Name: ____________________ **Tutorial group:** __________ **(number or day&time)**
CS1231 Discrete Structures **Quiz 1** **2023/24 Semester 2**

1. Let D be the set that contains precisely $-1, 0, 1, 2$. Which of the following propositions are true? Which are false? Briefly explain your answers.

   (a) $\forall x \in D\ (2x \geqslant 5 \rightarrow 2x \leqslant -5)$. [2 marks]

   Solution. True, because for any $x \in D$, we know $2x \leqslant 2 \times 2 = 4 < 5$, and thus the conditional is vacuously true.

   (b) $\forall x, y \in D\ (x + y < 2x \vee x + y < 2y)$. [2 marks]

   Solution. False: take $x = 0$ and $y = 0$.

   (c) $\exists x \in D\ (2x = 2 \vee x - 1 = 0)$. [2 marks]

   Solution. True, because $1 \in D$ and $2 \times 1 = 2$ (or $1 - 1 = 0$).

   (d) $\exists x \in D\ \forall y, z \in D\ x \neq y - z$. [3 marks]

   Solution. False, because given any $x \in D$, we can choose $y = x$ and $z = 0$, so that $y - z = x - 0 = x$.

   (e) $\forall x \in D\ (\neg\exists y \in D\ y = -x \rightarrow x = 2)$. [3 marks]

   Solution. True, because for all $x \in D$, if x is -1, 0 or 1, then $-x$ is 1, 0 or -1, so that $-x \in D$ and thus the conditional is true vacuously; else $x = 2$, in which case the conditional is also true.

   Comments.
   - Many misinterpreted the meaning of the proposition. According to Convention 2.2.8(3), the proposition reads
   $$\forall x \in D\ \big((\neg\exists y \in D\ y = -x) \rightarrow x = 2\big).$$
   This is *not* equivalent to $\forall x \in D\ \neg\exists y \in D\ \big(y = -x \rightarrow x = 2\big)$, which is a false proposition.
   - Some who interpreted the proposition correctly failed to explain all the cases.

2. Rewrite the following propositions symbolically. In your answers, you can use $\mathrm{Even}(n)$ and $\mathrm{Odd}(n)$ to stand for $\exists x \in \mathbb{Z}\ n = 2x$ and $\exists x \in \mathbb{Z}\ n = 2x + 1$ respectively.

   (a) Any two natural numbers whose product is odd must sum to a positive even integer. [2 marks]

   Solution. $\forall m, n \in \mathbb{N}\ \big(\mathrm{Odd}(mn) \rightarrow m + n > 0 \wedge \mathrm{Even}(m + n) \wedge m + n \in \mathbb{Z}\big)$.
   Another correct answer is $\forall m, n \in \mathbb{N}\ \big(\mathrm{Odd}(mn) \rightarrow m + n > 0 \wedge \mathrm{Even}(m + n)\big)$.

   (b) Being the square of some rational number is not a necessary condition for an integer to be even. [2 marks]

   Solution. $\neg\forall n \in \mathbb{Z}\ \big(\mathrm{Even}(n) \rightarrow \exists x \in \mathbb{Q}\ n = x^2\big)$.
   Some acceptable but less preferable answers include

   $$\begin{aligned}
   &\neg\forall n \in \mathbb{Z} && \exists x \in \mathbb{Q}\ \big(\mathrm{Even}(n) \rightarrow n = x^2\big),\\
   &\exists n \in \mathbb{Z} && \neg\exists x \in \mathbb{Q}\ \big(\mathrm{Even}(n) \rightarrow n = x^2\big),\\
   &\exists n \in \mathbb{Z} && \forall x \in \mathbb{Q}\ \neg\big(\mathrm{Even}(n) \rightarrow n = x^2\big),\\
   &\exists n \in \mathbb{Z} && \forall x \in \mathbb{Q}\ \big(\mathrm{Even}(n) \wedge n \neq x^2\big),\\
   &\exists n \in \mathbb{Z} && \neg\big(\mathrm{Even}(n) \rightarrow \exists x \in \mathbb{Q}\ n = x^2\big),\\
   &\exists n \in \mathbb{Z} && \big(\mathrm{Even}(n) \wedge \neg\exists x \in \mathbb{Q}\ n = x^2\big),\\
   &\exists n \in \mathbb{Z} && \big(\mathrm{Even}(n) \wedge \forall x \in \mathbb{Q}\ n \neq x^2\big).
   \end{aligned}$$

   Extra explanations. The given sentence is the negation of "Being the square of some rational number is a necessary condition for an integer to be even", which is intended to apply to all integers.

   Comments.
   - Very few got this completely correct. Some have the negation in the wrong place. Some wrote the wrong quantifiers. Some misinterpreted "is not a necessary condition" as "is a sufficient condition".
   - Some used predicates such as $\mathrm{Even}(n)$ like a number, as in "$\mathrm{Even}(n) = y$", where y is a number variable. One should *not* equate objects of different types: predicates take T or F after suitable substitutions, whereas number variables take numerical values.
   - Some used numerical terms such as x^2 like a predicate, as in "$x^2 \rightarrow \mathrm{Even}(n)$". We did not define what expressions like "$2^2 \rightarrow \mathrm{Even}(4)$" mean.

END OF QUIZ

3. Let p and q be propositional variables. Which of the following pairs of compound expressions are equivalent? Which are not? Prove that your answers are correct.

   (a) $p \wedge \neg q \to q \vee \neg p$ and $p \to q$. [3 marks]

   Solution. Yes, as shown below.

$$\begin{aligned} p \wedge \neg q \to q \vee \neg p &\equiv \neg(p \wedge \neg q) \vee q \vee \neg p && \text{by the logical identity on implication;} \\ &\equiv \neg p \vee q \vee q \vee \neg p && \text{by De Morgan's Laws and the Double Negative Law;} \\ &\equiv \neg p \vee q && \text{by the Idempotence Laws;} \\ &\equiv p \to q && \text{by the logical identity on implication.} \end{aligned}$$

   Alternatively, one can draw a truth table to show the equivalence.

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$q \vee \neg p$	$p \wedge \neg q \to q \vee \neg p$	$p \to q$
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

   Since the column for $p \wedge \neg q \to q \vee \neg p$ and the column for $p \to q$ are exactly the same, the two compound expressions are equivalent.

   (b) $(p \leftrightarrow q) \vee p$ and p. [3 marks]

   Solution. No, because if we substitute false propositions into p and q then $(p \leftrightarrow q) \vee p$ evaluates to T while p evaluates to F.

4. Mathematics student Eloise just successfully proved the fact that, for every non-negative real number x, there exists a real number y such that $y^2 = x$. She then moves on to consider the following proposition:

   for every real number a, if $a < 1$, then no real number z makes $z^2 + 2az + 1 = 0$.

   (i) Eloise attempts to prove this proposition as follows.

   We prove this by contraposition. Suppose $a \geqslant 1$. Then $a^2 - 1 = a \times a - 1 \geqslant a \times 1 - 1 \geqslant 1 \times 1 - 1 = 0$. So the fact just established gives $y \in \mathbb{R}$ such that $y^2 = a^2 - 1$. Define $z = y - a$. Then $z \in \mathbb{R}$ and

$$\begin{aligned} z^2 + 2az + 1 &= (y-a)^2 + 2a(y-a) + 1 && \text{as } z = y - a; \\ &= (y^2 - 2ay + a^2) + (2ay - 2a^2) + 1 = y^2 - a^2 + 1 \\ &= (a^2 - 1) - a^2 + 1 && \text{as } y^2 = a^2 - 1; \\ &= 0. \end{aligned}$$

   What is wrong with this attempt? [1 mark]

   Solution. This attempt shows the inverse/converse of the proposition, which may not be equivalent to the proposition being considered.

   Comments. Some got the contrapositive of the proposition wrong.

   (ii) Prove that this proposition is false. If you decide to use any facts about discriminants, then you should prove them as well. [2 marks]

   Solution. Let $a = -1$ and $z = 1$. Then $a = -1 < 1$ and $z^2 + 2az + 1 = 1^2 + 2 \times (-1) \times 1 + 1 = 0$. $\square$

   Comments. Many got full marks for this part. Some used discriminants without further explanation despite the warning.

END OF QUIZ