Prof. David Draper
Department of Statistics
University of California, Santa Cruz

STAT 131: Quiz 6 *[50 total points]*

Name: _____

In a problem you're working on, you need to simulate random draws from the following PDF for the continuous random variable Y:

$$\underline{f_Y(y)} = \left\{ \begin{array}{cc} \frac{3}{2} - y & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{array} \right\} . \quad (1)$$

$Y_1, Y_2, \cdots Y_n \sim$ pdf

(a) Sketch the PDF in equation (1) for y in the interesting range $[0, 1]$. *[10 points]*

(b) Work out the CDF $F_Y(y)$ for Y, specifying its values for all $-\infty < y < +\infty$, and sketch it in the interesting range $0 \le y \le 1$. *[10 points]*

$$F_Y(y) = \frac{3}{2}y - \frac{1}{2}y^2$$

(c) Work out the inverse CDF (quantile function) $F_Y^{-1}(p)$ for Y, specifying its values for all $0 < p < 1$, and sketch it for p in that range. *[10 points]*

$$p = \frac{3}{2}y - \frac{1}{2}y^2$$

$$2p = 3y - y^2$$

$$y^2 - 3y + 2p = 0$$

$$\left(y - \frac{3}{2}\right)^2 = \frac{9}{4} - 2p$$

$$y = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2p}$$

$$F_Y^{-1}(p) = \frac{3}{2} - \sqrt{\frac{9}{4} - 2p}$$

inverse CDF (quantile function) of p $0<p<1$

(d) Building on your result in part (c), explicitly specify how you can generate IID random draws from the PDF in equation (1). *[10 points]*

(d) inverse CDF sampling

1. Sample $p_1, p_2, p_3 \cdots p_n \sim$ Unif $(0,1)$ ← follows the distribution
2. Calculate $Y_1 = F_Y^{-1}(p_1)$, $Y_2 = F_Y^{-1}(p_2) \cdots, Y_n = F_Y^{-1}(p_n)$ inverse cdf value
3. Sample $Y_1, Y_2, \cdots Y_n \sim f_Y(y)$ or $\sim F_Y(y)$ ← iid random draws

Why this work?

We want to show $p \sim$ Unif $(0,1)$, $Y = F_Y^{-1}(p)$ ①;

show $Y \sim$ pdf $f_Y(y)$ or $Y \sim$ cdf $F_Y(y)$. ③

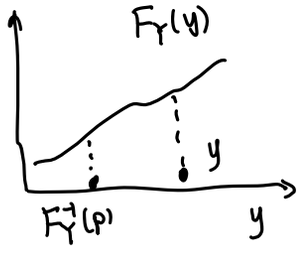
Y follows

$Pr(Y \le y) \stackrel{?}{=} F_Y(y)$ CDF

$Pr(Y \le y) = Pr(F_Y^{-1}(p) \le y)$ ..... ①

$= Pr(F_Y[F_Y^{-1}(p)] \le F_Y(y))$

---- .. ② F_Y is a cdf, it's increasing function



$= Pr(p \le F_Y(y))$

$= F_Y(y)$ ---- ③

(e) Once you have your random sample in part (d), briefly explain how you could graphically check whether it really *is* a sample from the PDF in equation (1). *[10 points]*

$Y_1, \cdots Y_n$ our samples.

graphically check whether they follows pdf in equation (1).

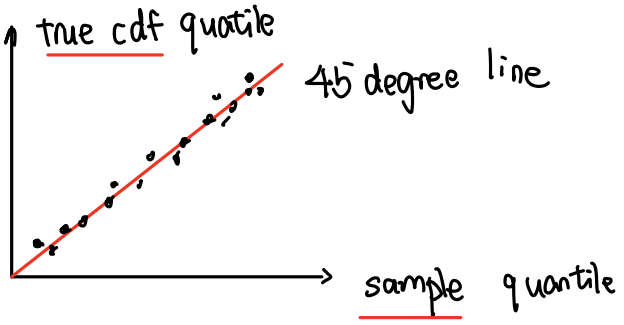
qq plot : quantile - quantile plot

$Y_1, Y_2, \cdots Y_n$ are our samples.

Calculate sample's quantile.

Compare our sample's quantile to true pdf quantile.

make qqplot.



If they're following 45 degree line, our sample quantile = true pdf quantile.

our iid sample ~ pdf. in (1).

$Y_1, Y_2, \cdots Y_n$.

median

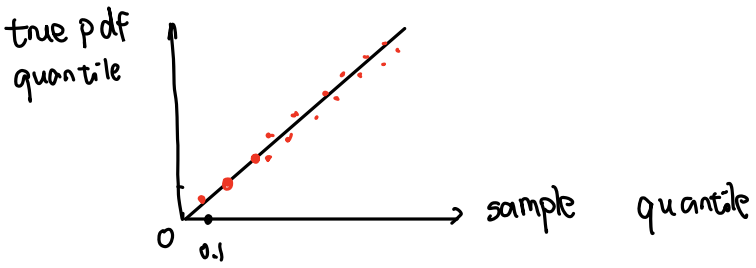
sample quantile: 0.1 → 0%, 5%, 15%, ... 40%, 50%, 60% ... 100%

= 0.15

true pdf quatile. $F_Y^{-1}(0), F_Y^{-1}(0.05), F_Y^{-1}(0.15) - F_Y^{-1}(0.5) \cdots$

$F_Y^{-1}(p)$ (inverse cdf / quantile function)

= 0.11 = 0.15


true pdf
quantile
0
0.1
sample
quantile