University of Konstanz
Faculty of Science (Physics)

Übungsblatt Nr. 4

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Aufgabe 1: Newtonsche Reibung

a) $F_{ges} = F_g + \kappa v^2$, also

$$m\ddot{z} = ma + \kappa \dot{z}^2$$

b) für die Endgeschwindigkeit gilt $a = 0\,\mathrm{m/s^2}$, also

$$\begin{aligned}
F_{ges} &= 0\,\frac{\mathrm{kg\,m}}{\mathrm{s^2}}\\
F_g + \kappa v_\infty^2 &= 0\,\frac{\mathrm{kg\,m}}{\mathrm{s^2}}\\
\kappa v_\infty^2 &= -F_g\\
v^2 &= -\frac{F_g}{\kappa}\\
|v_\infty| &= \sqrt{-\frac{F_g}{\kappa}} \quad | \quad \text{da die Fallschirmspringerin nach unten fällt}\\
v_\infty &= -\sqrt{-\frac{F_g}{\kappa}}\\
v_\infty &= -\sqrt{-\frac{6.5\cdot 10^1\,\mathrm{kg}\cdot(-1.0\cdot 10^1\,\mathrm{m/s})}{2.6\cdot 10^{-1}\,\mathrm{kg/m}}}\\
v_\infty &= -5.0\cdot 10^1\,\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

Die Endgeschwindigkeit beträgt demnach $-5.0\cdot 10^1\,\mathrm{m/s}$

c) $u \coloneqq \sqrt{-\frac{\kappa}{F_g}}v, \xi \coloneqq \sqrt{-\frac{m}{4\kappa a}}, \tau \coloneqq u - 1, \vartheta \coloneqq u + 1$, daraus folgt:

$v = \sqrt{-\frac{F_g}{\kappa}}, dv = \sqrt{-\frac{\kappa}{F_g}} du, du = d\tau, du = d\vartheta$ Es gilt:

$$
\begin{aligned}
F_{ges} &= F_g + \kappa v^2 \\
m\frac{dv}{dt} &= F_g + \kappa v^2 \\
dt &= \frac{m}{F_g + \kappa v^2} dv && | \text{ Substitution} \\
dt &= \frac{m}{F_g(1-u^2)} \sqrt{-\frac{F_g}{\kappa}} du \\
dt &= \sqrt{-\frac{m}{4\kappa a}} \cdot \frac{2}{1-u^2} du \\
dt &= \xi \cdot \frac{2}{u^2-1} du \\
\frac{1}{x} dt &= \cdot \frac{2}{u^2-1} du
\end{aligned}
$$

Da

$$
\begin{aligned}
\frac{1}{\tau} d\tau - \frac{1}{\vartheta} d\vartheta &= \frac{1}{u-1} du - \frac{1}{u+1} du \\
&= \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\
&= \left(\frac{u+1-(u-1)}{(u-1)(u+1)} \right) du \\
&= \frac{2}{u^2-1} du
\end{aligned}
$$

gilt:

$$
\begin{aligned}
\frac{1}{\xi} dt &= \frac{2}{u^2-1} du \\
\frac{1}{\xi} dt &= \frac{1}{\tau} d\tau - \frac{1}{\vartheta} d\vartheta \\
\int_{t_0}^{t_1} \frac{1}{\xi} dt &= \int_{\tau_0}^{\tau_1} \frac{1}{\tau} d\tau - \int_{\vartheta_0}^{\vartheta_1} \frac{1}{\vartheta} d\vartheta \\
\frac{1}{\xi} t &= \ln|\tau_1| - \ln|\tau_0| - (\ln|\vartheta_1| - \ln|\vartheta_0|)
\end{aligned}
$$

Durch Rücksubstituieren erhält man:

$$\frac{1}{\xi}t = \ln|u_1 - 1| - \ln|u_0 - 1| - (\ln|u_1 + 1| - \ln|u_0 + 1|)$$

$$\frac{1}{\xi}t = \ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right| - \ln\left|\underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} - 1\right|$$
$$- \left(\ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right| - \ln\left|\underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} + 1\right|\right)$$

$$\frac{1}{\xi}t = \ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right| - \ln|-1| - \left(\ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right| - \ln|1|\right)$$

$$\frac{1}{\xi}t = \ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right| - 0 - \left(\ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right| - 0\right)$$

$$\frac{1}{\xi}t = \ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right| - \ln\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right|$$

$$\frac{1}{\xi}t = \ln\frac{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right|}{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right|}$$

$$-\frac{1}{\xi}t = \ln\frac{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right|}{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right|}$$

$$\exp(-\frac{1}{\xi}t) = \frac{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1\right|}{\left|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1\right|}$$

$$\exp(-\frac{1}{\xi}t) = \frac{\left|v_1 + \sqrt{-\frac{F_g}{\kappa}}\right|}{\left|v_1 - \sqrt{-\frac{F_g}{\kappa}}\right|}$$

Nach der b) ist $|v_1| <= \sqrt{-\frac{F_g}{\kappa}}$ und da $a <= 0$, also $-v_1 <= \sqrt{-\frac{F_g}{\kappa}}$ und $v_1 - \sqrt{-\frac{F_g}{\kappa}} <= 0$:

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{\left|v_1 + \sqrt{-\frac{F_g}{\kappa}}\right|}{\left|v_1 - \sqrt{-\frac{F_g}{\kappa}}\right|}$$

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{v_1 + \sqrt{-\frac{F_g}{\kappa}}}{-v_1 + \sqrt{-\frac{F_g}{\kappa}}}$$

$$v_1 + \sqrt{-\frac{F_g}{\kappa}} = -v_1 \exp\left(-\frac{1}{\xi}t\right) + \sqrt{-\frac{F_g}{\kappa}} \exp\left(-\frac{1}{\xi}t\right)$$

$$v_1 \left(\exp\left(-\frac{1}{\xi}t\right) + 1\right) = \sqrt{-\frac{F_g}{\kappa}} \left(\exp\left(-\frac{1}{\xi}t\right) - 1\right)$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \cdot \frac{\exp\left(-\frac{1}{\xi}t\right) - 1}{\exp\left(-\frac{1}{\xi}t\right) + 1}$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\frac{1}{2\xi}t$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}}t \quad | \quad \text{einsetzen}$$

$$v_1 = -50\,\frac{\mathrm{m}}{\mathrm{s}} \tanh \sqrt{-\frac{0.26\,\mathrm{kg/m} - 10\,\mathrm{m/s^2}}{65\,\mathrm{kg}}t}$$

$$v_1 = -5.0 \cdot 10^1\,\frac{\mathrm{m}}{\mathrm{s}} \tanh\left(2 \cdot 10^{-1}\,\frac{1}{\mathrm{s}} \cdot t\right)$$

Noch Substiution mit $x \coloneqq -\sqrt{-\frac{\kappa a}{m}}t \implies dt = -\sqrt{-\frac{m}{\kappa a}}$

$$
\begin{aligned}
z &= \int_{t_0}^{t_1} \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}} t dv + z_0 \\
z &= \sqrt{-\frac{F_g}{\kappa}} \int_{0}^{t_1} \tanh -\sqrt{-\frac{\kappa a}{m}} t dv \\
z &= \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \int_{x_0}^{x_1} \tanh x dx \\
z &= \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln\cosh x_1 - \ln\cosh x_0 \\
z &= \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln\cosh -\sqrt{-\frac{\kappa a}{m}} t_1 - \ln\cosh \underbrace{-\sqrt{-\frac{\kappa g}{m}} t_0}_{=0} \\
z &= -\sqrt{\frac{m^2}{\kappa^2}} \ln\cosh -\sqrt{-\frac{\kappa a}{m}} t_1 - 0 \\
z &= -\frac{m}{\kappa} \ln\cosh -\sqrt{-\frac{\kappa a}{m}} t_1 \\
z &= -\frac{m}{\kappa} \ln\cosh \sqrt{-\frac{\kappa a}{m}} t_1 \\
z &= -\frac{6.5 \cdot 10^1\,\mathrm{kg}}{2.6 \cdot 10^{-1}\,\mathrm{kg/m}} \ln\cosh\left(2 \cdot 10^{-1}\,\frac{1}{\mathrm{s}} \cdot t_1\right) \\
z &= -2.50 \cdot 10^2\,\mathrm{m} \ln\cosh\left(2 \cdot 10^{-1}\,\frac{1}{\mathrm{s}} \cdot t_1\right)
\end{aligned}
$$

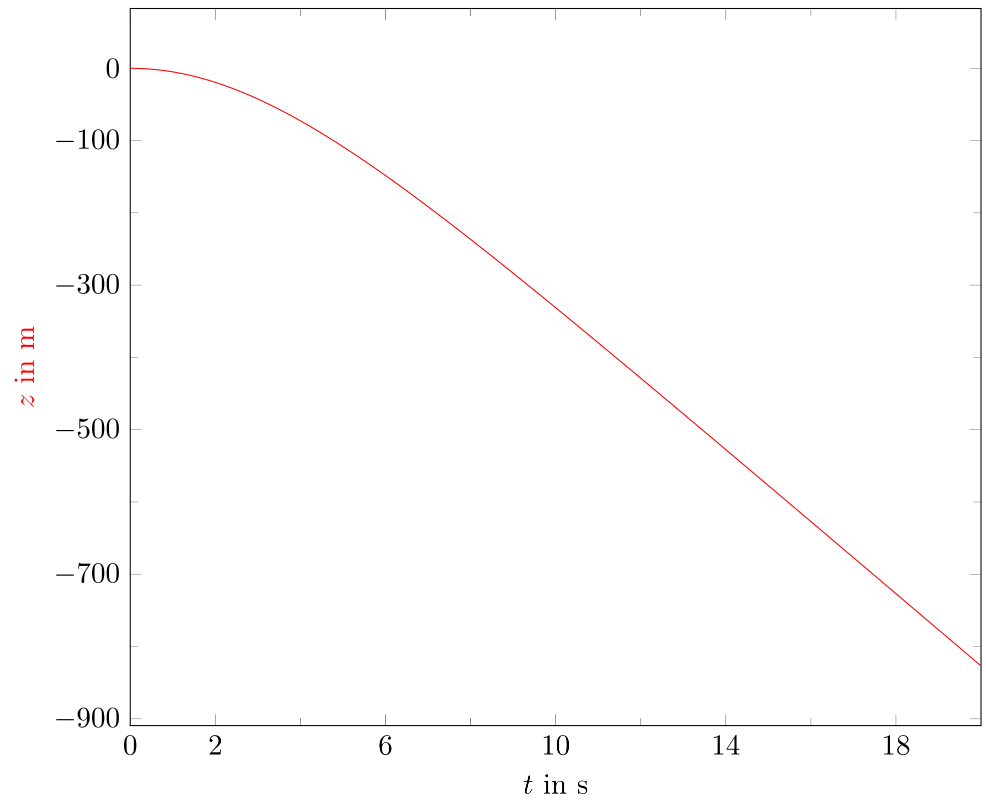
Für $v_0 = 0\,\mathrm{m/s}$ und $z_0 = 0\,\mathrm{m/s}$ gilt also: Die Geschwindigkeitsgleichung lautet:

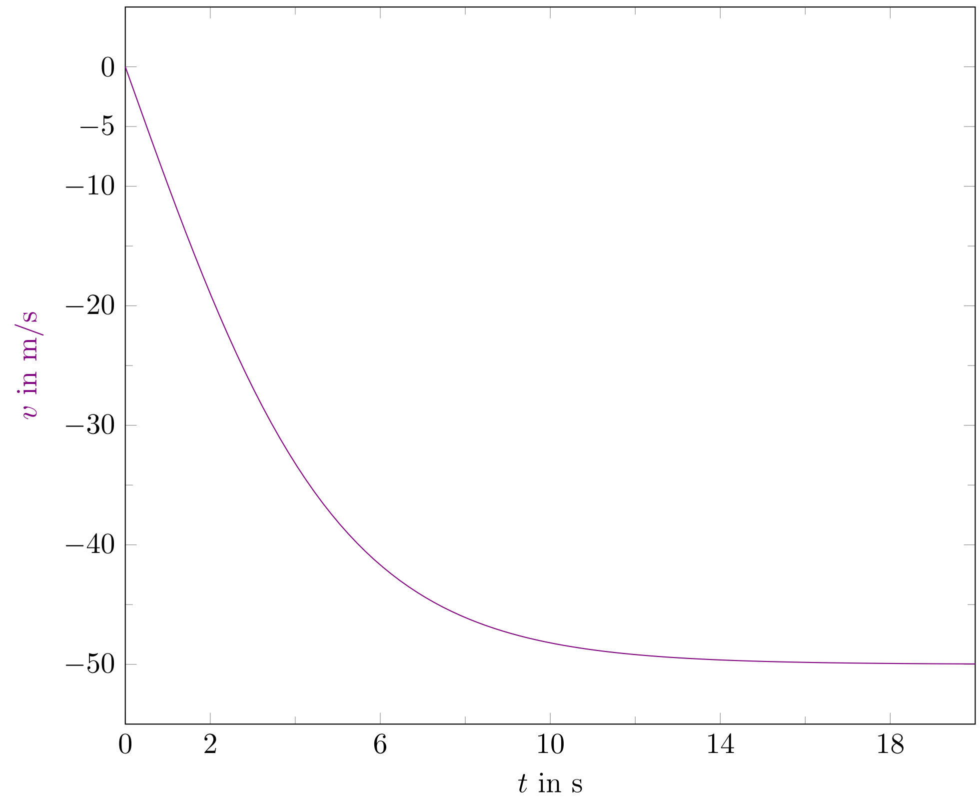
$$v = -5.0 \cdot 10^1\,\frac{\mathrm{m}}{\mathrm{s}} \tanh\left(2 \cdot 10^{-1}\,\frac{1}{\mathrm{s}} \cdot t\right)$$

Und die Strecke in Abhängikeit von der Zeit:

$$z = -2.50 \cdot 10^2\,\mathrm{m} \ln\cosh\left(2 \cdot 10^{-1}\,\frac{1}{\mathrm{s}} \cdot t_1\right)$$

d)


0
−100
−300
−500
−700
−900
z in m
0
2
6
10
14
18
t in s


0
−5
−10
−20
−30
−40
−50
v in m/s
0
2
6
10
14
18
t in s

e) Zeitpunkt zudem sie 95% ihrer Geschwindigkeit erreicht hat:

$$
\begin{aligned}
t &= \xi \ln \left| \frac{-0.95 \cdot \sqrt{\frac{\frac{\kappa}{F_g}}{\frac{\kappa}{F_g}}} - 1}{-0.95 \cdot \sqrt{\frac{\frac{\kappa}{F_g}}{\frac{\kappa}{F_g}}} + 1} \right| \\
&= \xi \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right| \\
&= \sqrt{\frac{6.5 \cdot 10^1\,\mathrm{kg}}{4 \cdot 2.6 \cdot 10^{-1}\,\mathrm{kg} \cdot 1.0 \cdot 10^1\,\mathrm{m/s^2}}} \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right| \\
t &\approx 9.1589\,\mathrm{s}
\end{aligned}
$$

in $z(t)$ eingesetzt

$$
\begin{aligned}
z &= -\frac{m}{\kappa} \ln \cosh - \sqrt{-\frac{\kappa a}{m}} t \\
z &\approx -\frac{6.5 \cdot 10^1\,\mathrm{kg}}{2.6 \cdot 10^{-1}\,\mathrm{kg/m}} \ln \cosh - \sqrt{\frac{2.6 \cdot 10^{-1}\,\mathrm{kg/m} \cdot 1.0 \cdot 10^1\,\mathrm{m/s^2}}{6.5 \cdot 10^1\,\mathrm{kg}}} 9.1589\,\mathrm{s} \\
z &\approx -2.9 \cdot 10^2\,\mathrm{m}
\end{aligned}
$$

Aufgabe 2: Zylinderkoordinate

a) $\rho = \sqrt{\left(\frac{v_{0,r}}{\omega_c}\right)^2 \sin^2 \omega_c t + \left(\frac{v_{0,r}}{\omega_c}\right)^2 \cos^2 \omega_c t} = \frac{v_{0,r}}{\omega_c}$,
$\varphi = \arccos \frac{\frac{v_{0,r}}{\omega_c} \cos \omega_c t}{\frac{v_{0,r}}{\omega_c}} = \arccos \cos \omega_c t = \omega_c t$,
$z = v_{0,z} t$
Die Koordinaten von $\vec{r}(t)$ in Zylinderkoordinaten sind also:

$$
\vec{r}(t) = \left(\frac{v_{0,r}}{\omega_c}, \omega_c t, v_{0,z} t \right)
$$

b)

$$
\begin{aligned}
\frac{d}{dt} \vec{r} &= \begin{pmatrix} \frac{d\rho}{dt} \cos \varphi - \frac{d\phi}{dt} \rho \sin \varphi \\ \frac{d\rho}{dt} \sin \varphi + \frac{d\phi}{dt} \rho \cos \varphi \\ \frac{dz}{dt} \end{pmatrix} \\
\vec{v} &= \begin{pmatrix} -\omega_c \rho \sin \varphi \\ \omega_c \rho \cos \varphi \\ v_{0,z} \end{pmatrix}
\end{aligned}
$$

und in Zylinderkoordinaten
$\rho' = \sqrt{(\omega_c \rho)^2 \sin^2 \phi + (\omega_c \rho)^2 \cos^2 \phi} = \omega_c \rho$,
$\varphi' = \arccos \frac{-\omega_c \rho}{\rho'} \sin \varphi = \arccos \cos \varphi + \frac{\pi}{2} = \varphi + \frac{\pi}{2}$,
$z' = v_{0,z}$
also in Zylinderkoordinaten:

$$
\vec{v} = \left(\omega_c \rho, \varphi + \frac{\pi}{2}, v_{0,z} \right) = \left(v_{0,r}, \omega_c t + \frac{\pi}{2}, v_{0,z} \right)
$$

$$\frac{d}{dt}\vec{v} = \begin{pmatrix} \frac{dv_{0,r}}{dt}\cos\omega_c t + \frac{\pi}{2} - \frac{d\omega_c t + \frac{\pi}{2}}{dt} v_{0,r}\sin\omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,r}}{dt}\sin\omega_c t + \frac{\pi}{2} + \frac{d\omega_c t + \frac{\pi}{2}}{dt} v_{0,r}\cos\omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,z}}{dt} \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} -\omega_c v_{0,r}\sin\omega_c t + \frac{\pi}{2} \\ \omega_c v_{0,r}\cos\omega_c t + \frac{\pi}{2} \\ 0 \end{pmatrix}$$

Also in Zylinderkoordinaten
$\rho'' = \sqrt{(\omega_c v_{0,r})^2 \sin^2\phi + (\omega_c v_{0,r})^2\cos^2\phi} = \omega_c v_{0,r}$,
$\varphi'' = \arccos\frac{-\omega_c v_{0,r}}{\omega_c v_{0,r}}\sin\omega_c t + \frac{pi}{2} = \arccos\cos\omega_c t + \pi = \omega_c t + \pi$,
$z'' = 0$
Also in Zylinderkoordinaten:

$$\vec{a} = (\omega_c v_{0,r}, \omega_c t + \pi, 0) = (\omega_c v_{0,r}, -\omega_c t, 0)$$