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(Physics)

Übungsblatt 06

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Aufgabe 1: Oberflächenladungsdichte

a)

$$\begin{aligned}
\varphi(z) &= \frac{1}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{\sigma}{\sqrt{z^2+\rho'^2}}\rho'\ \mathrm{d}\psi'\ \mathrm{d}\rho' + C\\
&= \frac{1}{4\pi\varepsilon_0}\int_0^R\frac{2\pi\rho'\sigma}{\sqrt{z^2+\rho'^2}}\ \mathrm{d}\rho' + C\\
&= \frac{\sigma}{2\varepsilon_0}\int_0^R\frac{\rho'}{\sqrt{z^2+\rho'^2}}\ \mathrm{d}\rho' + C\\
&= \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+\rho'^2}\right]_0^R + C\\
&= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right) + C\\
&= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right) + C
\end{aligned}$$

Dabei soll $\varphi(0) = 0$, also

$$\begin{aligned}
0 &\stackrel{!}{=} \varphi(0)\\
0 &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2}-0\right) + C\\
0 &= \frac{\sigma}{2\varepsilon_0}R + C\\
C &= -\frac{\sigma}{2\varepsilon_0}R
\end{aligned}$$

Also

$$\begin{aligned}
\varphi(z) &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z-R\right)\\
\varphi(z) &= -\frac{\sigma}{2\varepsilon_0}\left(z-\sqrt{z^2+R^2}+R\right)
\end{aligned}$$

b)

$$\begin{aligned}
\vec{E}(z) &= -\nabla\varphi(z)\\
&= \left(0, 0, \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\right)\\
&= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\vec{e}_z
\end{aligned}$$

x, y Koordinaten sind 0, da die Platte symmetrisch und somit in x, y-Richtungen sich die Felder ausgleichen.

c) Für $z \gg R$, also $\frac{R}{z} \sim 0$:

$$\begin{aligned}
\varphi(z) &= -\frac{\sigma}{4\varepsilon_0}\left(z - \sqrt{z^2+R^2} + R\right) \\
&= -\frac{\sigma}{2\varepsilon_0}\left(z - z\sqrt{1+\left(\frac{R}{z}\right)^2} + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}\left(z - z\left(1+\frac{\left(\frac{R}{z}\right)^2}{2}\right) + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}\left(z - z\left(1+\frac{(0)^2}{2}\right) + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}(z - z + R) \\
&\sim -\frac{R\sigma}{2\varepsilon_0}
\end{aligned}$$

und

$$\begin{aligned}
\vec{E}(z) &= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\vec{e}_z \\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{z\sqrt{1+\left(\frac{R}{z}\right)^2}}\right)\vec{e}_z \\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{\sqrt{1+\left(\frac{R}{z}\right)^2}}\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}\left(1 - \left(1 - \left(\frac{R}{z}\right)^2\right)\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}(1 - (1 - (0)))\,\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}(1 - 1)\,\vec{e}_z \\
&\sim 0
\end{aligned}$$

Für $R \to \infty$, also $\frac{z}{R} \sim 0$:

$$\begin{aligned}
\varphi(z) &= -\frac{\sigma}{4\varepsilon_0}\left(z - \sqrt{z^2+R^2} + R\right) \\
&= -\frac{\sigma}{2\varepsilon_0}\left(z - R\sqrt{1+\left(\frac{z}{R}\right)^2} + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}\left(z - R\left(1+\frac{\left(\frac{z}{R}\right)^2}{2}\right) + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}\left(z - R\left(1+\frac{(0)^2}{2}\right) + R\right) \\
&\sim -\frac{\sigma}{2\varepsilon_0}\left(z - R + R\right) \\
&\sim -\frac{z\sigma}{2\varepsilon_0}
\end{aligned}$$

und

$$\begin{aligned}
\vec{E}(z) &= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\vec{e}_z \\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{R\sqrt{1+\left(\frac{z}{R}\right)^2}}\right)\vec{e}_z \\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{R\sqrt{1+\left(\frac{z}{R}\right)^2}}\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}\left(1 - \left(\frac{z}{R} - \frac{z}{R}\left(\frac{R}{z}\right)^2\right)\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}\left(1 - (0-0)\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}\left(1 - 0\right)\vec{e}_z \\
&\sim \frac{\sigma}{2\varepsilon_0}\vec{e}_z
\end{aligned}$$

Aufgabe 2: Multipolentwicklung

a)

$$\begin{aligned}
\vec{p} &= \int_V \rho(\vec{r})\vec{r}\ \mathrm{d}^3r \\
&= \int_V q_1\delta(\vec{r}-\vec{r}_1)\vec{r}\ \mathrm{d}^3r + \int_V q_2\delta(\vec{r}-\vec{r}_2)\vec{r}\ \mathrm{d}^3r \\
&\quad + \int_V q_3\delta(\vec{r}-\vec{r}_3)\vec{r}\ \mathrm{d}^3r + \int_V q_4\delta(\vec{r}-\vec{r}_4)\vec{r}\ \mathrm{d}^3r \\
&= q_1\vec{r}_1 + q_2\vec{r}_2 + q_3\vec{r}_3 + q_4\vec{r}_4 \\
&= q_1\vec{r}_1 + q_2\vec{r}_2 - q_3\vec{r}_1 - q_4\vec{r}_2 \\
&= (q_1-q_3)\vec{r}_1 + (q_2-q_4)\vec{r}_2 \\
&= (q_1-q_3)(a,0,0) + (q_2-q_4)(0,a,0) \\
&= a((q_1-q_3),(q_2-q_4),0)
\end{aligned}$$

Für $\vec{p}=0$ muss also $q_1=q_3$ und $q_2=q_4$ gelten

b) für $k \neq l$ gilt:

$$\begin{aligned}
Q_{kl} &= \int_V \left(3r_kr_l - \delta_{kl}r^2\right)\rho(\vec{r})\ \mathrm{d}^3r \\
&= \int_V 3r_kr_l\rho(\vec{r})\ \mathrm{d}^3r \\
&= \int_V 3r_kr_lq_1\delta(\vec{r}-\vec{r}_1)\ \mathrm{d}^3r + \int_V 3r_kr_lq_2\delta(\vec{r}-\vec{r}_2)\ \mathrm{d}^3r \\
&\quad + \int_V 3r_kr_lq_3\delta(\vec{r}-\vec{r}_3)\ \mathrm{d}^3r + \int_V 3r_kr_lq_4\delta(\vec{r}-\vec{r}_4)\ \mathrm{d}^3r \\
&= 3r_{1_k}r_{1_l}q_1 + 3r_{2_k}r_{2_l}q_2 + 3r_{3_k}r_{3_l}q_3 + 3r_{4_k}r_{4_l}q_4
\end{aligned}$$

da bei r_1, r_2, r_3, r_4 nur eine Komponente nicht null ist, ist $r_{i_k}=0$ oder $r_{i_l}=0$.
also $Q_{kl} = 3\cdot 0q_1 + 3\cdot 0q_2 + 3\cdot 0q_3 + 3\cdot 0q_4 = 0$ für $k\neq l$.

Für $k=l=x$:

$$\begin{aligned}
Q_{xx} &= \int_V (3r_x^2 - \delta_{xx}r^2)\rho(\vec{r})\ \mathrm{d}^3r \\
&= \int_V (3r_x^2 - r^2)\rho(\vec{r})\ \mathrm{d}^3r \\
&= \int_V (3r_x^2-r^2)q_1\delta(\vec{r}-\vec{r}_1)\ \mathrm{d}^3r + \int_V (3r_x^2-r^2)q_2\delta(\vec{r}-\vec{r}_2)\ \mathrm{d}^3r \\
&\quad + \int_V (3r_x^2-r^2)q_3\delta(\vec{r}-\vec{r}_3)\ \mathrm{d}^3r + \int_V (3r_x^2-r^2)q_4\delta(\vec{r}-\vec{r}_4)\ \mathrm{d}^3r \\
&= (3r_{1_x}^2 - r_1^2)q_1 + (3r_{2_x}^2 - r_2^2)q_2 + (3r_{3_x}^2 - r_3^2)q_3 + (3r_{4_x}^2 - r_4^2)q_4 \\
&= (3a^2-a^2)q_1 + (3\cdot 0^2 - a^2)q_2 + (3(-a)^2 - a^2)q_3 + (3\cdot 0^2 - a^2)q_4 \\
&= 2a^2q_1 - a^2q_2 + 2a^2q_3 - a^2q_4 \\
&= a^2(2q_1 - q_2 + 2q_3 - q_4) \\
&= a^2(2(q_1+q_3) - (q_2+q_4))
\end{aligned}$$

Für $k = l = y$:

$$\begin{aligned}
Q_{yy} &= \int_V (3r_y^2 - \delta_{yy} r^2)\rho(\vec{r}) \, \mathrm{d}^3 r \\
&= \int_V (3r_y^2 - r^2)\rho(\vec{r}) \, \mathrm{d}^3 r \\
&= \int_V (3r_y^2 - r^2) q_1 \delta(\vec{r} - \vec{r}_1) \, \mathrm{d}^3 r + \int_V (3r_y^2 - r^2) q_2 \delta(\vec{r} - \vec{r}_2) \, \mathrm{d}^3 r \\
&\quad + \int_V (3r_y^2 - r^2) q_3 \delta(\vec{r} - \vec{r}_3) \, \mathrm{d}^3 r + \int_V (3r_y^2 - r^2) q_4 \delta(\vec{r} - \vec{r}_4) \, \mathrm{d}^3 r \\
&= (3r_{1_y}^2 - r_1^2) q_1 + (3r_{2_y}^2 - r_2^2) q_2 + (3r_{3_y}^2 - r_3^2) q_3 + (3r_{4_y}^2 - r_4^2) q_4 \\
&= (3 \cdot 0^2 - a^2) q_1 + (3 \cdot a^2 - a^2) q_2 + (3 \cdot 0^2 - a^2) q_3 + (3 \cdot (-a)^2 - a^2) q_4 \\
&= -a^2 q_1 + 2a^2 q_2 - a^2 q_3 + 2a^2 q_4 \\
&= a^2(-q_1 + 2q_2 - q_3 + 2q_4) \\
&= a^2(2(q_2 + q_4) - (q_1 + q_3))
\end{aligned}$$

Für $k = l = z$:

$$\begin{aligned}
Q_{zz} &= \int_V (3r_z^2 - \delta_{zz} r^2)\rho(\vec{r}) \, \mathrm{d}^3 r \\
&= \int_V (3r_z^2 - r^2)\rho(\vec{r}) \, \mathrm{d}^3 r \\
&= \int_V (3r_z^2 - r^2) q_1 \delta(\vec{r} - \vec{r}_1) \, \mathrm{d}^3 r + \int_V (3r_z^2 - r^2) q_2 \delta(\vec{r} - \vec{r}_2) \, \mathrm{d}^3 r \\
&\quad + \int_V (3r_z^2 - r^2) q_3 \delta(\vec{r} - \vec{r}_3) \, \mathrm{d}^3 r + \int_V (3r_z^2 - r^2) q_4 \delta(\vec{r} - \vec{r}_4) \, \mathrm{d}^3 r \\
&= (3r_{1_z}^2 - r_1^2) q_1 + (3r_{2_z}^2 - r_2^2) q_2 + (3r_{3_y}^2 - r_3^2) q_3 + (3r_{4_y}^2 - r_4^2) q_4 \\
&= (3 \cdot 0^2 - a^2) q_1 + (3 \cdot 0^2 - a^2) q_2 + (3 \cdot 0^2 - a^2) q_3 + (3 \cdot i_0^2 - a^2) q_4 \\
&= -a^2 q_1 - a^2 q_2 - a^2 q_3 - a^2 q_4 \\
&= a^2(-q_1 - q_2 - q_3 - q_4) \\
&= -a^2(q_1 + q_2 + q_3 + q_4)
\end{aligned}$$

$$a^2 \begin{pmatrix} 2(q_1 + q_3) - (q_2 + q_4) & 0 & 0 \\ 0 & 2(q_2 + q_4) - (q_1 + q_3) & 0 \\ 0 & 0 & -(q_1 + q_2 + q_3 + q_4) \end{pmatrix}$$

Damit alles Null:

$$\begin{aligned}
-(q_1 + q_2 + q_3 + q_4) &= 0 \\
-(q_2 + q_4) &= (q_1 + q_3) \qquad (1)
\end{aligned}$$

und

$$\begin{aligned}
2(q_1 + q_3) - (q_2 + q_4) &= 0 \\
2(q_1 + q_3) + (q_1 + q_3) &= 0 \\
3(q_1 + q_3) &= 0 \\
q_1 + q_3 &= 0 \\
q_1 &= -q_3
\end{aligned}$$

Also in (1) eingesetzt

$$
\begin{aligned}
-(q_2 + q_4) &= (q_1 + q_3) \\
-(q_2 + q_4) &= 0 \\
q_2 + q_4 &= 0 \\
q_2 &= -q_4
\end{aligned}
$$

Zum Überprüfen:

$$2(q_2 + q_4) - (q_1 + q_3) = 2 \cdot 0 + 0$$

bascht Also muss gelten $q_1 = -q_3$ und $q_2 = -q_4$

c) Aus a) muss folgen $q_1 = q_3$ und $q_2 = q_4$ wegen b) folgt aber auch $q_1 = -q_3 \overset{a)}{=} -q_1 \implies q_1 = q_3 = 0$ und $q_2 = -q_4 \overset{a)}{=} -q_2 \implies q_2 = q_4 = 0$ Also neeee nur für $q_1 = q_2 = q_3 = q_4 = 0$