Experimental runtime analysis of algorithms for sparse binary matrix-vector products

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Overview

- Goals
- Related Work
- Preliminaries
- Ideas
- Implementation
- Results
- Future Work

Goals

- Review currently available high-performance algorithms for SBM-V multiplication
- Discuss potential improvements/propose new algorithms
- Implement and benchmark the proposed algorithm
- Evaluate benchmark results

Related Work

- Experimentation on different architectures
  - Sparse and dense matrix multiplication hardware for heterogeneous multi-precision neural networks [4]
  - Sparse matrix multiplication on an associative processor [7]
  - Sparse Binary Matrix-Vector Multiplication on Neuromorphic Computers [5]
- Libraries
  - A Sparse Matrix Library in C++ for High Performance Architectures [1]
  - SparseX [2]
- General algorithms
  - Fast Sparse Matrix Multiplication [8]
  - Automatic Performance Tuning of Sparse Matrix Kernels [6]
  - Mailman algorithm [3]

Ideas

- Use appropriate data structure
  - CSR / Modified CSR
- Use different algorithm
  - Mailman
  - Precompute results to a map
  - Partial sum method

CSR / Modified CSR

- Efficient sparse matrix storage format
- For binary matrices V array can be omitted
- More efficient in storage if $NNZ < \frac{m(n-1)-1}{2}$
- Runtime depends on number of non-zero entries: $\mathcal{O}(NNZ)$

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix}$$

```
V         = [ 5 8 3 6 ]
COL_INDEX = [ 0 1 2 1 ]
ROW_INDEX = [ 0 1 2 3 4 ]
```

Figure: Example for matrix storage in CSR format. Source: Wikipedia

Mailman

- Works for matrices over fininte alphabets: $M \in \Sigma^{m \times n}$
- In binary case: $\Sigma = \{0, 1\}$
- Decompose $M = UP$, where
  - $U_n \in \Sigma^{n \times |\Sigma|^n}$ - contains all possible columns over Σ of length n
  - $P(i, j) = \delta(U^{(i)}, A^{(j)})$ - contains n ones, rest are zero
- Construct P - preprocess in $\mathcal{O}(nm)$
- U can be applied in $\mathcal{O}(4n)$ maximum - recursive construction and application
- Final runtime: $\mathcal{O}(mn/\log m)$

Precompute to map

- Use a map with rows in binary form as keys and corresponding sum result as value
- Precompute all key-value pairs, where row contains at most k ones
- Get result vector entries by querying the map
  - Hit $\rightarrow$ use the saved value
  - Miss $\rightarrow$ compute the value using the naive algorithm
- Multiplying the same vector with a different matrix does not require precomputation again
- Optimal value of k depends on sparsity
  - k too low $\rightarrow$ many misses
  - k too large $\rightarrow$ precomputation too expensive (can be offset by large amount of SPMV products with the same vector)

Partial sum method

- Each row corresponds to a partial sum of input vector elements
- $X = \{M_{(k)} | k \in \{0, 1, \dots, n\}\}$
- Find set of rows $K \subseteq X$, where $\bigoplus_{x \in K} x = y$, s.t. $y \in X \setminus K$
- Try to find as many rows like y as possible to reduce $\#$ of additions

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure: Number of additions reduced from 8 to 4

Implementation

- Software framework
- Software design

Software framework

- C++23 for main computations
- Python with NumPy, Pandas and Matplotlib for plot creation
- Makefile generates executable files
- GoogleTest for unit and performance tests
- Included OpenBLAS library to compare performance
- Codebase can be found on GitHub

Software design

- `IMatrix<T>` interface is implemented by multiple classes - uses templating to set data type stored
- Matrix types
  - `Matrix<N,M,T>` - general
  - `SparseMatrix<T>` - CSR storage format
  - `SparseBoolMatrix` - without *VALUES*
  - `RawBoolMatrix<N,M>` - for BLAS operations
  - `BitsetMatrix<N,M>` - uses an $N \times M$ bitset data structre
- RandomMatrixGenerator
- **MatrixProduct**
- Utils

Implemented algorithms

- Naive
- Optimized - using SparseBoolMatrix
- BLAS - using `cblas_dgemv` call
- Precompute to map

Tests

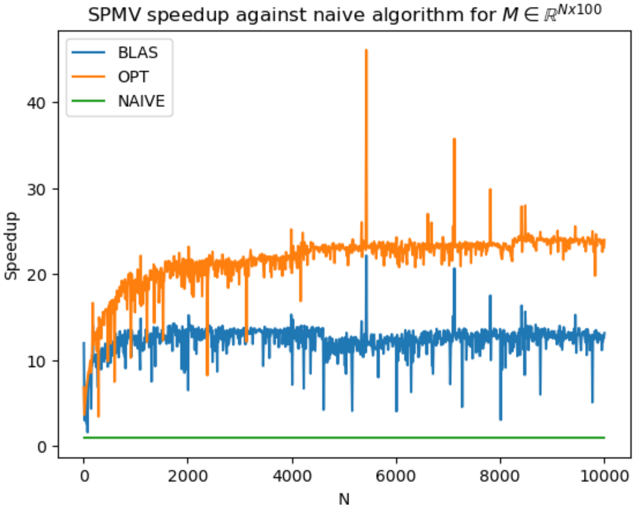
- Invoked using GoogleTest
- Unit tests for correctness
  - Matrix classes
  - Map key generation
  - Matrix product tests
- Performance tests - generate results to CSV for plotting
  - changing sparsity
  - excluding values below a certain threshold
  - changing matrix dimensions

Issues

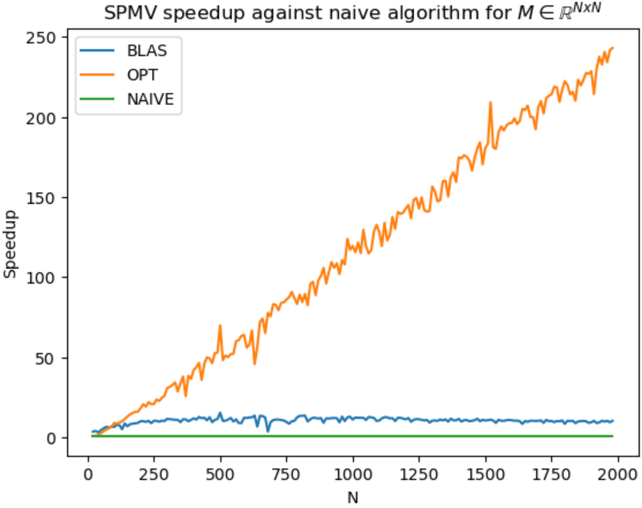
- Precompute to map $\rightarrow$ map access was too expensive in practice
- Partial sum method
  - Difficult to find optimal way to gather row sets, which fit the condition
  - Since addition is cheap (compared to multiplication), generally it is faster
  - More efficient for dense matrices, since number of additions is much higher
- Couldn't instantiate dimension tests at runtime, because of templating
  $\rightarrow$ `generate_tests.ipynb` helps to write test cases individually to file

Results I

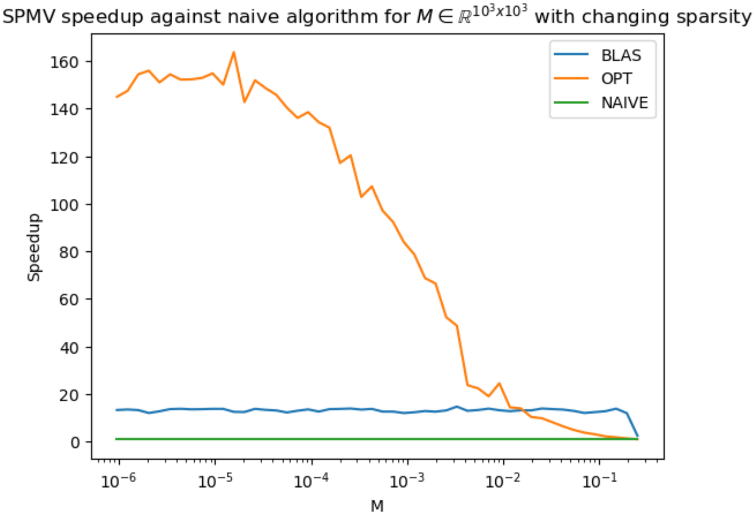
- Sparsity set to $1/(N + M)$



Results II

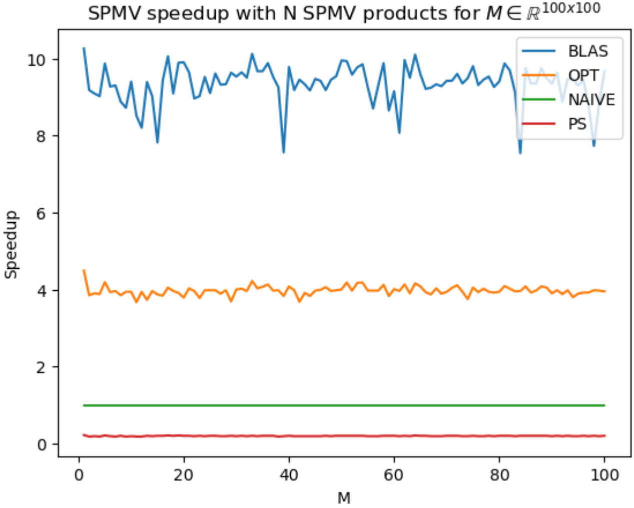

SPMV speedup against naive algorithm for $M \in \mathbb{R}^{NxN}$
BLAS
OPT
NAIVE
Speedup
250
200
150
100
50
0
0
250
500
750
1000
1250
1500
1750
2000
N

Results III


SPMV speedup against naive algorithm for $M \in \mathbb{R}^{10^3 x 10^3}$ with changing sparsity
Speedup
160
140
120
100
80
60
40
20
0
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
M
BLAS
OPT
NAIVE

Results IV

- Sparsity $= 0.5$



Conclusion & Future Work

- Main factor for runtime reduction is sparsity
- General requirement - efficient bit manipulation
- Implement map precompute with more efficient map data structure

Literature

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