1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 9, 16\}$.

   (i) Write down $(\mathbb{Z} \setminus A) \cap B$ in roster notation. [1 mark]

   **Solution.** $\{9, 16\}$.

   **Explanation.** $(\mathbb{Z} \setminus A) \cap B = \{\dots, -2, -1, 0, 6, 7, 8, \dots\} \cap \{1, 4, 9, 16\} = \{9, 16\}$.

   (ii) Write down $\{x + y : x, y \in A\}$ in roster notation. [1 mark]

   **Solution.** $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   **Explanation.** $\{x+y : x, y \in A\} = \{1+1, 1+2, 1+3, 1+4, 1+5, 2+1, 2+2, \dots, 3+1, \dots, 5+4, 5+5\} = \{2, 3, 4, \dots, 10\}$.

   (iii) Write down $\{x + y : x, y \in A\} \setminus \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$ in roster notation. [1 mark]

   **Solution.** $\{2, 3, 4, 5\}$.

   (iv) Is $\{x^2 : x \in A\} \subseteq B$? Explain your answer. [2 marks]

   **Solution.** <u>No</u>, because $25 \in \{x^2 : x \in A\}$ but $25 \notin B$.

   (v) Write down all partitions of $A$ that contains $\{1, 3, 5\}$. [2 marks]

   **Solution.** $\{\{1, 3, 5\}, \{2, 4\}\}$ and $\{\{1, 3, 5\}, \{2\}, \{4\}\}$.

   **Comments.** Many did not get this part right; cf. Tutorial Exercise 4.7.

2. Let $A$ and $B$ be sets. Prove that, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$. [4 marks]

   **Solution.** Assume $\mathcal{P}(A) = \mathcal{P}(B)$. Recall from Remark 4.2.4(3) that $A \subseteq A$ and $B \subseteq B$. So $A \in \mathcal{P}(A)$ and $B \in \mathcal{P}(B)$ by the definition of $\mathcal{P}$. It then follows from our assumption that $A \in \mathcal{P}(B)$ and $B \in \mathcal{P}(A)$. Unravelling the definition of $\mathcal{P}$, these give $A \subseteq B$ and $B \subseteq A$. Hence $A = B$ by Remark 4.2.4(2). $\square$

   **Alternative solution.** Let us prove the contrapositive. Assume $A \neq B$. Use the definition of set equality to find $z$ such that the biconditional $z \in A \Leftrightarrow z \in B$ is false. There are two cases.

   **Case 1: suppose $z \in A$ but $z \notin B$.** Then $\{z\} \subseteq A$ and $\{z\} \nsubseteq B$ by the definition of $\subseteq$. According to the definition of $\mathcal{P}$, these mean $\{z\} \in \mathcal{P}(A)$ and $\{z\} \notin \mathcal{P}(B)$, implying that $\mathcal{P}(A) \neq \mathcal{P}(B)$.

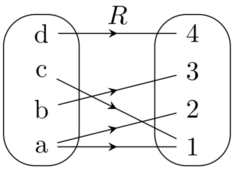
   **Case 2: suppose $z \in B$ but $z \notin A$.** Then $\{z\} \nsubseteq A$ and $\{z\} \subseteq B$ by the definition of $\subseteq$. According to the definition of $\mathcal{P}$, these mean $\{z\} \notin \mathcal{P}(A)$ and $\{z\} \in \mathcal{P}(B)$, implying that $\mathcal{P}(A) \neq \mathcal{P}(B)$.

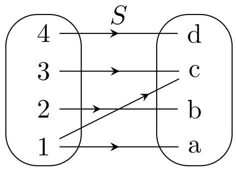
   So $\mathcal{P}(A) \neq \mathcal{P}(B)$ in all cases. $\square$

   **Comments.**
   - Many only rewrote the proposition to be proved, and did not make any progress.
   - Many mixed up elements ($\in$) and subsets ($\subseteq$). These may be different. For example, elements of $\mathcal{P}(\mathbb{N})$ are subsets of $\mathbb{N}$, *not* elements of $\mathbb{N}$; subsets of $\mathcal{P}(\mathbb{N})$ are *neither* elements *nor* subsets of $\mathbb{N}$.
   - Many deduced from $A \neq B$ that some element of $A$ is not an element of $B$. There is actually another case when some element of $B$ is not an element of $A$, which cannot be ignored.

MORE QUESTIONS AT THE BACK OF THIS PAGE

3. Let $A = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ and $B = \{1, 2, 3, 4\}$. The following are arrow diagrams of a relation $R$ from $A$ to $B$, and a relation $S$ from $B$ to $A$.




(i) Write down $R^{-1}$ is roster notation. [1 mark]

**Solution.** $R^{-1} = \{(1, \mathrm{a}), (1, \mathrm{c}), (2, \mathrm{a}), (3, \mathrm{b}), (4, \mathrm{d})\}$.

(ii) Write down $R \circ S$ in roster notation. [1 mark]

**Solution.** $R \circ S = \{(1, 1), (1, 2), (2, 3), (3, 1), (4, 4)\}$.

(iii) Is $R \circ S$ antisymmetric as a relation on $B$? Explain your answer. [2 marks]

**Solution.** <u>Yes</u>, because the only $x, y \in B$ that make $(x, y), (y, x) \in R \circ S$ are when $x$ and $y$ are both 1 or both 4.

(iv) Write down $S \circ R$ in roster notation. [1 mark]

**Solution.** $S \circ R = \{(\mathrm{a}, \mathrm{a}), (\mathrm{a}, \mathrm{b}), (\mathrm{a}, \mathrm{c}), (\mathrm{b}, \mathrm{c}), (\mathrm{c}, \mathrm{a}), (\mathrm{c}, \mathrm{c}), (\mathrm{d}, \mathrm{d})\}$.

(v) Is $S \circ R$ reflexive as relation on $A$? Explain your answer. [2 marks]

**Solution.** <u>No</u>, because $(\mathrm{b}, \mathrm{b}) \notin S \circ R$.

MORE QUESTIONS AT THE BACK OF THIS PAGE

4. Let $A = \{(x, y) \in \mathbb{R}^2 : x \geqslant 12 \text{ and } y > 31\}$. Define a relation $R$ on $A$ by setting, for all $(x_1, y_1), (x_2, y_2) \in A$,

$$(x_1, y_1)\ R\ (x_2, y_2) \quad \Leftrightarrow \quad (x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31).$$

(i) Prove that $R$ is an equivalence relation. [5 marks]

**Solution. (Reflexivity)** Let $(x, y) \in A$. Then $(x-12)(y-31) = (x-12)(y-31)$. So $(x, y)\ R\ (x, y)$ by the definition of $R$.

**(Symmetry)** Let $(x_1, y_1), (x_2, y_2) \in A$ such that $(x_1, y_1)\ R\ (x_2, y_2)$. Then $(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31)$ by the definition of $R$. This implies $(x_2 - 12)(y_1 - 31) = (x_1 - 12)(y_2 - 31)$. So $(x_2, y_2)\ R\ (x_1, y_1)$ by the definition of $R$.

**(Transitivity)** Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$ such that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$. Then, by the definition of $R$,

$$(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31), \text{ and} \tag{1}$$

$$(x_2 - 12)(y_3 - 31) = (x_3 - 12)(y_2 - 31). \tag{2}$$

**Case 1: suppose $x_2 \neq 12$.** As $(x_3, y_3) \in A$, we know also $y_3 \neq 31$. So $0 \neq (x_2 - 12)(y_3 - 31) = (x_3 - 12)(y_2 - 31)$ by (2). Cross-dividing (1) by (2) then gives

$$\frac{x_1 - 12}{x_3 - 12} = \frac{(x_1 - 12)(y_2 - 31)}{(x_3 - 12)(y_2 - 31)} = \frac{(x_2 - 12)(y_1 - 31)}{(x_2 - 12)(y_3 - 31)} = \frac{y_1 - 31}{y_3 - 31}.$$

Hence $(x_1 - 12)(y_3 - 31) = (x_3 - 12)(y_1 - 31)$.

**Case 2: suppose $x_2 = 12$.** Then, from (1) and (2), we have respectively

$$(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31) = 0 \cdot (y_1 - 31) = 0, \text{ and}$$
$$(x_3 - 12)(y_2 - 31) = (x_2 - 12)(y_3 - 31) = 0 \cdot (y_3 - 31) = 0.$$

However, as $(x_2, y_2) \in A$, we know $y_2 \neq 31$. So $x_1 - 12 = 0$ and $x_3 - 12 = 0$. Hence

$$(x_1 - 12)(y_3 - 31) = 0 \cdot (y_3 - 31) = 0 = 0 \cdot (y_1 - 31) = (x_3 - 12)(y_1 - 31).$$

It follows from the definition of $R$ that $(x_1, y_1)\ R\ (x_3, y_3)$ in all cases. □

**Alternative proof of transitivity.** Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$ such that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$. Then, by the definition of $R$,

$$(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31) \quad \text{and} \quad (x_2 - 12)(y_3 - 31) = (x_3 - 12)(y_2 - 31).$$

It follows that

$$\frac{(x_1 - 12)(y_2 - 31)}{y_1 - 31} = x_2 - 12 = \frac{(x_3 - 12)(y_2 - 31)}{y_3 - 31}.$$

$$\therefore \quad \frac{x_1 - 12}{y_1 - 31} = \frac{x_3 - 12}{y_3 - 31} \qquad \text{as } (x_2, y_2) \in A \text{ implies } y_2 \neq 31.$$

$$\therefore \quad (x_1 - 12)(y_3 - 31) = (x_3 - 12)(y_1 - 31).$$

$$\therefore \quad (x_1, y_1)\ R\ (x_3, y_3) \qquad \text{by the definition of } R.$$

□

**Yet another proof of transitivity.** Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$ such that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$. Then, by the definition of $R$,

$$(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31) \quad \text{and} \quad (x_2 - 12)(y_3 - 31) = (x_3 - 12)(y_2 - 31).$$

As $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$, we know that none of $y_1, y_2, y_3$ equals 31. Thus the equations above imply

$$\frac{x_1 - 12}{y_1 - 31} = \frac{x_2 - 12}{y_2 - 31} = \frac{x_3 - 12}{y_3 - 31}.$$

Hence $(x_1 - 12)(y_3 - 31) = (x_3 - 12)(y_1 - 31)$. It follows that $(x_1, y_1)\ R\ (x_3, y_3)$. □

**Comments.**
- Quite a number of students misread $(x_1 - 12)(y_2 - 31) = (x_2 - 12)(y_1 - 31)$ as $(x_1 - 12)(y_1 - 31) = (x_2 - 12)(y_2 - 31)$.
- Some did not realize that some $(x, y) \in A$ makes $x - 12 = 0$, and thus divided by zero.

(ii) Prove that there exists $y \in \mathbb{R}$ such that $(24, y) \in [(36, 49)]$. [2 marks]

**Solution.** Let $y = 40$. Then

$$(36 - 12)(y - 31) = (36 - 12)(40 - 31) = 24 \times 9 = 12 \times 9 \times 2 = 12 \times 18 = (24 - 12)(49 - 31).$$

So $(36, 49)\ R\ (24, y)$ by the definition of $R$. This means $(24, y) \in [(36, 49)]$ according to the definition of $[(36, 49)]$. □

**Comments.** Some only showed $(24, y) \in [(36, 49)]$ implies $y = 40$. This does not prove the proposition given; see Tutorial Exercise 3.2.

END OF QUIZ