Collaboration is encouraged; however, students must submit their own work.

Mathematics Department Kwantlen Polytechnic University

MATH 4280 Assignment #2 **Due on January 29**

Graph Theory & Applications Spring 2026

1. Assume that G is a graph with no 3-cycles (or triangles). It can be shown that $m \leq \lfloor \frac{n^2}{4} \rfloor$.

The followings are examples of such triangle free graphs and H achieves the stated upper bound.

Figure 1: Triangle-free graphs of order 7: G does not achieve the upper bound of $m = 12$ and H does

(a) In my examples, $n = 7$. For $n \in \{2, 3, 4, 5, 6\}$, find examples of triangle free graphs that achieve the upper bound stated on the number of edges. Try to find a common property that is shared with your graphs.

(b) For any arbitrary n, give an example of a graph with no 3–cycles that achieves the upper bound given on the number of edges.

2. A pair of non-increasing sequences $S_1 = \langle a_1, a_2, \ldots, a_r \rangle$ and $S_2 = \langle b_1, b_2, \ldots, b_s \rangle$ is said to be bi-graphical if there exists a bipartite graph G with biparticiton subsets $X = \{u_1, u_u, \ldots, u_r\}$ such that $\deg_G(u_i) = a_i$ for $1 \leq i \leq r$ and $Y = \{v_1, v_2, \ldots, v_s\}$ such that $\deg_G(v_j) = b_j$ for $1 \leq j \leq s$.

   It is shown that a pair of sequences S_1 and S_2 with $r \geq 2$, $0 < a_1 \leq s$ and $0 < b_1 \leq r$ is bigraphical if and only if the pair $\langle a_2, \ldots, a_r \rangle$ and $\langle b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_s \rangle$ is bigraphical.

   Use the stated proposition to determine whether or not each sequence is bi-graphical. If your answer is yes, present such a bipartite graph.

   (a) $S_1 = \langle 6, 5, 5, 5, 3, 2, 1, 1 \rangle$ and $S_2 = \langle 5, 5, 4, 3, 2 \rangle$

   (b) $S_1 = \langle 8, 6, 4, 4, 4, 4, 4 \rangle$ and $S_2 = \langle 6, 5, 4, 4, 4, 4, 3, 3, 1 \rangle$

3. (a) Draw two different graphs on 7 vertices such that the degree of each vertex is at least 3. Do your best to find a disconnected example, if possible. Could you find such a disconnected graph?

   (b) Repeat part (a) for graphs on 8 vertices such that the degree of each vertex is at least 4.

   (c) Now via your examples, try to make an observation on why finding disconnected graphs was impossible and use the idea to prove the following statement (or you can prove any way you want).

> A graph G with $\deg(v) \geq \dfrac{n-1}{2}$, $\forall v \in V(G)$, is connected.

   Hint for a possible approach: Pay attention to the distance between every pair of vertices in your graph examples above.

4. Determine whether or not the following pairs of graphs are isomorphic.

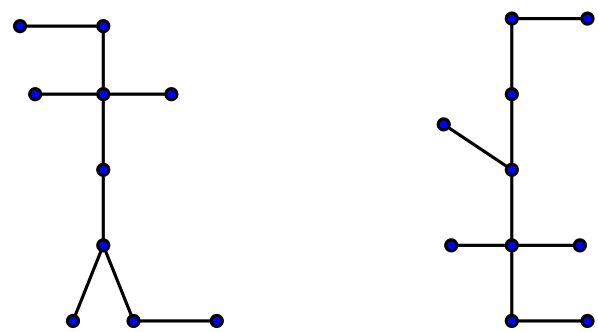

Figure 2:

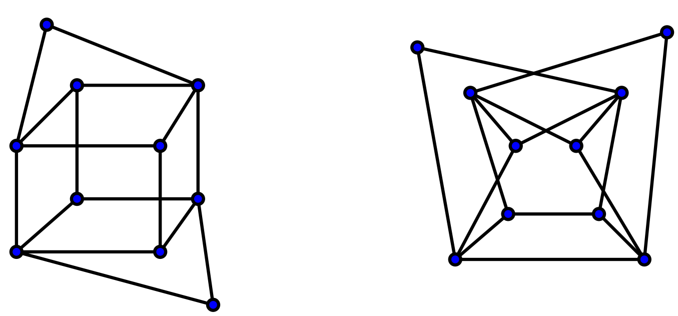

Figure 3:

5. (a) Find the complements of the paths P_i for $2 \leq i \leq 6$ and observe which ones are connected.

   (b) Formulate a conjecture for connectedness of complement of paths P_n based on your observations in part (a) and prove your conjecture.

   (c) Can a similar statement be formulated about trees in general?

6. There are several graph multiplication operations that produce new graphs from old. Given graphs G and H, one particular product, denote it as $*$, produces a graph $K = G * H$ with vertex set $V(G * H) = V(G) \times V(H)$ and edge set as follows:

$$E(G * H) = \Big\{ \big((u,a),(v,b)\big) \mid u = v \text{ and } ab \in E(H) \quad \text{or} \quad a = b \text{ and } uv \in E(G) \Big\}$$

(a) Draw the graphs $P_2 * K_3$ and $P3 * K_3$ and compare your drawings to the outputs from the notebook.

(b) Refer to the notebook and from the drop down menus find the product of graphs for all possible values of n_1 and n_2 and record the number of vertices and edges.

Check to see that number of edges follows the rule $m = n_1 m_2 + n_2 m_1$ where n_1 and m_1 are order and size of G and n_2 and m_2 are order and size of H. Think about why that is the case (a proof is not required).

7. (a) Find a bipartite graph so its complement is also bipartite.

   (a') Draw two bipartite graphs on at least five vertices and then draw their complements. Are the complements bipartite? You may use the notebook code to experiment with more examples.

   (c) Generalize your observations for bipartite graphs of proper orders and formulate it as a proposition. Prove your proposition.