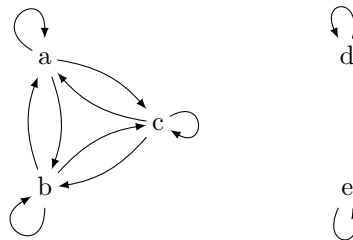


# Tutorial solutions for Chapter 6

Sometimes there are other correct answers.

6.1. (a)



(b) Yes, this  $R$  is **an equivalence relation**: as one can verify exhaustively, it is reflexive, symmetric, and transitive.

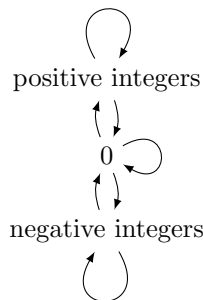
The equivalence classes are  $\{a, b, c\}$  and  $\{d\}$  and  $\{e\}$ .

(c) No, this  $R$  is **not a partial order**. It is not antisymmetric because, for example, we have  $(a, b) \in R$  and  $(b, a) \in R$ , but  $a \neq b$ . As  $R$  is not antisymmetric, it cannot be a partial order.

No, this  $R$  is **not a total order**, because it is not even a partial order.

**Moral.** It is often helpful to draw a diagram.

6.2. (a)



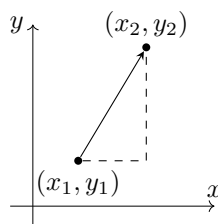
$R$  is **reflexive** because  $xx = x^2 \geq 0$  for all  $x \in \mathbb{Q}$ .

$R$  is **symmetric** because  $xy \geq 0$  implies  $yx \geq 0$  for all  $x, y \in \mathbb{Q}$ .

$R$  is **not antisymmetric** because  $1 \times 2 \geq 0$  and  $2 \times 1 \geq 0$  but  $1 \neq 2$ , for instance.

$R$  is **not transitive** because  $1 \times 0 \geq 0$  and  $0 \times (-1) \geq 0$  but  $1 \times (-1) = -1 < 0$ , for instance.

(b)



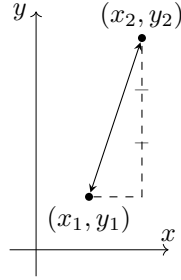
$S$  is **reflexive** because  $x \leq x$  and  $y \leq y$  for all  $(x, y) \in \mathbb{R}^2$ .

$S$  is **not symmetric** because  $(1, 1) S (2, 2)$  but  $(2, 2) \not S (1, 1)$ , for instance.

$S$  is **antisymmetric**. To see this, let  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  such that  $(x_1, y_1) S (x_2, y_2)$  and  $(x_2, y_2) S (x_1, y_1)$ . The former implies  $x_1 \leq x_2$  and  $y_2 \leq y_1$ , while the latter implies  $x_2 \leq x_1$  and  $y_2 \leq y_1$ . Thus  $x_1 = x_2$  and  $y_1 = y_2$ . Hence  $(x_1, y_1) = (x_2, y_2)$ .

$S$  is **transitive**. To see this, let  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$  such that  $(x_1, y_1) S (x_2, y_2)$  and  $(x_2, y_2) S (x_3, y_3)$ . These imply  $x_1 \leq x_2 \leq x_3$  and  $y_1 \leq y_2 \leq y_3$ . Thus  $x_1 \leq x_3$  and  $y_1 \leq y_3$ . Hence  $(x_1, y_1) S (x_3, y_3)$ .

6.3.



(a) (Reflexivity) Let  $(x, y) \in \mathbb{R}^2$ . Then  $3(x - x) = 3 \times 0 = 0 = y - y$ . So  $(x, y) R (x, y)$ .

(Symmetry) Let  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  such that  $(x_1, y_1) R (x_2, y_2)$ . In view of the definition of  $R$ , this means  $3(x_1 - x_2) = y_1 - y_2$ . Thus  $3(x_2 - x_1) = y_2 - y_1$ . So  $(x_2, y_2) R (x_1, y_1)$  by the definition of  $R$ .

(Transitivity) Let  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$  such that  $(x_1, y_1) R (x_2, y_2)$  and  $(x_2, y_2) R (x_3, y_3)$ . In view of the definition of  $R$ , this means  $3(x_1 - x_2) = y_1 - y_2$  and  $3(x_2 - x_3) = y_2 - y_3$ . Thus

$$3(x_1 - x_3) = 3(x_1 - x_2) + 3(x_2 - x_3) = (y_1 - y_2) + (y_2 - y_3) = y_1 - y_3.$$

So  $(x_1, y_1) R (x_3, y_3)$  by the definition of  $R$ .

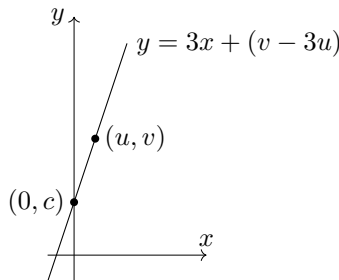
As  $R$  is reflexive, symmetric and transitive, it is an equivalence relation.

(b)  $[(u, v)]$  is the straight line with slope 3 passing through the point  $(u, v)$  in the plane  $\mathbb{R}^2$ .

**Explanations.**

$$\begin{aligned} [(u, v)] &= \{(x, y) \in \mathbb{R}^2 : (u, v) R (x, y)\} && \text{by the definition of } [(u, v)]; \\ &= \{(x, y) \in \mathbb{R}^2 : 3(u - x) = v - y\} && \text{by the definition of } R; \\ &= \{(x, y) \in \mathbb{R}^2 : y = 3x + (v - 3u)\}. \end{aligned}$$

From this, we see that  $[(u, v)]$  is a straight line with slope 3. We know  $(u, v) \in [(u, v)]$  by Lemma 6.2.4.



**Additional comment.** Note that every straight line with slope 3 is the equivalence class of some point in  $\mathbb{R}^2$ .

- (c) Let  $(u, v) \in \mathbb{R}^2$ . Define  $c = v - 3u$ . Then  $v - c = v - (v - 3u) = 3u = 3(u - 0)$ . So the definition of  $R$  tells us  $(u, v) R (0, c)$ . Hence  $(0, c) \in [(u, v)]$  by the definition of  $[(u, v)]$ .  $\square$

**How one may find this  $c$ .** Unravelling the definitions, we see that we want  $c \in \mathbb{R}$  such that  $(u, v) R (0, c)$ . This means  $3(u - 0) = v - c$ . Solving gives  $c = v - 3u$ . What we showed here is that the only  $c \in \mathbb{R}$  that can possibly make  $(0, c) \in [(u, v)]$  is  $v - 3u$ . As verified above, this choice of  $c$  actually works.

**Additional comment.** We do not need to explain how we found our  $c$  in a proof that such a  $c$  exists.

- 6.4. (0) By definition, every equivalence class is a subset of  $A$ . Every equivalence class  $[x]$  is nonempty because it contains  $x$  by Lemma 6.2.4.
- (1) Lemma 6.2.4 tells us every  $x \in A$  is in its own equivalence class  $[x]$ .
- (2) Let  $[x]$  and  $[y]$  be equivalence classes such that  $[x] \cap [y] \neq \emptyset$ . Take any  $z \in [x] \cap [y]$ . Then  $z \in [x]$  and  $z \in [y]$  by the definition of  $\cap$ . So Question 6.2.6 tells us  $[x] = [z] = [y]$ .

So  $A/\sim$  is a partition of  $A$ .  $\square$

**Alternative proof of (2) using only the definitions.** Assume  $[x] \cap [y] \neq \emptyset$ . Say, we have  $w \in [x] \cap [y]$ . This means  $x \sim w$  and  $y \sim w$  by the definition of  $[x]$  and  $[y]$ .

To show  $[x] = [y]$ , we need to prove both  $[x] \subseteq [y]$  and  $[y] \subseteq [x]$ . We will concentrate on the former; the latter is similar.

Take  $z \in [x]$ . Then  $x \sim z$  by the definition of  $[x]$ . By symmetry, we know from the first paragraph that  $w \sim x$ . Altogether we have  $y \sim w \sim x \sim z$ . So transitivity tells us  $y \sim z$ . Thus  $z \in [y]$  by the definition of  $[y]$ .  $\square$

- 6.5. (a) The symmetry of a relation  $R$  on  $A$  states that  $\forall x, y \in A \ (x R y \Rightarrow y R x)$ . The attempt claims that this implies  $\forall x, y \in A \ (x R y \wedge y R x)$ . This is not justified (and actually not true).
- (b) One counterexample is the relation  $R = \{(0, 0)\}$  on the set  $A = \{0, 1\}$ .

**Explanation.** The given proposition can be written semi-symbolically as

$$\forall \text{set } A \ \forall \text{relation } R \text{ on } A \ (\text{Sym}(A, R) \wedge \text{Tran}(A, R) \rightarrow \text{Refl}(A, R)),$$

where  $\text{Sym}(A, R)$  stands for “ $R$  is a symmetric relation on  $A$ ”, etc. So a counterexample consists of a set  $A$  and a relation  $R$  on  $A$  which makes the conditional proposition

$$\text{Sym}(A, R) \wedge \text{Tran}(A, R) \rightarrow \text{Refl}(A, R)$$

false. According to Example 1.4.24, the negation of this conditional proposition is

$$\text{Sym}(A, R) \wedge \text{Tran}(A, R) \wedge \neg \text{Refl}(A, R).$$

The relation given above satisfies this: it is symmetric and transitive, as one can verify exhaustively, but it is not reflexive because  $1 \in A$  but  $(1, 1) \notin R$ .

**Moral.** The compound expressions  $p \rightarrow q$  and  $p \wedge q$ , where  $p, q$  are propositional variables, are *not* equivalent, as one can verify using a truth table.

- 6.6. (a) This proposition is **false**. One counterexample is the relation  $R = \{(0, 0), (1, 1)\}$  on the set  $A = \{0, 1\}$ . It is both symmetric and antisymmetric, as one can verify exhaustively.
- (b) This proposition is also **false**. One counterexample is the divisibility relation on  $\mathbb{Z}$  from Example 6.1.8 and Example 6.4.5.

**Moral.** Antisymmetry is not the negation of symmetry. In fact, it is neither a necessary nor a sufficient condition for the negation of symmetry.

6.7. Let  $P(n)$  be the predicate

there exist no  $x_1, x_2, \dots, x_n \in A$  satisfying  $x_1 \neq x_2$  and

$$x_1 R x_2 \text{ and } x_2 R x_3 \text{ and } \dots \text{ and } x_{n-1} R x_n \text{ and } x_n R x_1$$

over  $\mathbb{Z}_{\geq 2}$ .

**(Base step)** The antisymmetry of  $R$  tells us there exist no  $x_1, x_2 \in A$  satisfying  $x_1 \neq x_2$  and  $x_1 R x_2$  and  $x_2 R x_1$ . So  $P(2)$  is true.

**(Induction step)** Let  $k \in \mathbb{Z}_{\geq 2}$  such that  $P(k)$  is true, i.e., there exist no  $x_1, x_2, \dots, x_k \in A$  satisfying  $x_1 \neq x_2$  and

$$x_1 R x_2 \text{ and } x_2 R x_3 \text{ and } \dots \text{ and } x_{k-1} R x_k \text{ and } x_k R x_1.$$

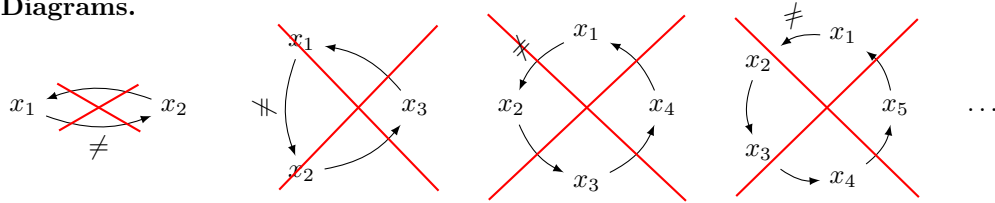
Suppose  $P(k+1)$  is false. This gives  $x_1, x_2, \dots, x_k, x_{k+1} \in A$  satisfying  $x_1 \neq x_2$  and

$$x_1 R x_2 \text{ and } x_2 R x_3 \text{ and } \dots \text{ and } x_{k-1} R x_k \text{ and } x_k R x_{k+1} \text{ and } x_{k+1} R x_1.$$

In view of the transitivity of  $R$ , the last two conditions imply  $x_k R x_1$ . This, together with the rest of the conditions above, contradict the induction hypothesis. So  $P(k+1)$  is true.

Hence  $\forall n \in \mathbb{Z}_{\geq 2}$   $P(n)$  is true by MI. □

**Diagrams.**



## Further exercises

6.8. (a)  $R$  is **not reflexive** because  $0 \times 0 = 0 \not\leq 0$ .

$R$  is **symmetric** because  $xy > 0$  implies  $yx > 0$  for all  $x, y \in \mathbb{Q}$ .

$R$  is **not antisymmetric** because  $1 \times 2 > 0$  and  $2 \times 1 > 0$  but  $1 \neq 2$ , for instance.

$R$  is **transitive**. To see this, suppose  $x, y, z \in \mathbb{Q}$  such that  $xy > 0$  and  $yz > 0$ . As  $xy > 0$ , either  $x, y$  are both positive, or  $x, y$  are both negative.

- Suppose  $x > 0$  and  $y > 0$ . Then  $z > 0$  too as  $yz > 0$ . It follows that  $xz > 0$ .
- Suppose  $x < 0$  and  $y < 0$ . Then  $z < 0$  too as  $yz > 0$ . It follows that  $xz > 0$ .

(b)  $S$  is **not reflexive** because  $x \neq x + 1$  for all  $x \in \mathbb{Z}$ .

$S$  is **not symmetric** because  $y = x + 1$  implies  $x = y - 1 \neq y + 1$  for all  $x, y \in \mathbb{Z}$ .

$S$  is **antisymmetric** vacuously because if  $y = x + 1$  and  $x = y + 1$ , then  $x = y + 1 = (x + 1) + 1 = x + 2 \neq x$ , which is a contradiction.

$S$  is **not transitive** because if  $y = x + 1$  and  $z = y + 1$ , then  $z = (x + 1) + 1 = x + 2 \neq x + 1$ .

$$6.9. \quad R = R^{-1}$$

- $$\begin{aligned} \Leftrightarrow \quad & \forall (x, y) \in A^2 \quad ((x, y) \in R \Leftrightarrow (x, y) \in R^{-1}) \quad \text{as } R, R^{-1} \subseteq A^2; \\ \Leftrightarrow \quad & \forall (x, y) \in A^2 \quad ((x, y) \in R \Leftrightarrow (y, x) \in R) \quad \text{by the definition of } R^{-1}; \\ \Leftrightarrow \quad & \forall (x, y) \in A^2 \quad ((x, y) \in R \Rightarrow (y, x) \in R) \quad \text{by Exercise 3.3;} \\ \Leftrightarrow \quad & R \text{ is symmetric} \quad \text{by the definition of symmetry.} \end{aligned}$$

6.10. (a) (Reflexivity) Let  $a \in \mathbb{Z}$ . Note that  $a - a = 0 = 2 \times 0$ , where  $0 \in \mathbb{Z}$ . So  $a - a$  is even, and thus  $a R a$  by the definition of  $R$ .

(Symmetry) Let  $a, b \in \mathbb{Z}$  such that  $a R b$ . Then  $a - b$  is even by the definition of  $R$ . Use the definition of even integers to find  $x \in \mathbb{Z}$  such that  $a - b = 2x$ . Then  $b - a = 2(-x)$ , where  $-x \in \mathbb{Z}$ . So  $b - a$  is even, and thus  $b R a$  by the definition of  $R$ .

(Alternative proof of symmetry) Let  $a, b \in \mathbb{Z}$  such that  $a R b$ . Then  $a - b$  is even by the definition of  $R$ . So Exercise 3.2.6(1) implies  $b - a = -(a - b)$  is also even. Thus  $b R a$  by the definition of  $R$ .

(Yet another proof of symmetry) We know  $R^{-1} = R$  from Tutorial Exercise 5.4(a). So  $R$  is symmetric by Tutorial Exercise 6.9.

(Transitivity) Let  $a, b, c \in \mathbb{Z}$  such that  $a R b$  and  $b R c$ . Then  $a - b$  and  $b - c$  are both even by the definition of  $R$ . Use the definition of even integers to find  $x, y \in \mathbb{Z}$  such that  $a - b = 2x$  and  $b - c = 2y$ . Then  $a - c = (a - b) + (b - c) = 2x + 2y = 2(x + y)$ , where  $x + y \in \mathbb{Z}$ . So  $a - c$  is even, and thus  $a R c$  by the definition of  $R$ .

(Alternative proof of transitivity) We know  $R \circ R = R \subseteq R$  from Tutorial Exercise 5.4(b) and Remark 4.2.4(3). So  $R$  is transitive by Exercise 6.1.10.

As  $R$  is reflexive, symmetric, and transitive, it is an equivalence relation.  $\square$

- (b)  $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\} = [2] = [-2] = [4] = [-4] = \dots$   
 $[1] = \{\dots, -3, -1, 1, 3, 5, \dots\} = [-1] = [3] = [-3] = [5] = \dots$