**Name:** ________________________ **Tutorial group:** __________ **(number or day&time)**
**CS1231 Discrete Structures** **Quiz 2** **2023/24 Semester 2**

1. Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$.

   (i) Write down $\{x \in A : x^2 + 2x = 0\}$ in roster notation. [1 mark]

   **Solution.** $\{-2, 0\}$.

   (ii) Write down the power set of the set you wrote in (i) in roster notation. [1 mark]

   **Solution.** $\{\{\}, \{-2\}, \{0\}, \{-2, 0\}\}$.

   (iii) Is $\{x^2 : x \in A\} \subseteq B$? Explain your answer. [2 marks]

   **Solution.** No, because $0 \in \{x^2 : x \in A\}$ but $0 \notin B$.

   (iv) Write down $A \setminus B$ in roster notation. [1 mark]

   **Solution.** $\{-2, -1, 0\}$.

   (v) Write down in roster notation a set $C$ such that $\{A \setminus B,\ C\}$ is a partition of $A$. [1 mark]

   **Solution.** $\{1, 2\}$.

2. Let $A$ and $B$ be sets. Prove that, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$ for some set $C$, then $A = B$. [5 marks]

**Solution.** Let $C$ be a set such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$.

To show that $A \subseteq B$, take any $x \in A$.

**Case 1: assume $x \in C$.** Then $x \in A \cap C = B \cap C \subseteq B$ by Example 4.3.9(1).

**Case 2: assume $x \notin C$.** Note that Example 4.3.9(2) implies $x \in A \subseteq A \cup C = B \cup C$. So the definition of $\cup$ tells us $x \in B$ or $x \in C$. As $x \notin C$ by assumption, we deduce that $x \in B$.

So $x \in B$ in all cases.

The proof for $B \subseteq A$ is similar. □

**Alternative proof by contraposition.** Suppose $A \neq B$. Take any set $C$. As $A \neq B$, either there is some $x \in A$ such that $x \notin B$, or there is some $x \in B$ such that $x \notin A$. We focus on the former case here; the latter case can similarly be handled. Fix $x \in A$ such that $x \notin B$.

**Case 1: assume $x \in C$.** Then $x \in A \cap C$ as $x \in A$, but $x \notin B \cap C$ because $x \notin B$. So $A \cap C \neq B \cap C$.

**Case 2: assume $x \notin C$.** Then $x \in A \cup C$ as $x \in A$, but $x \notin B \cup C$ because $x \notin B$ and $x \notin C$. So $A \cup C \neq B \cup C$.

Hence $A \cap C \neq B \cap C$ or $A \cup C \neq B \cup C$ in all cases. □

**Alternative proof using set identities.** Let $C$ be a set such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$. Then

$$\begin{aligned}
A &= A \cap (A \cup C) && \text{by the Absorption Laws;}\\
&= A \cap (B \cup C) && \text{as } A \cup C = B \cup C;\\
&= (A \cap B) \cup (A \cap C) && \text{by the Distributive Laws;}\\
&= (B \cap A) \cup (B \cap C) && \text{by the Commutative Laws, as } A \cap C = B \cap C;\\
&= B \cap (A \cup C) && \text{by the Distributive Laws;}\\
&= B \cap (B \cup C) && \text{as } A \cup C = B \cup C;\\
&= B && \text{by the Absorption Laws.}
\end{aligned}$$

□

**Comments.**

- Some used a truth table in their attempts. This is possible, but additional explanations is required. Moreover, it is tricky to get the explanations right. For example, some used a truth table to show that

$$\forall x\ \big((x \in A \cap C \leftrightarrow x \in B \cap C) \wedge (x \in A \cup C \leftrightarrow x \in B \cup C) \rightarrow (x \in A \leftrightarrow x \in B)\big),$$

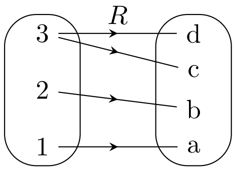
  but what you are asked to show is

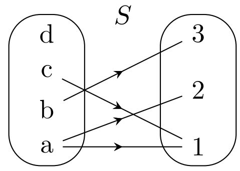
$$\forall x\ (x \in A \cap C \leftrightarrow x \in B \cap C) \wedge \forall x\ (x \in A \cup C \leftrightarrow x \in B \cup C) \rightarrow \forall x\ (x \in A \leftrightarrow x \in B);$$

  the two are different, and additional explanation is required to link the two.
- Some used the Inclusion–Exclusion Rule. This applies to only finite sets, but our sets may not be finite. In addition, it shows only $|A| = |B|$, not $A = B$.
- Some deduced from $A \neq B$ that $A \not\subseteq B$. There is a second case where $B \not\subseteq A$, which cannot be ignored altogether.

MORE QUESTIONS AT THE BACK OF THIS PAGE

3. Let $A = \{1, 2, 3\}$ and $B = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. The following are arrow diagrams of a relation $R$ from $A$ to $B$, and a relation $S$ from $B$ to $A$.




(i) Write down $R^{-1}$ is roster notation. [1 mark]

**Solution.** $R^{-1} = \{(\mathrm{a}, 1), (\mathrm{b}, 2), (\mathrm{c}, 3), (\mathrm{d}, 3)\}$.

(ii) Write down $R \circ S$ in roster notation. [1 mark]

**Solution.** $R \circ S = \{(\mathrm{a}, \mathrm{a}), (\mathrm{a}, \mathrm{b}), (\mathrm{b}, \mathrm{c}), (\mathrm{b}, \mathrm{d}), (\mathrm{c}, \mathrm{a})\}$.

(iii) Is $R \circ S$ antisymmetric as a relation on $B$? Explain your answer. [2 marks]

**Solution.** Yes, because the only $x, y \in B$ that make $(x, y), (y, x) \in R \circ S$ are when $x$ and $y$ are both a.

**Comments.** Some wrote that there are no $x, y \in B$ such that $x\ (R \circ S)\ y$ and $y\ (R \circ S)\ x$. This is not true because a $(R \circ S)$ a and a $(R \circ S)$ a.

(iv) Write down $S \circ R$ in roster notation. [1 mark]

**Solution.** $S \circ R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$.

(v) Is $S \circ R$ reflexive as relation on $A$? Explain your answer. [2 marks]

**Solution.** No, because $(2, 2) \notin S \circ R$.

4. Define a relation $R$ on $\mathbb{Z}$ by setting, for all $a, b \in \mathbb{Z}$,

$$a \mathrel{R} b \quad \Leftrightarrow \quad a + b \text{ is even.}$$

(i) Prove that $R$ is an equivalence relation, based only on what we have established in this course. [5 marks]

**Solution. (Reflexivity)** Let $a \in \mathbb{Z}$. Then $a + a = 2a$, which is even. So $a \mathrel{R} a$ by the definition of $R$.

**(Symmetry)** Let $a, b \in \mathbb{Z}$ such that $a \mathrel{R} b$. Then $a + b$ is even by the definition of $R$. This implies $b + a$ is even too, because $+$ is commutative. So $b \mathrel{R} a$ by the definition of $R$.

**(Transitivity)** Let $a, b, c \in \mathbb{Z}$ such that $a \mathrel{R} b$ and $b \mathrel{R} c$. Then, by the definition of $R$, both $a + b$ and $b + c$ are even. Use the definition of even to find $x, y \in \mathbb{Z}$ such that $a + b = 2x$ and $b + c = 2y$. Then

$$a + c = (2x - b) + (2y - b) = 2x + 2y - 2b = 2(x + y - b),$$

where $x + y - b \in \mathbb{Z}$. So $a + c$ is even by the definition of even. It follows from the definition of $R$ that $a \mathrel{R} c$. $\square$

**Comments.** Some claimed that $a \mathrel{R} b$ if and only if $a, b$ are both odd or both even for all $a, b \in \mathbb{Z}$ without any explanation. A proof is needed for this.

(ii) Write down $[1231] \cap \{7, 8, 9, 10, 11, 12\}$ in roster notation, where $[1231]$ denotes the equivalence class of 1231 with respect to the equivalence relation $R$. [2 marks]

**Solution.** $\{7, 9, 11\}$.

**Extra explanation.** All of $7+1231$, $9+1231$, and $11+1231$ are even, but none of $8+1231$, $10+1231$, and $12 + 1231$ is even.

**Comments.** Some wrote $\{(1231, 7), (1231, 9), (1231, 11)\}$. By definition, with respect to an equivalence relation on a set $A$, elements of equivalence classes are elements of $A$, not elements of $A \times A$.

END OF QUIZ