Tutorial solutions for Chapter 9

Sometimes there are other correct answers.

9.1. As A and B have the same cardinality, there is a bijection $B \to A$. Since A is countable, we deduce from Proposition 9.2.7 that B is also countable. $\square$

9.2. If $A = \varnothing$, then $A \cup B = \varnothing \cup B = B$ as $\varnothing$ is an identity for $\cup$, and thus $A \cup B$ is countable because B is countable. Similarly, if $B = \varnothing$, then $A \cup B = A$ and is thus countable by assumption. So let us assume $A \neq \varnothing$ and $B \neq \varnothing$.

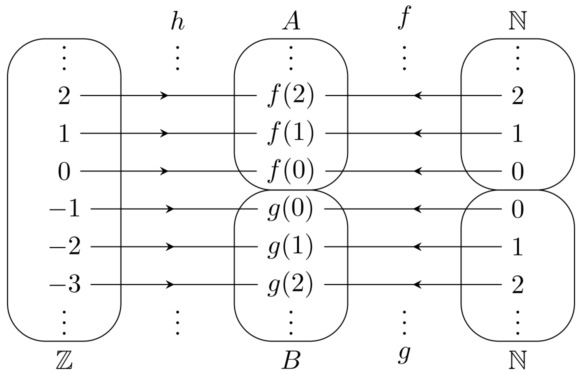
Use Lemma 9.1.2 to find surjections $f \colon \mathbb{N} \to A$ and $g \colon \mathbb{N} \to B$. Define the function $h \colon \mathbb{Z} \to A \cup B$ by setting, for each $x \in \mathbb{Z}$,

$$h(x) = \begin{cases} f(x), & \text{if } x \geqslant 0; \\ g(-x-1), & \text{if } x < 0. \end{cases}$$

If $x \in \mathbb{Z}$ such that $x \geqslant 0$, then x is in the domain of f, and so we can indeed apply f to x. If $x \in \mathbb{Z}$ such that $x < 0$, then $-x-1 > -0-1 = -1$, and thus $-x-1 \geqslant 0$ as $-x-1 \in \mathbb{Z}$; so we can indeed apply g to $-x-1$.

In view of Proposition 9.2.7, it suffices to verify the surjectivity of h. Let $y \in A \cup B$. Then $y \in A$ or $y \in B$ by the definition of $A \cup B$. If $y \in A$, then the surjectivity of f gives $v \in \mathbb{N}$ such that $y = f(v) = h(v)$. If $y \in B$, then the surjectivity of g gives $w \in \mathbb{N}$ such that $y = g(w) = g\big(-(-w-1)-1\big) = h(-w-1)$ as $-w-1 \leqslant -0-1 = -1 < 0$. So we have $x \in \mathbb{Z}$ which makes $y = h(x)$ in all cases. $\square$

Figure.



Intuition. Focus on the case when both A and B are nonempty; the other cases can be handled more easily. By countability, the elements of A and the elements of B can be listed out (possibly with repetition) respectively as

$$f(0), f(1), f(2), f(3), \ldots \quad \text{and} \quad g(0), g(1), g(2), g(3), \ldots.$$

So all elements of $A \cup B$ appear in the "list"

$$\ldots, g(3), g(2), g(1), g(0), f(0), f(1), f(2), f(3), \ldots$$

of shape $\mathbb{Z}$. Since $\mathbb{Z}$ is countable, this implies $A \cup B$ is countable too.

Alternative proof using surjections that does not use the countability of $\mathbb{Z}$. If $A = \varnothing$, then $A \cup B = \varnothing \cup B = B$ as $\varnothing$ is an identity for $\cup$, and thus $A \cup B$ is countable because B is countable. Similarly, if $B = \varnothing$, then $A \cup B = A$ and is thus countable by assumption. So let us assume $A \neq \varnothing$ and $B \neq \varnothing$.

Use Lemma 9.1.2 to find surjections $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. Define the function $h\colon \mathbb{N} \to A \cup B$ by setting, for each $x \in \mathbb{N}$,

$$h(x) = \begin{cases} f(\frac{x}{2}), & \text{if } x \text{ is even;} \\ g(\frac{x-1}{2}), & \text{if } x \text{ is odd.} \end{cases}$$

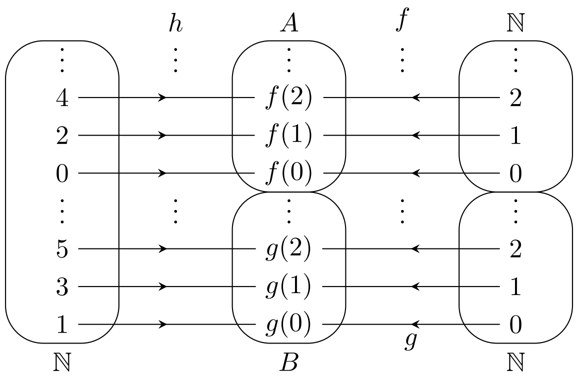
From Proposition 3.2.21 and Proposition 3.2.17, we know every natural number is either odd or even, but not both. If $x \in \mathbb{N}$ that is even, say $x = 2v$ where $v \in \mathbb{Z}$, then $x/2 = v \in \mathbb{Z}$ and $x/2 \geqslant 0/2 = 0$; thus we can indeed apply f to $x/2$. If $x \in \mathbb{N}$ that is odd, say $x = 2w + 1$ where $w \in \mathbb{Z}$, then $(x-1)/2 = w \in \mathbb{Z}$ and

$$\begin{aligned} & \frac{x-1}{2} \geqslant \frac{0-1}{2} = -\frac{1}{2}, \\ \therefore \quad & \frac{x-1}{2} \geqslant 0 \qquad \text{as } \frac{x-1}{2} \in \mathbb{Z}; \end{aligned}$$

thus we can indeed apply g to $(x-1)/2$.

In view of Lemma 9.1.2, it suffices to verify the surjectivity of h. Let $y \in A \cup B$. Then $y \in A$ or $y \in B$ by the definition of $A \cup B$. If $y \in A$, then the surjectivity of f gives $v \in \mathbb{N}$ such that $y = f(v) = f(\frac{2v}{2}) = h(2v)$ as $2v$ is even. If $y \in B$, then the surjectivity of g gives $w \in \mathbb{N}$ such that $y = g(w) = g(\frac{(2w+1)-1}{2}) = h(2w+1)$ as $2w + 1$ is odd. So we have $x \in \mathbb{N}$ which makes $y = h(x)$ in all cases, as required. $\square$

Figure.



Intuition. Focus on the case when both A and B are nonempty; the other cases can be handled more easily. By countability, the elements of A and the elements of B can be listed out (possibly with repetition) respectively as

$$f(0), f(1), f(2), f(3), \ldots \quad \text{and} \quad g(0), g(1), g(2), g(3), \ldots.$$

So all elements of $A \cup B$ appear in the sequence

$$f(0), g(0), f(1), g(1), f(2), g(2), f(3), g(3), \ldots.$$

Hence $A \cup B$ is countable too.

Alternative proof using injections. Use the countability of A and B to find injections $f\colon A \to \mathbb{N}$ and $g\colon B \to \mathbb{N}$. Define $h\colon A \cup B \to \mathbb{N}$ by setting, for each $x \in A \cup B$,

$$h(x) = \begin{cases} 2\,f(x), & \text{if } x \in A; \\ 2\,g(x) + 1, & \text{if } x \notin A. \end{cases}$$

Note that, if $x \in A \cup B$ but $x \notin A$, then $x \in B$ by the definition of $A \cup B$, and thus we can indeed apply g to x. It remains to prove that h is injective.

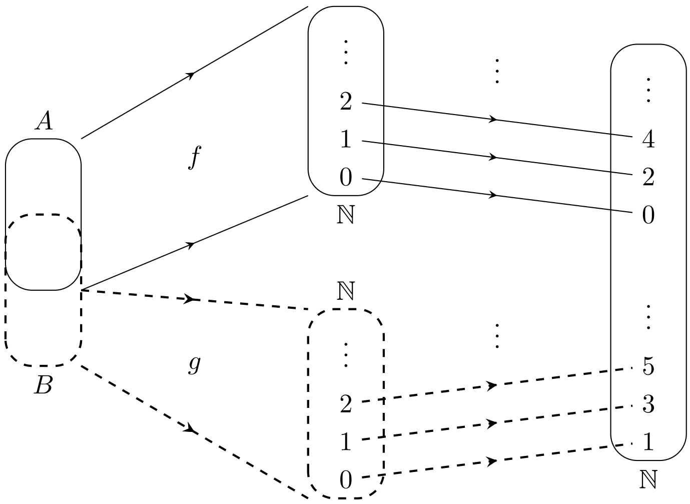
Let $x_1, x_2 \in A \cup B$ such that $h(x_1) = h(x_2)$. Say $y = h(x_1) = h(x_2)$. We know from Proposition 3.2.21 that y is either even or odd.

Case 1: suppose y is even. Then y is not odd by Proposition 3.2.17. This means $y \neq 2w+1$ for any $w \in \mathbb{Z}$. As $y = h(x_1) = h(x_2)$, we deduce that $x_1, x_2 \in A$ by the definition of h. This implies $2\,f(x_1) = h(x_1) = h(x_2) = 2\,f(x_2)$. So $f(x_1) = f(x_2)$. As f is injective, we conclude that $x_1 = x_2$.

Case 2: suppose y is odd. Then y is not even by Proposition 3.2.17. This means $y \neq 2w$ for any $w \in \mathbb{Z}$. As $y = h(x_1) = h(x_2)$, we deduce that $x_1, x_2 \notin A$ by the definition of h. This implies $2\,g(x_1) + 1 = h(x_1) = h(x_2) = 2\,g(x_2) + 1$. So $g(x_1) = g(x_2)$. As g is injective, we conclude that $x_1 = x_2$.

Hence $x_1 = x_2$ in all cases. $\square$

Diagram.



Intuition. As A and B are countable, we have a way f to represent each element $x \in A$ by $f(x) \in \mathbb{N}$, and a way g to represent each element $x \in B$ by $g(x) \in \mathbb{N}$. So we can represent each element $x \in A \cup B$ by the natural number $f(x)$ or $g(x)$, except when x is in both A and B, in which case we choose to represent x by $f(x)$ to avoid ambiguity. This demonstrates the countability of $A \cup B$.

One reason why small sets should be closed under (binary) union. We may view two sets as equivalent if and only if their (symmetric) difference is small. In particular, if S is a set and A, B are small sets, then S should be equivalent to $S \setminus A$, and $S \setminus A$ should be equivalent to $(S \setminus A) \setminus B = S \setminus (A \cup B)$. Any notion of equivalence should be (reflexive, symmetric, and) transitive. So if S is a set and A, B are small sets, then S should be equivalent to $S \setminus (A \cup B)$ by transitivity; this indicates that $A \cup B$ should also be small.

9.3. Fix countable sets $A_0, A_1, A_2, \ldots$. Let $P(n)$ be the predicate "$\bigcup_{i=0}^{n} A_i$ is countable" over $\mathbb{N}$.

(Base step) $P(0)$ is true because $\bigcup_{i=0}^{n} A_i = A_0$, which is countable by assumption.

(Induction step) Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., that $\bigcup_{i=0}^{k} A_i$ is countable.

$$\begin{aligned}
\bigcup_{i=0}^{k+1} A_i &= A_0 \cup A_1 \cup \cdots \cup A_k \cup A_{k+1} && \text{by the definition of } \bigcup_{i=0}^{k+1} A_i; \\
&= \Big(\bigcup_{i=0}^{k} A_i\Big) \cup A_{k+1} && \text{by the definition of } \bigcup_{i=0}^{k} A_i.
\end{aligned}$$

Here $\bigcup_{i=0}^{k} A_i$ is countable by the induction hypothesis, and A_{k+1} is countable by assumption. So Exercise 9.2 tells us $\left(\bigcup_{i=0}^{k} A_i\right) \cup A_{k+1} = \bigcup_{i=0}^{k+1} A_i$ is also countable. This shows $P(k+1)$ is true.

Hence $\forall n \in \mathbb{N} \;\; P(n)$ is true by MI. □

Alternative proof. Let $P(n)$ be the predicate

$$\bigcup_{i=0}^{n} A_i \text{ is countable for all countable sets } A_0, A_1, \ldots, A_n$$

over $\mathbb{N}$.

(Base step) $P(0)$ is true because, if A_0 is a countable set, then $\bigcup_{i=0}^{0} A_i = A_0$ is countable.

(Induction step) Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., that $\bigcup_{i=0}^{k} A_i$ is countable for all countable sets $A_0, A_1, \ldots, A_k$. Take countable sets $B_0, B_1, \ldots, B_{k+1}$. Then $B_k \cup B_{k+1}$ is countable by Exercise 9.2. So $\bigcup_{i=0}^{k+1} = B_0 \cup B_1 \cup \cdots \cup (B_k \cup B_{k+1})$ is countable too by the induction hypothesis. This shows $P(k+1)$ is true.

Hence $\forall n \in \mathbb{N} \;\; P(n)$ is true by MI. □

Additional comment. In the first proof above, it does not matter much whether we put the universal quantifier for $A_0, A_1, A_2, \ldots$ into $P(n)$ or not. This is *not* true in the alternative proof.

9.4. (a) Suppose $A_0, A_1, A_2, \ldots$ are countable sets. Exercise 9.3 only tells us the following sets are countable:

$$\bigcup_{i=0}^{0} A_i, \quad \bigcup_{i=0}^{1} A_i, \quad \bigcup_{i=0}^{2} A_i, \quad \bigcup_{i=0}^{3} A_i, \quad \bigcup_{i=0}^{4} A_i, \quad \bigcup_{i=0}^{5} A_i, \quad \bigcup_{i=0}^{6} A_i, \quad \bigcup_{i=0}^{7} A_i, \quad \ldots$$

There is no specific reason why any of these should be equal to $\bigcup_{i=0}^{\infty} A_i$. Even given the fact that the union of all the sets displayed above is equal to $\bigcup_{i=0}^{\infty} A_i$, it is still not clear why the countability of these individual sets imply the countability of the union; in fact, this is essentially what one is trying to prove here.

Extra argument. One can prove that, if $A_0, A_1, A_2, \ldots$ are finite sets, then $\bigcup_{i=0}^{n} A_i$ is finite for every $n \in \mathbb{N}$. As deductions are supposed to depend on the forms of the propositions involved alone according to Terminology 3.1.11, if what is claimed were indeed correct, then the same argument would allow one to deduce that $\bigcup_{i=0}^{\infty} A_i$ is finite whenever $A_0, A_1, A_2, \ldots$ are finite. However, this conclusion is not true: let if we let $A_i = \{i\}$ for each $i \in \mathbb{N}$, then each A_i is finite, but $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$, which is infinite.

Moral. A predicate that is satisfied by arbitrarily large finite sets may not be satisfied by infinite sets. There is a gap between "arbitrarily large finite" and "infinite".

(b) If $A_i = \varnothing$ for every $i \in \mathbb{N}$, then

$$\begin{aligned}
\bigcup_{i=0}^{\infty} A_i &= \{x : x \in A_i \text{ for some } i \in \mathbb{N}\} && \text{by the definition of } \bigcup_{i=0}^{\infty} A_i; \\
&= \{x : x \in \varnothing \text{ for some } i \in \mathbb{N}\} && \text{as each } A_i = \varnothing; \\
&= \varnothing && \text{as } \varnothing \text{ has no element by definition,}
\end{aligned}$$

and thus $\bigcup_{i=0}^{\infty} A_i$ is countable by Exercise 8.3.9 and Proposition 9.1.3. So suppose we have $k \in \mathbb{N}$ such that $A_k \neq \varnothing$.

For each $i \in \mathbb{N}$, define

$$B_i = \begin{cases} A_i, & \text{if } A_i \neq \varnothing; \\ A_k, & \text{if } A_i = \varnothing. \end{cases}$$

Then each $B_i \neq \varnothing$ by construction. Moreover, each $A_i \subseteq B_i$ by Remark 4.2.4(2) and Proposition 4.2.7. It follows that

$$\begin{aligned} \bigcup_{i=0}^{\infty} A_i &= \{x : x \in A_i \text{ for some } i \in \mathbb{N}\} && \text{by the definition of } \bigcup_{i=0}^{\infty} A_i; \\ &\subseteq \{x : x \in B_i \text{ for some } i \in \mathbb{N}\} && \text{as each } A_i \subseteq B_i; \\ &= \bigcup_{i=0}^{\infty} B_i && \text{by the definition of } \bigcup_{i=0}^{\infty} B_i. \end{aligned}$$

Therefore, in view of Proposition 9.2.5, it suffices to show that $\bigcup_{i=0}^{\infty} B_i$ is countable.

For each $i \in \mathbb{N}$, use Lemma 9.1.2 to find a surjection $f_i \colon \mathbb{N} \to B_i$. Define $g \colon \mathbb{N} \times \mathbb{N} \to \bigcup_{i=0}^{\infty} B_i$ by setting $g(i,j) = f_i(j)$ for each $(i,j) \in \mathbb{N} \times \mathbb{N}$. We claim that g is a surjection. Proposition 9.2.7 will then tell us the countability of $\bigcup_{i=0}^{\infty} B_i$.

Let $x \in \bigcup_{i=0}^{\infty} B_i$. Use the definition of $\bigcup_{i=0}^{\infty} B_i$ to find $i \in \mathbb{N}$ such that $x \in B_i$. Then the surjectivity of f_i gives $j \in \mathbb{N}$ such that $x = f_i(j) = g(i,j)$. □

Intuition. Ignore all the A_i's that are empty. By countability, the elements of each A_i can be listed out (possibly with repetition) as

$$f_i(0), f_i(1), f_i(2), f_i(3), \ldots.$$

So all elements of $\bigcup_{i=0}^{\infty} A_i$ appear in the array

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \iddots \\ f_3(0) & f_3(1) & f_3(2) & f_3(3) & \ldots \\ f_2(0) & f_2(1) & f_2(2) & f_2(3) & \ldots \\ f_1(0) & f_1(1) & f_1(2) & f_1(3) & \ldots \\ f_0(0) & f_0(1) & f_0(2) & f_0(3) & \ldots \end{array}$$

of shape $\mathbb{N} \times \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, this implies $\bigcup_{i=0}^{\infty} A_i$ is countable too.

Alternative proof. For each $i \in \mathbb{N}$, use the countability of A_i to find an injection $f_i \colon A_i \to \mathbb{N}$. Define $g \colon \bigcup_{i=0}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$ by setting, for each $x \in \bigcup_{i=0}^{\infty} A_i$,

$g(x) = (i_x, f_{i_x}(x))$ where i_x is the smallest element of $\{i \in \mathbb{N} : x \in A_i\}$.

(Well-definedness) Let us verify that the definition of g above indeed assigns to every element of $\bigcup_{i=0}^{\infty} A_i$ exactly one element of $\mathbb{N} \times \mathbb{N}$. Take $x \in \bigcup_{i=0}^{\infty} A_i$. Then the definition of $\bigcup_{i=0}^{\infty} A_i$ tells us $x \in A_i$ for some $i \in \mathbb{N}$. So $\{i \in \mathbb{N} : x \in A_i\}$ is nonempty, and thus must have a smallest element, say i_x, by the Well-Ordering Principle. Such an i_x must be unique because, if i is another smallest element of this set, then $i_x \leqslant i$ and $i \leqslant i_x$ by the smallest-ness of i_x and i respectively, and thus $i_x = i$. As $x \in A_{i_x}$, we can indeed apply f_{i_x} to x to get an element of $\mathbb{N}$.

(Injectivity) Let $x, y \in \bigcup_{i=0}^{\infty} A_i$ such that $g(x) = g(y)$. Say $g(x) = (i,j) = g(y)$. Then the definition of g tells us that $x, y \in A_i$ and $f_i(x) = j = f_i(y)$. So $x = y$ by the injectivity of f_i.

Now g is an injection $\bigcup_{i=0}^{\infty} A_i \to \mathbb{N} \times \mathbb{N}$. Recall from Theorem 9.1.9 that $\mathbb{N} \times \mathbb{N}$ is countable. So $\bigcup_{i=0}^{\infty} A_i$ must also be countable by Proposition 9.2.7. □

Extra explanation. Here is a procedure that gives the function g defined in the alternative proof above.

1. Input $x \in \bigcup_{i=0}^{\infty} A_i$.
2. Search for the smallest $i \in \mathbb{N}$ such that $x \in A_i$.
3. Apply f_i to x to get $j \in \mathbb{N}$.
4. Output (i, j).

Intuition. As each A_i is countable, we have a way f_i to represent each element $x \in A_i$ by $f_i(x) \in \mathbb{N}$. So we can represent each element $x \in \bigcup_{i=0}^{\infty} A_i$ by $(i, f_i(x)) \in \mathbb{N} \times \mathbb{N}$ where $i \in \mathbb{N}$ such that $x \in A_i$, except when x is in multiple A_i's, in which case we choose the smallest such i to avoid ambiguity. We can in turn represent this ordered pair $(i, f_i(x))$ by a natural number in view of the countability of $\mathbb{N} \times \mathbb{N}$. Combining the two gives a representation of the elements of $\bigcup_{i=0}^{\infty} A_i$ by natural numbers, which demonstrates the countability of $\bigcup_{i=0}^{\infty} A_i$.

Moral. Even wrong proofs may have true conclusions.

9.5. (a) **True**, because this is the contrapositive of Proposition 9.2.5.

(b) **True**, as shown below.

We prove this by contraposition. Let A, B be sets such that $A \setminus B$ is countable. We will show that either A is countable or B is uncountable. If B is uncountable, then there is nothing to prove. So suppose B is countable. As $A \setminus B$ and B are both countable, we know $(A \setminus B) \cup B$ is also countable by Exercise 9.2. Note that Further Exercise 4.9(a) tells us $(A \setminus B) \cup B = A \cup B$. Thus $A \cup B$ is countable. As $A \subseteq A \cup B$ by Tutorial Exercise 4.6, Proposition 9.2.5 implies A is countable too. $\square$

Intuition. As mentioned in Exercise 9.2, one may view countable sets as small sets, and uncountable set as large sets.

(a) This proposition is intuitive because, if a set B has a large subset A, then B should itself be large.

(b) This proposition is intuitive because, if we take away a small set B from a large set A, then the result $A \setminus B$ should remain large.

9.6. Let A be an infinite set. Apply Proposition 9.2.3 to find $A_0 \subseteq A$ that has the same cardinality as $\mathbb{N}$. Since A_0 has the same cardinality as $\mathbb{N}$, we have a bijection $A_0 \to \mathbb{N}$, say f. Recall from Theorem 9.3.1 that there is no surjection $A_0 \to \mathcal{P}(A_0)$. So, in view of Exercise 7.4.4(1), there can be no surjection $\mathbb{N} \to \mathcal{P}(A_0)$. Then Lemma 9.1.2 tells us $\mathcal{P}(A_0)$ is uncountable because $\mathcal{P}(A_0) \neq \varnothing$ by Further Exercise 4.9(c). Therefore, in view of Exercise 9.5(a), it suffices to verify that $\mathcal{P}(A_0) \subseteq \mathcal{P}(A)$.

Let $S \in \mathcal{P}(A_0)$. Then the definition of $\mathcal{P}(A_0)$ tells us $S \subseteq A_0 \subseteq A$. Thus $S \subseteq A$ by the transitivity of $\subseteq$ from Example 6.1.4. This means $S \in \mathcal{P}(A)$ according to the definition of $\mathcal{P}(A)$. $\square$

Intuition. On the one hand, as verified above, the power set of a larger set is larger. On the other hand, we know from Proposition 9.2.3 that the countable infinity is the smallest infinity. Since the power set of the countable infinite set $\mathbb{N}$ is already large, the power set of any infinite set should also be large.

Further exercises

9.7. Apply the definition of countability to find injections $g_A \colon A \to \mathbb{N}$ and $g_B \colon B \to \mathbb{N}$. Consider the function $f \colon A \times B \to \mathbb{N} \times \mathbb{N}$ satisfying $f(x, y) = \big(g_A(x), g_B(y)\big)$ for all

$(x, y) \in A \times B$. This function is injective because if $(x_1, y_1), (x_2, y_2) \in A \times B$ such that $f(x_1, y_1) = f(x_2, y_2)$, then

$$\begin{array}{rlcl l}
& \big(g_A(x_1), g_B(y_1)\big) & = & \big(g_A(x_2), g_B(y_2)\big) & \text{by the definition of } f; \\
\therefore & g_A(x_1) = g_A(x_2) & \text{and} & g_B(y_1) = g_B(y_2) & \text{by the definition of ordered pairs;} \\
\therefore & x_1 = x_2 & \text{and} & y_1 = y_2 & \text{as } g_A, g_B \text{ are injective;} \\
\therefore & (x_1, y_1) & = & (x_2, y_2) & \text{by the definition of ordered pairs.}
\end{array}$$

Now we have an injection $f \colon A \times B \to \mathbb{N} \times \mathbb{N}$, of which the codomain $\mathbb{N} \times \mathbb{N}$ is countable by Theorem 9.1.9. So, in view of Proposition 9.2.7, the domain $A \times B$ must also be countable. □

Intuition. As A and B are countable, each element of A can be represented by a natural number, and each element of B can be represented by a natural number. So each ordered pair in $A \times B$ can be represented by a pair of natural numbers. Each pair of natural numbers can in turn be represented by a natural number in view of the countability of $\mathbb{N} \times \mathbb{N}$. Combining the two gives a representation of the elements of $A \times B$ by natural numbers, which demonstrates the countability of $A \times B$.

Alternative proof using surjections. If $A = \varnothing$ or $B = \varnothing$, then

$$A \times B = \{(x, y) : x \in A \text{ and } x \in B\} = \varnothing,$$

which we know is countable by Exercise 8.3.9 and Proposition 9.1.3. So suppose $A \neq \varnothing$ and $B \neq \varnothing$.

Use the countability of A and B to find surjections $h_A \colon \mathbb{N} \to A$ and $h_B \colon \mathbb{N} \to B$. Consider the function $f \colon \mathbb{N} \times \mathbb{N} \to A \times B$ satisfying $f(m, n) = \big(h_A(m), h_B(n)\big)$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. We verify that this f is surjective. As $\mathbb{N} \times \mathbb{N}$ is countable, Proposition 9.2.7 will then tell us $A \times B$ is countable.

Let $(x, y) \in A \times B$. By the definition of $A \times B$, this means $x \in A$ and $y \in B$. Use the surjectivity of h_A and h_B to find $m, n \in \mathbb{N}$ such that $h_A(m) = x$ and $h_B(n) = y$. Then $(m, n) \in \mathbb{N} \times \mathbb{N}$ by the definition of $\mathbb{N} \times \mathbb{N}$, and $f(m, n) = \big(h_A(m), h_B(n)\big) = (x, y)$ by the definition of f, as required. □

Intuition. Focus on the case when both A and B are nonempty; the other cases can be handled more easily. By countability, the elements of A and the elements of B can be listed out (possibly with repetition) respectively as

$$f(0), f(1), f(2), f(3), \ldots \quad \text{and} \quad g(0), g(1), g(2), g(3), \ldots$$

So all elements of $A \times B$ appear in the array

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
(f(3), g(0)) & (f(3), g(1)) & (f(3), g(2)) & (f(3), g(3)) & \ldots \\
(f(2), g(0)) & (f(2), g(1)) & (f(2), g(2)) & (f(2), g(3)) & \ldots \\
(f(1), g(0)) & (f(1), g(1)) & (f(1), g(2)) & (f(1), g(3)) & \ldots \\
(f(0), g(0)) & (f(0), g(1)) & (f(0), g(2)) & (f(0), g(3)) & \ldots
\end{array}$$

of shape $\mathbb{N} \times \mathbb{N}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, this implies $A \times B$ is countable too.

9.8. (a) (Existence) Since line 2.1.1 is reached, the condition on line 2.1 is satisfied. So some $x \in A$ makes $m = f(x)$.

(Uniqueness) Let $x_1, x_2 \in A$ such that $m = f(x_1)$ and $m = f(x_2)$. Then $f(x_1) = m = f(x_2)$. So $x_1 = x_2$ by the injectivity of f. □

(b) Let $P(n)$ be the sentence

at the start of the nth iteration of the loop, the images $g(0), g(1), \ldots, g(\ell_n - 1)$ are all defined, and $\{g(0), g(1), \ldots, g(\ell_n - 1)\} = \{x \in A : f(x) < n\}$.

(Base step) Note that $\ell_0 = 0$ from line 1. So the list $g(0), g(1), \ldots, g(\ell_0 - 1)$ is empty. Hence we do not need to check that any image of g is defined, and

$$\{g(0), g(1), \ldots, g(\ell_0 - 1)\} = \varnothing = \{x \in A : f(x) < 0\}.$$

This shows $P(0)$.

(Induction step) Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., at the start of the nth iteration of the loop, the images $g(0), g(1), \ldots, g(\ell_k - 1)$ are all defined, and $\{g(0), g(1), \ldots, g(\ell_k - 1)\} = \{x \in A : f(x) < k\}$.

Case 1: suppose $k = f(x)$ for some $x \in A$. Then the kth iteration of the loop defines $g(\ell_k)$ on line 2.1.1 and makes $\ell_{k+1} = \ell_k + 1$ on line 2.1.2. So, the induction hypothesis implies that, at the start of the $(k+1)$th iteration of the loop, the images $g(0), g(1), \ldots, g(\ell_{k+1} - 1)$ are all defined. Moreover,

$$\begin{aligned}
&\{g(0), g(1), \ldots, g(\ell_{k+1} - 1)\} \\
&= \{g(0), g(1), \ldots, g(\ell_k + 1 - 1)\} \\
&= \{g(0), g(1), \ldots, g(\ell_k - 1)\} \cup \{g(\ell_k)\} \\
&= \{x \in A : f(x) < k\} \cup \{x \in A : f(x) = k\} \\
&\qquad\qquad\qquad\qquad \text{by the induction hypothesis and (a);} \\
&= \{x \in A : f(x) < k + 1\}.
\end{aligned}$$

Case 2: suppose $k \neq f(x)$ for any $x \in A$. Then $\ell_{k+1} = \ell_k$, and

$$\begin{aligned}
&\{g(0), g(1), \ldots, g(\ell_{k+1} - 1)\} \\
&= \{g(0), g(1), \ldots, g(\ell_k - 1)\} \\
&= \{x \in A : f(x) < k\} && \text{by the induction hypothesis;} \\
&= \{x \in A : f(x) < k + 1\} && \text{as } k \neq f(x) \text{ for any } x \in A.
\end{aligned}$$

So $P(k+1)$ is true in all cases.

Hence $\forall n \in \mathbb{N}\ \ P(n)$ is true by MI. □

(c) (F1) follows from (b), while (F2) follows from (a).

(Surjectivity) Let $a \in A$. Define $n = f(a) + 1$. Then

$$\begin{aligned}
a &\in \{x \in A : f(x) < n\} && \text{as } f(a) = n - 1 < n; \\
&= \{g(0), g(1), \ldots, g(\ell_n - 1)\} && \text{by (b).}
\end{aligned}$$

In particular, we know $a = g(\ell)$ for some $\ell \in \mathbb{N}$.

(Injectivity) Let $\ell_i, \ell_j \in \mathbb{N}$ such that $g(\ell_i) = g(\ell_j)$. Suppose $g(\ell_i)$ is defined in the n_ith iteration of the loop, and $g(\ell_j)$ in the n_jth iteration. Then $n_i = f(g(\ell_i)) = f(g(\ell_j)) = n_j$ as $g(\ell_i) = g(\ell_j)$, in view of how the images $g(\ell)$'s are defined. This says $g(\ell_i)$ and $g(\ell_j)$ are defined in the same iteration of the loop. Since each iteration defines at most one image of g, we deduce that $\ell_i = \ell_j$. □