Tutorial solutions for Chapter 5

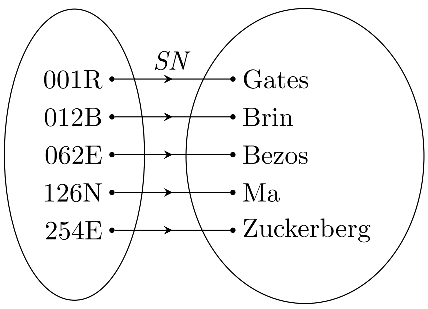
Sometimes there are other correct answers.

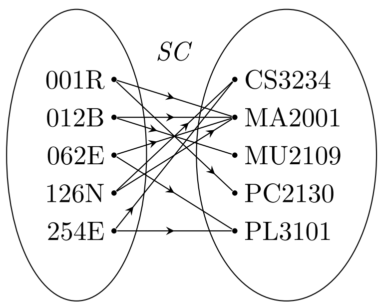
5.1. (a) $C \times G = \{(\text{MA1100}, \text{A}), (\text{MA1100}, \text{B}), (\text{MA1100}, \text{C}),$
$(\text{CS1231}, \text{A}), (\text{CS1231}, \text{B}), (\text{CS1231}, \text{C})\}.$

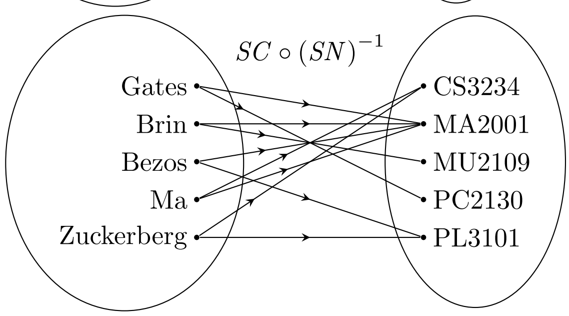
(b) $C \times G \times S = \{(\text{MA1100}, \text{A}, +), (\text{MA1100}, \text{A}, -),$
$(\text{MA1100}, \text{B}, +), (\text{MA1100}, \text{B}, -),$
$(\text{MA1100}, \text{C}, +), (\text{MA1100}, \text{C}, -),$
$(\text{CS1231}, \text{A}, +), (\text{CS1231}, \text{A}, -),$
$(\text{CS1231}, \text{B}, +), (\text{CS1231}, \text{B}, -),$
$(\text{CS1231}, \text{C}, +), (\text{CS1231}, \text{C}, -)\}.$

(c) $\mathcal{P}(\mathcal{P}(\varnothing)) \times S = \{\varnothing, \{\varnothing\}\} \times S$ by Tutorial Exercise 4.3;
$= \{(\varnothing, +), (\varnothing, -), (\{\varnothing\}, +), (\{\varnothing\}, -)\}.$

5.2. (a)



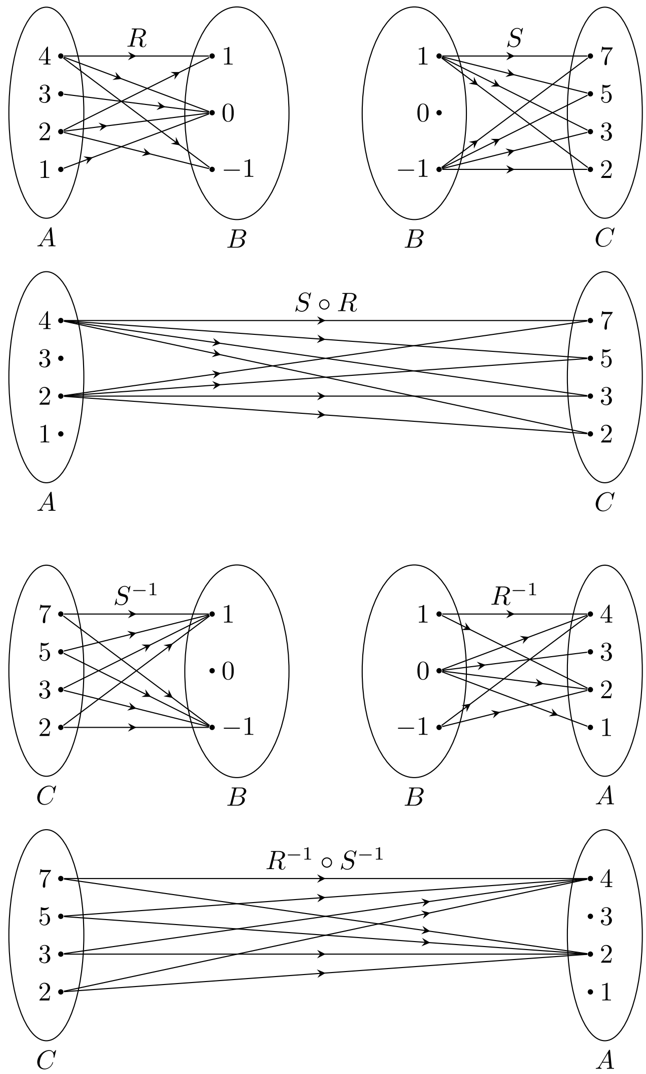




(b) $x \; SC \circ (SN)^{-1} \; y$ says

"x is (the name of) a student who is enrolled in the course y".

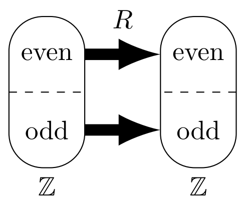
5.3.



Additional information.

$$
\begin{aligned}
R &= \{(1,0),(2,-1),(2,0),(2,1),(3,0),(4,-1),(4,0),(4,1)\}.\\
S &= \{(-1,2),(-1,3),(-1,5),(-1,7),(1,2),(1,3),(1,5),(1,7)\}.\\
S \circ R &= \{(2,2),(2,3),(2,5),(2,7),(4,2),(4,3),(4,5),(4,7)\}.\\
R^{-1} &= \{(0,1),(-1,2),(0,2),(1,2),(0,3),(-1,4),(0,4),(1,4)\}.\\
S^{-1} &= \{(2,-1),(3,-1),(5,-1),(7,-1),(2,1),(3,1),(5,1),(7,1)\}.\\
R^{-1} \circ S^{-1} &= \{(2,2),(3,2),(5,2),(7,2),(2,4),(3,4),(5,4),(7,4)\} = (S \circ R)^{-1}.
\end{aligned}
$$

5.4.



(a) ($\Rightarrow$) Let $(a, b) \in R^{-1}$. Then $(b, a) \in R$ by the definition of R^{-1}. In view of the definition of R, this means $b - a$ is even. Use the definition of even integers to find $x \in \mathbb{Z}$ such that $b - a = 2x$. Then $a - b = 2(-x)$ where $-x \in \mathbb{Z}$. So $a - b$ is even by the definition of even integers. According to the definition of R, this means $a\,R\,b$. Thus $(a, b) \in R$.

($\Leftarrow$) Let $(a, b) \in R$. Then $a - b$ is even by the definition of R. Use the definition of even integers to find $x \in \mathbb{Z}$ such that $a - b = 2x$. Then $b - a = 2(-x)$ where $-x \in \mathbb{Z}$. So $b - a$ is even by the definition of even integers. According to the definition of R, this means $b\,R\,a$. Thus $(a, b) \in R^{-1}$ by the definition of R^{-1}. $\square$

Alternative proof. For all (a, b),

$$\begin{array}{lll} & (a,b) \in R^{-1} & \\ \Leftrightarrow & (b,a) \in R & \text{by the definition of } R^{-1}; \\ \Leftrightarrow & b - a \text{ is even} & \text{by the definition of } R; \\ \Leftrightarrow & a - b \text{ is even} & \text{by Exercise 3.2.6(1),} \\ & & \text{as } a - b = -(b - a) \text{ and } b - a = -(a - b); \\ \Leftrightarrow & (a,b) \in R & \text{by the definition of } R. \end{array}$$

$\square$

Additional comments. Note that what we need to show here is essentially

$$\forall a, b \in \mathbb{Z} \ \ \big((a, b) \in R^{-1} \leftrightarrow (a, b) \in R\big).$$

In view of the definition of R^{-1}, this is equivalent to

$$\forall a, b \in \mathbb{Z} \ \ \big((b, a) \in R \leftrightarrow (a, b) \in R\big).$$

Therefore, applying Tutorial Exercise 3.3 to the predicate $P(x, y) =$ "$(y, x) \in R$", we see that for this exercise it suffices to show

$$\forall a, b \in \mathbb{Z} \ \ \big((b, a) \in R \rightarrow (a, b) \in R\big),$$

or in other words, in view of the definition of R^{-1},

$$\forall a, b \in \mathbb{Z} \ \ \big((a, b) \in R^{-1} \rightarrow (a, b) \in R\big).$$

As mentioned in Tutorial Exercise 3.3, all these are related to the symmetry of the relation R, a notion to be introduced in Chapter 6.

(b) ($\Rightarrow$) Let $(a, c) \in R \circ R$. Use the definition of $R \circ R$ to find $b \in \mathbb{Z}$ such that $(a, b), (b, c) \in R$. In view of the definition of R, this means both $a - b$ and $b - c$ are even. Apply the definition of even integers to find $x, y \in \mathbb{Z}$ such that $a - b = 2x$ and $b - c = 2y$. Then $a - c = (a - b) + (b - c) = 2x + 2y = 2(x + y)$, where $x + y \in \mathbb{Z}$. So $a - c$ is even. According to the definition of R, this means $a\,R\,c$. Thus $(a, c) \in R$.

($\Leftarrow$) Let $(a, b) \in R$. Note that $b - b = 0 = 2 \times 0$, which is even. So $(b, b) \in R$ by the definition of R. As $(a, b), (b, b) \in R$, we deduce that $(a, b) \in R \circ R$ by the definition of $R \circ R$. $\square$

Moral. The proof for the $\Rightarrow$-part and the proof of the $\Leftarrow$-part may be completely different.

5.5. By the definition of relation composition, both $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are relations from A to D, i.e., they are both subsets of $A \times D$. So it suffices show that $(w, z) \in T \circ (S \circ R)$ if and only if $(w, z) \in (T \circ S) \circ R$ for all $(w, z) \in A \times D$.

($\Rightarrow$) Let $(a, d) \in T \circ (S \circ R)$. Apply the definition of $\circ$ to find $c \in C$ such that $(a, c) \in S \circ R$ and $(c, d) \in T$. Applying the definition of $\circ$ again, we get $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. With $c \in C$ satisfying $(b, c) \in S$ and $(c, d) \in T$, we know

$(b,d) \in T \circ S$ by the definition of $\circ$. With $b \in B$ satisfying $(a,b) \in R$ and $(b,d) \in T \circ S$, the definition of $\circ$ tells us $(a,d) \in (T \circ S) \circ R$.

($\Leftarrow$) Let $(a,d) \in (T \circ S) \circ R$. Apply the definition of $\circ$ to find $b \in B$ such that $(a,b) \in R$ and $(b,d) \in (T \circ S)$. Applying the definition of $\circ$ again, we get $c \in C$ such that $(b,c) \in S$ and $(c,d) \in T$. With $b \in B$ satisfying $(a,b) \in R$ and $(b,c) \in S$, we know $(a,c) \in S \circ R$ by the definition of $\circ$. With $c \in C$ satisfying $(a,c) \in S \circ R$ and $(c,d) \in T$, the definition of $\circ$ tells us $(a,d) \in T \circ (S \circ R)$. $\square$

Alternative proof. By the definition of relation composition, both $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are relations from A to D, i.e., they are both subsets of $A \times D$. So it suffices show that $(w,z) \in T \circ (S \circ R)$ if and only if $(w,z) \in (T \circ S) \circ R$ for all $(w,z) \in A \times D$. Given any $(w,z) \in A \times D$, by the definition of $\circ$,

$$\begin{aligned}
&(w,z) \in T \circ (S \circ R) \\
\Leftrightarrow\quad & (w,y) \in S \circ R \text{ and } (y,z) \in T \text{ for some } y \in C \\
\Leftrightarrow\quad & (w,x) \in R \text{ and } (x,y) \in S \text{ for some } x \in B \text{ and } (y,z) \in T \text{ for some } y \in C \\
\Leftrightarrow\quad & (w,x) \in R \text{ and } (x,y) \in S \text{ and } (y,z) \in T \text{ for some } x \in B \text{ and } y \in C \\
\Leftrightarrow\quad & (w,x) \in R \text{ and } (x,y) \in S \text{ and } (y,z) \in T \text{ for some } y \in C \text{ and } x \in B \\
\Leftrightarrow\quad & (w,x) \in R \text{ and } (x,z) \in T \circ S \text{ for some } x \in B \\
\Leftrightarrow\quad & (w,z) \in (T \circ S) \circ R. \qquad \square
\end{aligned}$$

Additional comment. One can extract an explanation of the highlighted step above from the first proof.

5.6. $V = \{\mathrm{a,b,c,d,e}\}$, $\quad D = \{(\mathrm{b,b}), (\mathrm{b,c}), (\mathrm{c,a}), (\mathrm{c,d}), (\mathrm{d,b}), (\mathrm{d,c}), (\mathrm{e,d}), (\mathrm{e,e})\}$.
$W = \{1,2,3,4,5,6\}$, $\quad E = \{\{3,3\}, \{4,4\}, \{6,6\}, \{1,3\}, \{1,5\}, \{2,5\}, \{3,5\}, \{3,4\}\}$.

Additional explanations. The edges in the left drawing have directions. So it is the drawing of the directed graph (V,D). Since (V,D) is a directed graph, the elements of D are ordered pairs.

The edges in the right drawing have no direction. So it is the drawing of the undirected graph (W,E). Since (W,E) is an undirected graph, the elements of E are sets.

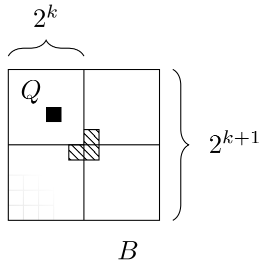
5.7. Let $P(n)$ be the predicate

> if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos

over $\mathbb{Z}^+$.

(Base step) $P(1)$ is true because such a board itself is an L-tromino.

(Induction step) Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true. Let B be a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Divide B into four $2^k \times 2^k$ quadrants. Let Q be the quadrant containing the removed square. Remove one L-tromino from the centre of B in a way such that each quadrant other than Q has one square removed. We are left with four $2^k \times 2^k$ checkerboards, each with one square removed. By the induction hypothesis, each quadrant can be covered by L-trominos. Hence B can be covered by L-trominos. This shows $P(k+1)$ is true.



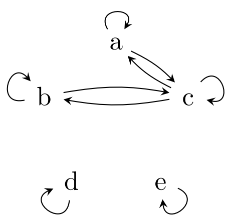
Hence $\forall n \in \mathbb{Z}^+ \;\; P(n)$ is true by MI. $\square$

Further exercises

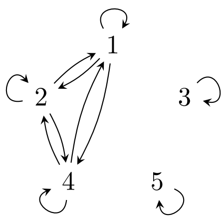
5.8. (a)

$$R = \{(\mathrm{a},1),(\mathrm{a},2),(\mathrm{b},4),(\mathrm{c},1),(\mathrm{c},2),(\mathrm{c},4),(\mathrm{d},5),(\mathrm{e},3)\}.$$
$$R^{-1} = \{(1,\mathrm{a}),(2,\mathrm{a}),(4,\mathrm{b}),(1,\mathrm{c}),(2,\mathrm{c}),(4,\mathrm{c}),(5,\mathrm{d}),(3,\mathrm{e})\}.$$
$$R^{-1}\circ R = \{(\mathrm{a},\mathrm{a}),(\mathrm{a},\mathrm{c}),(\mathrm{b},\mathrm{b}),(\mathrm{b},\mathrm{c}),(\mathrm{c},\mathrm{a}),(\mathrm{c},\mathrm{b}),(\mathrm{c},\mathrm{c}),(\mathrm{d},\mathrm{d}),(\mathrm{e},\mathrm{e})\}.$$
$$R\circ R^{-1} = \{(1,1),(1,2),(1,4),(2,1),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4),(5,5)\}.$$

(b) $R^{-1}\circ R$:



$R\circ R^{-1}$:



Additional remark. We are asked for drawings of directed graphs here. So we refer to Definition 5.3.5(5), and thus Definition 5.3.3.

5.9. By the definition of relation inverse, we know R^{-1} is a relation from B to A, and thus $(R^{-1})^{-1}$ is a relation from A to B. Given any $(x,y)\in A\times B$,

$$\begin{aligned}(x,y)\in(R^{-1})^{-1} &\Leftrightarrow (y,x)\in R^{-1} && \text{by the definition of } (R^{-1})^{-1};\\ &\Leftrightarrow (x,y)\in R && \text{by the definition of } R^{-1}.\end{aligned}$$ $\square$

Alternative proof. By the definition of relation inverse, we know R^{-1} is a relation from B to A. So

$$\begin{aligned}(R^{-1})^{-1} &= \{(x,y)\in A\times B : (y,x)\in R^{-1}\} && \text{by the definition of } (R^{-1})^{-1};\\ &= \{(x,y)\in A\times B : (x,y)\in R\} && \text{by the definition of } R^{-1};\\ &= R.\end{aligned}$$ $\square$