University of Konstanz
Faculty of Science
(Mathematics and Statistics)

Übungsblatt 8

Elias Gestrich

Aufgabe 1: Taylor-Entwicklung I

$$
\begin{aligned}
\partial_x f &= \frac{1(x+y)-(x-y)\cdot 1}{(x+y)^2} = \frac{2y}{(x+y)^2} \\
\partial_y f &= \frac{(-1)\cdot(x+y)-(x-y)\cdot 1}{(x+y)^2} = -\frac{2x}{(x+y)^2} \\
\partial_x^2 f &= -\frac{4y}{(x+y)^3} \\
\partial_y\partial_x f &= \frac{2(x+y)^2 - 2y\cdot 2\cdot(x+y)}{(x+y)^4} = \frac{2(x+y)-4y}{(x+y)^3} = \frac{2(x-y)}{(x+y)^3} \\
\partial_x\partial_y f &= -\frac{2(x+y)^2 - 2x\cdot 2\cdot(x+y)}{(x+y)^4} = -\frac{2(x+y)-4x}{(x+y)^3} = -\frac{2(-x+y)}{(x+y)^3} = \frac{2(x-y)}{(x+y)^3} \\
\partial_y^2 f &= \frac{4x}{(x+y)^3}
\end{aligned}
$$

0. Glied:

$$\sum_{|\alpha|=0} \frac{\partial^\alpha f(1,1)}{\alpha!}\xi^\alpha = f(1,1) = 0$$

1. Glied:

$$\sum_{|\alpha|=1} \frac{\partial^\alpha f(1,1)}{\alpha!}\xi^\alpha = \frac{\partial_x f(1,1)}{1!}\xi_x + \frac{\partial_y f(1,1)}{1!}\xi_y = \frac{2}{2^2}\xi_x - \frac{2}{2^2}\xi_y = \frac{1}{2}\xi_x - \frac{1}{2}\xi_y = \frac{1}{2}\langle(1,-1),\xi\rangle$$

2. Glied:

$$
\begin{aligned}
\sum_{|\alpha|=1} \frac{\partial^\alpha f(1,1)}{\alpha!}\xi^\alpha &= \frac{\partial_x^2 f(1,1)}{2!0!}\xi_x^2 + \frac{\partial_x\partial_y f(1,1)}{1!1!}\xi_x\xi_y + \frac{\partial_y^2 f(1,1)}{0!2!}\xi_x^2 \\
&= \frac{\frac{4}{2^3}}{2}\xi_x^2 + 0 + \frac{\frac{4}{2^3}}{2}\xi_y^2 \\
&= \frac{1}{4}\xi_x^2 + \frac{1}{4}\xi_y^2 \\
&= \frac{1}{4}\langle\xi,\xi\rangle \\
&= \frac{1}{4}\|\xi\|_2^2
\end{aligned}
$$

Also ist die Taylor-Entwicklung der Funktion f im Punkt $x_0 = (1,1)$ bis einschließlich den Gliedern 2. Ordnung:

$$T_{x_0}f(x_0+\xi) = 0 + \frac{1}{2}\langle(1,-1),\xi\rangle + \frac{1}{4}\|\xi\|_2^2$$

$$T_{x_0}f(x) = 0 + \frac{1}{2}\langle(1,-1),(x-x_0)\rangle + \frac{1}{4}\|(x-x_0)\|_2^2$$

Aufgabe 2: Taylor-Entwicklung II

$$\begin{aligned}
\partial_x f &= 0 + (y+\cos(y))\cos(x) \\
\partial_y f &= (1-\sin(y))\sin(x) + 0 \\
\partial_x^2 f &= -(y+\cos(y))\sin(x) \\
\partial_y\partial_x f &= (1-\sin(y))\cos(x) \\
\partial_x\partial_y f &= (1-\sin(y))\cos(x) \\
\partial_y^2 f &= -\cos(y)\sin(x)
\end{aligned}$$

0. Glied:

$$\sum_{|\alpha|=0} \frac{\partial^\alpha f\left(\frac{\pi}{2},0\right)}{\alpha!}\xi^\alpha = f\left(\frac{\pi}{2},0\right) = (0+1)\cdot 1 = 1$$

1. Glied:

$$\sum_{|\alpha|=1} \frac{\partial^\alpha f\left(\frac{\pi}{2},0\right)}{\alpha!}\xi^\alpha = \frac{\partial_x f\left(\frac{\pi}{2},0\right)}{1!}\xi_x + \frac{\partial_y f\left(\frac{\pi}{2},0\right)}{1!}\xi_y = (0+1)\cdot 0\cdot\xi_x + (1-0)\cdot 1\cdot\xi_y = \langle(0,1),\xi\rangle$$

2. Glied:

$$\begin{aligned}
\sum_{|\alpha|=2} \frac{\partial^\alpha f\left(\frac{\pi}{2},0\right)}{\alpha!}\xi^\alpha &= \frac{\partial_x^2 f\left(\frac{\pi}{2},0\right)}{2!0!}\xi_x^2 + \frac{\partial_x\partial_y f\left(\frac{\pi}{2},0\right)}{1!1!}\xi_x\xi_y + \frac{\partial_y^2 f\left(\frac{\pi}{2},0\right)}{0!2!}\xi_y^2 \\
&= \frac{-(0+1)\cdot 1}{2}\xi_x^2 + (1-0)\cdot 0\xi_x\xi_y - \frac{1\cdot 1}{2}\xi_y^2 \\
&= -\frac{1}{2}\xi_x^2 - \frac{1}{2}\xi_y^2 \\
&= -\frac{1}{2}\|\xi\|_2^2
\end{aligned}$$

Also ist die Taylor-Entwicklung der Funktion f im Punkt $x_0 = \left(\frac{\pi}{2},0\right)$ bis einschließlich den Gliedern 2. Ordnung:

$$T_{x_0}f(x_0+\xi) = 1 + \langle(0,1),\xi\rangle - \frac{1}{2}\|\xi\|_2^2$$

$$T_{x_0}f(x) = 1 + \langle(0,1),(x-x_0)\rangle - \frac{1}{2}\|(x-x_0)\|_2^2$$