Name: ________________ **Tutorial group:** ________ **(number or day&time)**

CS1231 Discrete Structures **Quiz 1** **2023/24 Semester 1**

1. Let D be the set that contains precisely $1, 3, 5, 7, 9$. Which of the following propositions are true? Which are false? Briefly explain your answers.

   (a) $\forall x, y \in D\ (y - x > 10 \to y - x > 20)$. [2 marks]

   Solution. True, because for any $x, y \in D$, we know $y - x \leqslant 10$, and thus the conditional is vacuously true.

   (b) $\forall x, y, z \in D\ xy \neq z$. [2 marks]

   Solution. False: take $x = 3$ and $y = 3$ and $z = 9$.

   (c) $\exists x \in D\ (x \leqslant 10 \vee \exists y \in \mathbb{Z}\ x = 2y - 3)$. [2 marks]

   Solution. True, because given any $x \in D$, we see that $x \leqslant 10$ and/or $\exists y \in \mathbb{Z}\ x = 2y - 3$.

   (d) $\exists y \in \mathbb{N}\ \forall x \in D\ y = x^2$. [3 marks]

   Solution. False, as explained below.

   - If $y = 1$, then choose $x = 3$, so that $y = 1 \neq 9 = 3^2 = x^2$.
   - If $y \neq 1$ then choose $x = 1$, so that $y \neq 1 = 1^2 = x^2$.

   Alternative solution. Suppose the proposition is true. Fix $y \in \mathbb{N}$ such that $\forall x \in D\ y = x^2$. As $1, 3 \in D$, this implies $1 = 1^2 = y = 3^2 = 9$. However, we know $1 \neq 9$. This is a contradiction. So the proposition must be false.

   Comments.
   - Many misinterpreted the proposition as $\forall x \in D\ \exists y \in \mathbb{N}\ y = x^2$. The two are *not* equivalent; see Warning 2.4.4.
   - Many tried to explain the falsity of the proposition by producing $y \in \mathbb{N}$ and $x \in D$ such that $y \neq x^2$. This shows only the falsity of $\forall y \in \mathbb{N}\ \forall x \in D\ y = x^2$.

   (e) $\forall x \in D\ (\exists y \in D\ y = 4x + 5 \to x = 3)$. [3 marks]

   Solution. False, because if $x = 1$, then $x \in D$ and $\exists y \in D\ y = 4x + 5$ is true with witness 9, but $x = 3$ is false.

   Comments. Many misinterpreted the meaning of the proposition. According to Convention 2.2.8(3), the proposition reads
   $$\forall x \in D\ \big((\exists y \in D\ y = 4x + 5) \to x = 3\big).$$
   This is *not* equivalent to $\forall x \in D\ \exists y \in D\ (y = 4x + 5 \to x = 3)$, which is true.

2. Rewrite the following propositions symbolically.

   (a) For two distinct integers to have the same square, it is necessary that one of them is positive and the other is negative. [2 marks]

   Solution. $\forall x, y \in \mathbb{Z}\ \big(x \neq y \wedge x^2 = y^2 \to (x > 0 \wedge y < 0) \vee (y > 0 \wedge x < 0)\big)$.
   Another correct answer is $\forall x, y \in \mathbb{Z}\ \big(x \neq y \to \big(x^2 = y^2 \to (x > 0 \wedge y < 0) \vee (y > 0 \wedge x < 0)\big)\big)$.

   (b) Not every positive integer is the sum of the squares of two integers. [2 marks]

   Solution. $\neg\forall n \in \mathbb{Z}^+\ \exists x, y \in \mathbb{Z}\ n = x^2 + y^2$.
   Another correct answer is $\neg\forall n \in \mathbb{Z}\ (n > 0 \to \exists x, y \in \mathbb{Z}\ n = x^2 + y^2)$. Other acceptable but less preferable answers include $\exists n \in \mathbb{Z}^+\ \forall x, y \in \mathbb{Z}\ n \neq x^2 + y^2$ and $\neg\forall n \in \mathbb{Z}\ \exists x, y \in \mathbb{Z}\ (n > 0 \to n = x^2 + y^2)$ and $\exists n \in \mathbb{Z}\ (n > 0 \wedge \forall x, y \in \mathbb{Z}\ n \neq x^2 + y^2)$ and $\exists n \in \mathbb{Z}\ \forall x, y \in \mathbb{Z}\ (n > 0 \wedge n \neq x^2 + y^2)$.

3. Let p, q, r be propositional variables. Which of the following pairs of compound expressions are equivalent? Which are not? Prove that your answer is correct.

   (a) $p \wedge (q \to r)$ and $p \leftrightarrow (q \to r)$. [3 marks]

   Solution. No, because if we substitute false propositions into p and r, and substitute a true proposition into q, then $p \wedge (q \to r)$ evaluates to F while $p \leftrightarrow (q \to r)$ evaluates to T.

   Comments. Some wrote that the two are evaluate to different truth values when p and $q \to r$ both evaluate to F, without explaining how to make $q \to r$ evaluate to F. Without this explanation, it is not clear whether $q \to r$ can actually evaluate to F.

   (b) $p \vee q \to \neg p$ and $\neg p$. [3 marks]

   Solution. Yes, as shown below.
   $$\begin{aligned} p \vee q \to \neg p &\equiv \neg(p \vee q) \vee \neg p && \text{by the logical identity on implication;}\\ &\equiv (\neg p \wedge \neg q) \vee \neg p && \text{by De Morgan's Laws;}\\ &\equiv \neg p && \text{by the Absorption Laws.}\end{aligned}$$
   Alternatively, one can draw a truth table to show the equivalence.

p	q	$p \vee q$	$\neg p$	$p \vee q \to \neg p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	T

   Since the column for $p \vee q \to \neg p$ and the column for $\neg p$ are exactly the same, the two compound expressions are equivalent.

4. Consider the proposition "any rational number x that is at least 1 must satisfy $x^2 \geqslant 1$".

   (i) Someone tries to prove this proposition as follows.

   > Suppose not. Then we get a rational number, say x, such that $x \geqslant 1$ but $x^2 < 1$. Since $x^2 < 1$, we know $(x-1)(x+1) = x^2 - 1 < 0$. This tells us $x - 1 < 0 < x + 1$ or $x + 1 < 0 < x - 1$. The latter implies $x + 1 < x - 1$ and thus $1 < -1$. This contradiction proves the required proposition.

   What is wrong with this attempt? [1 mark]

   Solution. This attempt reaches a contradiction only in the case when $x + 1 < 0 < x - 1$. It does not explain why there is a contradiction also in the case when $x - 1 < 0 < x + 1$.

   Comments.
   - Some claimed that the proposition to be proven is $\forall x \in \mathbb{Q}\ (x \geqslant 1 \wedge x^2 \geqslant 1)$, but actually it is $\forall x \in \mathbb{Q}\ (x \geqslant 1 \to x^2 \geqslant 1)$; see Example 2.2.11(1).
   - Some wrote that the author proved the wrong proposition, while some others wrote that the author negated the proposition wrongly; cf. Tutorial Exercises 3.1 and 3.2. First, the attempt is not complete and thus does not prove any proposition. Second, the proposition was negated correctly to $\exists x \in \mathbb{Q}\ (x \geqslant 1 \wedge x^2 < 1)$. Some wrongly claimed that the negation is $\exists x \in \mathbb{Q}\ (x \geqslant 1 \to x^2 < 1)$ or $\exists x \in \mathbb{Q}\ (x < 1 \to x^2 < 1)$.

   (ii) Give a correct proof of this proposition. [2 marks]

   Solution. Suppose not. Then we get a rational number, say x, such that $x \geqslant 1$ but $x^2 < 1$. Since $x^2 < 1$, we know $(x-1)(x+1) = x^2 - 1 < 0$. This tells us $x - 1 < 0 < x + 1$ or $x + 1 < 0 < x - 1$. The former implies $x < 1$, which contradicts the hypothesis that $x \geqslant 1$. The latter implies $x + 1 < x - 1$ and thus $1 < -1$, which is not true. So we have a contradiction in all cases. □

   Alternative solution. Let x be a rational number such that $x \geqslant 1$. Then $x \cdot x \geqslant x \cdot 1$ as $x \geqslant 1 > 0$. So $x^2 \geqslant x \geqslant 1$ as $x \geqslant 1$. □

   Another alternative solution. Let x be a rational number such that $x \geqslant 1$. Define $y = x - 1$. On the one hand, we know $y = x - 1 \geqslant 1 - 1 = 0$ as $x \geqslant 1$. So $2y \geqslant 0$ too. On the other hand, we know $x = (x - 1) + 1 = y + 1$. Combining the two, we deduce that
   $$x^2 = (y+1)^2 = y^2 + 2y + 1 \geqslant 0 + 0 + 1 = 1. \qquad \square$$

   Yet another solution. Let x be a rational number such that $x \geqslant 1$. Then $x - 1 \geqslant 1 - 1 = 0$ and $x + 1 \geqslant 1 + 1 = 2 > 0$. So $x^2 - 1 = (x-1)(x+1) \geqslant 0$. This implies $x^2 \geqslant 1$. □

   Comments. Many failed to give a proof. For example, some deduced from $(x-1)(x+1) < 0$ directly $-1 < x < 1$, which is not justified by anything we have established in this course. Similarly, others tried to square both sides of $x \geqslant 1$ to get $x^2 \geqslant 1$. Some assumed that the proposition given is true.