = \sqrt{-\frac{F_g}{\kappa}}, dv = \sqrt{-\frac{\kappa}{F_g}} du, du = d\tau, du = d\vartheta \text{ Es gilt:}$$

$$F_{ges} = F_g + \kappa v^2$$

$$m \frac{dv}{dt} = F_g + \kappa v^2$$

$$dt = \frac{m}{F_g + \kappa v^2} dv$$

| Substitution

$$dt = \frac{m}{F_g(1-u^2)} \sqrt{-\frac{F_g}{\kappa}} du$$

$$dt = \sqrt{-\frac{m}{4\kappa a}} \cdot \frac{2}{1-u^2} du$$

$$dt = \xi \cdot \frac{2}{u^2-1} du$$

$$\frac{1}{x} dt = \frac{2}{u^2-1} du$$

Da

$$\begin{aligned} \frac{1}{\tau} d\tau - \frac{1}{\vartheta} d\vartheta &= \frac{1}{u-1} du - \frac{1}{u+1} du \\ &= \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\ &= \left(\frac{u+1 - (u-1)}{(u-1)(u+1)} \right) du \\ &= \frac{2}{u^2-1} du \end{aligned}$$

gilt:

$$\begin{aligned} \frac{1}{\xi} dt &= \frac{2}{u^2-1} du \\ \frac{1}{\xi} dt &= \frac{1}{\tau} d\tau - \frac{1}{\vartheta} d\vartheta \\ \int_{t_0}^{t_1} \frac{1}{\xi} dt &= \int_{\tau_0}^{\tau_1} \frac{1}{\tau} d\tau - \int_{\vartheta_0}^{\vartheta_1} \frac{1}{\vartheta} d\vartheta \\ \frac{1}{\xi} t &= \ln |\tau_1| - \ln |\tau_0| - (\ln |\vartheta_1| - \ln |\vartheta_0|) \end{aligned}$$

Durch Rücksubstituieren erhält man:

$$\begin{aligned}
\frac{1}{\xi}t &= \ln|u_1 - 1| - \ln|u_0 - 1| - (\ln|u_1 + 1| - \ln|u_0 + 1|) \\
\frac{1}{\xi}t &= \ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right| - \ln \left| \underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} - 1 \right| \\
&\quad - \left(\ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right| - \ln \left| \underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} + 1 \right| \right) \\
\frac{1}{\xi}t &= \ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right| - \ln|-1| - \left(\ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right| - \ln|1| \right) \\
\frac{1}{\xi}t &= \ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right| - 0 - \left(\ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right| - 0 \right) \\
\frac{1}{\xi}t &= \ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right| - \ln \left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right| \\
\frac{1}{\xi}t &= \ln \frac{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right|}{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right|} \\
-\frac{1}{\xi}t &= \ln \frac{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right|}{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right|} \\
\exp\left(-\frac{1}{\xi}t\right) &= \frac{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 + 1 \right|}{\left| \sqrt{-\frac{\kappa}{F_g}}v_1 - 1 \right|} \\
\exp\left(-\frac{1}{\xi}t\right) &= \frac{\left| v_1 + \sqrt{-\frac{F_g}{\kappa}} \right|}{\left| v_1 - \sqrt{-\frac{F_g}{\kappa}} \right|}
\end{aligned}$$

Nach der b) ist $|v_1| \leq \sqrt{-\frac{F_g}{\kappa}}$ und da $a \leq 0$, also $-v_1 \leq \sqrt{-\frac{F_g}{\kappa}}$ und $v_1 - \sqrt{-\frac{F_g}{\kappa}} \leq 0$:

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{\left|v_1 + \sqrt{-\frac{F_g}{\kappa}}\right|}{\left|v_1 - \sqrt{-\frac{F_g}{\kappa}}\right|}$$

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{v_1 + \sqrt{-\frac{F_g}{\kappa}}}{-v_1 + \sqrt{-\frac{F_g}{\kappa}}}$$

$$v_1 + \sqrt{-\frac{F_g}{\kappa}} = -v_1 \exp\left(-\frac{1}{\xi}t\right) + \sqrt{-\frac{F_g}{\kappa}} \exp\left(-\frac{1}{\xi}t\right)$$

$$v_1 \left(\exp\left(-\frac{1}{\xi}t\right) + 1 \right) = \sqrt{-\frac{F_g}{\kappa}} \left(\exp\left(-\frac{1}{\xi}t\right) - 1 \right)$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \cdot \frac{\exp\left(-\frac{1}{\xi}t\right) - 1}{\exp\left(-\frac{1}{\xi}t\right) + 1}$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\frac{1}{2\xi}t$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}}t \quad | \quad \text{einsetzen}$$

$$v_1 = -50 \frac{\text{m}}{\text{s}} \tanh \sqrt{-\frac{0.26 \text{ kg/m} - 10 \text{ m/s}^2}{65 \text{ kg}}}t$$

$$v_1 = -5.0 \cdot 10^1 \frac{\text{m}}{\text{s}} \tanh \left(2 \cdot 10^{-1} \frac{1}{\text{s}} \cdot t \right)$$

Noch Substitution mit $x := -\sqrt{-\frac{\kappa a}{m}}t \implies dt = -\sqrt{-\frac{m}{\kappa a}}$

$$z = \int_{t_0}^{t_1} \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}}t dv + z_0$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \int_0^{t_1} \tanh -\sqrt{-\frac{\kappa a}{m}}t dv$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \int_{x_0}^{x_1} \tanh x dx$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln \cosh x_1 - \ln \cosh x_0$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln \cosh -\sqrt{-\frac{\kappa a}{m}}t_1 - \underbrace{\ln \cosh -\sqrt{-\frac{\kappa a}{m}}t_0}_{=0}$$

$$z = -\sqrt{\frac{m^2}{\kappa^2}} \ln \cosh -\sqrt{-\frac{\kappa a}{m}}t_1 - 0$$

$$z = -\frac{m}{\kappa} \ln \cosh -\sqrt{-\frac{\kappa a}{m}}t_1$$

$$z = -\frac{m}{\kappa} \ln \cosh \sqrt{-\frac{\kappa a}{m}}t_1$$

$$z = -\frac{6.5 \cdot 10^1 \text{ kg}}{2.6 \cdot 10^{-1} \text{ kg/m}} \ln \cosh \left(2 \cdot 10^{-1} \frac{1}{\text{s}} \cdot t_1\right)$$

$$z = -2.50 \cdot 10^2 \text{ m} \ln \cosh \left(2 \cdot 10^{-1} \frac{1}{\text{s}} \cdot t_1\right)$$

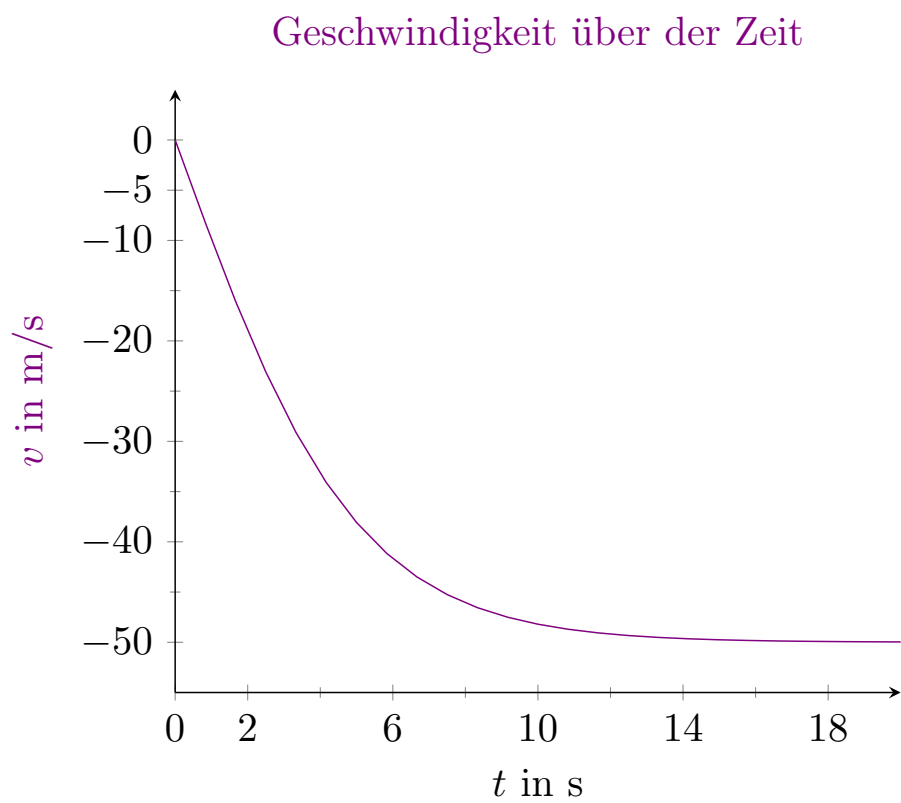
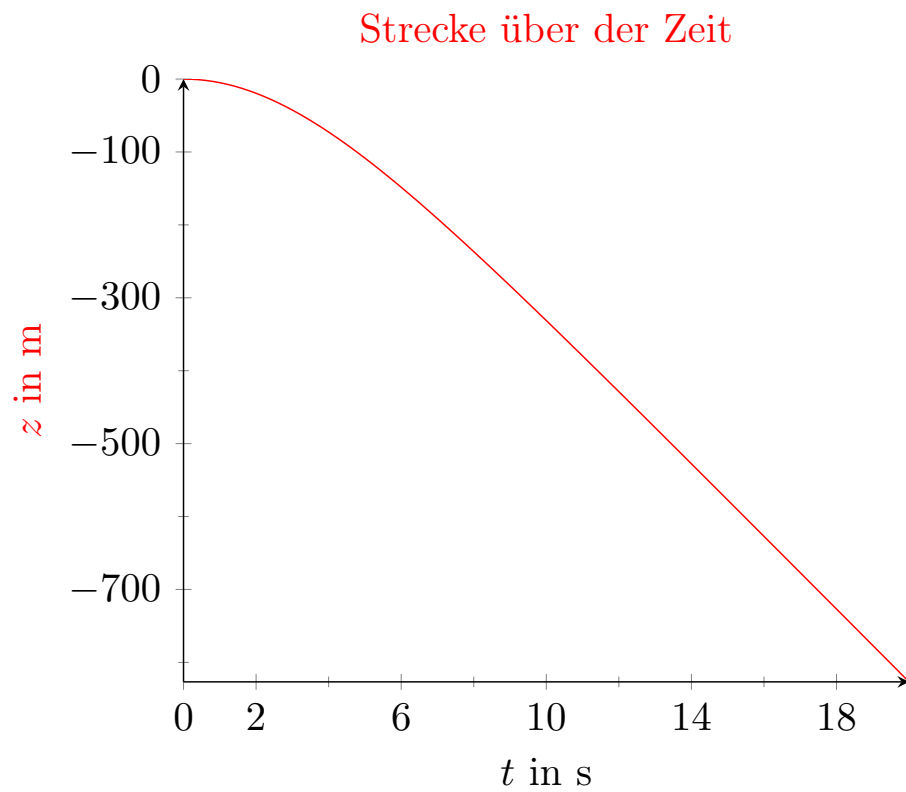
Für $v_0 = 0 \text{ m/s}$ und $z_0 = 0 \text{ m/s}$ gilt also: Die Geschwindigkeitsgleichung lautet:

$$v = -5.0 \cdot 10^1 \frac{\text{m}}{\text{s}} \tanh \left(2 \cdot 10^{-1} \frac{1}{\text{s}} \cdot t\right)$$

Und die Strecke in Abhängigkeit von der Zeit:

$$z = -2.50 \cdot 10^2 \text{ m} \ln \cosh \left(2 \cdot 10^{-1} \frac{1}{\text{s}} \cdot t_1\right)$$

d)



e) Zeitpunkt zudem sie 95% ihrer Geschwindigkeit erreicht hat:

$$\begin{aligned}
 t &= \xi \ln \left| \frac{-0.95 \cdot \sqrt{\frac{\kappa}{\frac{F_g}{\kappa}}} - 1}{-0.95 \cdot \sqrt{\frac{\kappa}{\frac{F_g}{\kappa}}} + 1} \right| \\
 &= \xi \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right| \\
 &= \sqrt{\frac{6.5 \cdot 10^1 \text{ kg}}{4 \cdot 2.6 \cdot 10^{-1} \text{ kg} \cdot 1.0 \cdot 10^1 \text{ m/s}^2}} \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right| \\
 t &\approx 9.1589 \text{ s}
 \end{aligned}$$

in $z(t)$ eingesetzt

$$\begin{aligned}
 z &= -\frac{m}{\kappa} \ln \cosh - \sqrt{-\frac{\kappa a}{m}} t \\
 z &\approx -\frac{6.5 \cdot 10^1 \text{ kg}}{2.6 \cdot 10^{-1} \text{ kg/m}} \ln \cosh - \sqrt{\frac{2.6 \cdot 10^{-1} \text{ kg/m} \cdot 1.0 \cdot 10^1 \text{ m/s}^2}{6.5 \cdot 10^1 \text{ kg}}} 9.1589 \text{ s} \\
 z &\approx -2.9 \cdot 10^2 \text{ m}
 \end{aligned}$$

Aufgabe 2: Zylinderkoordinaten

a) $\rho = \sqrt{\left(\frac{v_{0,r}}{\omega_c}\right)^2 \sin^2 \omega_c t + \left(\frac{v_{0,r}}{\omega_c}\right)^2 \cos^2 \omega_c t} = \frac{v_{0,r}}{\omega_c},$

$$\varphi = \arccos \frac{\frac{v_{0,r}}{\omega_c} \cos \omega_c t}{\frac{v_{0,r}}{\omega_c}} = \arccos \cos \omega_c t = \omega_c t,$$

$$z = v_{0,z} t$$

Die Koordinaten von $\vec{r}(t)$ in Zylinderkoordinaten sind also:

$$\vec{r}(t) = \left(\frac{v_{0,r}}{\omega_c}, \omega_c t, v_{0,z} t \right)$$

b)

$$\begin{aligned}
 \frac{d}{dt} \vec{r} &= \begin{pmatrix} \frac{d\rho}{dt} \cos \varphi - \frac{d\varphi}{dt} \rho \sin \varphi \\ \frac{d\rho}{dt} \sin \varphi + \frac{d\varphi}{dt} \rho \cos \varphi \\ \frac{dz}{dt} \end{pmatrix} \\
 \vec{v} &= \begin{pmatrix} -\omega_c \rho \sin \varphi \\ \omega_c \rho \cos \varphi \\ v_{0,z} \end{pmatrix}
 \end{aligned}$$

und in Zylinderkoordinaten

$$\rho' = \sqrt{(\omega_c \rho)^2 \sin^2 \varphi + (\omega_c \rho)^2 \cos^2 \varphi} = \omega_c \rho,$$

$$\varphi' = \arccos \frac{-\omega_c \rho}{\rho'} \sin \varphi = \arccos \cos \varphi + \frac{\pi}{2} = \varphi + \frac{\pi}{2},$$

$$z' = v_{0,z}$$

also in Zylinderkoordinaten:

$$\vec{v} = \left(\omega_c \rho, \varphi + \frac{\pi}{2}, v_{0,z} \right) = \left(v_{0,r}, \omega_c t + \frac{\pi}{2}, v_{0,z} \right)$$

$$\frac{d}{dt}\vec{v} = \begin{pmatrix} \frac{dv_{0,r}}{dt} \cos \omega_c t + \frac{\pi}{2} - \frac{d\omega_c t + \frac{\pi}{2}}{dt} v_{0,r} \sin \omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,r}}{dt} \sin \omega_c t + \frac{\pi}{2} + \frac{d\omega_c t + \frac{\pi}{2}}{dt} v_{0,r} \cos \omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,z}}{dt} \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} -\omega_c v_{0,r} \sin \omega_c t + \frac{\pi}{2} \\ \omega_c v_{0,r} \cos \omega_c t + \frac{\pi}{2} \\ 0 \end{pmatrix}$$

Also in Zylinderkoordinaten

$$\rho'' = \sqrt{(\omega_c v_{0,r})^2 \sin^2 \phi + (\omega_c v_{0,r})^2 \cos^2 \phi} = \omega_c v_{0,r},$$

$$\varphi'' = \arccos \frac{-\omega_c v_{0,r}}{\omega_c v_{0,r}} \sin \omega_c t + \frac{\pi}{2} = \arccos \cos \omega_c t + \pi = \omega_c t + \pi,$$

$$z'' = 0$$

Also in Zylinderkoordinaten:

$$\vec{a} = (\omega_c v_{0,r}, \omega_c t + \pi, 0) = (\omega_c v_{0,r}, -\omega_c t, 0)$$