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Gruppe 3, Jonathan Weihing

University of Konstanz
Faculty of Science
(Mathematics and Statistics)

Übungsblatt 7

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$$
\begin{aligned}
\partial_i \|x\| &= \partial_i \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \\
&= (2x_i) \cdot \frac{1}{2} \left(\sum_{i=1}^{n} x_i^2\right)^{-\frac{1}{2}} \\
&= \frac{(x_i)}{\|x\|}
\end{aligned}
$$

$$
\begin{aligned}
\partial_i \|x\|^2 &= \partial_i \left(\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\right)^2 \\
\partial_i \|x\|^2 &= \partial_i \sum_{i=1}^{n} x_i^2 \\
&= (2x_i)
\end{aligned}
$$

Aufgabe 1: Wärmeleitungskern

$$\begin{aligned}
\Delta F &= \sum_{i=1}^{n} \partial_i^2 F \\
&= \sum_{i=1}^{n} \partial_i^2 t^{-\frac{n}{2}} \exp\left(-\frac{\|x\|^2}{4t}\right) \\
&= \sum_{i=1}^{n} \partial_i - \frac{2x_i}{4t} t^{-\frac{n}{2}} \exp\left(-\frac{\|x\|^2}{4t}\right) \\
&= \sum_{i=1}^{n} \left(-\frac{2}{4t} t^{-\frac{n}{2}} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) + \left(\frac{4x_i^2}{16t^2} t^{-\frac{n}{2}} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) \\
&= \left(-\frac{1}{2} t^{-\frac{n}{2}-1} \exp\left(-\frac{\|x\|^2}{4t}\right)\right)\left(\sum_{i=1}^{n} 1\right) + \left(\frac{1}{4} t^{-\frac{n}{2}-2} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) \sum_{i=1}^{n} x_i^2 \\
&= \left(-\frac{n}{2} t^{-\frac{n}{2}-1} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) + \left(\frac{\|x\|^2}{4} t^{-\frac{n}{2}-2} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) \\
&= \left(\frac{\|x\|^2 t^{-\frac{n}{2}-2}}{4} - \frac{nt^{-\frac{n}{2}-1}}{2}\right) \exp\left(-\frac{\|x\|^2}{4t}\right)
\end{aligned}$$

und

$$\begin{aligned}
\frac{\partial F}{\partial t} &= \left(-\frac{n}{2} t^{-\frac{n}{2}-1} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) - \left(\left(-\frac{\|x\|^2}{4t^2}\right) t^{-\frac{n}{2}} \exp\left(-\frac{\|x\|^2}{4t}\right)\right) \\
&= \left(\frac{\|x\|^2 t^{-\frac{n}{2}-2}}{4} - \frac{nt^{-\frac{n}{2}-1}}{2}\right) \exp\left(-\frac{\|x\|^2}{4t}\right)
\end{aligned}$$

also $\Delta F = \frac{\partial F}{\partial t}$, also $\Delta F - \frac{\partial F}{\partial t} = 0$

Aufgabe 2: Harmonische Funktionen

Für $n = 2$:

$$\sum_{i=1}^{2} \partial_i^2 \ln|x| = \sum_{i=1}^{2} \partial_i \frac{x_i}{|x|} \cdot \frac{1}{|x|} = \sum_{i=1}^{2} \partial_i \frac{x_i}{|x|^2} = \sum_{i=1}^{2} \frac{1 \cdot |x|^2 - x_i \cdot 2x_i}{|x|^4} = \frac{2 \cdot |x|^2 - 2|x|^2}{|x|^4} = 0$$

Für $n > 2$

$$\begin{aligned}
\sum_{i=1}^{n} \partial_i^2 |x|^{2-n} &= \sum_{i=1}^{n} \partial_i \frac{x_i}{|x|} |x|^{1-n} \\
&= \sum_{i=1}^{n} \partial_i \frac{x_i}{|x|^n} \\
&= \sum_{i=1}^{n} \frac{|x|^n - x_i \cdot \frac{x_i}{|x|} \cdot n |x|^{n-1}}{|x|^{2n}} \\
&= \sum_{i=1}^{n} \frac{|x|^n - n x_i^2 |x|^{n-2}}{|x|^{2n}} \\
&= \frac{n |x|^n - n |x|^2 |x|^{n-2}}{|x|^{2n}} \\
&= \frac{n |x|^n - n |x|^n}{|x|^{2n}} \\
&= 0
\end{aligned}$$