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# Übungsblatt Nr. 3

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## **Aufgabe 1:** Newtonsche Reibung

a) $F_{ges} = F_g + \kappa v^2$

b) für die Endgeschwindigkeit gilt $a = 0\,\mathrm{m/s^2}$, also

$$\begin{aligned}
F_{ges} &= 0\,\frac{\mathrm{kg\,m}}{\mathrm{s}^2}\\
F_g + \kappa v_\infty^2 &= 0\,\frac{\mathrm{kg\,m}}{\mathrm{s}^2}\\
\kappa v_\infty^2 &= -F_g\\
v^2 &= -\frac{F_g}{\kappa}\\
|v_\infty| &= \sqrt{-\frac{F_g}{\kappa}}
\end{aligned}$$

c) $u \coloneqq \sqrt{-\frac{\kappa}{F_g}}v, \xi \coloneqq \sqrt{-\frac{m}{4\kappa a}}, \tau \coloneqq u-1, \vartheta \coloneqq u+1$, daraus folgt:
$v = \sqrt{-\frac{F_g}{\kappa}}, dv = \sqrt{-\frac{\kappa}{F_g}}du, du = d\tau, du = d\vartheta$ Es gilt:

$$\begin{aligned}
F_{ges} &= F_g + \kappa v^2\\
m\frac{dv}{dt} &= F_g + \kappa v^2\\
dt &= \frac{m}{F_g + \kappa v^2}dv && \mid \text{Substitution}\\
dt &= \frac{m}{F_g(1-u^2)}\sqrt{-\frac{F_g}{\kappa}}du\\
dt &= \sqrt{-\frac{m}{4\kappa a}}\cdot\frac{2}{1-u^2}du\\
dt &= \xi\cdot\frac{2}{u^2-1}du\\
\frac{1}{x}dt &= \cdot\frac{2}{u^2-1}du
\end{aligned}$$

Da

$$
\begin{aligned}
\frac{1}{\tau}d\tau - \frac{1}{\vartheta}d\vartheta &= \frac{1}{u-1}du - \frac{1}{u+1}du \\
&= \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du \\
&= \left(\frac{u+1-(u-1)}{(u-1)(u+1)}\right)du \\
&= \frac{2}{u^2-1}du
\end{aligned}
$$

gilt:

$$
\begin{aligned}
\frac{1}{\xi}dt &= \frac{2}{u^2-1}du \\
\frac{1}{\xi}dt &= \frac{1}{\tau}d\tau - \frac{1}{\vartheta}d\vartheta \\
\int_{t_0}^{t_1}\frac{1}{\xi}dt &= \int_{\tau_0}^{\tau_1}\frac{1}{\tau}d\tau - \int_{\vartheta_0}^{\vartheta_1}\frac{1}{\vartheta}d\vartheta \\
\frac{1}{\xi}t &= \ln|\tau_1| - \ln|\tau_0| - (\ln|\vartheta_1| - \ln|\vartheta_0|)
\end{aligned}
$$

Durch Rücksubstituieren erhält man:

$$
\begin{aligned}
\frac{1}{\xi}t &= \ln|u_1-1| - \ln|u_0-1| - (\ln|u_1+1| - \ln|u_0+1|) \\
\frac{1}{\xi}t &= \ln|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1| - \ln|\underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} - 1| \\
&\quad - \left(\ln|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1| - \ln|\underbrace{\sqrt{-\frac{\kappa}{F_g}}v_0}_{=0} + 1|\right) \\
\frac{1}{\xi}t &= \ln|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1| - \ln|-1| - \left(\ln|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1| - \ln|1|\right) \\
\frac{1}{\xi}t &= \ln|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1| - 0 - \left(\ln|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1| - 0\right) \\
\frac{1}{\xi}t &= \ln|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1| - \ln|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1| \\
\frac{1}{\xi}t &= \ln\frac{|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1|}{|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1|} \\
-\frac{1}{\xi}t &= \ln\frac{|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1|}{|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1|} \\
\exp(-\frac{1}{\xi}t) &= \frac{|\sqrt{-\frac{\kappa}{F_g}}v_1 + 1|}{|\sqrt{-\frac{\kappa}{F_g}}v_1 - 1|} \\
\exp(-\frac{1}{\xi}t) &= \frac{|v_1 + \sqrt{-\frac{F_g}{\kappa}}|}{|v_1 - \sqrt{-\frac{F_g}{\kappa}}|}
\end{aligned}
$$

Nach der b) ist $|v_1| <= \sqrt{-\frac{F_g}{\kappa}}$ und da $a <= 0$, also $-v_1 <= \sqrt{-\frac{F_g}{\kappa}}$ und $v_1 - \sqrt{-\frac{F_g}{\kappa}} <= 0$:

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{|v_1 + \sqrt{-\frac{F_g}{\kappa}}|}{|v_1 - \sqrt{-\frac{F_g}{\kappa}}|}$$

$$\exp\left(-\frac{1}{\xi}t\right) = \frac{v_1 + \sqrt{-\frac{F_g}{\kappa}}}{-v_1 + \sqrt{-\frac{F_g}{\kappa}}|}$$

$$-v_1 \exp\left(-\frac{1}{\xi}t\right) + sqrt - \frac{F_g}{\kappa}\exp\left(-\frac{1}{\xi}t\right) = v_1 + \sqrt{-\frac{F_g}{\kappa}}$$

$$v_1\left(\exp\left(-\frac{1}{\xi}t\right) + 1\right) = \sqrt{-\frac{F_g}{\kappa}}\left(\exp\left(-\frac{1}{\xi}t\right) - 1\right)$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \cdot \frac{\exp\left(-\frac{1}{\xi}t\right) - 1}{\exp\left(-\frac{1}{\xi}t\right) + 1}$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\frac{1}{2\xi}t$$

$$v_1 = \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}}t$$

Noch Substiution mit $x := -\sqrt{-\frac{\kappa a}{m}}t \implies dt = -\sqrt{-\frac{m}{\kappa a}}$

$$z = \int_{t_0}^{t_1} \sqrt{-\frac{F_g}{\kappa}} \tanh -\sqrt{-\frac{\kappa a}{m}} t dv + z_0$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \int_0^{t_1} \tanh -\sqrt{-\frac{\kappa a}{m}} t dv$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \int_{x_0}^{x_1} \tanh x dx$$

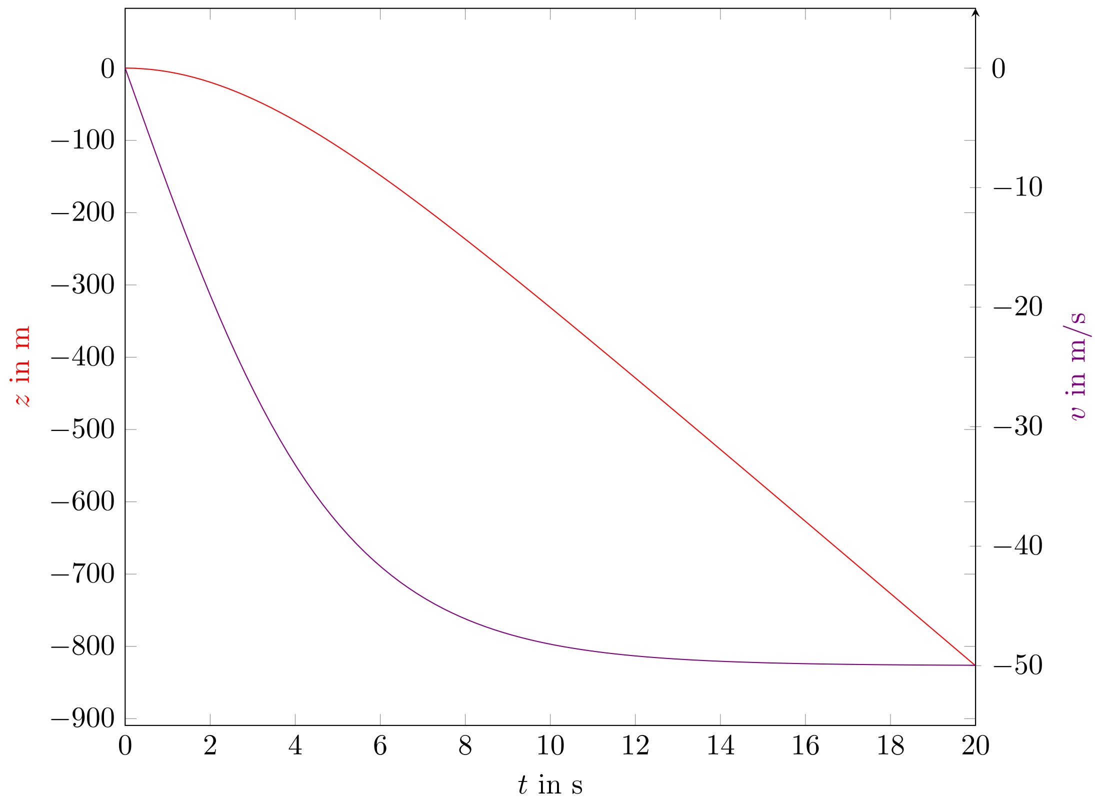
$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln\cosh x_1 - \ln\cosh x_0$$

$$z = \sqrt{-\frac{F_g}{\kappa}} \cdot \left(-\sqrt{-\frac{m}{\kappa a}}\right) \ln\cosh -\sqrt{-\frac{\kappa a}{m}}t_1 - \ln\cosh \underbrace{-\sqrt{-\frac{\kappa g}{m}}t_0}_{=0}$$

$$z = -\sqrt{\frac{m^2}{\kappa^2}} \ln\cosh -\sqrt{-\frac{\kappa a}{m}}t_1 - 0$$

$$z = -\frac{m}{\kappa} \ln\cosh -\sqrt{-\frac{\kappa a}{m}}t_1$$

d)




e) Zeitpunkt zudem sie 95% ihrer Geschwindigkeit erreicht hat:

$$t = \xi \ln \left| \frac{-0.95 \cdot \sqrt{\frac{\frac{\kappa}{F_g}}{\frac{\kappa}{F_g}}} - 1}{-0.95 \cdot \sqrt{\frac{\frac{\kappa}{F_g}}{\frac{\kappa}{F_g}}} + 1} \right|$$

$$= \xi \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right|$$

$$= \sqrt{\frac{6.5 \cdot 10^1\,\mathrm{kg}}{4 \cdot 2.6 \cdot 10^{-1}\,\mathrm{kg} \cdot 1.0 \cdot 10^1\,\mathrm{m/s^2}}} \ln \left| \frac{-0.95 - 1}{-0.95 + 1} \right|$$

$$t \approx 9.1589\,\mathrm{s}$$

in $z(t)$ eingesetzt

$$z = -\frac{m}{\kappa} \ln \cosh - \sqrt{-\frac{\kappa a}{m}} t$$

$$z \approx -\frac{6.5 \cdot 10^1\,\mathrm{kg}}{2.6 \cdot 10^{-1}\,\mathrm{kg/m}} \ln \cosh - \sqrt{\frac{2.6 \cdot 10^{-1}\,\mathrm{kg/m} \cdot 1.0 \cdot 10^1\,\mathrm{m/s^2}}{6.5 \cdot 10^1\,\mathrm{kg}}} 9.1589\,\mathrm{s}$$

$$z \approx -2.9 \cdot 10^2\,\mathrm{m}$$

## **Aufgabe 2:** Zylinderkoordinate

a) $\rho = \sqrt{\left(\frac{v_{0,r}}{\omega_c}\right)^2 \sin^2 \omega_c t + \left(\frac{v_{0,r}}{\omega_c}\right)^2 \cos^2 \omega_c t} = \frac{v_{0,r}}{\omega_c}$,

$\varphi = \arccos \frac{\frac{v_{0,r}}{\omega_c} \cos \omega_c t}{\frac{v_{0,r}}{\omega_c}} = \arccos \cos \omega_c t = \omega_c t$,

$z = v_{0,z}t$
Die Koordinaten von $\vec{r}(t)$ in Zylinderkoordinaten sind also:

$$\vec{r}(t) = \left(\frac{v_{0,r}}{\omega_c}, \omega_c t, v_{0,z}t\right)$$

b)

$$\frac{d}{dt}\vec{r} = \begin{pmatrix} \frac{d\rho}{dt}\cos\varphi - \frac{d\phi}{dt}\rho\sin\varphi \\ \frac{d\rho}{dt}\sin\varphi + \frac{d\phi}{dt}\rho\cos\varphi \\ \frac{dz}{dt} \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} -\omega_c\rho\sin\varphi \\ \omega_c\rho\cos\varphi \\ v_{0,z} \end{pmatrix}$$

und in Zylinderkoordinaten
$\rho' = \sqrt{(\omega_c\rho)^2\sin^2\phi + (\omega_c\rho)^2\cos^2\phi} = \omega_c\rho$,
$\varphi' = \arccos\frac{-\omega_c\rho}{\rho'}\sin\varphi = \arccos\cos\varphi + \frac{\pi}{2} = \varphi + \frac{\pi}{2}$,
$z' = v_{0,z}$
also in Zylinderkoordinaten:

$$\vec{v} = \left(\omega_c\rho, \varphi + \frac{\pi}{2}, v_{0,z}\right) = \left(v_{0,r}, \omega_c t + \frac{\pi}{2}, v_{0,z}\right)$$

$$\frac{d}{dt}\vec{v} = \begin{pmatrix} \frac{dv_{0,r}}{dt}\cos\omega_c t + \frac{\pi}{2} - \frac{d\omega_c t + \frac{\pi}{2}}{dt}v_{0,r}\sin\omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,r}}{dt}\sin\omega_c t + \frac{\pi}{2} + \frac{d\omega_c t + \frac{\pi}{2}}{dt}v_{0,r}\cos\omega_c t + \frac{\pi}{2} \\ \frac{dv_{0,z}}{dt} \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} -\omega_c v_{0,r}\sin\omega_c t + \frac{\pi}{2} \\ \omega_c v_{0,r}\cos\omega_c t + \frac{\pi}{2} \\ 0 \end{pmatrix}$$

Also in Zylinderkoordinaten
$\rho'' = \sqrt{(\omega_c v_{0,r})^2\sin^2\phi + (\omega_c v_{0,r})^2\cos^2\phi} = \omega_c v_{0,r}$,
$\varphi'' = \arccos\frac{-\omega_c v_{0,r}}{\omega_c v_{0,r}}\sin\omega_c t + \frac{pi}{2} = \arccos\cos\omega_c t + \pi = \omega_c t + \pi$,
$z'' = 0$
Also in Zylinderkoordinaten:

$$\vec{a} = (\omega_c v_{0,r}, \omega_c t + \pi, 0) = (\omega_c v_{0,r}, -\omega_c t, 0)$$