Arc Length in Polar Coordinates

If a curve is given in polar coordinates $r = f(\theta)$, an integral for the length of the curve can be derived using the arc length formula for a parametric curve. Regard θ as the parameter. The parametric arc length formula becomes

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Now $x = r\cos\theta$ and $y = r\sin\theta$, so

$$\frac{dx}{d\theta} = -r\sin\theta + \left(\frac{dr}{d\theta}\right)\cos\theta,$$

$$\frac{dy}{d\theta} = r\cos\theta + \left(\frac{dr}{d\theta}\right)\sin\theta,$$

Square and add, using the fact that $(\cos\theta)^2 + (\sin\theta)^2 = 1$:

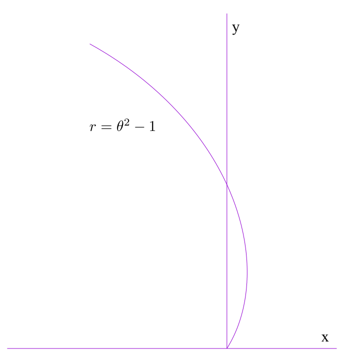
$$\begin{array}{rcl}
\left(\dfrac{dx}{d\theta}\right)^2 & = & r^2(\sin\theta)^2 - 2r\left(\dfrac{dr}{d\theta}\right)\sin\theta\cos\theta + \left(\dfrac{dr}{d\theta}\right)^2(\cos\theta)^2 \\
\left(\dfrac{dy}{d\theta}\right)^2 & = & r^2(\cos\theta)^2 + 2r\left(\dfrac{dr}{d\theta}\right)\cos\theta\sin\theta + \left(\dfrac{dr}{d\theta}\right)^2(\sin\theta)^2 \\
\hline
\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 & = & r^2 \qquad\qquad\qquad\qquad\qquad\qquad + \left(\dfrac{dr}{d\theta}\right)^2
\end{array}$$

Hence,

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Note: As with other arc length computations, it's pretty easy to come up with polar curves which lead to integrals with non-elementary antiderivatives. In that case, the best you might be able to do is to approximate the integral using a calculator or a computer.

Example. Find the length of the curve $r = \theta^2 - 1$ from $\theta = 1$ to $\theta = 2$.



$$\frac{dr}{d\theta} = 2\theta.$$

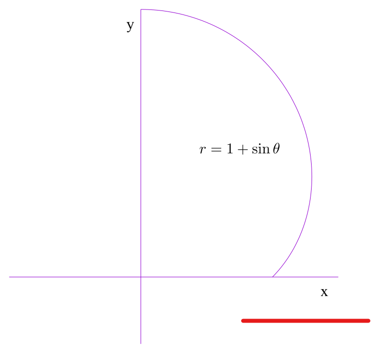
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (\theta^2 - 1)^2 + 4\theta^2 = \theta^4 - 2\theta^2 + 1 + 4\theta^2 = \theta^4 + 2\theta^2 + 1 = (\theta^2 + 1)^2.$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \theta^2 + 1.$$

The length is

$$\int_1^2 (\theta^2 + 1)\, d\theta = \left[\frac{1}{3}\theta^3 + \theta\right]_1^2 = \frac{10}{3} = 3.33333\ldots. \quad \square$$

Example. Find the length of the cardiod $r = 1 + \sin\theta$ for $\theta = 0$ to $\theta = \frac{\pi}{2}$.



$$\frac{dr}{d\theta} = \cos\theta.$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + \sin\theta)^2 + (\cos\theta)^2 = 1 + 2\sin\theta + (\sin\theta)^2 + (\cos\theta)^2 = 2 + 2\sin\theta.$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}\sqrt{1 + \sin\theta}.$$

I'll do the antiderivative separately:

$$\int \sqrt{2}\sqrt{1 + \sin\theta}\, d\theta = \sqrt{2}\int \frac{\sqrt{1 + \sin\theta}\sqrt{1 - \sin\theta}}{\sqrt{1 - \sin\theta}}\, d\theta = \sqrt{2}\int \frac{\sqrt{1 - (\sin\theta)^2}}{\sqrt{1 - \sin\theta}}\, d\theta =$$

$$\sqrt{2}\int \frac{\sqrt{(\cos\theta)^2}}{\sqrt{1 - \sin\theta}}\, d\theta = \sqrt{2}\int \frac{\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta =$$

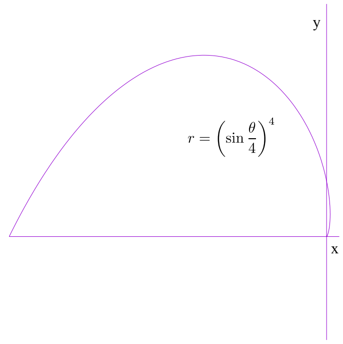
$$\left[u = 1 - \sin\theta, \quad du = -\cos\theta\, d\theta, \quad d\theta = \frac{du}{-\cos\theta}\right]$$

$$\sqrt{2}\int \frac{\cos\theta}{\sqrt{u}} \cdot \frac{du}{-\cos\theta} = -\sqrt{2}\int \frac{1}{\sqrt{u}}\, du = -\sqrt{2} \cdot 2\sqrt{u} + c = -2\sqrt{2}\sqrt{1 - \sin\theta} + c.$$

The length is

$$\int_0^{\pi/2} \sqrt{2}\sqrt{1 + \sin\theta}\, d\theta = \left[-2\sqrt{2}\sqrt{1 - \sin\theta}\right]_0^{\pi/2} = 2\sqrt{2} = 2.82842\ldots. \quad \square$$

Example. Find the length of the polar curve $r = \left(\sin\frac{\theta}{4}\right)^4$ for $\theta = 0$ to $\theta = \pi$.



$$\frac{dr}{d\theta} = \left(\sin \frac{\theta}{4}\right)^3 \cos \frac{\theta}{4}.$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(\sin \frac{\theta}{4}\right)^8 + \left(\sin \frac{\theta}{4}\right)^6 \left(\cos \frac{\theta}{4}\right)^2 = \left(\sin \frac{\theta}{4}\right)^6 \left[\left(\sin \frac{\theta}{4}\right)^2 + \left(\cos \frac{\theta}{4}\right)^2\right] = \left(\sin \frac{\theta}{4}\right)^6.$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \left(\sin \frac{\theta}{4}\right)^3.$$

The length is

$$\int_0^\pi \left(\sin \frac{\theta}{4}\right)^3 \, d\theta.$$

I'll do the antiderivative separately:

$$\int \left(\sin \frac{\theta}{4}\right)^3 \, d\theta \int \left(\sin \frac{\theta}{4}\right)^2 \sin \frac{\theta}{4} \, d\theta = \int \left(1 - \left(\cos \frac{\theta}{4}\right)^2\right) \sin \frac{\theta}{4} \, d\theta =$$

$$\left[u = \cos \frac{\theta}{4}, \quad du = -\frac{1}{4} \sin \frac{\theta}{4} \, d\theta, \quad d\theta = -4 \frac{du}{\sin \frac{\theta}{4}}\right]$$

$$-4 \int (1 - u^2) \left(\sin \frac{\theta}{4}\right) \cdot \frac{du}{\sin \frac{\theta}{4}} = -4 \int (1 - u^2) \, du = -4 \left(u - \frac{1}{3} u^3\right) + c = -4 \cos \frac{\theta}{4} + \frac{4}{3} \left(\cos \frac{\theta}{4}\right)^3 + c.$$

So

$$\int_0^\pi \left(\sin \frac{\theta}{4}\right)^3 \, d\theta = \left[-4 \cos \frac{\theta}{4} + \frac{4}{3} \left(\cos \frac{\theta}{4}\right)^3\right]_0^\pi = \frac{8}{3} - \frac{5\sqrt{2}}{3} = 0.30964 \ldots. \quad \square$$
