NATIONAL UNIVERSITY OF SINGAPORE

CS1231 — DISCRETE STRUCTURES

(Semester 2: AY2023/24)

Time allowed: 2 hours

INSTRUCTIONS TO STUDENTS

1. Do **NOT** turn over this cover page and do **NOT** start writing until your invigilator tells you to do so.
2. On the right, write down your Student Number and, for each digit or letter in there, shade the corresponding circle **COMPLETELY** in the grid using ink or pencil.
3. Do not write your name.
4. This assessment paper contains **EIGHT** questions. It comprises **TEN** pages excluding this cover page.
5. Answer **ALL** questions.
6. Write your answers in the spaces provided.
7. If you need extra space for your answers, then use the back cover.
8. You may leave your numerical answers as products and quotients of expressions of the form n, n^r, $P(n,r)$, $n!$ or $\binom{n}{r}$ where $n, r \in \mathbb{N}$.
9. This is an **OPEN BOOK** assessment. You may refer to any materials on physical paper.
10. The use of handheld calculators is allowed.

STUDENT NUMBER

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Examiner's use only	
Question	Marks
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8	
Total	

1. For $r \in \mathbb{R}^+$, denote by $\mathrm{I}(r)$ the closed interval $\{a \in \mathbb{R} : -r \leqslant a \leqslant r\}$. Define

$$A = \{(s, s^2) : s \in \mathrm{I}(4)\} \quad \text{and} \quad B = \{(t^4 - t^2, t^3 - t) : t \in \mathrm{I}(2)\}.$$

Consider the relation f from A to B defined by

$$f = \big\{\big((2t, 4t^2), (t^4 - t^2, t^3 - t)\big) : t \in \mathrm{I}(2)\big\}.$$

(i) Prove that f is a function $A \to B$. [5 marks]

Solution. **(F1)** Take any $(s, s^2) \in A$, where $s \in \mathrm{I}(4)$. Then $-4 \leqslant s \leqslant 4$ by the definition of $\mathrm{I}(4)$. Let $t = s/2$. Then $s = 2t$ and $-2 \leqslant t \leqslant 2$. So $t \in \mathrm{I}(2)$. Thus $(t^4 - t^2, t^3 - t) \in B$ and

$$\big((s, s^2), (t^4 - t^2, t^3 - t)\big) = \big((2t, 4t^2), (t^4 - t^2, t^3 - t)\big) \in f$$

by the definition of f.

(F2) Let $x \in A$ and $y_1, y_2 \in B$ satisfying $(x, y_1), (x, y_2) \in f$. Then the definition of f gives $t_1, t_2 \in \mathrm{I}(2)$ such that

$$(x, y_1) = \big((2t_1, 4{t_1}^2), ({t_1}^4 - {t_1}^2, {t_1}^3 - t_1)\big) \quad \text{and} \quad (x, y_2) = \big((2t_2, 4{t_2}^2), ({t_2}^4 - {t_2}^2, {t_2}^3 - t_2)\big).$$

Looking at the first coordinates, we see $(2t_1, 4{t_1}^2) = x = (2t_2, 4{t_2}^2)$. This implies $2t_1 = 2t_2$, and thus $t_1 = t_2$. It follows that

$$y_1 = ({t_1}^4 - {t_1}^2, {t_1}^3 - t_1) = ({t_2}^4 - {t_2}^2, {t_2}^3 - t_2) = y_2.$$

$\square$

1. (Cont'd from the previous page)

(ii) Is f surjective? Prove that your answer is correct. [3 marks]

Solution. Yes, as shown below.

Let $(t^4 - t^2, t^3 - t) \in B$, where $t \in \mathrm{I}(2)$. Define $s = 2t$. As $-2 \leqslant t \leqslant 2$, we know $-4 \leqslant s \leqslant 4$. So $(s, s^2) \in A$ and

$$f(s, s^2) = f(2t, 4t^2) = (t^4 - t^2, t^3 - t).$$

$\square$

(iii) Is f injective? Prove that your answer is correct. [3 marks]

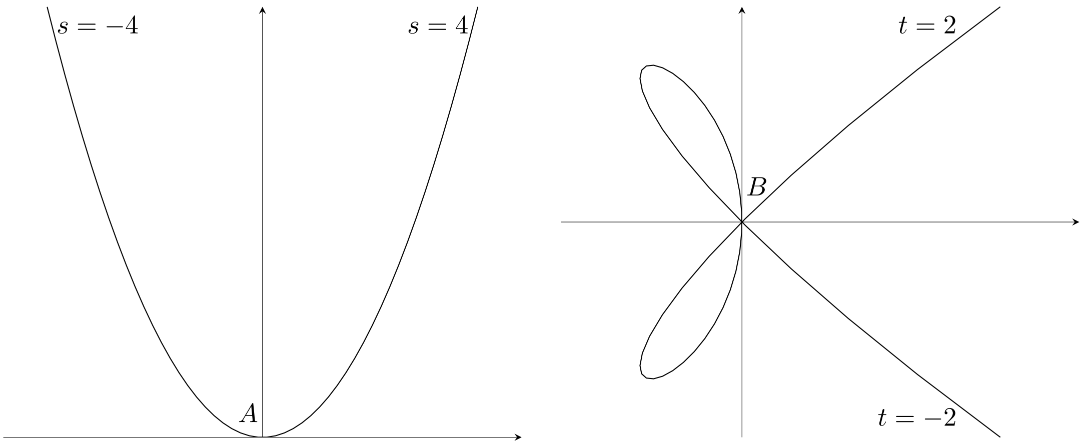
Solution. No, as shown below.

Note that

$$\begin{aligned} f(2,4) &= f(2 \times 1, 4 \times 1^2) = (1^4 - 1^2, 1^3 - 1) = (0,0) \\ &= \big((-1)^4 - (-1)^2, (-1)^3 - (-1)\big) = f\big(2 \times (-1), 4 \times (-1)^2\big) = f(-2,4), \end{aligned}$$

but $(2,4) \neq (-2,4)$. $\square$

Diagrams (extra).



2. (a) Is it true that, for all sets A and B, every function $f\colon A \to B$, when considered as a subset of $A \times B$, has the same cardinality as its domain A? Prove that your answer is correct.

[4 marks]

Solution. Yes, as shown below.

Define $g\colon A \to f$ by setting $g(x) = (x, f(x))$ for each $x \in A$.

(Surjectivity) Let $(x, y) \in f$. Then $y = f(x)$ and thus $g(x) = (x, f(x)) = (x, y)$.

(Injectivity) Let $x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$. Then $(x_1, f(x_1)) = (x_2, f(x_2))$ by the definition of g. This implies $x_1 = x_2$.

We see above that there is a bijection $A \to f$. So A has the same cardinality as f. $\square$

Alternative proof. Define $h\colon f \to A$ by setting $h(x, y) = x$ for each $(x, y) \in A$.

(Surjectivity) Let $x \in A$. Then $(x, f(x)) \in f$ and $h(x, f(x)) = x$.

(Injectivity) Let $(x_1, y_1), (x_2, y_2) \in f$ such that $h(x_1, y_1) = h(x_2, y_2)$. Then $x_1 = x_2$ by the definition of h. Also $y_1 = f(x_1) = f(x_2) = y_2$. So $(x_1, y_1) = (x_2, y_2)$.

We see above that there is a bijection $f \to A$. So f has the same cardinality as A. $\square$

(b) Is it true that, for all sets A and B, every function $f\colon A \to B$, when considered as a subset of $A \times B$, has the same cardinality as its codomain B? Prove that your answer is correct.

[4 marks]

Solution. No, as shown below.

Consider $A = \{-1, 1\}$ and $B = \{0\}$. Define $f\colon A \to B$ by setting $f(-1) = 0 = f(1)$. Then $|f| = |\{(-1, 0), (1, 0)\}| = 2 \neq 1 = |\{0\}| = |B|$. So f does not have the same cardinality as B. $\square$

3. Is it true that, for all sets A, if there is a surjection f from A to some countable set B, then A is countable? Prove that your answer is correct. [4 marks]

Solution. No, as shown below.

Consider the function $f\colon \mathcal{P}(\mathbb{N}) \to \{1\}$ such that $f(x) = 1$ for every $x \in \mathcal{P}(\mathbb{N})$. This is a surjection because $\varnothing \in \mathcal{P}(\mathbb{N})$ and $f(\varnothing) = 1$. Now $\{1\}$ is a finite set, as witnessed by $\mathrm{id}_{\{1\}}$. So it is countable by Proposition 9.1.3. However, we know from Corollary 9.3.2 that $\mathcal{P}(\mathbb{N})$ is uncountable. □

4. There are 30 sets, of which 20 are countable and 24 are infinite. How many of them are both countable and infinite? Explain your answer. [3 marks]

Solution. Let C be the set of all countable sets and I be the set of all infinite sets here. Note that all sets are either countable or infinite because finite sets are countable by Proposition 9.1.3. So $|C \cup I| = 30$. Also, by the Inclusion–Exclusion Rule,

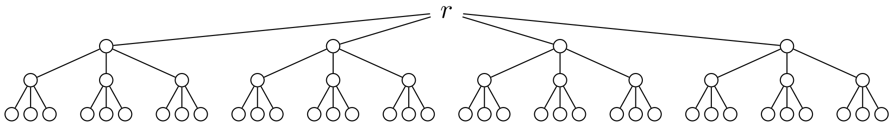
$$\begin{aligned} |C \cup I| &= |C| + |I| - |C \cap I| \\ &= 20 + 24 - |C \cap I|. \end{aligned}$$

Hence, the number of sets that are both countable and infinite here is

$$|C \cap I| = 20 + 24 - |C \cup I| = 20 + 24 - 30 = 14.$$

Alternative solution. There are 30 sets, of which 24 are infinite. So the Difference Rule tells us $30 - 24 = 6$ of these are finite. All these 6 sets are countable in view of Proposition 9.1.3. The rest are infinite. Since there are 20 countable sets here, the number of countable infinite sets here is $20 - 6 = 14$ by the Difference Rule.

5. The following is a drawing of a tree T with root r.



Calculate the number of subgraphs of T that satisfy all of the following: (a) it contains the vertex r, (b) when r is considered the root, it has height 3 and all parents have exactly two children, and (c) when r is considered the root, all the terminal vertices of the subgraph are terminal vertices of T. Show your working, where it is indicated where each term comes from. [2 marks]

Solution.

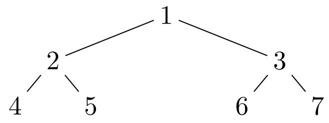
- There is 1 way to choose top vertex r.
- There are $\binom{4}{2}$ ways to choose the two children of the root.
- For each of these 2 children, there are $\binom{3}{2}$ ways to choose its two children.
- For each of these $2 \times 2 = 4$ children, there are again $\binom{3}{2}$ ways to choose its two children.

By the General Multiplication Rule, the number of such sequences of choices is

$$\binom{4}{2}\binom{3}{2}^2\binom{3}{2}^4 = \frac{4 \times 3}{2 \times 1} \times 3^2 \times 3^4 = 4374.$$

Each such sequence of choices gives rise to a required subgraph, and each required subgraph is constructed from exactly one such sequence of choices. Therefore, the number of such graphs is also 4374.

6. The following is a drawing of a graph G.

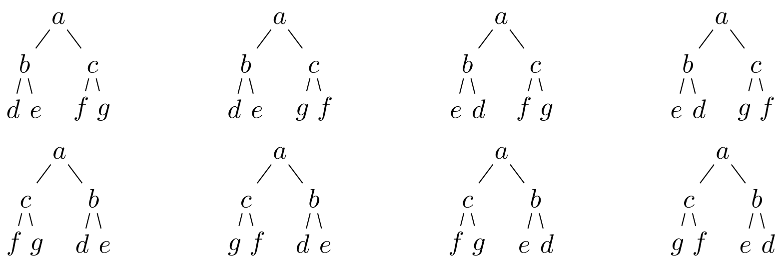


Calculate the number of graphs that satisfy all of the following: (a) it is isomorphic to G, and (b) it has the same vertices as G. Show your working, where it is indicated where each term comes from. [2 marks]

Solution.

- There are 7 ways to choose a new label $x_1 \in \{1, 2, \ldots, 7\}$ for the vertex that is labelled 1 in G.
- There are $7 - 1 = 6$ ways to choose a new label $x_2 \in \{1, 2, \ldots, 7\} \setminus \{x_1\}$ for the vertex that is labelled 2 in G.
- There are $7 - 2 = 5$ ways to choose a new label $x_3 \in \{1, 2, \ldots, 7\} \setminus \{x_1, x_2\}$ for the vertex that is labelled 3 in G.
- $\ldots$
- There are $7 - 6 = 1$ ways to choose a new label $x_7 \in \{1, 2, \ldots, 7\} \setminus \{x_1, x_2, \ldots, x_6\}$ for the vertex that is labelled 7 in G.

By the General Multiplication Rule, the number of such sequences of choices is $7 \times 6 \times \cdots \times 1 = 7!$. Each such sequence of choices gives rise to a required graph, and each required graph is constructed from exactly 8 such sequences:



are drawings of the same graph. Therefore, the number of such graphs is $7!/8 = 630$.

Alternative solution.

- There are 7 ways to choose a new label $x_1 \in \{1, 2, \ldots, 7\}$ for the vertex that is labelled 1 in G.
- There are $\binom{7-1}{2} = \binom{6}{2}$ ways to choose a subset $\{x_2, x_3\} \subseteq \{1, 2, \ldots, 7\} \setminus \{x_1\}$ of new labels for the set $\{2, 3\}$ of vertices in G, where $x_2 < x_3$.
- There are $\binom{6-2}{2} = \binom{4}{2}$ ways to choose a subset $\{x_4, x_5\} \subseteq \{1, 2, \ldots, 7\} \setminus \{x_1, x_2, x_3\}$ of size 2 to make edges x_2x_4 and x_2x_5.
- There are $\binom{4-2}{2} = \binom{2}{2}$ ways to choose a set $\{x_6, x_7\} \subseteq \{1, 2, \ldots, 7\} \setminus \{x_1, x_2, \ldots, x_5\}$ of size 2 to make edges x_3x_6 and x_3x_7.

By the General Multiplication Rule, the number of such sequences of choices is $7 \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 630$. Each such sequence of choices gives rise to a required graph, and each required graph is constructed from exactly one such sequence. Therefore, the number of such graphs is also 630.

7. Consider the undirected graph G where

$$\begin{aligned} &\mathrm{V}(G) = \{2,4,5,7,8\}, \quad \text{and} \\ &\mathrm{E}(G) = \{ab : a, b \in \mathrm{V}(G), \text{ and either } a+4<b \text{ or } b+4<a\}. \end{aligned}$$

(i) Draw G. Label the vertices in your drawing. [1 mark]

Solution. 7—2—8 4 5

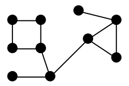
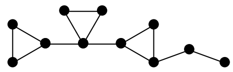
(ii) How many connected components does G have? [1 mark]

Solution. 3.

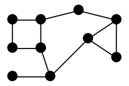
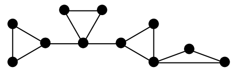
(iii) What is the length of the longest path in G? [1 mark]

Solution. 2.

8. A *tuberoot* is defined to be a connected loopless undirected finite graph in which no two distinct cycles share a common vertex. Here are drawings of two tuberoots.

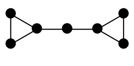
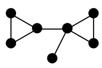
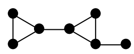
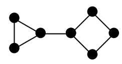



The following are *not* drawings of tuberoots.

(i) How many tuberoots are there with exactly 7 vertices and exactly 2 cycles if we count isomorphic tuberoots as one? Explain your answer. [3 marks]

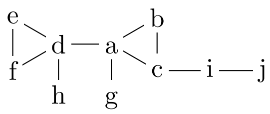
Solution. There are exactly 4, as listed below.

For each cycle C in a tuberoot G, define the undirected graph $G \lhd C$ as follows:

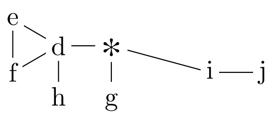
$\mathrm{V}(G \lhd C) = (\mathrm{V}(G) \setminus \mathrm{V}(C)) \cup \{\ast\}$, where $\ast$ is a new vertex;
$\mathrm{E}(G \lhd C) = \{e \in \mathrm{E}(G) : e \cap \mathrm{V}(C) = \varnothing\} \cup \{\ast b : b \notin \mathrm{V}(C) \text{ and } ab \in \mathrm{E}(G) \text{ for some } a \in \mathrm{V}(C)\}$.

(ii) The following is a drawing of a tuberoot G.



Draw $G \lhd C$, where C is the cycle abca. [1 mark]

Solution.



8. (Cont'd from the previous page)

(iii) Let G be a tuberoot and C be a cycle in G. Prove that $G \lhd C$ is connected. [4 marks]

Solution. Pick any $u, v \in \mathrm{V}(G \lhd C)$.

Case 1: suppose $u, v \in \mathrm{V}(G) \setminus \mathrm{V}(C)$. Use the connectedness of G to find a path $P = x_0 x_1 \dots x_n$ in G where $x_0 = u$ and $x_n = v$.

Case 1.1: suppose $\mathrm{V}(P) \cap \mathrm{V}(C) = \varnothing$. Then P is a path in $G \lhd C$ in view of the definition of $G \lhd C$.

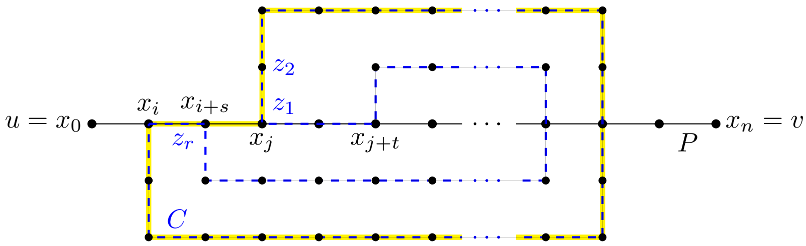
Case 1.2: suppose $\mathrm{V}(P) \cap \mathrm{V}(C) \neq \varnothing$. We claim that $\mathrm{V}(P) \cap \mathrm{V}(C) = \{x_i, x_{i+1}, \dots, x_{i+s}\}$ and $\mathrm{E}(P) \cap \mathrm{E}(C) = \{x_i x_{i+1}, x_{i+1} x_{i+2}, \dots, x_{i+s-1} x_{i+s}\}$ for some $i, s \in \mathbb{N}$ satisfying $0 \leqslant i \leqslant i + s \leqslant n$. Suppose not. Invoke the Well-Ordering Principle to find $i, j, s, t \in \mathbb{N}$ with $i \leqslant i + s < j \leqslant j + t$ such that

$$\begin{aligned} &x_i x_{i+1}, x_{i+1} x_{i+2}, \dots, x_{i+s-1} x_{i+s} \in \mathrm{E}(C) \text{ and} \\ &x_{i+s} x_{i+s+1}, x_{i+s+1} x_{i+s+2}, \dots, x_{j-1} x_j \notin \mathrm{E}(C) \text{ and} \\ &x_j x_{j+1}, x_{j+1} x_{j+2}, \dots, x_{j+t-1} x_{j+t} \in \mathrm{E}(C). \end{aligned}$$

Say $C = z_1 z_2 \dots z_\ell z_1$, where $z_1 = x_j$ and $z_r = x_{i+s}$, and $z_1 z_2 \notin \mathrm{E}(P)$. Note that $x_{i+s} x_{i+s+1} \in \mathrm{E}(P) \setminus \mathrm{E}(C)$ and $z_{r-1} z_r \in \mathrm{E}(C)$, where $x_{i+s} = z_r$. So $x_{i+s}, x_{i+s+1}, z_{r-1}$ must be three different vertices. This tells us that

$$x_{i+s} x_{i+s+1} \dots x_{j-1} x_j z_2 z_3 \dots z_r$$

is a cycle in G. It is distinct from C, because it contains the edge $x_{i+s} x_{i+s+1} \notin \mathrm{E}(C)$, but it shares the vertex x_{i+s} with C. This contradicts the fact that G is a tuberoot, and thus proves the claim.



Use the claim to obtain $i, s \in \mathbb{N}$ such that $0 \leqslant i \leqslant i + s \leqslant n$ and $\mathrm{V}(P) \cap \mathrm{V}(C) = \{x_i, x_{i+1}, \dots, x_{i+s}\}$. Notice $x_0 = u \notin \mathrm{V}(C)$ and $x_n = v \notin \mathrm{V}(C)$. Since $x_i, x_{i+s} \in \mathrm{V}(C)$, actually $0 < i \leqslant i + s < n$. So $x_0 x_1 \dots x_{i-1} \circledast x_{i+s+1} x_{i+s+2} \dots x_n$ is a path between u and v in $G \lhd C$.

Case 2: suppose exactly one of u, v is in $\mathrm{V}(G) \setminus \mathrm{V}(C)$. Say $u \in \mathrm{V}(G) \setminus \mathrm{V}(C)$ and $v = \circledast$. In view of the connectedness of G, there is a path in G between u and some vertex in C. Use the Well-Ordering Principle to find such a path $P = x_0 x_1 \dots x_n$ of shortest length, where $x_0 = u$ and $x_n \in \mathrm{V}(C)$. By the minimality of the length of P, we know $x_0, x_1, \dots, x_{n-1} \notin \mathrm{V}(C)$. Then $x_0 x_1 \dots x_{n-1} \circledast$ is a path in $G \lhd C$ between u and v by the definition of $G \lhd C$.

Case 3: suppose $u, v \notin \mathrm{V}(G) \setminus \mathrm{V}(C)$. Then $u = \circledast = v$ by the definition of $G \lhd C$. So $\circledast$ is a path between u and v in $G \lhd C$.

In all cases, we have a path between u and v in $G \lhd C$. $\square$

8. (Cont'd from the previous page)

(iv) Let G be a tuberoot and C be a cycle in G. Prove that the cycles in $G \lhd C$ are precisely the cycles in G excluding C. [3 marks]

Solution. Only vertices in C and edges containing a vertex in C are affected in the construction of $G \lhd C$ from G. As no distinct cycles share a common vertex in the tuberoot G, the graph $G \lhd C$ inherits all the cycles in G except C.

Let D be any cycle in $G \lhd C$.

Case 1: suppose $\ast \notin \mathrm{V}(D)$. Then D is actually a cycle in G because all the new edges added when constructing $G \lhd C$ from G contain $\ast$. Since $G \lhd C$ no longer has the vertices (and the edges) in C, we know $D \neq C$.

Case 2: suppose $\ast \in \mathrm{V}(D)$. Say $D = b_1 b_2 \dots b_m b_1$, where $b_1 = \ast$ and $\ell \geqslant 3$. This tells us $b_2 \neq b_m$ and $\ast b_2, \ast b_m \in \mathrm{E}(G \lhd C)$. Use the definition of $G \lhd C$ to find $a_2, a_m \in \mathrm{V}(C)$ such that $a_2 b_2, a_m b_m \in \mathrm{E}(G)$. Say $C = z_1 z_2 \dots z_\ell z_1$, where $z_1 = a_m$ and $z_k = a_2$. Now a_2, b_2, b_m are three distinct vertices. So $a_2 b_2 b_3 \dots b_m a_m z_2 z_3 \dots z_k$ is a cycle in G. It is distinct from C because it contains the vertex b_2, but it shares the vertex a_2 with C. This contradicts the hypothesis that G is a tuberoot. $\square$

(v) Let G be a tuberoot and C be a cycle in G. Explain why

$$|\mathrm{V}(G \lhd C)| = |\mathrm{V}(G)| - |\mathrm{V}(C)| + 1 \quad \text{and} \quad |\mathrm{E}(G \lhd C)| = |\mathrm{E}(G)| - |\mathrm{E}(C)|.$$

[2 marks]

Solution. By the Difference Rule and the Addition Rule,

$$|\mathrm{V}(G \lhd C)| = |(\mathrm{V}(G) \setminus \mathrm{V}(C)) \cup \{\ast\}| = |\mathrm{V}(G)| - |\mathrm{V}(C)| + |\{\ast\}| = |\mathrm{V}(G)| - |\mathrm{V}(C)| + 1.$$

When constructing $G \lhd C$ from G, all the edges in C are removed, and each edge $ab \in \mathrm{E}(G)$ where $a \in \mathrm{V}(C)$ and $b \notin \mathrm{V}(C)$ is replaced by an edge $\ast b \in \mathrm{E}(G \lhd C)$. We show that this replacement process is injective, which will imply

$$\begin{aligned}
|\mathrm{E}(G)| &= |\{e \in \mathrm{E}(G) : e \cap \mathrm{V}(C) = \varnothing\}| + |\{e \in \mathrm{E}(G) : |e \cap \mathrm{V}(C)| = 1\}| \\
&\quad + |\{e \in \mathrm{E}(G) : e \subseteq \mathrm{V}(C)\}| \\
&= |\{e \in \mathrm{E}(G) : e \cap \mathrm{V}(C) = \varnothing\}| \\
&\quad + |\{\ast b \in \mathrm{E}(G) : b \notin \mathrm{V}(C) \text{ and } ab \in \mathrm{E}(G) \text{ for some } a \in \mathrm{V}(C)\}| \\
&\quad + |\{e \in \mathrm{E}(G) : e \subseteq \mathrm{V}(C)\}| \\
&= |\mathrm{E}(G \lhd C)| + |\mathrm{E}(C)|
\end{aligned}$$

by the Addition Rule and the definition of $G \lhd C$, and thus $|\mathrm{E}(G \lhd C)| = |\mathrm{E}(G)| - |\mathrm{E}(C)|$.

Let $a_1, a_2 \in \mathrm{V}(C)$ and $b_1, b_2 \in \mathrm{V}(G) \setminus \mathrm{V}(C)$ such that $a_1 b_1, a_2 b_2 \in \mathrm{E}(G)$ and $\ast b_1 = \ast b_2$. Then $b_1 = b_2$ because $b_1 \neq \ast \neq b_2$. Suppose $a_1 \neq a_2$. Say $C = z_1 z_2 \dots z_\ell z_1$, where $z_1 = a_1$ and $z_k = a_2$. Note that a_1, a_2, b_1 are three different vertices. So $b_1 z_1 z_2 \dots z_k b_2$ is a cycle in G. It is distinct from C because it contains the vertex b_1, but it shares the vertex a_1 with C. This contradicts the hypothesis that G is a tuberoot. $\square$

8. (Cont'd from the previous page)

(vi) Prove by induction on the number c of cycles in G that, for every tuberoot G,

$$|\mathrm{E}(G)| = |\mathrm{V}(G)| - 1 + c.$$

[4 marks]

Solution. (Base step) Let G be a tuberoot with no cycle. By the definition of tuberoots, we know G is loopless and connected. So G is acyclic and is thus a tree. It follows from Theorem 12.1.10 that $|\mathrm{E}(G)| = |\mathrm{V}(G)| - 1 = |\mathrm{V}(G)| - 1 + 0$.

(Induction step) Let $k \in \mathbb{N}$ such that the proposition is true for all tuberoots with exactly k cycles. Consider a tuberoot G with exactly $k+1$ cycles. Pick any cycle C in G. We know from (iii) that $G \lhd C$ is connected. Since G has no loop and the construction of $G \lhd C$ does not introduce any loop, we know that $G \lhd C$ is again loopless. Being a tuberoot, no distinct cycles in G share a common vertex. Since all cycles in $G \lhd C$ are cycles in G by (iv), the same is true in $G \lhd C$. Therefore, we deduce that $G \lhd C$ is a tuberoot. By (iv) again, the tuberoot $G \lhd C$ has exactly one fewer cycles than G. So $G \lhd C$ has exactly k cycles. The induction hypothesis then tells us $|\mathrm{E}(G \lhd C)| = |\mathrm{V}(G \lhd C)| - 1 + k$. By this and (v),

$$\begin{aligned} |\mathrm{E}(G)| &= |\mathrm{E}(G \lhd C)| + |\mathrm{E}(C)| = |\mathrm{V}(G \lhd C)| - 1 + k + |\mathrm{E}(C)| \\ &= |\mathrm{V}(G)| - |\mathrm{V}(C)| + 1 - 1 + k + |\mathrm{E}(C)| = |\mathrm{V}(G)| - 1 + (k+1). \end{aligned} \qquad \square$$

Alternative way to complete the induction step. Let $k \in \mathbb{N}$ such that the proposition is true for all tuberoots with exactly k cycles. Consider a tuberoot G_{k+1} with exactly $k+1$ cycles. Pick any cycle C in G_{k+1}. Take $e \in \mathrm{E}(C)$. Let G_k be the graph obtained from G_{k+1} by removing the edge e, i.e.,

$$\mathrm{V}(G_k) = \mathrm{V}(G_{k+1}) \quad \text{and} \quad \mathrm{E}(G_k) = \mathrm{E}(G_{k+1}) \setminus \{e\}.$$

As G_{k+1} is a connected and e comes from a cycle in G_{k+1}, we can show that G_k is connected by following the proof of Theorem 12.1.4. As G_k is a subgraph of the loopless graph G_{k+1} where no two distinct cycles share a common vertex, we know G_k also has no loop, and no two distinct cycles in G_k share a common vertex too. These show that G_k is a tuberoot.

Now we count the number of cycles in G_k. In the construction from G_{k+1} to G_k, only vertices and edges in C are affected. As no distinct cycles share a common vertex in the tuberoot G_{k+1}, the graph G_k inherits all the cycles in G_{k+1} except C. Conversely, all cycles in G_k are cycles in G_{k+1} because G_k is a subgraph of G_{k+1}. So the cycles in G_k are precisely the cycles in G_{k+1} excluding C. As G_{k+1} has $k+1$ cycles, we deduce that G_k has k cycles. Hence

$$\begin{aligned} |\mathrm{E}(G_{k+1})| &= |\mathrm{E}(G_k)| + |\{e\}| && \text{by the Addition Rule;} \\ &= |\mathrm{E}(G_k)| + 1 \\ &= |\mathrm{V}(G_k)| - 1 + k + 1 && \text{by the induction hypothesis;} \\ &= |\mathrm{V}(G_{k+1})| - 1 + (k+1). \end{aligned} \qquad \square$$

END OF PAPER

Use this page if and only if you need extra space for your answers, in which case (1) indicate the question number(s) clearly below, and (2) indicate in the original answer space that you are using this page for that question. Do *not* use this space for rough work.

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