Solutions to selected exercises

1a, page 4

Consider a vacuously true condition proposition, say $p \to q$. By the definition of vacuously true, the hypothesis p is false. In particular, it is not the case that p is true and q is false. So the conditional proposition $p \to q$ is true by the definition of $\to$.

1b, page 4

In the notation of Definition 1.3.8, we have $p = (a \wedge b)$ and $q = (c \vee d)$ here. So the converse is $(c \vee d) \to (a \wedge b)$, the inverse is $\big(\neg(a \wedge b)\big) \to \big(\neg(c \vee d)\big)$, and the contrapositive is $\big(\neg(c \vee d)\big) \to \big(\neg(a \wedge b)\big)$.

1c, page 5

A proposition cannot be both true and not not false, because by definition no proposition can be both true and false.

A proposition cannot be both not true and not false, because by definition no proposition can be neither true nor false.

1d, page 6

According to Convention 1.4.3, the compound expression $\neg p \to \neg q$ stands for $(\neg p) \to (\neg q)$.

- $\neg p$ evaluates to F according to the definition of $\neg$.
- $\neg q$ evaluates to T according to definition of $\neg$.
- $\neg p \to \neg q$ evaluates to T according to the definition of $\to$.

In summary,

$$\begin{array}{ccccc} \neg & p & \to & \neg & q \\ & \mathrm{T} & & & \mathrm{F} \\ \mathrm{F} & & & \mathrm{T} & \\ & & \mathrm{T} & & \end{array}$$

1e, page 6

(1) P and P evaluate to the same truth value under any substitution of propositions into the propositional variables.

(2) If P and Q evaluate to the same truth value under any substitution of propositions into the propositional variables, then so do Q and P.

(3) If P and Q evaluate to the same truth value under any substitution of propositions into the propositional variables, and Q and R evaluate to the same truth value under any substitution of propositions into the propositional variables, then so do P and R.

✎ 1f, page 8

No, as the following truth table shows.

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	F	T	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

Actually, to justify a "no" answer, it suffices to give *one* row of this truth table in which the columns for $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ are different.

✎ 1g, page 9

One can prove the Associativity, the Commutativity, the Identity, and the Annihilator parts of Theorem 1.4.21 using the truth tables below.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

p	q	$p \vee q$	$q \vee p$	$p \wedge q$	$q \wedge p$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

p	T	C	$p \vee C$	$p \wedge T$	$p \vee T$	$p \wedge C$
T	T	F	T	T	T	F
F	T	F	F	F	T	F

Alternatively, one can prove the associativity of $\wedge$ from that of $\vee$ as follows:

$$
\begin{aligned}
(p \wedge q) \wedge r &\equiv (\neg\neg p \wedge \neg\neg q) \wedge \neg\neg r && \text{by the Double Negative Law;} \\
&\equiv \neg(\neg p \vee \neg q) \wedge \neg\neg r && \text{by De Morgan's Laws;} \\
&\equiv \neg\big((\neg p \vee \neg q) \vee \neg r\big) && \text{by De Morgan's Laws;} \\
&\equiv \neg\big(\neg p \vee (\neg q \vee \neg r)\big) && \text{by the associativity of } \vee; \\
&\equiv \neg\neg p \wedge \neg(\neg q \vee \neg r) && \text{by De Morgan's Laws;} \\
&\equiv \neg\neg p \wedge (\neg\neg q \wedge \neg\neg r) && \text{by De Morgan's Laws;} \\
&\equiv p \wedge (q \wedge r) && \text{by the Double Negative Law.}
\end{aligned}
$$

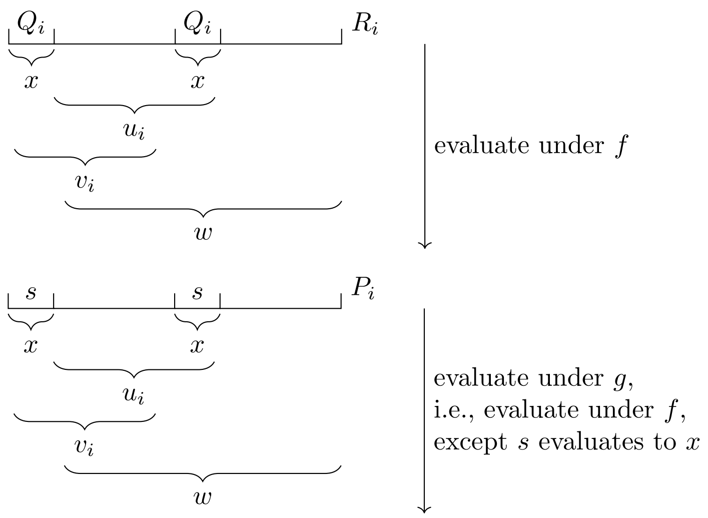
One can also prove the commutativity of $\wedge$ from that of $\vee$ as follows:

$$\begin{aligned} p \wedge q &\equiv \neg\neg p \wedge \neg\neg q && \text{by the Double Negative Law;} \\ &\equiv \neg(\neg p \vee \neg q) && \text{by De Morgan's Laws;} \\ &\equiv \neg(\neg q \vee \neg p) && \text{by the commutativity of } \vee; \\ &\equiv \neg\neg q \wedge \neg\neg p && \text{by De Morgan's Laws;} \\ &\equiv q \wedge p && \text{by the Double Negative Law.} \end{aligned}$$

1h, page 9

Let P_1, P_2, Q_1, Q_2 be compound expressions such that $P_1 \equiv P_2$ and $Q_1 \equiv Q_2$. Denote by R_1, R_2 the compound expressions obtained from P_1, P_2 by respectively substituting Q_1, Q_2 into a propositional variable s. We want to argue that $R_1 \equiv R_2$. For this, consider an arbitrary substitution f of propositions into the propositional variables. We want to show that R_1 and R_2 evaluate to the same truth value under f. Since $Q_1 \equiv Q_2$, the compound expressions evaluate to the same truth value, say x, under f. Let g denote the substitution that is the same as f, except a proposition of truth value x is substituted into the propositional variable s. Then evaluating R_1, R_2 under f gives the same results as evaluating P_1, P_2 under g respectively. As $P_1 \equiv P_2$, these evaluate to the same truth value under g. It follows that R_1 and R_2 evaluate to the same truth value under f, as required.

Diagram.



2a, page 17

(1) False: note that $\forall x \in \mathbb{Z} \;\; \neg(x > 0 \wedge x < 1)$ is true, but $\forall x \in \mathbb{R} \;\; \neg(x > 0 \wedge x < 1)$ is false.

(2) True, because every integer is a real number.

(3) True, because every integer is a real number.

(4) False: note that $\exists x \in \mathbb{R} \;\; (x > 0 \wedge x < 1)$ is true, but $\exists x \in \mathbb{R} \;\; (x > 0 \wedge x < 1)$ is false.

There are many other counterexamples for (1) and (4).

2b, page 18

(3) Note that the following are true.

$$\begin{array}{rcll} \neg\forall x \in D \;\; P(x) & \leftrightarrow & \neg\forall x \; \big(x \in D \rightarrow P(x)\big) & \text{by Definition 2.2.1(5).} \\ \neg\forall x \; \big(x \in D \rightarrow P(x)\big) & \leftrightarrow & \exists x \; \neg\big(x \in D \rightarrow P(x)\big) & \text{by part (1).} \\ \exists x \; \neg\big(x \in D \rightarrow P(x)\big) & \leftrightarrow & \exists x \; \big(x \in D \wedge \neg P(x)\big) & \text{by Example 1.4.24.} \\ \exists x \; \big(x \in D \wedge \neg P(x)\big) & \leftrightarrow & \exists x \in D \;\; \neg P(x) & \text{by Definition 2.2.4(5).} \end{array}$$

From these, we deduce that $\neg\forall x \in D \quad P(x)$ is true if and only if $\exists x \in D \quad \neg P(x)$ is true. $\square$

(4) Note that the following are true.

$$\begin{array}{rcll}
\neg\exists x \in D \quad P(x) & \leftrightarrow & \neg\exists x \, \big(x \in D \wedge P(x)\big) & \text{by Definition 2.2.4(5).} \\
\neg\exists x \, \big(x \in D \wedge P(x)\big) & \leftrightarrow & \forall x \, \neg\big(x \in D \wedge P(x)\big) & \text{by part (2).} \\
\forall x \, \neg\big(x \in D \wedge P(x)\big) & \leftrightarrow & \forall x \, \big(\neg(x \in D) \vee \neg P(x)\big) & \text{by De Morgan's Laws.} \\
\forall x \, \big(\neg(x \in D) \vee \neg P(x)\big) & \leftrightarrow & \forall x \, \big(x \in D \to \neg P(x)\big) & \text{by the Implication part of Theorem 1.4.21.} \\
\forall x \, \big(x \in D \to \neg P(x)\big) & \leftrightarrow & \forall x \in D \quad \neg P(x) & \text{by Definition 2.2.1(5).}
\end{array}$$

From these, we deduce that $\neg\exists x \in D \quad P(x)$ is true if and only if $\forall x \in D \quad \neg P(x)$ is true. $\square$

2c, page 21

(3) We have the following equivalences by Theorem 2.3.1.

$$\begin{array}{rcl}
\neg\exists x \; \exists y \; Q(x,y) & \leftrightarrow & \forall x \; \neg\exists y \; Q(x,y). \\
\forall x \; \neg\exists y \; Q(x,y) & \leftrightarrow & \forall x \; \forall y \; \neg Q(x,y).
\end{array}$$

From these, we deduce that $\neg\exists x \; \exists y \; Q(x,y)$ is true if and only if $\forall x \; \forall y \; \neg Q(x,y)$ is true. $\square$

(4) We have the following equivalences by Theorem 2.3.1.

$$\begin{array}{rcl}
\neg\exists x \; \forall y \; Q(x,y) & \leftrightarrow & \forall x \; \neg\forall y \; Q(x,y). \\
\forall x \; \neg\forall y \; Q(x,y) & \leftrightarrow & \forall x \; \exists y \; \neg Q(x,y).
\end{array}$$

From these, we deduce that $\neg\exists x \; \forall y \; Q(x,y)$ is true if and only if $\forall x \; \exists y \; \neg Q(x,y)$ is true. $\square$

2d, page 21

(1) True.

Additional explanations.

- This proposition reads "there exists x in D such that, for every y in E, $xy = 0$".
- Alternatively, one can express this as "there is an element x of D which, when multiplied to any element y of E, gives a product of 0".
- This is true because if we choose the element 0 of D, then no matter which element y of E we multiply it to, we get a product of 0.

(2) True.

Additional explanations.

- This proposition reads "for every y in E, there is x in D such that $xy = 0$".
- Alternatively, one can express this as "no matter which element y of E is given, one can always multiply it to an element x of D to get 0".
- This is true because no matter which element y of E is given, we can always multiply it to the element 0 of D to get 0.

(3) False.

Additional explanations.

- This proposition reads "there exists x in D such that, for every y in E, $xy < 0$".

- Alternatively, one can express this as "there is an element x of D which, when multiplied to any element y of E, gives a negative product".
- This is false because no element x of D, when multiplied to any element y of E, always gives a negative product: let us consider all the elements of D one by one.
  - For the element -1 of D, when we multiply it to the element -1 of E, we get a non-negative product.
  - For the elements 0 and 1 of D, when we multiply them to the element 1 of E, we get non-negative products.

  There are other counterexamples.

(4) True.

Additional explanations.
- This proposition reads "for every y in E, there exists x in D such that $xy < 0$".
- Alternatively, one can express this as "one can make any element y of E negative by multiplying it to some element x of D."
- This is true, as one can verify exhaustively for each element of E.
  - For the elements 1 and 2 of E, one can multiply them to the element -1 of D to make a negative product.
  - For the elements -1 and -2 of E, one can multiply them to the element 1 of D to make a negative product.

(5) True.

Additional explanations.
- This proposition reads "there exist x_1, x_2 in D such that $x_1 + x_2 = 2$".
- Alternatively, one can express this as "there are two (possibly equal) elements x_1, x_2 of D whose sum is 2".
- This is true because 1 is an element of D and $1 + 1 = 2$.

(6) False.

Additional explanations.
- In view of Theorem 2.4.10(1), the negation $\neg\forall y_1, y_2 \in E \;\; y_1 = y_2$ of this proposition is equivalent to $\exists y_1, y_2 \in E \;\; y_1 \neq y_2$.
- This negation reads "there exist y_1, y_2 in E such that $y_1 \neq y_2$".
- Alternatively, one can express this negation as "there are two different elements y_1, y_2 of E".
- This negation is true: for example, the numbers 1 and 2 are different elements of E.
- So the given proposition is false, because its negation is true.

✎ 3a, page 26

No. In the case when p is a false proposition and q is a true proposition, we have both $p \to q$ and $\neg p$ true, but $\neg q$ is false.

✎ 3b, page 29

(1) Suppose n is even. Use the definition of even integers to find an integer x such that $n = 2x$. Then $-n = -2x = 2(-x)$ where $-x$ is an integer. So $-n$ is even. □

(2) Suppose n is odd. Use the definition of odd integers to find an integer x such that $n = 2x + 1$. Then $-n = -(2x + 1) = 2(-x - 1) + 1$ where $-x - 1$ is an integer. So n is odd. □

✎ 3c, page 30

(1) We deduced n is odd from the assumption that n is not even. This uses Proposition 3.2.21.

(2) We deduced n^2 is not even from our knowledge that n^2 is odd. This uses Proposition 3.2.17.

✎ 3d, page 31

Proof. We show that $\forall x \in \mathbb{Z}\ \ 2x \neq 1$. Take any $z \in \mathbb{Z}$. By trichotomy, we have $z > 0$ or $z = 0$ or $z < 0$.

Case 1: suppose $z > 0$. Then $z \in \mathbb{N}$. So

$$\begin{array}{lll} & z \geqslant 1 & \text{by the discreteness of } \mathbb{N}; \\ \therefore & 2z \geqslant 2 \times 1 = 2 > 1 & \\ \therefore & 2z \neq 1 & \text{by the irreflexivity of } <. \end{array}$$

Case 2: suppose $z \leqslant 0$. Then $2z \leqslant 2 \times 0 = 0 < 1$. So $2z \neq 1$ by the irreflexivity of $<$. □

✎ 3e, page 33

Let us read the two steps in Strong MI as follows.

(base step) show that $P(i)$ is true for each integer i satisfying $b \leqslant i \leqslant c$;

(induction step) for each integer $k \geqslant c$, use the hypothesis that $P(i)$ is true for each integer i satisfying $b \leqslant i \leqslant k$ to show that $P(k+1)$ is also true.

Consider the case when $c = b - 1$.

- For the base step, there is nothing to prove, because no integer i satisfies $b \leqslant i \leqslant c = b - 1$.
- Consider the induction step when $k = c$. The hypothesis we can use is $P(i)$ is true for each integer i satisfying $b \leqslant i \leqslant c = b - 1$. However, there is no such i. So there is no hypothesis one can use to show $P(b)$. In other words, one simply needs to show that $P(b)$ is true here (without assuming anything).
- In the induction step when $k = c + 1$, one needs to show $P(b) \to P(b+1)$.
- In the induction step when $k = c + 2$, one needs to show $P(b) \wedge P(b+1) \to P(b+2)$.
- Etc.

We deduce that $P(b), P(b+1), P(b+2), \ldots$ are all true by a series of modus ponens.

✎ 3f, page 33

Let n and d be positive integers such that $d \mid n$. Use the definition of divisibility to find $k \in \mathbb{Z}$ such that $n = dk$. As $d > 0$ and $n > 0$, we know $k > 0$ too. Thus $k \geqslant 1$ as k is an integer. Hence $n = dk \geqslant d \times 1 = d$. □

✎ 3g, page 34

By definition, the prime number q has exactly two positive factors. Example 3.3.4 and Remark 3.3.8(1) give two distinct positive factors 1 and q of q. So p must equal 1 or q. Since $p \neq 1$ by Remark 3.3.8(1), we deduce that $p = q$. □

4a, page 38

(1) We want to prove that $E = \mathbb{Z}^+$, where $E = \{x+1 : x \in \mathbb{Z}_{\geqslant 0}\}$.

Proof. ($\Rightarrow$) Let $z \in E$. Use the definition of E to find $x \in \mathbb{Z}_{\geqslant 0}$ such that $x + 1 = z$. Then $x \in \mathbb{Z}$ and $x \geqslant 0$ by the definition of $\mathbb{Z}_{\geqslant 0}$. As $x \in \mathbb{Z}$, we know $x + 1 \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under $+$. As $x \geqslant 0$, we know $x + 1 \geqslant 0 + 1 = 1 > 0$. So $z = x + 1 \in \mathbb{Z}^+$ by the definition of $\mathbb{Z}^+$.

($\Leftarrow$) Let $z \in \mathbb{Z}^+$. Then $z \in \mathbb{Z}$ and $z > 0$. Define $x = z - 1$. As $z \in \mathbb{Z}$, we know $x \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under $-$. As $z > 0$, we know $x = z - 1 > 0 - 1 = -1$, and thus $x \geqslant 0$ as $x \in \mathbb{Z}$. So $x \in \mathbb{Z}_{\geqslant 0}$ by the definition of $\mathbb{Z}_{\geqslant 0}$. Hence the definition of E tells us $z = x + 1 \in E$. $\square$

(2) We want to prove that $F = \mathbb{Z}$, where $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$.

Proof. ($\Rightarrow$) Let $z \in F$. Use the definition of F to find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $x - y = z$. Then $x, y \in \mathbb{Z}$ by the definition of $\mathbb{Z}_{\geqslant 0}$. So $z = x - y \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $-$.

($\Leftarrow$) Let $z \in \mathbb{Z}$.

- Case 1: suppose $z \geqslant 0$. Let $x = z$ and $y = 0$. Then $x, y \in \mathbb{Z}_{\geqslant 0}$. So $z = z - 0 = x - y \in F$ by the definition of F.
- Case 2: suppose $z < 0$. Let $x = 0$ and $y = -z$. Then $x, y \in \mathbb{Z}_{\geqslant 0}$ as $z < 0$. So $z = 0 - (-z) = x - y \in F$ by the definition of F.

So $z \in F$ in all the cases. $\square$

4b, page 39

- $\{1\} \in C$ but $\{1\} \not\subseteq C$;
- $\{2\} \notin C$ but $\{2\} \subseteq C$;
- $\{3\} \in C$ and $\{3\} \subseteq C$; and
- $\{4\} \notin C$ and $\{4\} \not\subseteq C$.

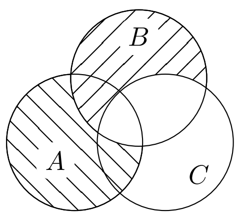
4c, page 42

Let $z \in A$. In particular, we know $z \in A$ or $z \in B$. So $z \in A \cup B$ by the definition of $\cup$.

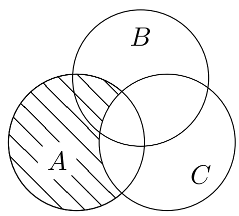
4d, page 43

Assume $A \subseteq B$ and $A \subseteq C$. Take any $z \in A$. Then $z \in B$ and $z \in C$ as $A \subseteq B$ and $A \subseteq C$ by assumption. So $z \in B \cap C$ by the definition of $\cap$. As the choice of z in A was arbitrary, this shows $A \subseteq B \cap C$. $\square$

4e, page 43



$(A \setminus B) \cup (B \setminus C)$



$A \setminus C$

No. For a counterexample, let $A = C = \varnothing$ and $B = \{1\}$. Then

$$(A \setminus B) \cup (B \setminus C) = \varnothing \cup \{1\} = \{1\} \neq \varnothing = A \setminus C.$$

✎ 4f, page 43

Ideas. (1) The set of all sets?

(2) $\left\{\left\{\left\{\{\{\ldots\ \ldots\}\}\right\}\right\}\right\}$?

✎ 4g, page 43

Maybe, but is it better?

Proof. Take any set R. Split into two cases.

- Case 1: assume $R \in R$. Then $\neg(R \notin R)$. So $\neg(R \in R \Rightarrow R \notin R)$. Hence
$$\exists x\ \neg(x \in R \quad \Leftrightarrow \quad x \notin x).$$
- Case 2: assume $R \notin R$. Then $\neg(R \in R)$. So $\neg(R \notin R \Rightarrow R \in R)$. Hence
$$\exists x\ \neg(x \in R \quad \Leftrightarrow \quad x \notin x).$$

In either case, we showed $\neg\forall x\ (x \in R \Leftrightarrow x \notin x)$. So the proof is finished. □