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Exercise 1: Analyzing an Algorithm

a) A suitable loop invariant $\mathcal{INV}$ is:

   (i) $A[i, \ldots, n]$ is sorted non-descended

   (ii) $\forall k, l \in \mathbb{N} : 1 \leq k < i \wedge i \leq j \leq n \implies A[k] \leq A[l]$

b) Prove through Induction:

   base case: $i = n$

   (i) $A[i, \ldots, n] = A[n, \ldots, n] = A[n]$, an one-element array is always sorted

   (ii) $\forall k, l \in \mathbb{N} : 1 \leq k < i \wedge i \leq l \leq n : A[k] \leq A[l]$
   $\iff \forall k, l \in \mathbb{N} : 1 \leq k < n \wedge n \leq j \leq n \implies A[k] \leq A[l]$. Let $A[m]$ be the greatest element in A, that means $A[(m+1) - 1] \geq A[m+1]$, so it would be moved to the next index, after increasing i, j, m, by one this would repeat until $j = n$ and $A[m]$ was moved to the last index.

   induction step: $i \curvearrowright i - 1$

   induction hypothesis

   (i) $A[i, \ldots, n]$ is sorted non-descended

   (ii) $\forall k, l \in \mathbb{N} : 1 \leq k < i \wedge i \leq j \leq n \implies A[k] \leq A[j]$

   Because of the induction hypothesis we now that $A[i, \ldots, n]$ is sorted non-descending (i) and that $A[i-1] \leq A[i]$ (ii), so $A[i-1, \ldots, n]$ is sorted non-descending. Let $A[m]$ be the greatest element in $A[1, \ldots, i]$, that means $A[(m+1) - 1] \geq A[m+1]$, so it would be moved to the next index, after increasing i, j, m, by one this would repeat until $j = i$ and $A[m]$ was moved to the index i. So that $\forall k \in \mathbb{N} : 1 \leq k < i : A[k] \leq A[i]$. And because we know that $\forall k, l \in \mathbb{N} : 1 \leq k < i \wedge i \leq j \leq n \implies A[k] \leq A[j]$, because of the induction hypothesis, we have now proven (i) and (ii). ■

c) (i) We have proven, that $\mathcal{INV}$ is a loop invariant.

   (ii) In the beginning the Array is unsorted, but the sorted area is empty, so the $\mathcal{INV}$ is true

   (iii) after the $n-1$th iteration the Array $A[1, \ldots, n]$ is sorted non-descending because the Array $A[2, \ldots, n]$ is sorted non-descending and the element $A[1]$ is not greater than every element in the Array $A[2, \ldots, n]$.

(iv) the loop terminates after $n-1$ iterations

Exercise 2: Hoar-Logic - Analysis

a) $x - 2 < 0 \iff x < 2$

b) $z - 5 > 5 \iff z > 10$

Exercise 3: Dynamic Programming - Knapsack

B	0	1	2	3	4	5	6
I_1	0	2	4	6	7	9	11
I_2	0	2	4	6	7	9	11
I_3	0	1	4	5	7	9	10
I_4	0	1	4	5	7	9	10
using only other items	0	1	4	5	7	8	9