Tutorial solutions for Chapter 3

Sometimes there are other correct answers.

3.1. One can rewrite the proposition to be proved symbolically as

$$\forall x \in \mathbb{R} \;\; x^2 \geqslant 0.$$

To prove this by contradiction (i.e., using Technique 3.2.16), one starts with the negation, which is equivalent to

$$\exists x \in \mathbb{R} \;\; x^2 < 0$$

by Theorem 2.3.1. However, the author started instead with

$$\forall x \in \mathbb{R} \;\; x^2 < 0.$$

This is not justified.

Moral 1. Proofs of true propositions need not be correct.

Moral 2. Negate propositions carefully when proving by contradiction.

3.2. (a) The argument given proves $\forall x \in \mathbb{R} \;\; (x^2 + 9 = 6x \to x = 3)$. However, what we want to prove is $\exists x \in \mathbb{R} \;\; (x^2 + 9 = 6x)$. The author did not explain how one can derive what we want to prove from what is proved.

The situation is different if the author proves $\forall x \in \mathbb{R} \;\; (x = 3 \to x^2 + 9 = 6x)$ instead, because instantiating x to 3 here (in the sense of Example 3.1.15) immediately gives a real number x satisfying $x^2 + 9 = 6x$, namely 3.

(b) Note that 3 is real number and $3^2 + 9 = 18 = 6 \times 3$. □

Moral. Do not mix up a conditional proposition and its converse: as shown in Theorem 1.4.12(3), the two may not be equivalent.

Additional information. Strictly speaking, proofs do not need to contain information about how one came up with them. However, sometimes authors still include this information because this can be useful to the reader.

3.3. **Proof.** ((i) $\to$ (ii)) Assume (i) is true. Pick elements z and w of A. Instantiating (i) with z, w tells us $P(z, w) \to P(w, z)$ is true. Instantiating (i) with w, z tells us $P(w, z) \to P(z, w)$ is true. Combining the two using Tutorial Exercise 1.4, we deduce that $P(z, w) \leftrightarrow P(w, z)$ is true. As the choice of the elements z and w from A was arbitrary, we have shown (ii).

((ii) $\to$ (i)) Assume (ii) is true. Pick elements z and w of A. Then $P(z, w) \leftrightarrow P(w, z)$ by (ii). So Tutorial Exercise 1.4 tells us $P(z, w) \to P(w, z)$ is true. As the choice of the elements z and w from A was arbitrary, we have shown (i). □

- To prove (i) $\leftrightarrow$ (ii), we split into (i) $\to$ (ii) and (ii) $\to$ (i) according to Technique 3.2.9.

- We applied universal instantiation from Example 3.1.15, together with Technique 3.2.9, in the proof of (i) $\to$ (ii).
- We applied universal instantiation from Example 3.1.15 in the proof of (ii) $\to$ (i).

Another proof. ((i) $\to$ (ii)) Assume (i) is true. Pick elements z and w of A. As $P(x, y)$ is a predicate on A, the sentence $P(z, w)$ is a proposition. So it is either true or false.

Case 1: suppose $P(z, w)$ is true. Then $P(w, z)$ is also true by (i).

Case 2: suppose $P(z, w)$ is false. If $P(w, z)$ is true, then $P(z, w)$ is also true by (i), which contradicts our supposition. So $P(w, z)$ is false.

Hence $P(z, w)$ and $P(w, z)$ are either both true or both false in all cases. Thus $P(z, w) \leftrightarrow P(w, z)$ is true by the definition of $\leftrightarrow$. As the choice of the elements z and w from A was arbitrary, we have shown (ii).

((ii) $\to$ (i)) Assume (ii) is true. Pick elements z and w of A such that $P(z, w)$ is true. We know $P(z, w) \leftrightarrow P(w, z)$ by (ii). According to the definition of $\leftrightarrow$, this means $P(z, w)$ and $P(w, z)$ have the same truth value. Since $P(z, w)$ is true, it must be the case that $P(w, z)$ is also true. As the choice of the elements z and w from A satisfying $P(z, w)$ was arbitrary, we have shown (i). $\square$

Yet another proof of (ii) $\to$ (i). Assume (ii) is true. Pick elements z and w of A. Then $P(z, w) \leftrightarrow P(w, z)$ by (ii). The definition of $\leftrightarrow$ now tells us that $P(z, w)$ and $P(w, z)$ are either both true or both false.

Case 1: suppose $P(z, w)$ and $P(w, z)$ are both true. Then $P(z, w) \to P(w, z)$ is true by the definition of $\to$.

Case 2: suppose $P(z, w)$ and $P(w, z)$ are both false. Then $P(z, w) \to P(w, z)$ is (vacuously) true by the definition of $\to$.

Hence $P(z, w) \to P(w, z)$ is true in all cases. As the choice of the elements z and w from A was arbitrary, we have shown (i). $\square$

Moral. One can instantiate a universal proposition (in the sense of Example 3.1.15) in more than one way.

Additional remark 1. In fact, one *must* instantiate (i) in more than one way in this proof because the compound expressions $p \to q$ and $p \leftrightarrow q$, where p, q are propositional variables, are *not* equivalent.

Additional remark 2. The sentence $\forall x, y \in A \ \big(P(x, y) \to P(y, x)\big)$ in (i) may not be equivalent to $\forall x, y \in A \ \ P(x, y) \to \forall x, y \in A \ \ P(y, x)$. For example, in the case when $A = \mathbb{Z}$ and $P(x, y)$ is $x < y$, the former sentence is false, while the latter is (vacuously) true.

Additional remark 3. The sentence "As the choice of ... was arbitrary, ..." is sometimes omitted in proofs, as this should be clear to the (experienced) reader.

Further exercises

3.4. The argument considers only three real numbers: -1, 0, and 1. It does not explain why $x^2 - 6x + 7 > 0$ is true for other real numbers, for example, for $x = 2$.

3.5. Suppose the proposition is not true. Let m, n, k be integers such that $m^2 + n^2 = k^2$, but m and n are both odd. Use the definition of odd integers to find $x, y \in \mathbb{Z}$ such that $m = 2x + 1$ and $n = 2y + 1$. Then

$$\begin{aligned} k^2 &= m^2 + n^2 = (2x + 1)^2 + (2y + 1)^2 \\ &= (4x^2 + 4x + 1) + (4y^2 + 4y + 1) = 2(2x^2 + 2x + 2y^2 + 2y + 1), \end{aligned}$$

where $2x^2+2x+2y^2+2y+1$ is an integer. So k^2 is even by the definition of even integers. Thus, in view Proposition 3.2.8 we know k is even. Use the definition of even integers to find an integer z such that $k=2z$. Then $(4x^2+4x+1)+(4y^2+4y+1)=k^2=(2z)^2=4z^2$. Hence

$$\begin{aligned}
z^2-(x^2+x+y^2+y) &= \frac{4z^2-(4x^2+4x+4y^2+4y)}{4} \\
&= \frac{(4x^2+4x+1)+(4y^2+4y+1)-(4x^2+4x+4y^2+4y)}{4} = \frac{1}{2},
\end{aligned}$$

which is not an integer. However, we know that $z^2-(x^2+x+y^2+y)$ is an integer because x, y and z are. This is a contradiction. Therefore, the proposition must be true. $\square$