Tutorial solutions for Chapter 1

Sometimes there are other correct answers.

1.1. All of these are true, except the proposition in (c), which is false.

Additional explanations.

(a) $p \wedge q \to r$

F F T

F

T

(b) $p \leftrightarrow q \vee \neg r$

F F T

F

F

T

(c) $p \wedge \big(q \vee (\neg r \to p)\big)$

F F T F

F ?

F

(d) Treating p, q, r as propositional variables,

$$\big(q \vee r \vee (\neg p \leftrightarrow q)\big) \wedge (q \vee r) \equiv q \vee r$$

by the absorption logical identities. Since r is a true proposition, the given proposition is also true.

1.2. (a) $t \leftrightarrow c \wedge a$.

(b) $t \to a$.

(c) $t \to c$.

1.3. (a) (i)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

(ii)

p	$p \wedge p$
T	T
F	F

(iii)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(iv)

T	C	$\neg T$
T	F	F

(b) (i)

$$\begin{aligned}
p \vee (q \wedge r) &\equiv \neg\neg p \vee (\neg\neg q \wedge \neg\neg r) && \text{by the Double Negative Law;}\\
&\equiv \neg\neg p \vee \neg(\neg q \vee \neg r) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big(\neg p \wedge (\neg q \vee \neg r)\big) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big((\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\big) && \text{by (a)(i);}\\
&\equiv \neg\big(\neg(p \vee q) \vee \neg(p \vee r)\big) && \text{by De Morgan's Laws;}\\
&\equiv \neg\neg\big((p \vee q) \wedge (p \vee r)\big) && \text{by De Morgan's Laws;}\\
&\equiv (p \vee q) \wedge (p \vee r) && \text{by the Double Negative Law.}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\neg\big(p \vee (q \wedge r)\big) &\equiv \neg p \wedge \neg(q \wedge r) && \text{by De Morgan's Laws;}\\
&\equiv \neg p \wedge (\neg q \vee \neg r) && \text{by De Morgan's Laws;}\\
&\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r) && \text{by (a)(i);}\\
&\equiv \neg(p \vee q) \vee \neg(p \vee r) && \text{by De Morgan's Laws;}\\
&\equiv \neg\big((p \vee q) \wedge (p \vee r)\big) && \text{by De Morgan's Laws.}
\end{aligned}$$

Thus $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(ii)

$$\begin{aligned}
p \vee p &\equiv \neg\neg p \vee \neg\neg p && \text{by the Double Negative Law;}\\
&\equiv \neg(\neg p \wedge \neg p) && \text{by De Morgan's Laws;}\\
&\equiv \neg\neg p && \text{by (a)(ii);}\\
&\equiv p && \text{by the Double Negative Law.}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
p \vee p &\equiv \neg\neg(p \vee p) && \text{by the Double Negative Law;}\\
&\equiv \neg(\neg p \wedge \neg p) && \text{by De Morgan's Laws;}\\
&\equiv \neg(\neg p) && \text{by (a)(ii);}\\
&\equiv p && \text{by the Double Negative Law.}
\end{aligned}$$

(iii)

$$\begin{aligned}
p \vee (p \wedge q) &\equiv (p \vee p) \wedge (p \vee q) && \text{by (b)(i);}\\
&\equiv p \wedge (p \vee q) && \text{by (b)(ii);}\\
&\equiv p && \text{by (a)(iii).}
\end{aligned}$$

One can alternatively proceed as follows:

$$\begin{aligned}
p \vee (p \wedge q) &\equiv (p \wedge T) \vee (p \wedge q) && \text{as } T \text{ is an identity for } \wedge;\\
&\equiv p \wedge (T \vee q) && \text{by the Distributive Laws;}\\
&\equiv p \wedge T && \text{as } T \text{ is annihilating with respect to } \vee;\\
&\equiv p && \text{as } T \text{ is an identity for } \wedge.
\end{aligned}$$

(iv)

$$\begin{aligned}
\neg C &\equiv \neg\neg T && \text{by (a)(iv);}\\
&\equiv T && \text{by the Double Negative Law.}
\end{aligned}$$

Additional explanations. Let us provide a slightly more intuitive explanation of why the Absorption Law $p \wedge (p \vee q)$ is true. The other Absorption Law can be explained in a similar way.

Consider propositions p and q. By the definition of $\vee$, if p is true, then $p \vee q$ must be true as well. So, for p and $p \vee q$ to be both true, it is sufficient to have p true. Of course, this is also necessary. Therefore, we have $p \wedge (p \vee q)$ true if and only if p is true.

1.4. Let us use a truth table:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

As the column for $p \leftrightarrow q$ is exactly the same as the column for $(p \to q) \wedge (q \to p)$, the two compound expressions are equivalent.

Alternative solution. According to our definition of $\leftrightarrow$ in Definition 1.3.11, under a substitution of propositions into the propositional variables p and q, we know $p \leftrightarrow q$ evaluates to true if and only if p and q evaluate to the same truth value. As propositions by definition have exactly one truth value amongst "true" and "false", the latter in turn is equivalent to p and q both evaluating to true or both evaluating to false. This means the compound expression $(p \wedge q) \vee (\neg p \wedge \neg q)$ evaluates to true under this substitution. All these show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$. Hence

$$\begin{aligned}
&p \leftrightarrow q \\
&\equiv (p \wedge q) \vee (\neg p \wedge \neg q) && \text{by the above;} \\
&\equiv \big((p \wedge q) \vee \neg p\big) \wedge \big((p \wedge q) \vee \neg q\big) && \text{by the Distributive Laws;} \\
&\equiv \big((p \vee \neg p) \wedge (q \vee \neg p)\big) \wedge \big((p \vee \neg q) \wedge (q \vee \neg q)\big) && \text{by the Distributive Laws;} \\
&\equiv \big(T \wedge (q \vee \neg p)\big) \wedge \big((p \vee \neg q) \wedge T\big) && \text{by the logical identities on negation;} \\
&\equiv (q \vee \neg p) \wedge (p \vee \neg q) && \text{as } T \text{ is an identity for } \wedge; \\
&\equiv (p \to q) \wedge (q \to p) && \text{by the logical identity on the implication.}
\end{aligned}$$

Here T denotes a tautology.

1.5. Let T be a tautology.

$$\begin{aligned}
&(p \to q) \wedge (q \to r) \to (p \to r) \\
&\equiv \neg\big((\neg p \vee q) \wedge (\neg q \vee r)\big) \vee (\neg p \vee r) && \text{by the logical identity on the implication;} \\
&\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee \neg p \vee r && \text{by De Morgan's Laws;} \\
&\equiv (p \wedge \neg q) \vee \neg p \vee (q \wedge \neg r) \vee r && \text{by De Morgan's Laws;} \\
&\equiv \big((p \vee \neg p) \wedge (\neg q \vee \neg p)\big) \vee \big((q \vee r) \wedge (\neg r \vee r)\big) && \text{by the Distributive Laws;} \\
&\equiv \big(T \wedge (\neg q \vee \neg p)\big) \vee \big((q \vee r) \wedge T\big) && \text{by the logical identities on negation;} \\
&\equiv \neg q \vee \neg p \vee q \vee r && \text{as } T \text{ is the identity for } \wedge; \\
&\equiv T \vee \neg p \vee r && \text{by the logical identities on negation;} \\
&\equiv T && \text{as } T \text{ is annihilating with respect to } \vee.
\end{aligned}$$

Alternatively, one can use the following truth table:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

1.6. No. Let us substitute a false proposition into p and a true proposition into q. Then $p \to q$ evaluates to T, while $q \to p$ evaluates to F. So $(p \to q) \to (q \to p)$ evaluates to F. This shows the compound expression given is not a tautology.

Additional comment 1. There is no other substitution under which this expression evaluates to F.

Additional comments 2. To determine whether the given expression is a tautology, one may want to draw a complete truth table. However, to prove that this expression is not a tautology, it suffices to identify one row of the truth table in which it evaluates to F. This is essentially what the solution above is doing.

One may use the logical identities to show that the expression given is equivalent to $p \vee \neg q$. While $p \vee \neg q$ does not look like a tautology, this does not logically imply it is not a tautology: we need to demonstrate a way to substitute propositions into p and q which makes $p \vee \neg q$ evaluate to F; cf. Note 1.4.28. One may find it easier to come up with such a substitution for the equivalent expression $p \vee \neg q$ than for the original expression.

1.7. By the definition of tautologies, the compound expression $P \leftrightarrow Q$ is a tautology means $P \leftrightarrow Q$ evaluates to true no matter what propositions we substitute into its propositional variables. According to the definition of $\leftrightarrow$, this in turn means P and Q evaluate to the same truth value no matter what propositions we substitute into their propositional variables. This is exactly what the definition of $P \equiv Q$ says. $\square$

Alternative solution using Notation 3.1.19.

$P \leftrightarrow Q$ is a tautology

$\Leftrightarrow$	$P \leftrightarrow Q$ evaluates to T no matter what propositions we substitute into its propositional variables	by the definition of tautologies;
$\Leftrightarrow$	P and Q evaluate to the same truth value no matter what propositions we substitute into their propositional variables	by the definition of $\leftrightarrow$;
$\Leftrightarrow$	$P \equiv Q$	by the definition of $P \equiv Q$. $\square$

Additional comment. Some students try to prove the proposition by comparing (certain rows in) the truth table for $P \leftrightarrow Q$ and the "truth table" for $P \equiv Q$, where P and Q take different truth values. *There is no such truth table for* $P \equiv Q$. If we replace P and Q by propositions, say $1 + 1 = 2$ and $2 + 3 = 4$, so that they get some truth values, then the result looks like $(1+1 = 2) \equiv (2+3 = 4)$. Our definition does not give any meaning to this: it covers only the case when $\equiv$ is used between two compound expressions, not two propositions.

Further exercises

1.8. (Note that, since we are given two compound expressions, we are referring to equivalence in the sense of Definition 1.4.7 here.)

(a) These are not equivalent.

Let us substitute a false proposition into p, q and r. On the one hand, we know $p \to q$ evaluates to T, and so $(p \to q) \to r$ evaluates to F. On the other hand, we know $q \to r$ evaluates to T, and so $p \to (q \to r)$ evaluates to T. So the two compound expressions are not equivalent.

Additional comment. One can also substitute a false proposition into p and r, and a true proposition into q. No other substitution works.

(b) These are equivalent.

Let us use a truth table:

p	q	r	$p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow r$	$q \leftrightarrow r$	$p \leftrightarrow (q \leftrightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	F	T	F
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	F	T	F

As the column for $(p \leftrightarrow q) \leftrightarrow r$ is exactly the same as the column for $p \leftrightarrow (q \leftrightarrow r)$, the two compound expressions are equivalent.

Alternative solution. On the one hand,

$$
\begin{aligned}
&(p \leftrightarrow q) \leftrightarrow r \\
&\equiv \big((p \rightarrow q) \wedge (q \rightarrow p) \rightarrow r\big) \\
&\quad \wedge \big(r \rightarrow (p \rightarrow q) \wedge (q \rightarrow p)\big) && \text{by Tutorial Exercise 1.4;} \\
&\equiv \big(\neg\big((\neg p \vee q) \wedge (\neg q \vee p)\big) \vee r\big) \\
&\quad \wedge \big(\neg r \vee \big((\neg p \vee q) \wedge (\neg q \vee p)\big)\big) && \text{by Theorem 1.4.14;} \\
&\equiv \big(\big((p \wedge \neg q) \vee (q \wedge \neg p)\big) \vee r\big) \\
&\quad \wedge \big(\neg r \vee \big((\neg p \vee q) \wedge (\neg q \vee p)\big)\big) && \text{by De Morgan's Laws;} \\
&\equiv \big((p \vee q \vee r) \wedge (p \vee \neg p \vee r) \wedge (\neg q \vee q \vee r) \wedge (\neg q \vee \neg p \vee r)\big) \\
&\quad \wedge \big((\neg r \vee \neg p \vee q) \wedge (\neg r \vee \neg q \vee p)\big) && \text{by the Distributive Laws;} \\
&\equiv \big((p \vee q \vee r) \wedge (T \vee r) \wedge (T \vee r) \wedge (\neg q \vee \neg p \vee r)\big) \\
&\quad \wedge \big((\neg r \vee \neg p \vee q) \wedge (\neg r \vee \neg q \vee p)\big) && \text{by Example 1.4.19;} \\
&\equiv \big((p \vee q \vee r) \wedge T \wedge T \wedge (\neg q \vee \neg p \vee r)\big) \\
&\quad \wedge \big((\neg r \vee \neg p \vee q) \wedge (\neg r \vee \neg q \vee p)\big) && \text{as } T \text{ is annihilating with respect to } \vee; \\
&\equiv (p \vee q \vee r) \wedge (\neg q \vee \neg p \vee r) \\
&\quad \wedge (\neg r \vee \neg p \vee q) \wedge (\neg r \vee \neg q \vee p) && \text{as } T \text{ is an identity with respect to } \wedge,
\end{aligned}
$$

where T is a tautology. On the other hand,

$$
\begin{aligned}
&p \leftrightarrow (q \leftrightarrow r) \\
&\equiv \big(p \rightarrow (q \leftrightarrow r)\big) \wedge \big((q \leftrightarrow r) \rightarrow p\big) && \text{by Tutorial Exercise 1.4;} \\
&\equiv \big((q \leftrightarrow r) \rightarrow p\big) \wedge \big(p \rightarrow (q \leftrightarrow r)\big) && \text{as } \wedge \text{ is commutative;} \\
&\equiv (q \leftrightarrow r) \leftrightarrow p && \text{by Tutorial Exercise 1.4;} \\
&\equiv (q \vee r \vee p) \wedge (\neg r \vee \neg q \vee p) \\
&\quad \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg r \vee q) && \text{by an argument similar to the one above.}
\end{aligned}
$$

Therefore, in view of the Associative Laws and the Commutative Laws for $\wedge$ and $\vee$, we know $(p \leftrightarrow q) \leftrightarrow r \equiv p \leftrightarrow (q \leftrightarrow r)$.

Discussion. We saw that our convention of writing neither $p \rightarrow q \rightarrow r$ nor $p \leftrightarrow q \leftrightarrow r$ is *not* mainly due to non-associativity, because $\leftrightarrow$ is actually associative, as we showed. The real reason is that we would like to avoid these being misinterpreted as $(p \rightarrow q) \wedge (q \rightarrow r)$ and $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ respectively; cf. Notation 3.1.19.

1.9. (a)

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r)$	$(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Since the column for $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ consists entirely of T's, we see that $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ is a tautology.

Alternatively, one can use the logical identities as follows.

$$
\begin{aligned}
& (p \vee q) \wedge (p \to r) \wedge (q \to r) \to r \\
& \equiv (p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r) \to r && \text{by the logical identity on implication;} \\
& \equiv (p \vee q) \wedge \big((\neg p \wedge \neg q) \vee r)\big) \to r && \text{by the Distributive Law;} \\
& \equiv (p \vee q) \wedge \big(\neg(p \vee q) \vee r)\big) \to r && \text{by De Morgan's Laws;} \\
& \equiv \big((p \vee q) \wedge \neg(p \vee q)\big) \vee \big((p \vee q) \wedge r\big) \to r && \text{by the Distributive Law;} \\
& \equiv \big((p \vee q) \wedge r\big) \to r && \text{by the logical identities on negation and identities;} \\
& \equiv \neg\big((p \vee q) \wedge r\big) \vee r && \text{by the logical identity on implication;} \\
& \equiv \big(\neg(p \vee q) \vee \neg r\big) \vee r && \text{by De Morgan's Laws;} \\
& \equiv \neg(p \vee q) \vee (\neg r \vee r) && \text{by the Associative Laws;} \\
& \equiv T && \text{by the logical identities on negation and identities,}
\end{aligned}
$$

where T is a tautology.

(b)

p	C	$\neg p$	$\neg p \to C$	$(\neg p \to C) \to p$
T	F	F	T	T
F	F	T	F	T

Since the column for $(\neg p \to C) \to p$ consists entirely of T's, we see that $(\neg p \to C) \to p$ is a tautology.

Alternatively, we may use the logical identities as follows:

$$
\begin{aligned}
(\neg p \to C) \to p &\equiv \neg(p \vee C) \vee p && \text{by the logical identity on implication;} \\
&\equiv \neg p \vee p && \text{by the logical identities on negation and identities;} \\
&\equiv T && \text{by the logical identity on negation,}
\end{aligned}
$$

where T is a tautology.

1.10. Suppose $P \equiv Q$.

Consider any substitution of propositions into the propositional variables in P and Q. We split into two cases.

- Suppose P evaluates to T under this substitution. Then Q evaluates to T too because $P \equiv Q$. So both $\neg P$ and $\neg Q$ evaluate to F according to the definition of $\neg$.
- Suppose P evaluates to F under this substitution. Then Q evaluates to F too because $P \equiv Q$. So both $\neg P$ and $\neg Q$ evaluate to T according to the definition of $\neg$.

Thus $\neg P$ and $\neg Q$ evaluate to the same truth value in all cases.

Since this is true regardless of which substitution we pick, we conclude that $\neg P \equiv \neg Q$. □

Alternative solution. Assume $P \equiv Q$.

Consider any substitution of propositions into the propositional variables in P and Q. We know P and Q evaluate to the same truth value because $P \equiv Q$ by assumption. It then follows from the definition of $\neg$ that $\neg P$ and $\neg Q$ evaluate to the same truth value.

Since this is true regardless of which substitution we pick, we conclude that $\neg P \equiv \neg Q$. $\square$