Name: ________________________ **Tutorial group:** __________ **(number or day&time)**
CS1231 Discrete Structures **Quiz 1** **2022/23 Semester 2**

1. Let D be the set that contains precisely $1, 3, 5, 7, 9$. Which of the following propositions are true? Which are false? Briefly explain your answers.

   (a) $\forall x \in D\ (x + 1 \in D \rightarrow x < 0)$. [2 marks]

   Solution. True, because given any $x \in D$, we know $x + 1 \notin D$, and thus the conditional is vacuously true.

   (b) $\exists x, y \in D\ x + y = 18$. [2 marks]

   Solution. True: take $x = 9$ and $y = 9$.

   (c) $\forall x \in D\ \exists y \in \mathbb{Z}\ (y = 2x \vee x \geqslant 0)$. [2 marks]

   Solution. True, because given any $x \in D$, we see that $2x \in \mathbb{Z}$ and/or $x \geqslant 0$.

   (d) $\exists x \in \mathbb{Z}\ \forall y \in D\ y = 2x + 1$. [3 marks]

   Solution. False, as explained below.

   - If $x = 0$, then choose $y = 3$, so that $y = 3 \neq 1 = 2x + 1$.
   - If $x \neq 0$ then choose $y = 1$, so that $y = 1 = 2 \times 0 + 1 \neq 2x + 1$.

   Alternative solution. Suppose the proposition is true. Fix $x \in \mathbb{Z}$ such that $\forall y \in D\ y = 2x + 1$. As $1, 3 \in D$, this implies $1 = 2x + 1 = 3$. However, we know $1 \neq 3$. This is a contradiction. So the proposition must be false.

   Comments. Many did not give specific ways to substitute values into y to make the proposition false. Without these, it is not clear why such substitutions exist.

   (e) $\forall x \in D\ \big((\exists y \in \mathbb{N}\ x + 2y = 3) \rightarrow x = 1\big)$. [3 marks]

   Solution. False, because if $x = 3$, then $x \in D$ and $\exists y \in \mathbb{N}\ x + 2y = 3$ is true with witness 0, but $x = 1$ is false.

2. Rewrite the following propositions symbolically. In your answers, you can use $\text{Even}(n)$ and $\text{Odd}(n)$ to stand for $\exists x \in \mathbb{Z}\ n = 2x$ and $\exists x \in \mathbb{Z}\ n = 2x + 1$ respectively.

   (a) For two integers to have the same square, it is necessary that either both are even or both are odd. [2 marks]

   Solution. $\forall x, y \in \mathbb{Z}\ \big(x^2 = y^2 \rightarrow \big(\text{Even}(x) \wedge \text{Even}(y)\big) \vee \big(\text{Odd}(x) \wedge \text{Odd}(y)\big)\big)$.

   Two acceptable but less preferable answers are $\forall x, y \in \mathbb{Z}\ \big(x^2 = y^2 \rightarrow \big(\text{Even}(x) \leftrightarrow \text{Even}(y)\big)\big)$ and $\forall x, y \in \mathbb{Z}\ \big(x^2 = y^2 \rightarrow \big(\text{Odd}(x) \leftrightarrow \text{Odd}(y)\big)\big)$

   Comments.
   - Some has the $\rightarrow$ pointing in the wrong direction. See Terminology 1.3.3 and Tutorial Exercise 1.2(b) for the meaning of the word "necessary" in mathematics.
   - Many wrote $\exists x, y \in \mathbb{Z}$ instead of $\forall x, y \in \mathbb{Z}$. The proposition given means "for x, y to have the same square, it is necessary ..." is true for *all* integers x, y. So the correct quantifier here is $\forall$.
   - According to Convention 1.4.3 and Convention 2.2.8(3), all brackets given in the answers above are necessary. Many omitted at least one pair of them.

   (b) An integer is non-negative if and only if it is the sum of the squares of four integers. [2 marks]

   Solution. $\forall n \in \mathbb{Z}\ (n \geqslant 0 \leftrightarrow \exists x, y, z, w \in \mathbb{Z}\ n = x^2 + y^2 + z^2 + w^2)$.
   It is also acceptable to split $\leftrightarrow$ into $\rightarrow$ and $\leftarrow$.

   Comments.
   - Some wrote $\forall n \in \mathbb{Z}\ \exists x, y, z, w \in \mathbb{Z}\ (n \geqslant 0 \leftrightarrow n = x^2 + y^2 + z^2 + w^2)$, which is not what the given proposition expresses. In general, one can find predicates $P(n)$ and $Q(n, z)$ on $\mathbb{Z}$ such that $\forall n \in \mathbb{Z}\ (P(n) \leftrightarrow \exists x \in \mathbb{Z}\ Q(n, x))$ and $\forall n \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ (P(n) \leftrightarrow Q(n, x))$ are not equivalent: for example, one can take $P(n)$ and $Q(n, x)$ to be $n \neq n$ and $n = x$ respectively, in which case the former proposition is false, say, with counterexample $n = 0$, while the latter proposition is true by picking $x = n + 1$ no matter which $n \in \mathbb{Z}$ is given.
   - Many wrote $\exists n \in \mathbb{Z}$ instead of $\forall n \in \mathbb{Z}$. The proposition given means "n is non-negative if and only if ..." is true for *all* integers n. So the correct quantifier here is $\forall$.
   - According to Convention 2.2.8(3), the pair of brackets given in the answer above is necessary. Many omitted them.

3. Let p, q, r be propositional variables. Which of the following pairs of compound expressions are equivalent? Which are not? Prove that your answer is correct.

   (a) $p \wedge q \to r$ and $p \to (q \to r)$. [3 marks]

   Solution. Yes, as shown below.

$$\begin{aligned} p \wedge q \to r &\equiv \neg(p \wedge q) \vee r && \text{by the logical identity on implication;} \\ &\equiv \neg p \vee \neg q \vee r && \text{by De Morgan's Laws;} \\ &\equiv \neg p \vee (q \to r) && \text{by the logical identity on implication;} \\ &\equiv p \to (q \to r) && \text{by the logical identity on implication.} \end{aligned}$$

   Alternatively, one can draw a truth table to show the equivalence.

p	q	r	$p \wedge q$	$p \wedge q \to r$	$q \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

   (b) $p \wedge q \leftrightarrow p$ and $p \wedge q$. [3 marks]

   Solution. No, because if we substitute false propositions into p and q, then $p \wedge q \leftrightarrow p$ evaluates to T while $p \wedge q$ evaluates to F.

   Comments. Some proved using the logical identities that $p \wedge q \leftrightarrow p$ is equivalent to $p \to q$, and then deduce directly that it is not equivalent to $p \wedge q$. Simply noticing $p \to q$ and $p \wedge q$ are different does not logically imply that they are not equivalent; see Note 1.4.24.

4. Consider the proposition "a rational number x satisfies $x^2 + 2 = 3x$ if and only if $x = 1$ or $x = 2$".

   (i) Someone tries to prove this proposition as follows.

   Let x be a rational number such that $x^2 + 2 = 3x$. Then

$$\begin{aligned} & x^2 - 3x + 2 = 0. \\ \therefore\quad & (x-1)(x-2) = 0. \\ \therefore\quad & x = 1 \quad \text{or} \quad x = 2. \end{aligned}$$

   What is wrong with this attempt? [1 mark]

   Solution. This attempt shows only $\forall x \in \mathbb{R}\ \big(x^2 + 2 = 3x \to x = 1 \vee x = 2\big)$. It does not explain why $\forall x \in \mathbb{R}\ \big(x = 1 \vee x = 2 \to x^2 + 2 = 3x\big)$.

   Comments. Some wrote that the author proved the "if" part, while s/he actually proved the "only if" part; see Terminology 1.3.3.

   (ii) Give a correct proof of this proposition. [2 marks]

   Solution. Let x be a rational number. Then

$$\begin{aligned} & x^2 + 2 = 3x \\ \Leftrightarrow\quad & x^2 - 3x + 2 = 0 \\ \Leftrightarrow\quad & (x-1)(x-2) = 0 \\ \Leftrightarrow\quad & x = 1 \quad \text{or} \quad x = 2. \end{aligned}$$

   Comments. One can split the proof into a $\Rightarrow$ part and a $\Leftarrow$ part, but including only one of these is not enough.

END OF QUIZ