数值分析理论作业

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问题 1. §3.T27

证明: 数据 $\{(x_i, y_i)\}_{i=1}^m$ 的最小二乘拟合函数 $p(x) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1}$ 恰是经过点集 $\{(x_i, y_i)\}_{i=1}^m$ 的 Lagrange 插值多项式.

记经过点集 $\{(x_i, y_i)\}_{i=1}^m$ 的 Lagrange 插值多项式为 $q(x) \in P_{m-1}$, 则 $q(x_i) = y_i, i = 1, 2, \cdots, m$, 则

$$0 \le \frac{1}{m}\sum_{i=1}^m (p(x_i) - y_i)^2 = \min_{f(x)\in P_{m-1}} \frac{1}{m}\sum_{i=1}^m (f(x_i) - y_i)^2 \le \frac{1}{m}\sum_{i=1}^m (q(x_i) - y_i)^2 = 0$$

于是 $\frac{1}{m}\sum_{i=1}^m (p(x_i) - y_i)^2 = 0$, 所以 $p(x_i) = y_i = q(x_i), i = 1, 2, \cdots, m$, 再由 $\deg(p(x) - q(x)) \le m - 1$ 得到 $p(x) = q(x)$, 于是 $p(x)$ 是经过点集 $\{(x_i, y_i)\}_{i=1}^m$ 的 Lagrange 插值多项式.

问题 2. §3.T28

记 $P_2 = Span\{1, x, x^2\}$ 是二次多项式空间, 求 $p(x) \in P_2$ 使得

$$\int_0^2 (x-1)^2 [|x-1|^3 - p(x)]^2 \mathrm{d}x$$

达到最小.

对积分区间平移, 题目变为求 $p(x) \in P_2$

$$\mathcal{L}(p) = \int_0^2 (x-1)^2 [|x-1|^3 - p(x)]^2 \mathrm{d}x = \int_{-1}^1 x^2 [|x|^3 - p(x+1)]^2 \mathrm{d}x$$

达到最小.

设 $p(x+1) = a_0 + a_1 x + a_2 x^2$, 则

$$\begin{cases} \dfrac{\partial}{\partial a_0}\mathcal{L}(p) = \displaystyle\int_{-1}^1 2x^2 [|x|^3 - p(x+1)]\mathrm{d}x = 0, \\ \dfrac{\partial}{\partial a_1}\mathcal{L}(p) = \displaystyle\int_{-1}^1 2x^3 [|x|^3 - p(x+1)]\mathrm{d}x = 0, \\ \dfrac{\partial}{\partial a_2}\mathcal{L}(p) = \displaystyle\int_{-1}^1 2x^4 [|x|^3 - p(x+1)]\mathrm{d}x = 0, \end{cases}$$

整理得到方程组

$$\begin{cases} \frac{1}{3}a_0 + \frac{1}{5}a_2 = \frac{1}{6}, \\ \frac{1}{5}a_1 = 0, \\ \frac{1}{5}a_0 + \frac{1}{7}a_2 = \frac{1}{8}, \end{cases}$$

解得 $a_0 = -5/32, a_1 = 0, a_2 = -35/32$, 故

$$p(x) = -\frac{5}{32} - \frac{35}{32}(x-1)^2 = -\frac{5}{4} + \frac{35}{16}x - \frac{35}{32}x^2.$$