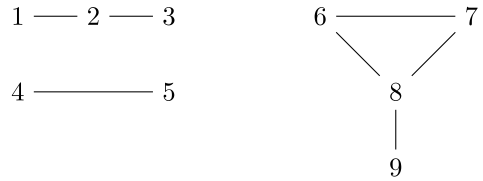
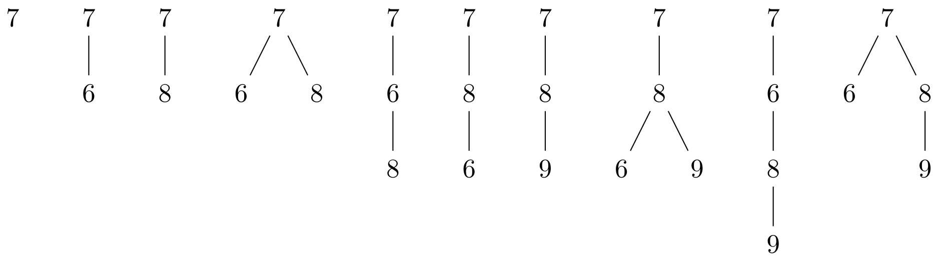
Tutorial solutions for Chapter 12

Sometimes there are other correct answers.

12.1. (a)



(b)



(c)

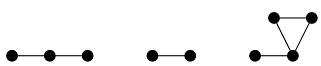
- There are $\binom{9}{3}$ ways to choose three vertices for ; amongst these three vertices, there are $\binom{3}{1}$ ways to choose the middle vertex.
- There are $\binom{9-3}{2} = \binom{6}{2}$ ways left to choose two vertices for .
- There are $\binom{9-3-2}{3} = \binom{4}{3}$ ways left to choose three vertices for ; amongst these three vertices, there are $\binom{3}{1}$ ways to choose where to attach the final vertex.

By the Multiplication Rule, the number of such graphs is

$$\binom{9}{3}\binom{3}{1}\binom{6}{2}\binom{4}{3}\binom{3}{1} = \frac{9\times 8\times 7}{3\times 2\times 1}\times 3\times \frac{6\times 5}{2\times 1}\times \frac{4\times 3\times 2}{3\times 2\times 1}\times 3 = 45360.$$

Alternative solution.

- There are $9!$ ways to assign $1, 2, \ldots, 9$ to the 9 vertices in the graph drawn below without repetition.



(∗)

- Each assignment of distinct labels to the vertices of gives the same graph as exactly one other assignment. More specifically, if a, b, c are distinct elements of $\{1, 2, \ldots, 9\}$, then

$$a \text{ — } b \text{ — } c \qquad\text{and}\qquad c \text{ — } b \text{ — } a$$

are drawings of the same graph

$$(\{a, b, c\}, \{ab, bc\}),$$

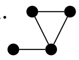
and no other assignment gives a drawing of this graph.

- Each assignment of distinct labels to the vertices of $\bullet\!\!-\!\!\bullet$ gives the same graph as exactly one other assignment. More specifically, if a, b, c are distinct elements of $\{1, 2, \ldots, 9\}$, then

$$a \text{ --- } b \qquad\qquad \text{and} \qquad\qquad b \text{ --- } a$$

  are drawings of the same graph

$$(\{a, b\}, \{ab\}),$$

  and no other assignment gives a drawing of this graph.
- Each assignment of distinct labels to the vertices of  gives the same graph as exactly one other assignment. More specifically, if a, b, c are distinct elements of $\{1, 2, \ldots, 9\}$, then

$$\begin{tikzcd}[column sep=small] & a \arrow[rr, dash] \arrow[dr, dash] & & b \arrow[dl, dash] \\ d \arrow[rr, dash] & & c & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}[column sep=small] & b \arrow[rr, dash] \arrow[dr, dash] & & a \arrow[dl, dash] \\ d \arrow[rr, dash] & & c & \end{tikzcd}$$

  are drawings of the same graph

$$(\{a, b, c, d\}, \{ab, bc, ca, cd\}),$$

  and no other assignment gives a drawing of this graph.
- Hence, by the Multiplication Rule, each graph that has the same vertices as G and are isomorphic to G can be obtained from exactly $2 \times 2 \times 2 = 8$ assignments of the labels $1, 2, \ldots, 9$ to the vertices in $(*)$
- Therefore, the number of such graphs is $9!/8 = 45360$.

12.2. (a) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 3, 16$ respectively, as listed below.

$(n = 1)$ 1

$(n = 2)$ 1 — 2

$(n = 3)$ 1 — 2 — 3   1 — 3 — 2   2 — 1 — 3

$(n = 4)$ 1 — 2 — 3 — 4   1 — 2 — 4 — 3   1 — 3 — 2 — 4

1 — 3 — 4 — 2   1 — 4 — 2 — 3   1 — 4 — 3 — 2

2 — 1 — 3 — 4   2 — 1 — 4 — 3   2 — 3 — 1 — 4

2 — 4 — 1 — 3   3 — 1 — 2 — 4   3 — 2 — 1 — 4

$$\begin{tikzcd}[column sep=small, row sep=tiny] & & 3 \\ 2 \arrow[r, dash] & 1 \arrow[ur, dash] \arrow[dr, dash] & \\ & & 4 \end{tikzcd} \quad \begin{tikzcd}[column sep=small, row sep=tiny] & & 3 \\ 1 \arrow[r, dash] & 2 \arrow[ur, dash] \arrow[dr, dash] & \\ & & 4 \end{tikzcd} \quad \begin{tikzcd}[column sep=small, row sep=tiny] & & 2 \\ 1 \arrow[r, dash] & 3 \arrow[ur, dash] \arrow[dr, dash] & \\ & & 4 \end{tikzcd} \quad \begin{tikzcd}[column sep=small, row sep=tiny] & & 2 \\ 1 \arrow[r, dash] & 4 \arrow[ur, dash] \arrow[dr, dash] & \\ & & 3 \end{tikzcd}$$

Additional comment. We can check whether any tree is missing or double-counted via some extra counting. For example, consider trees of the shape

$$\bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet$$

whose vertices are precisely $1, 2, 3, 4$. There are $4!$ ways to assign $1, 2, 3, 4$ to the four vertices here, say, from left to right, without repetition. Each such assignment gives the same graph as exactly one other assignment. More specifically, if a, b, c, d are distinct elements of $\{1, 2, 3, 4\}$, then

$$a \text{ --- } b \text{ --- } c \text{ --- } d \qquad\qquad \text{and} \qquad\qquad d \text{ --- } c \text{ --- } b \text{ --- } a$$

are drawings of the same graph

$$(\{1, 2, 3, 4\}, \{ab, bc, cd\}),$$

and no other assignment gives a drawing of this graph. This tells us that the number of such vertex assignments is exactly twice the number of such graphs. Thus there are exactly $4!/2 = 12$ such graphs; this number matches with what we had in the solutions.

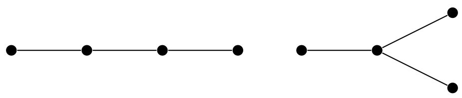
Extra reading. It is known that, for each $n \in \mathbb{Z}^+$, there are exactly n^{n-2} trees whose vertices are precisely $1, 2, \ldots, n$. For a proof, see the note "Joyal's Proof of Cayley's Formula" by Gyu Eun Lee and Doron Zeilberger at `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/JoyalCayley.pdf`.

(b) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 1, 2$ respectively.

$(n = 1)$

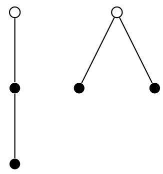
$(n = 2)$

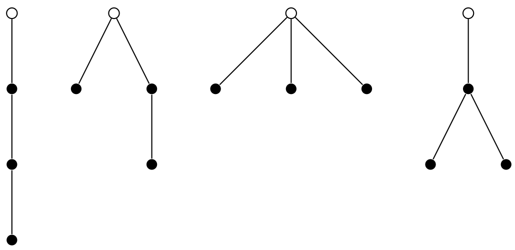
$(n = 3)$

$(n = 4)$ 

Additional comment. These are the "shapes" of the trees we got in (a).

(c) For $n = 1, 2, 3, 4$, the numbers are $1, 1, 2, 4$ respectively.

$(n = 1)$ $(n = 2)$  $(n = 3)$ 

$(n = 4)$ 

Additional comment. These are the different ways to choose roots for the trees we got in (b).

Moral. It is best if we can count without counting. Nevertheless, it is sometimes easier to count by counting.

12.3. $(\Rightarrow)$ Suppose G is a tree. Let $u, v \in \mathrm{V}(G)$ such that $uv \notin \mathrm{E}(G)$. If $u = v$, then uv is a loop and thus $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$ is cyclic. So suppose $u \neq v$. Use connectedness to find a path P between u and v in G. Note that P must have length at least 2 because $uv \notin \mathrm{E}(G)$. So $\big(\mathrm{V}(P), \mathrm{E}(P) \cup \{uv\}\big)$ is a cycle in $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$, making $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$ cyclic.

$(\Leftarrow)$ We prove by contraposition. Suppose G is not a tree. As G is acyclic, this tells us G is unconnected. Find $u, v \in \mathrm{V}(G)$ between which there is no path in G. Note that $u \neq v$ because $(\{u\}, \{\})$ is a path between u and u in G. Also $uv \notin \mathrm{E}(G)$.

Suppose, towards a contradiction, we have a cycle in $(\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\})$, say,

$$C = x_1 x_2 \ldots x_\ell x_1.$$

Note that $uv \in \mathrm{E}(C)$ because G is acyclic. Renaming the x's if needed, we may assume $u = x_1$ and $v = x_\ell$. Then, as $\ell \geqslant 3$, we have a path $x_1 x_2 \ldots x_\ell$ between u and v in G, which contradicts our choice of u, v. $\square$

Alternative proof for $\Leftarrow$. Assume that adding any new edge makes G cyclic. We show G is a tree, i.e., it is both acyclic and connected. We know that G is acyclic by hypothesis. For connectedness, pick any $u, v \in \mathrm{V}(G)$.

Case 1: suppose $uv \in \mathrm{E}(G)$. There are two subcases.

Case 1.1: suppose $u = v$. Then $(\{u\}, \{\})$ is a path between u and v in G.

Case 1.2: suppose $u \neq v$. Then uv is a path between u and v in G.

Case 2: suppose $uv \notin \mathrm{E}(G)$. Consider the graph

$$\hat{G} = (\mathrm{V}(G), \mathrm{E}(G) \cup \{uv\}).$$

By assumption, we know $\hat{G}$ is cyclic. So the definition of cyclic graphs tells us that $\hat{G}$ either has a loop or has a cycle.

Case 2.1: suppose $\hat{G}$ has a loop. As G is acyclic, it has no loop. Now $\hat{G}$ has a loop, and the only difference between G and $\hat{G}$ is the additional edge uv in $\hat{G}$. So it must be the case that uv is a loop, i.e., that $u = v$. In this case, we have the path $(\{u\}, \{\})$ between u and v in G.

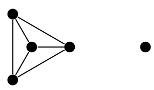
Case 2.2: suppose $\hat{G}$ has a cycle. As G is a acyclic, it has no cycle. Now $\hat{G}$ has a cycle, and the only difference between G and $\hat{G}$ is the additional edge uv in $\hat{G}$. So it must be the case that some/all cycles in $\hat{G}$ have uv in it. Consider a cycle $x_1x_2 \ldots x_\ell x_1$ in $\hat{G}$. Renaming the x's if needed, we may assume $u = x_1$ and $v = x_\ell$. Then, as $\ell \geqslant 3$, we have the path $x_1x_2 \ldots x_\ell$ between u and v in G.

So there is a path between u and v in G in all cases. □

12.4. As G is finite, it has only finitely many connected components. Let $H_1, H_2, \ldots, H_k$ list all the connected components of G without repetition. As G is unconnected, we know $k > 1$. By the definition of connected components, each H_i is connected. In addition, each H_i is acyclic because G is acyclic. So each H_i is a tree. It follows that

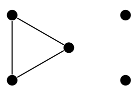
$\lvert\mathrm{E}(G)\rvert$	
$= \lvert\mathrm{E}(H_1) \cup \mathrm{E}(H_2) \cup \cdots \cup \mathrm{E}(H_k)\rvert$	by the additional comment in Tutorial Exercise 11.4(b);
$= \lvert\mathrm{E}(H_1)\rvert + \lvert\mathrm{E}(H_2)\rvert + \cdots + \lvert\mathrm{E}(H_k)\rvert$	by the Addition Rule;
$= \lvert\mathrm{V}(H_1)\rvert - 1 + \lvert\mathrm{V}(H_2)\rvert - 1 + \cdots + \lvert\mathrm{V}(H_k)\rvert - 1$	by Theorem 12.1.10;
$= \lvert\mathrm{V}(H_1)\rvert + \lvert\mathrm{V}(H_2)\rvert + \cdots + \lvert\mathrm{V}(H_k)\rvert - k$	
$= \lvert\mathrm{V}(H_1) \cup \mathrm{V}(H_2) \cup \ldots \mathrm{V}(H_k)\rvert - k$	by the Addition Rule;
$= \lvert\mathrm{V}(G)\rvert - k$	by Tutorial Exercise 11.4(a);
$< \lvert\mathrm{V}(G)\rvert - 1$	as $k > 1$. □

12.5. No. One counterexample is the graph G drawn below.



This graph is not connected, but $\lvert\mathrm{E}(G)\rvert = 6 \geqslant 4 = 5 - 1 = \lvert\mathrm{V}(G)\rvert - 1$.

12.6. No. One counterexample is the graph G drawn below.



This graph is cyclic, but $\lvert\mathrm{E}(G)\rvert = 3 \leqslant 4 = 5 - 1 = \lvert\mathrm{V}(G)\rvert - 1$.

12.7. We proceed by strong induction on the number of vertices in T.

(Base step) Let T be a rooted tree with exactly one vertex. Then this vertex is terminal, and there is no internal vertex. So

$$\text{number of internal vertices} = 0 = 1 - 1 = \text{number of terminal vertices} - 1.$$

(Induction step) Let $k \in \mathbb{Z}^+$ such that the theorem is true for all rooted trees with at most k vertices in which every internal vertex has exactly two children. Consider a rooted tree T with exactly $k+1$ vertices in which every internal vertex has exactly two children. Let r be the root of T. Then r must be internal because otherwise T has exactly one vertex, but $1 < 1+1 \leqslant k+1$, which is a contradiction. So r has exactly two children by assumption, say u and v.

Observe that urv is a path between u and v in T. By Proposition 12.1.3, this is the only path between u and v in T. Therefore, in the graph

$$H = (\mathrm{V}(T) \setminus \{r\}, \mathrm{E}(T) \setminus \{ru, rv\}),$$

there is no path between u and v, and thus the vertices u, v are in different connected components by Theorem 11.3.7. Let H_u be a connected component of H containing u, and let H_v be a connected component of H containing v. Being connected components, we know H_u and H_v are each connected in particular. Also H_u and H_v are acyclic because they are subgraphs of the acyclic graph T. So both H_u and H_v are trees. We will treat u and v as the roots of H_u and H_v respectively. Denote by t_u and t_v the number of terminal vertices in H_u and H_v respectively. As neither H_u nor H_v has the vertex r in it, both H_u and H_v have at most k vertices.

There is a path P between the root r and any other vertex x in T by the connectedness of T. As u and v are the only children of the root r in T, any such path P has either u or v in it. Deleting the vertex r and the edge ru or rv depending on whether u or v is in P, we obtain a path between either u or v and the same vertex x in H. In view of Proposition 12.1.3, one can obtain any path between u or v and a vertex in H in this way.

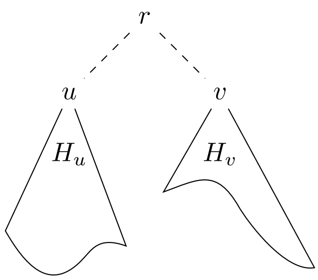
One consequence of the previous paragraph is that, in view of Theorem 11.3.7 and Tutorial Exercise 11.4(a), any vertex $x \in \mathrm{V}(T) \setminus \{r\}$ is either in H_u or in H_v. Another consequence is that, for all vertices x, y in H_u, we know x is a parent of y in T if and only if x is a parent of y in H_u; similarly for H_v. Thus, since r is not the child of any vertex in T, every internal vertex has exactly two children both in H_u and in H_v. Applying the induction hypothesis to the rooted trees H_u and H_v, we deduce that H_u has $t_u - 1$ internal vertices and H_v has $t_v - 1$ internal vertices.

On the one hand, all the terminal vertices in H_u and in H_v are terminal vertices in T. Moreover, the rooted tree T has no other terminal vertex because r is not a terminal vertex in T as discussed above. So T has exactly $t_u + t_v$ terminal vertices. On the other hand, all the internal vertices in H_u and in H_v are internal vertices in T. In addition to these, the rooted tree T has exactly one more internal vertex r. So T has exactly $(t_u - 1) + (t_v - 1) + 1 = t_u + t_v - 1$ internal vertices. So

$$(\text{number of internal vertices in } T) = (\text{number of terminal vertices in } T) - 1.$$

This completes the induction. □

Diagram.



Alternative proof. As preparation, first observe that the following propositions are true for all finite rooted trees T with root r and height h in which every internal vertex has exactly two children.

(0) If $rx_1x_2\ldots x_\ell$ is a path of length ℓ and z is a child of x_ℓ, where $\ell \in \mathbb{N}$, then $rx_1x_2\ldots x_\ell z$ is a path of length $\ell+1$.

(1) If $h = 0$, then r is the only vertex, and thus r is a terminal vertex in T.

(2) If $h \geqslant 1$, then there are paths $rx_1x_2\ldots x_{h-1}v$ and $rx_1x_2\ldots x_{h-1}w$ of length h where v, w are distinct terminal vertices in T.

Now we proceed to prove the theorem by induction on the number of terminal vertices in T. Note that, by (1) and (2) above, such a T must have at least one terminal vertex.

(Base step) Let T be a finite rooted tree with exactly one terminal vertex in which every internal vertex has exactly two children. Then (2) tells us T must have height 0. So T must have height 0. Now the root is the only vertex, and it is a terminal vertex in T by (1). Therefore, in T,

$$\text{number of internal vertices} = 0 = 1 - 1 = \text{number of terminal vertices} - 1.$$

(Induction step) Let $k \in \mathbb{Z}^+$ such that the theorem is true for all rooted trees with exactly k terminal vertices. Consider a rooted tree T with exactly $k+1$ terminal vertices in which every internal vertex has exactly two children. Let h be the height of T. Since the number of vertices in T is at least $k+1 \geqslant 1+1 = 2$, we know from (1) that $h \geqslant 1$. Apply (2) to find paths $x_0x_1\ldots x_{h-1}v$ and $x_0x_1\ldots x_{h-1}w$ of length h where x_0 is the root and v, w are distinct terminal vertices in T. As x_1 is a child of the root, we know the root is not a terminal vertex in T. So neither v nor w is the root in T.

Consider the rooted tree T_0 that is obtained from T by removing the vertices v, w and the edges $x_{h-1}v$ and $x_{h-1}w$. Let us verify that T_0 is indeed a tree.

The graph T_0 is acyclic because it is a subgraph of an acyclic graph T. In view of Lemma 11.1.10, to show connectedness, it suffices to prove that, given any vertex z in T_0, there is a path between the root and z in T_0. Pick any vertex z in T_0. Use the connectedness of T to find a path $y_0y_1\ldots y_\ell$ in T where y_0 is the root and $y_\ell = z$. We know both v and w are not any of $y_0, y_1, \ldots, y_{\ell-1}$ because v, w are terminal vertices in T, but $y_0, y_1, \ldots, y_{\ell-1}$ are all internal vertices. We also know v and w are not y_ℓ because $y_\ell = z \in \mathrm{V}(T_0) = \mathrm{V}(T) \setminus \{v, w\}$. So $y_0y_1\ldots y_\ell$ is actually a path in T_0 between the root and z.

The rooted tree T_0 inherits all the internal vertices and all the terminal vertices of T, except that

- the two terminal vertices v, w in T are not in T_0; and
- the internal vertex x_{h-1} in T becomes a terminal vertex in T_0 because all its children in T, namely v and w, are not in T_0.

In particular, if u is an internal vertex in T_0, then u is an internal vertex in T, and u retains in T_0 both of its children in T because $u \neq x_{h-1}$. Also,

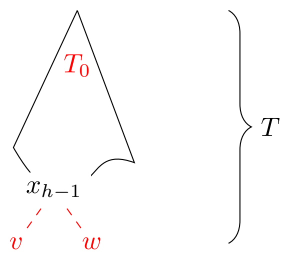
$$\begin{aligned}(\text{number of terminal vertices in } T_0) &= (\text{number of terminal vertices in } T) - 2 + 1 \\ &= (k+1) - 2 + 1 = k.\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\text{number of internal vertices in } T) \\
&= (\text{number of internal vertices in } T_0) + 1 && \text{because of } x_{h-1}; \\
&= (\text{number of terminal vertices in } T_0) - 1 + 1 && \text{by the induction hypothesis;} \\
&= k && \text{by the calculation above;} \\
&= (\text{number of terminal vertices in } T) - 1.
\end{aligned}$$

This completes the induction. □

Diagram.



Explanation of why every finite rooted tree has a height (extra materials). Let T be a finite rooted tree, and n be the number of vertices in T. In T, there is no path of length n, because by Further Exercise 11.7 any such path has $n+1$ vertices, which T does not have. So there is $\ell \in \mathbb{N}$ such that T contains no path of length ℓ between the root and some vertex. Apply the Well-Ordering Principle to find the smallest such $\ell \in \mathbb{N}$. Note that $\ell \neq 0$ because $(\{r\}, \{\})$, where r is the root, is a path of length 0 in T. Then, in view of the smallestness of ℓ, we know T has a path of length $\ell - 1$ between the root and some vertex. So $\ell - 1$ is the height of T.

Explanation of (0) (extra materials). Let $x_0x_1 \ldots x_\ell$ be a path in T where all the x's are different and $x_0 = r$. Take any child z of x_ℓ in T. We know $z \neq x_0$ because z has a parent, but x_0 does not. We also know that z is not any of $x_1, x_2, \ldots, x_\ell$, because if $z = x_i$ where $i \in \{1, 2, \ldots, \ell\}$, then the path $x_0x_1x_2 \ldots x_{i-1}x_i$ in T shows that

$$\begin{aligned}
& x_{i-1} = x_\ell && \text{as these are both parent of } z; \\
\therefore\quad & i - 1 = \ell && \text{as all the } x\text{'s are different;} \\
\therefore\quad & i = \ell + 1,
\end{aligned}$$

which is not possible since $i \in \{1, 2, \ldots, \ell\}$. Thus $x_0x_1 \ldots x_\ell z$ is a path of length $\ell + 1$ in T.

Explanation of (1) (extra materials). We argue by contraposition. Suppose the root r is not a terminal vertex. Then it must have a child, which cannot be the root itself by the definition of children. Let u be a vertex that is not the root r in T. Then connectedness give a path between r and u in T. Any such path must have length at least 1 as $r \neq u$. So $h \geqslant 1$ by the definition of height.

Explanation of (2) (extra materials). Suppose $h \geqslant 1$. Work in T. The definition of height gives a path $x_0x_1 \ldots x_{h-1}v$ where $x_0, x_1, \ldots, x_{h-1}, v$ are all different and $x_0 = r$. This path has length h. The existence of such a path shows that x_{h-1} is a parent of v. So x_{h-1} is an internal vertex. By assumption, every internal vertex has exactly two children. Hence the internal vertex x_{h-1} must have a child, say w, that is not v. Then $x_0x_1 \dots x_{h-1}w$ is a path of length h by part (0). Both v, w are terminal vertices because if, say, the vertex v had a child z, then part (0) would tell us $x_0x_1 \dots x_{h-1}vz$ is a path of length $h+1$, which contradicts the maximality of h as the height.

Further exercises

12.8. We prove this by contraposition. Assume G is connected.

Case 0: suppose G has no vertex. Then G cannot have any edge. So $|\mathrm{E}(G)| = 0 \geqslant -1 = 0 - 1 = |\mathrm{V}(G)| - 1$.

Case 1: suppose G has at least one vertex and G is acyclic. Then G is a tree. So $|\mathrm{E}(G)| = |\mathrm{V}(G)| - 1$ by Theorem 12.1.10.

Case 2: suppose G is cyclic. Then $|\mathrm{E}(G)| \geqslant |\mathrm{V}(G)| > |\mathrm{V}(G)| - 1$.

Hence $|\mathrm{E}(G)| \geqslant |\mathrm{V}(G)| - 1$ in all cases. □

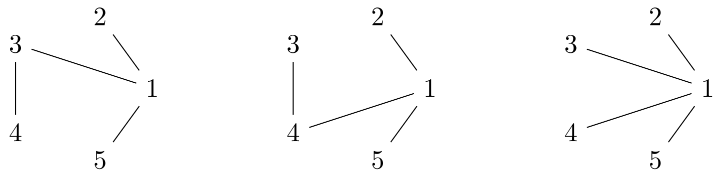
12.9. We prove this by contraposition. Assume G is acyclic.

Case 1: suppose G is connected. Then G is a tree. So $|\mathrm{E}(G)| = |\mathrm{V}(G)| - 1$ by Theorem 12.1.10.

Case 2: suppose G is not connected. So $|\mathrm{E}(G)| < |\mathrm{V}(G)| - 1$ by Tutorial Exercise 12.4.

Hence $|\mathrm{E}(G)| \leqslant |\mathrm{V}(G)| - 1$ in all cases. □

12.10. (a)



(b) Note that G is itself a spanning tree of G. So G has at least one spanning tree. To show that G has at most one spanning tree, it suffices to prove that any spanning tree of G must be equal to G.

Let T be any spanning tree of G. The definition of spanning trees already tells us $\mathrm{V}(G) = \mathrm{V}(T)$. We know $\mathrm{E}(T) \subseteq \mathrm{E}(G)$ as T is a subgraph of G.

Finally, we prove $\mathrm{E}(G) \subseteq \mathrm{E}(T)$ by contradiction. Suppose we have an edge $e \in \mathrm{E}(G) \setminus \mathrm{E}(T)$. Then $(\mathrm{V}(T), \mathrm{E}(T) \cup \{e\})$ is cyclic by Tutorial Exercise 12.3 because T is a tree. However, this is a subgraph of an acyclic graph G. So we have the required contradiction. □

(c)* Assume G has exactly one spanning tree, say T. It suffices to show that $G = T$. The definition of spanning trees already tells us $\mathrm{V}(G) = \mathrm{V}(T)$. We know $\mathrm{E}(T) \subseteq \mathrm{E}(G)$ as T is a subgraph of G. We prove $\mathrm{E}(G) \subseteq \mathrm{E}(T)$ by contradiction.

Suppose we have an edge $uv \in \mathrm{E}(G) \setminus \mathrm{E}(T)$. Being a tree, the spanning tree T is connected. Let $x_0x_1 \dots x_k$ be a path in T where $x_0 = u$ and $x_k = v$. Note that $k \geqslant 1$ because G has no loop and $x_0x_k = uv \in \mathrm{E}(G)$. As $x_0x_1 \in \mathrm{E}(T)$ but $uv \notin \mathrm{E}(T)$, we know $x_0x_1 \neq uv$. We will show that $\hat{G} = (\mathrm{V}(G), \mathrm{E}(G) \setminus \{x_0x_1\})$ remains connected. With this connectedness, we will be able to use the Tree-Growing Algorithm to find a spanning tree $\hat{T}$ of $\hat{G}$ and hence of G. This $\hat{T} \neq T$, because $x_0x_1 \in \mathrm{E}(T)$ but $x_0x_1 \notin \mathrm{E}(\hat{T})$. So G has two different spanning trees, contradicting our initial assumption.

To show the connectedness of $\hat{G}$, we imitate the proof of Theorem 12.1.4. Take any $a, b \in \mathrm{V}(\hat{G})$. Use the connectedness of T to find a path $P = y_0y_1 \dots y_\ell$ in T where $y_0 = a$ and $y_\ell = b$.

Case 1: suppose $x_0x_1 \notin \mathrm{E}(P)$. Then P is a path between a and b in $\hat{G} = (\mathrm{V}(G), \mathrm{E}(G) \setminus \{x_0x_1\})$.

Case 2: suppose $x_0x_1 \in \mathrm{E}(P)$. Swapping a and b if needed, say $x_0 = y_r$ and $x_1 = y_{r+1}$. Now in $\hat{G} = (\mathrm{V}(G), \mathrm{E}(G) \setminus \{x_0x_1\})$,

- between a and y_r there is a path $y_0y_1 \ldots y_r$;
- between y_r and y_{r+1} there is a path $ux_kx_{k-1} \ldots x_1$;
- between y_{r+1} and b there is a path $y_{r+1}y_{r+2} \ldots y_\ell$.

So two applications of Lemma 11.1.10 give us a path between a and b in $\hat{G}$. □

12.11. There is no path of length $|\mathrm{V}(T)|$ in T because this would require $|\mathrm{V}(T)|+1$ vertices in T by Further Exercise 11.7, which T does not have. So there is $\ell \in \mathbb{N}$ such that T contains no path of length ℓ. Apply the Well-Ordering Principle to find the smallest such $\ell \in \mathbb{N}$. Note that $\ell \neq 0$ because T has at least one vertex. In view of the smallestness of ℓ, the tree T has a path of length $\ell - 1$, say $x_0x_1 \ldots x_{\ell-1}$.

We claim that x_0x_1 is the only edge containing x_0 in T. Suppose not. Say $x_0y \in \mathrm{E}(T)$, where $y \neq x_1$. If $y = x_i$, where $y \in \{2, 3, \ldots, \ell - 1\}$, then $x_0x_1 \ldots x_ix_0$ is a cycle in T, contradicting the acyclicity of T. So $y \neq x_i$ for any $y \in \{1, 2, \ldots, \ell - 1\}$. This makes $yx_0x_1 \ldots x_{\ell-1}$ a path of length ℓ in T, contradicting the choice of ℓ. This contradiction proves the claim.

In a similar way, one can prove that $x_{\ell-2}x_{\ell-1}$ is the only edge containing $x_{\ell-1}$ in T.

Note that T is a tree and is thus connected. So, since T has at least two vertices, it has a path of length at least one. This implies $\ell \geqslant 2$. As $\ell - 1 \geqslant 2 - 1 = 1$, we know $x_0 \neq x_{\ell-1}$. □

Alternative solution. We prove this by contraposition. Assume T has at most one vertex that is in exactly one edge.

Case 1: suppose T is not connected. Then T is not a tree.

Case 2: suppose T is connected. Suppose T has exactly n vertices $v_1, v_2, \ldots, v_n$. Since T has at least two vertices, every vertex must be in at least one edge in T in view of connectedness. By assumption, every vertex except possibly one must be in at least two edges in T. So

$$1 + 2(|\mathrm{V}(T)| - 1) \leqslant \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|\mathrm{E}(T)|$$

by the Handshaking Lemma (i.e., Further Exercise 11.9). It follows that

$$|\mathrm{E}(T)| \geqslant \frac{1 + 2(|\mathrm{V}(T)| - 1)}{2} = |\mathrm{V}(T)| - \frac{1}{2} > |\mathrm{V}(T)| - 1.$$

Thus T is cyclic by Further Exercise 12.8. This implies T is not a tree.

Hence T is not a tree in all cases. □