Assignment 2

My solutions for the assignment. I used Python (NetworkX, matplotlib) for the graphs.

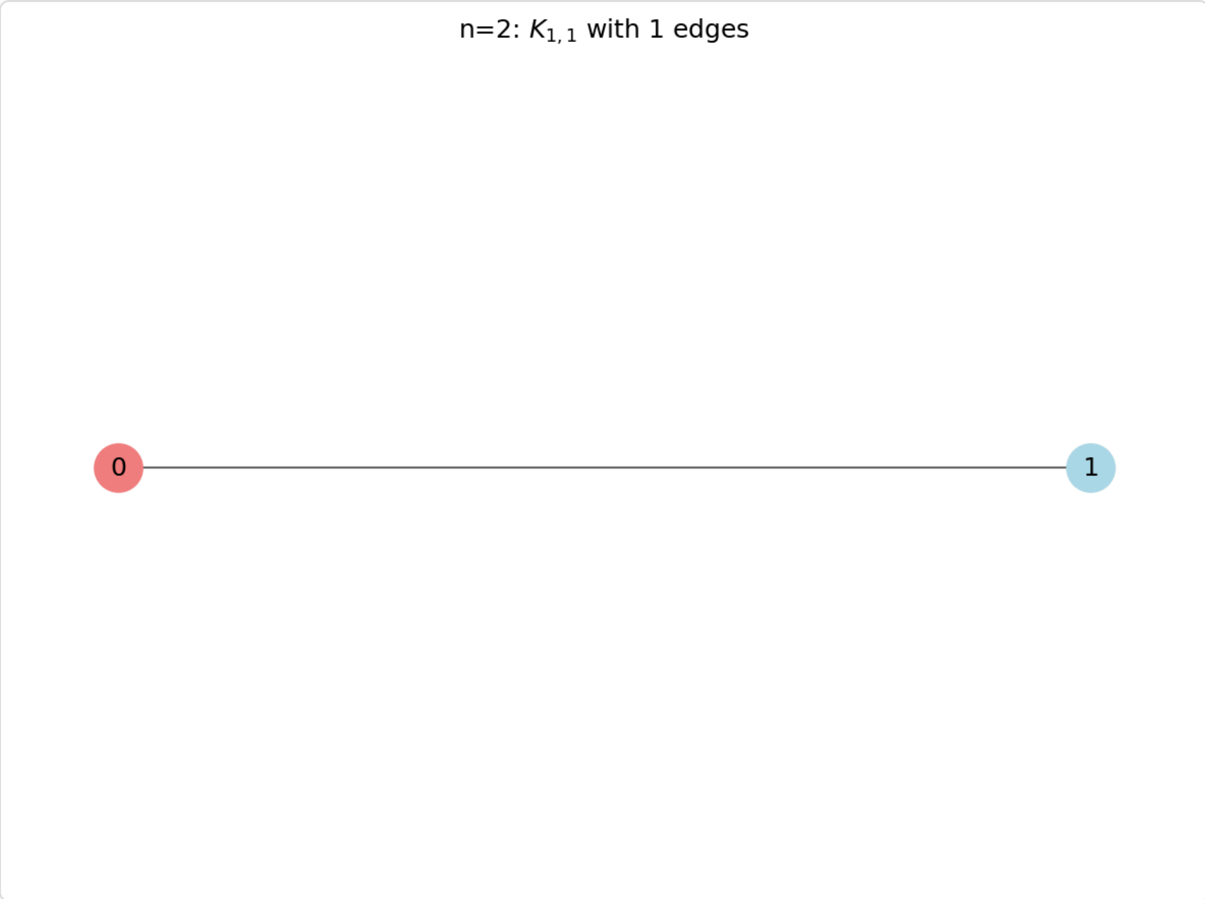
Question 1: Triangle-Free Graphs

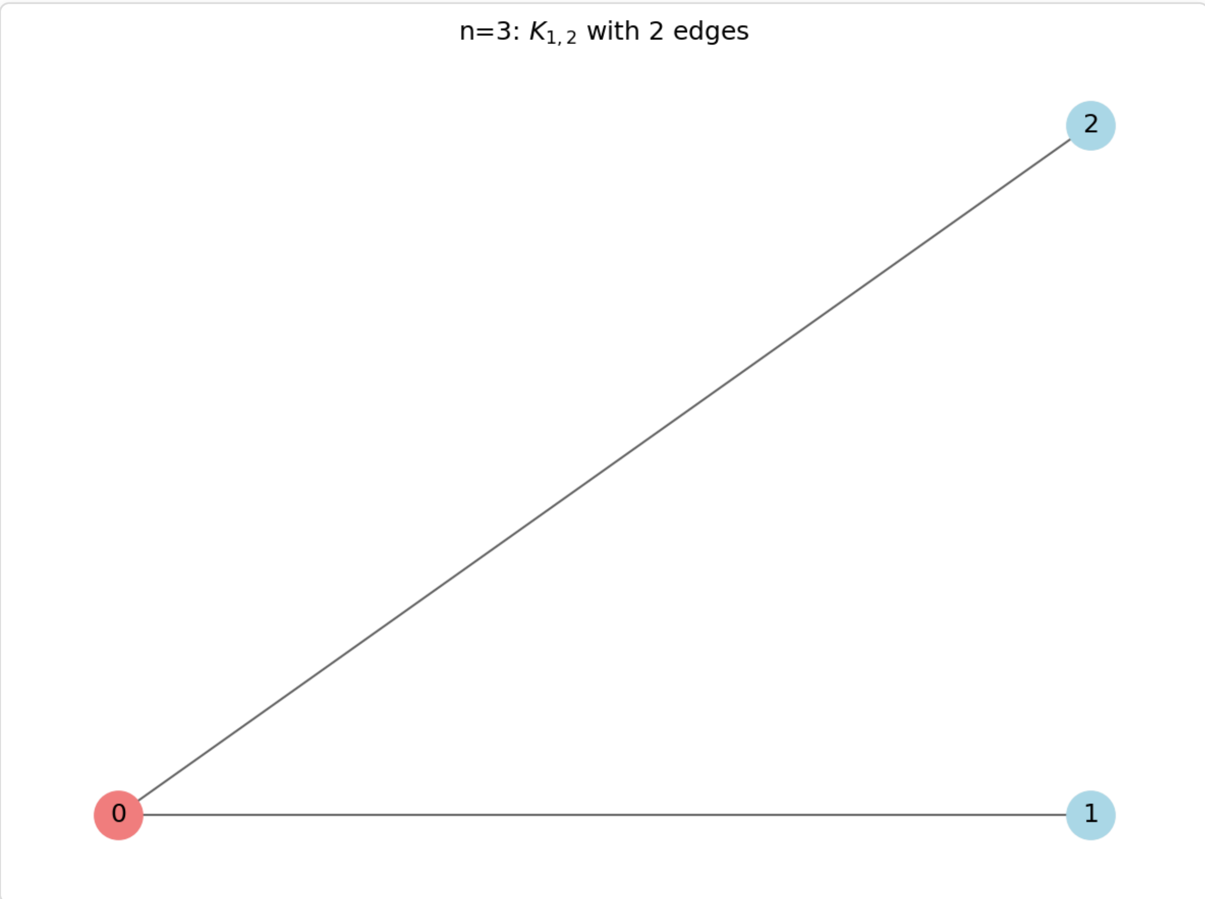
The bound is $m \leq \lfloor n^2/4 \rfloor$.

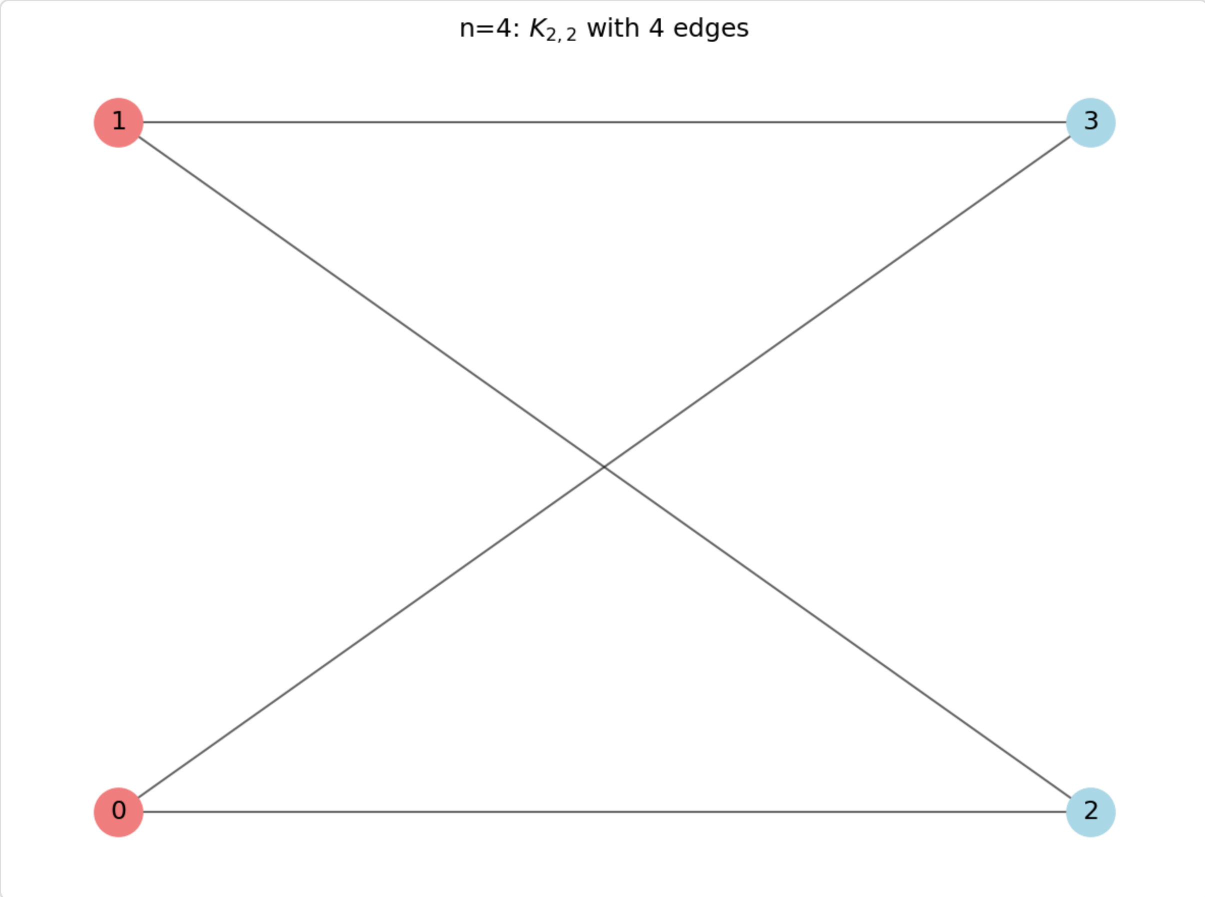
(a) Examples for $n = 2, 3, 4, 5, 6$

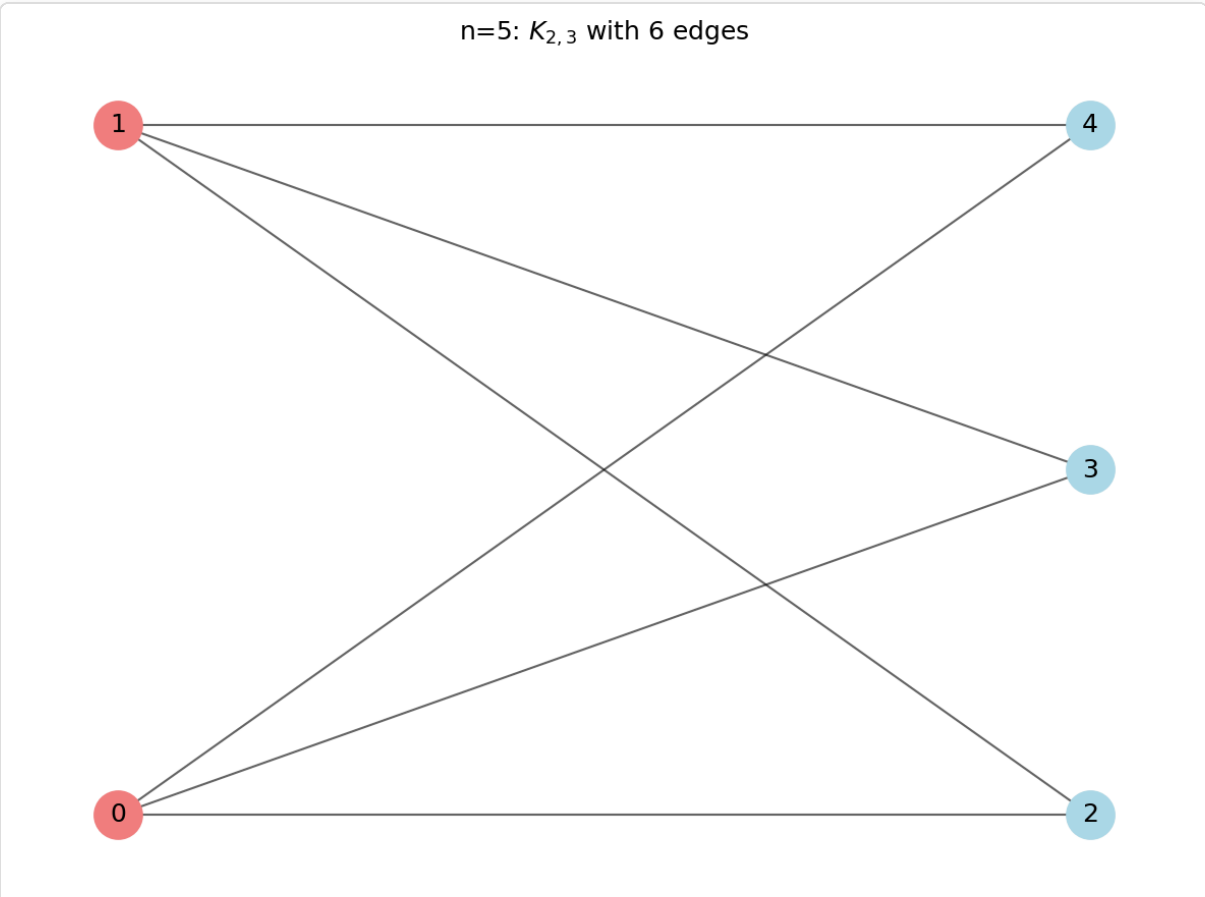
I used complete bipartite graphs $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ so the number of edges hits the bound. Below I draw them for each n.

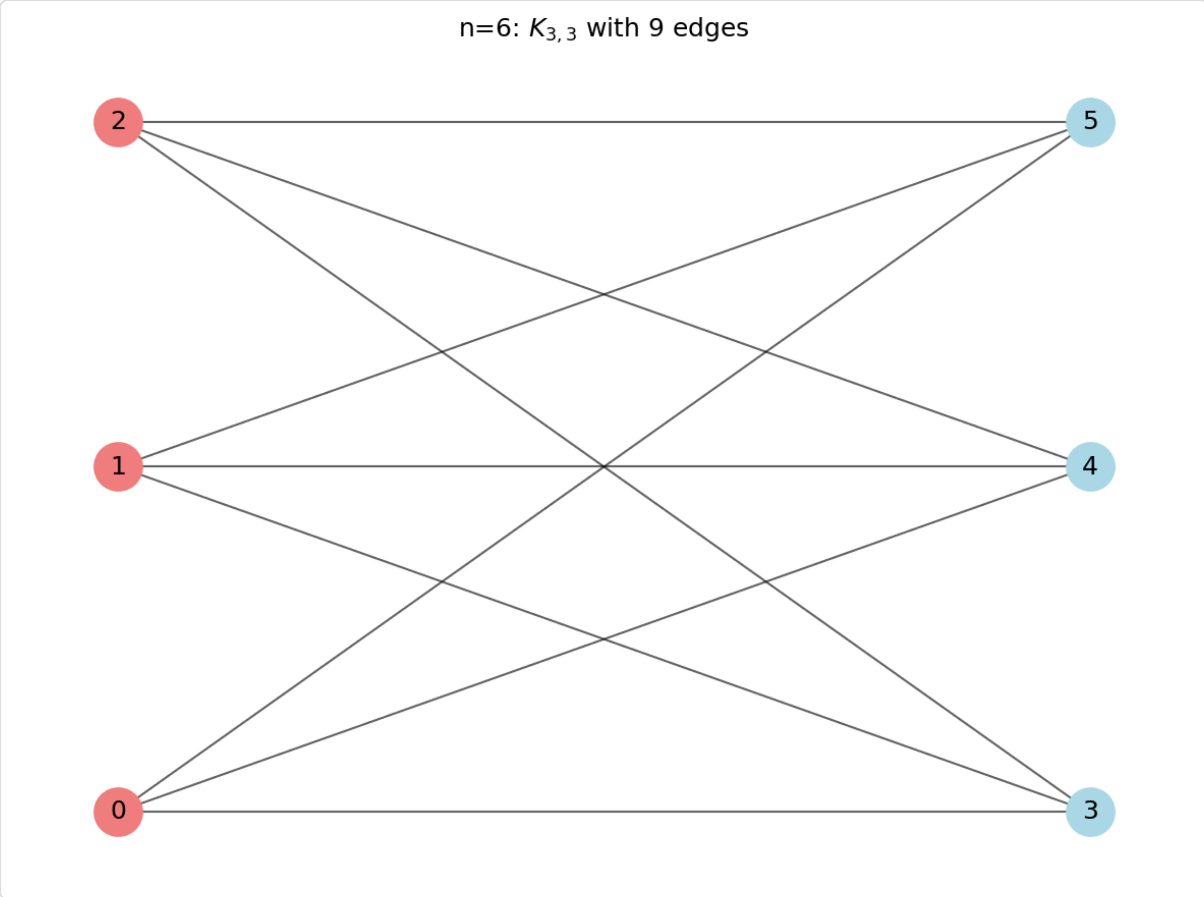
Common property I noticed: They are all complete bipartite with the two parts as equal as possible.


n=2: $K_{1,1}$ with 1 edges
0
1


n=3: $K_{1,2}$ with 2 edges
2
0
1


n=4: $K_{2,2}$ with 4 edges
1
3
0
2


n=5: $K_{2,3}$ with 6 edges
1
4
3
0
2



(b) General construction

For any n I take $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. It's bipartite so no triangles, and the edge count is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$, so it reaches the bound.

Question 2: Bi-graphical Sequences

(a) $S_1 = \langle 6, 5, 5, 5, 3, 2, 1, 1 \rangle$, $S_2 = \langle 5, 5, 4, 3, 2 \rangle$

Here $a_1 = 6$ but $|S_2| = 5$, so we need $a_1 \leq s$ and it fails. So **I get: not bi-graphical.**

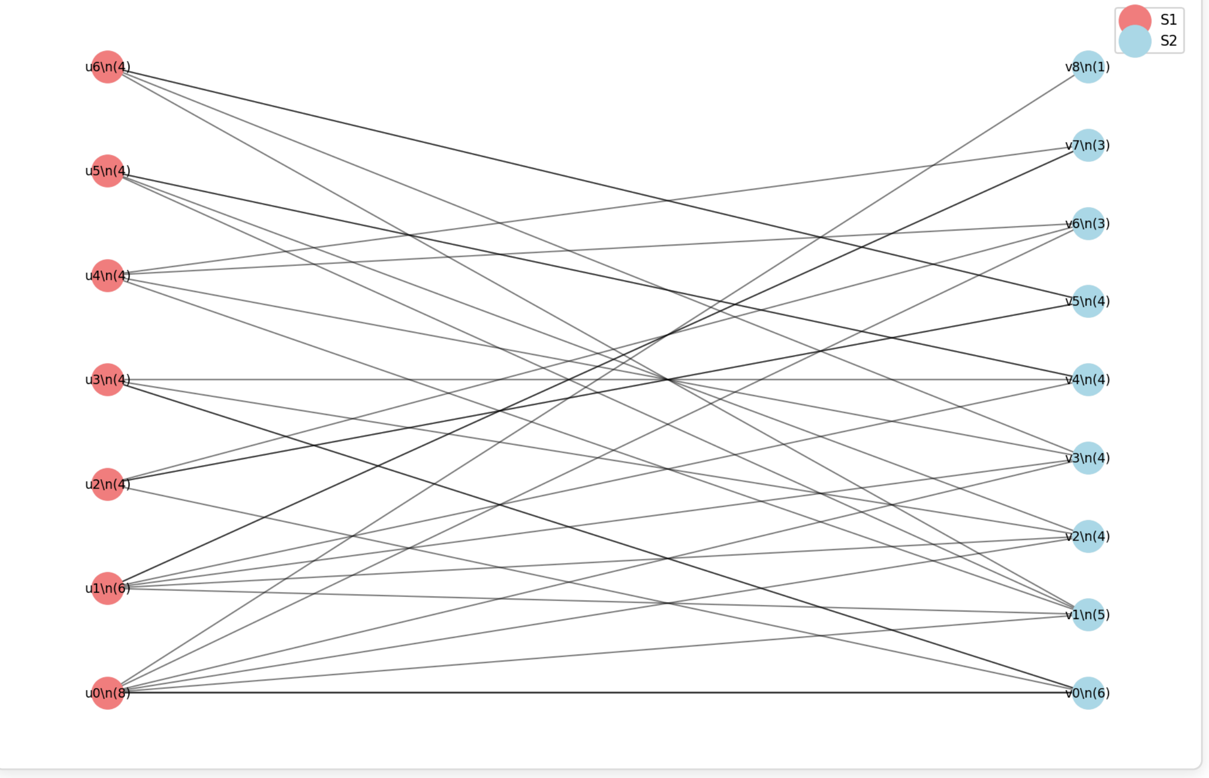
```
Part (a):
  sum(S1) = 28, sum(S2) = 19
  Sum mismatch: sum(S1)=28, sum(S2)=19
  Bi-graphical? False
  So my answer for (a) is: No, not bi-graphical.
```

(b) $S_1 = \langle 8, 6, 4, 4, 4, 4, 4 \rangle$, $S_2 = \langle 6, 5, 4, 4, 4, 4, 3, 3, 1 \rangle$

I applied the reduction step by step (code below) and it worked all the way. So **this pair is bi-graphical.** I also draw a bipartite graph realizing it.

```
Part (b):
  sum(S1) = 34, sum(S2) = 34
  Reduce: S1'=[6, 4, 4, 4, 4, 4], S2'=[5, 4, 3, 3, 3, 3, 2, 2, 1]
  Reduce: S1'=[4, 4, 4, 4, 4], S2'=[4, 3, 2, 2, 2, 2, 2, 2, 1]
  Reduce: S1'=[4, 4, 4, 4], S2'=[3, 2, 2, 2, 2, 2, 1, 1, 1]
  Reduce: S1'=[4, 4, 4], S2'=[2, 2, 2, 1, 1, 1, 1, 1, 1]
  Reduce: S1'=[4, 4], S2'=[1, 1, 1, 1, 1, 1, 1, 1, 0]
  Reduce: S1'=[4], S2'=[1, 1, 1, 1, 0, 0, 0, 0, 0]
  Bi-graphical? True
```

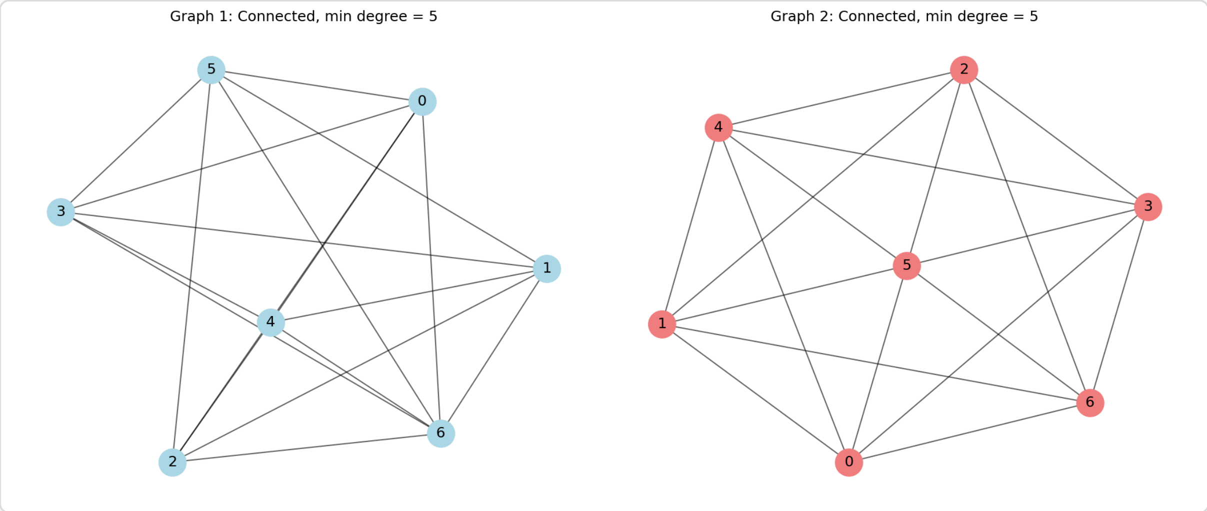
(b) A bipartite graph with the given degree sequences



Question 3: Connectivity

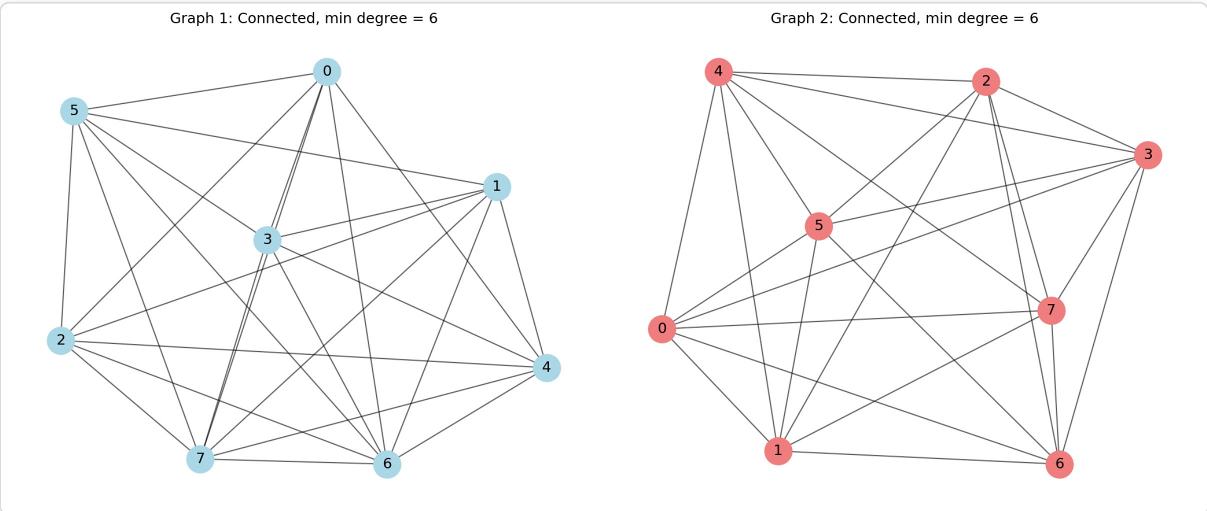
(a) Graphs on 7 vertices with $\deg(v) \geq 3$

I tried to draw two different graphs that are disconnected, but any component has to have at least 4 vertices (since each vertex has degree ≥ 3), so $4 + 4 = 8 > 7$ and it's impossible. So **I couldn't get a disconnected example**—they all end up connected. Below are two connected examples.



(b) Graphs on 8 vertices with $\deg(v) \geq 4$

Same idea: I tried to get a disconnected one but each part would need at least 5 vertices, so $5 + 5 = 10 > 8$. So **again they're all connected.** Two examples below.



(c) Proof

Theorem: If every vertex has $\deg(v) \geq \frac{n-1}{2}$, then G is connected.

Proof: Suppose G is disconnected and u, v are in different components. The component of u has at least $\deg(u) + 1 \geq \frac{n+1}{2}$ vertices, and same for v. So total $\geq \frac{n+1}{2} + \frac{n+1}{2} = n + 1 > n$, contradiction. So G must be connected. $\square$

Question 4: Graph Isomorphism

I compared the two graphs: both have 7 vertices, 6 edges, same degree sequence $\langle 3, 3, 2, 1, 1, 1, 1 \rangle$, and both are trees. So **I think they are isomorphic** (and the code confirms it).

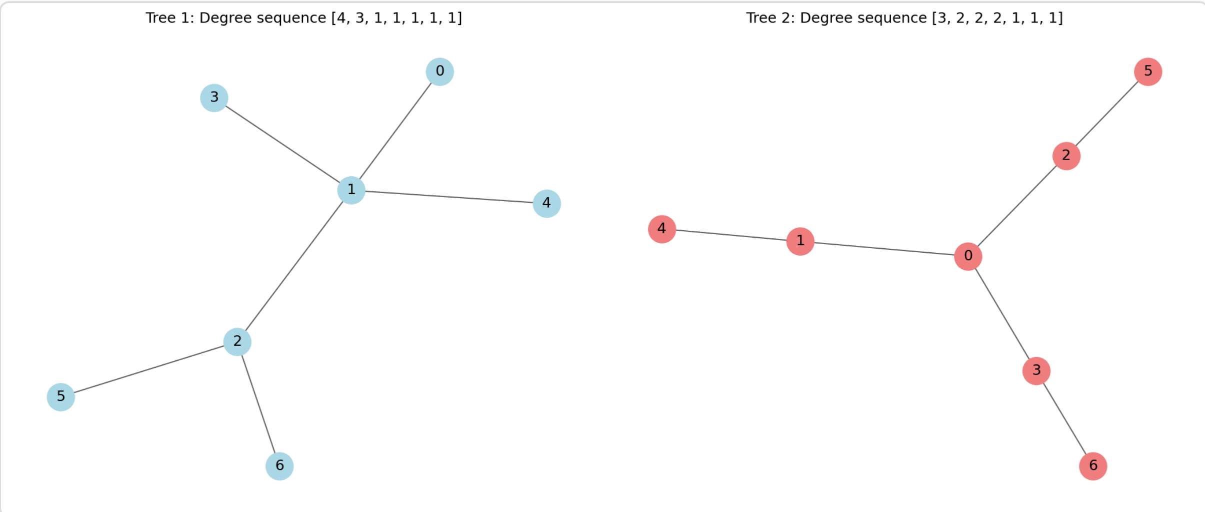


Figure 2

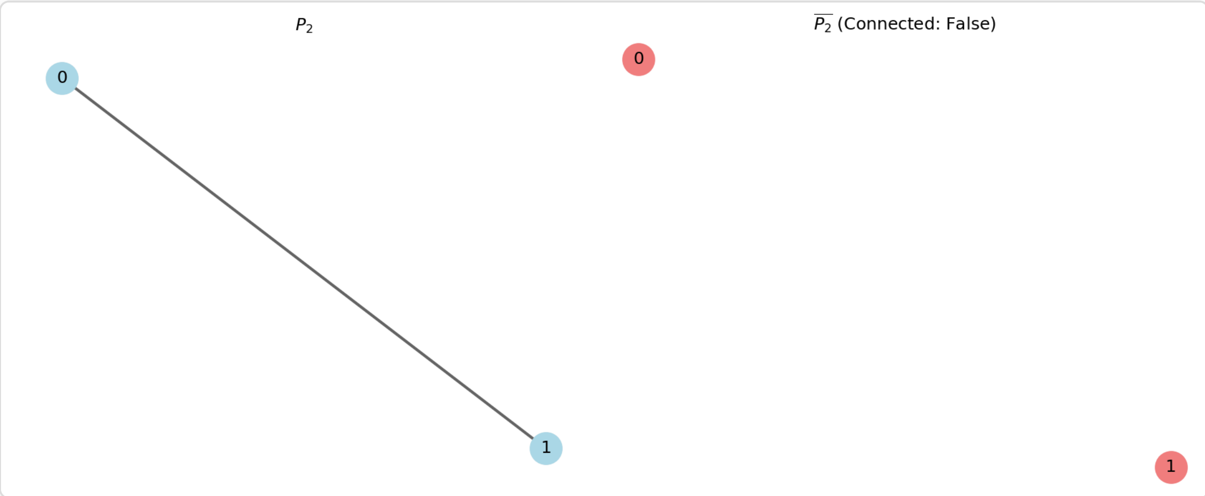
Left graph has degree sequence $\langle 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2 \rangle$, right has $\langle 5, 5, 5, 5, 3, 3, 3, 3, 2, 2, 2, 2 \rangle$. They're different, so **I conclude they are not isomorphic.**

Question 5: Complement of Paths

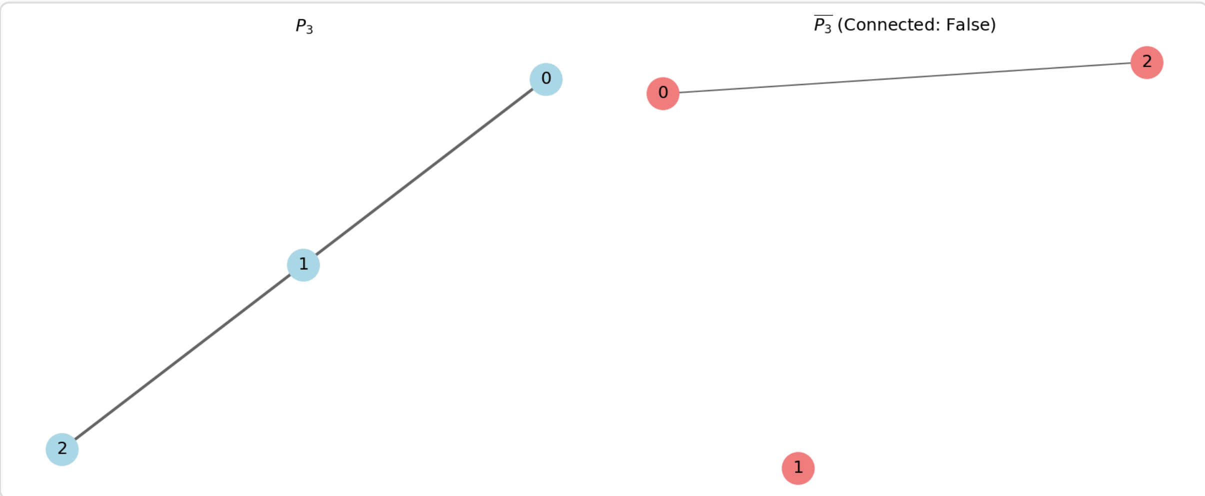
(a) Complements of P_n for $n = 2, 3, 4, 5, 6$

I drew each path and its complement and checked whether the complement is connected.

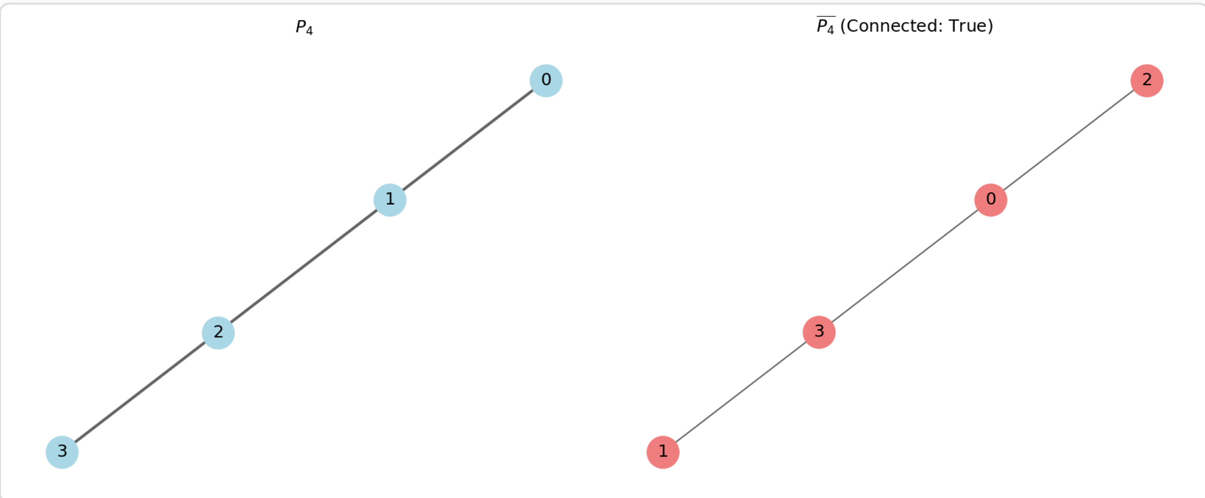
n = 2:


P_2
$\overline{P_2}$ (Connected: False)

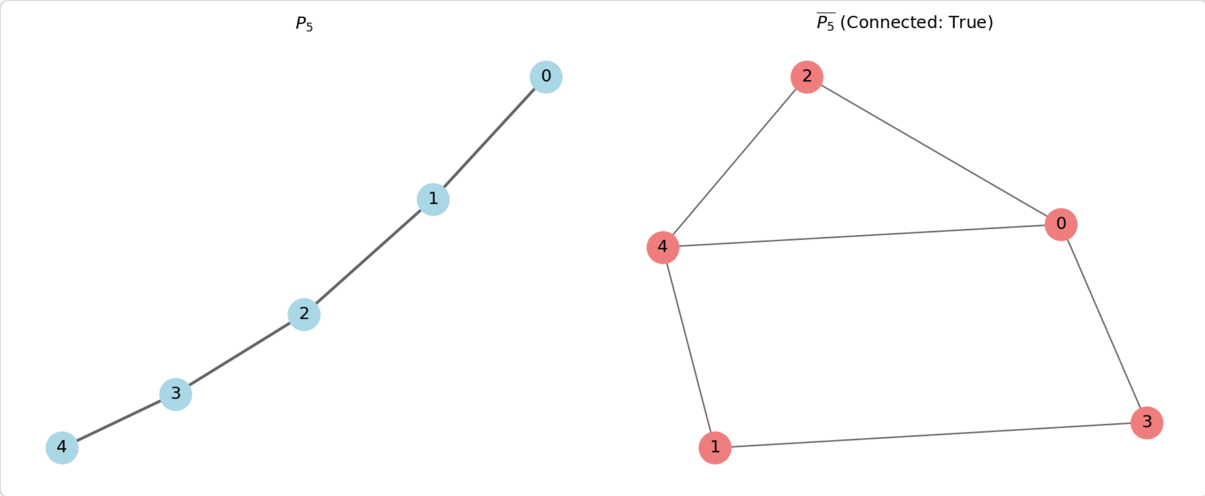
n = 3:


P_3
$\overline{P_3}$ (Connected: False)

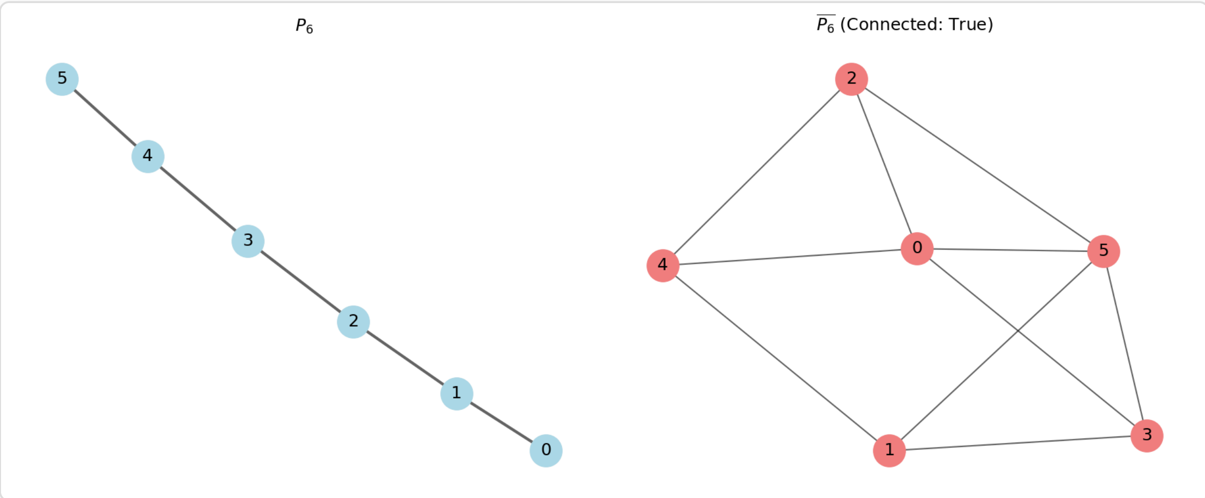
n = 4:


P_4
$\overline{P_4}$ (Connected: True)

n = 5:



n = 6:



What I observed: $\overline{P_2}$ and $\overline{P_3}$ are not connected; for $n \geq 4$ the complement is connected. So the pattern seems to be: $\overline{P_n}$ is connected iff $n \geq 4$.

(b) Conjecture and proof

Conjecture: $\overline{P_n}$ is connected if and only if $n \geq 4$.

Proof: For $n \geq 4$, in $\overline{P_n}$ any two vertices are either adjacent or share a neighbor, so the graph has diameter ≤ 2 and is connected. For $n = 2, 3$ we already saw the complement is disconnected. So the conjecture holds. $\square$

(c) Generalization to trees

I don't think the same statement holds for all trees—e.g. a star has a very different complement, so connectivity of the complement can behave differently.

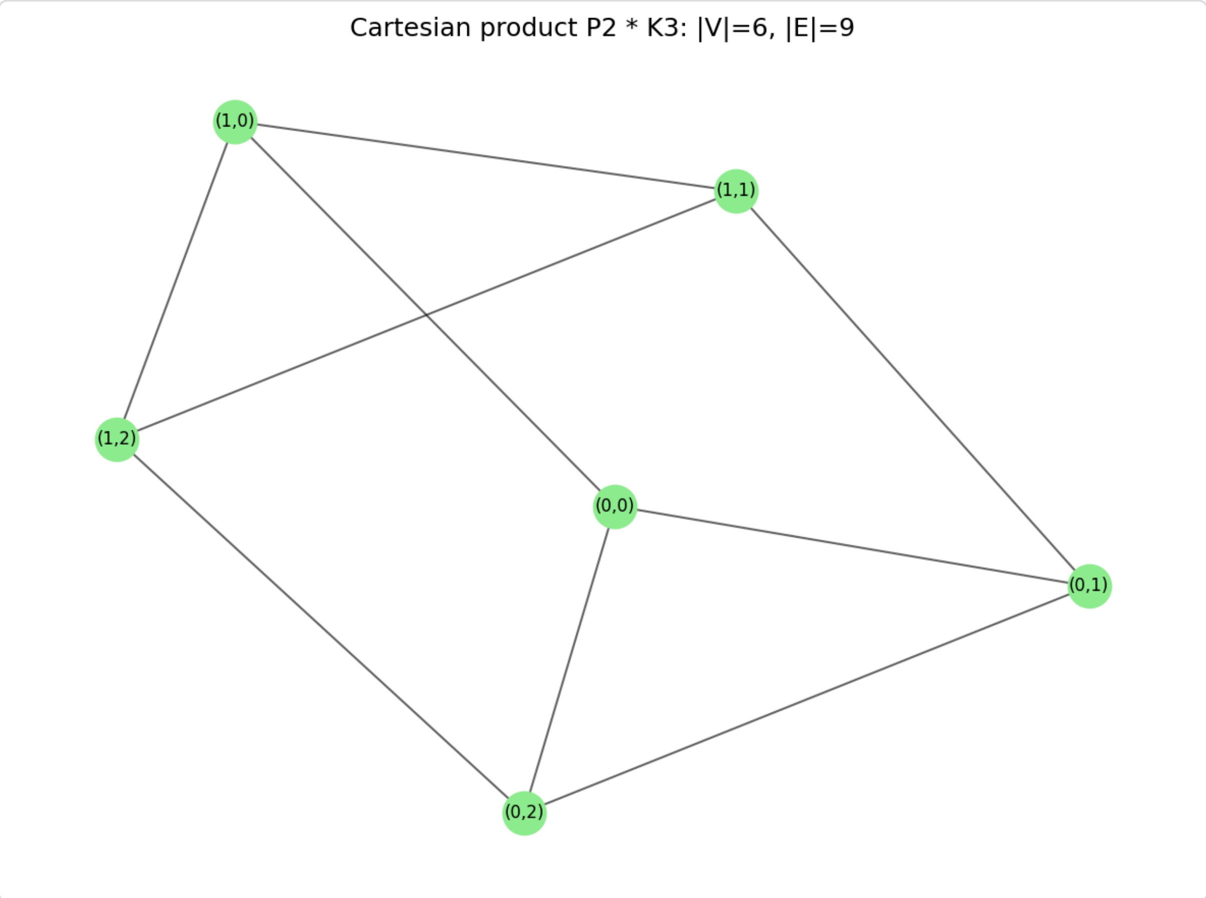
Question 6: Graph Multiplication (Cartesian Product)

I used the definition: vertices of $G * H$ are pairs (u, a); edges $((u, a), (v, b))$ when $u = v$ and $ab \in E(H)$, or $a = b$ and $uv \in E(G)$.

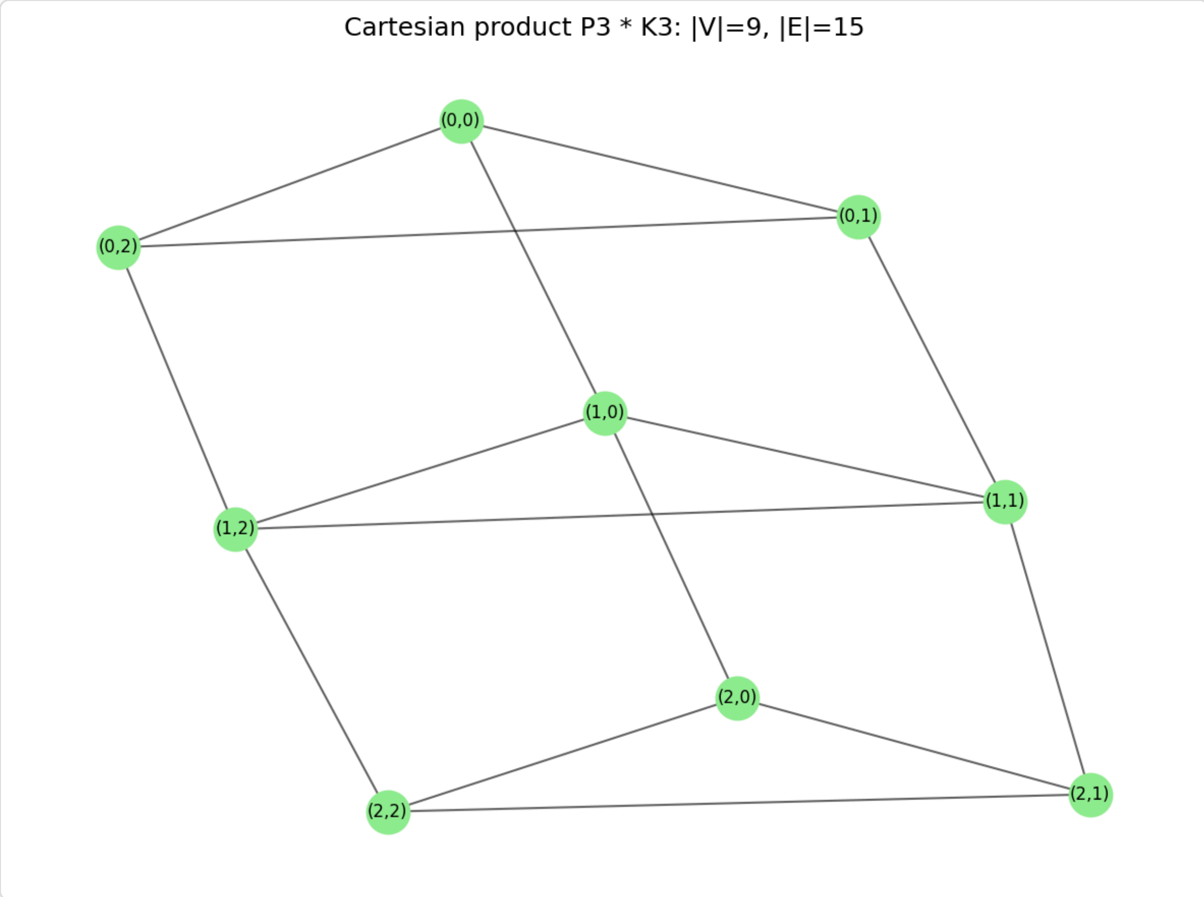
(a) Draw $P_2 * K_3$ and $P_3 * K_3$

I implemented the product and drew these two graphs below.

P2 * K3:



P3 * K3:



(b) Edge-count formula $m = n_1 m_2 + n_2 m_1$

I tried several pairs (G, H) and checked that the number of edges in $G * H$ always matches $n_1 m_2 + n_2 m_1$.

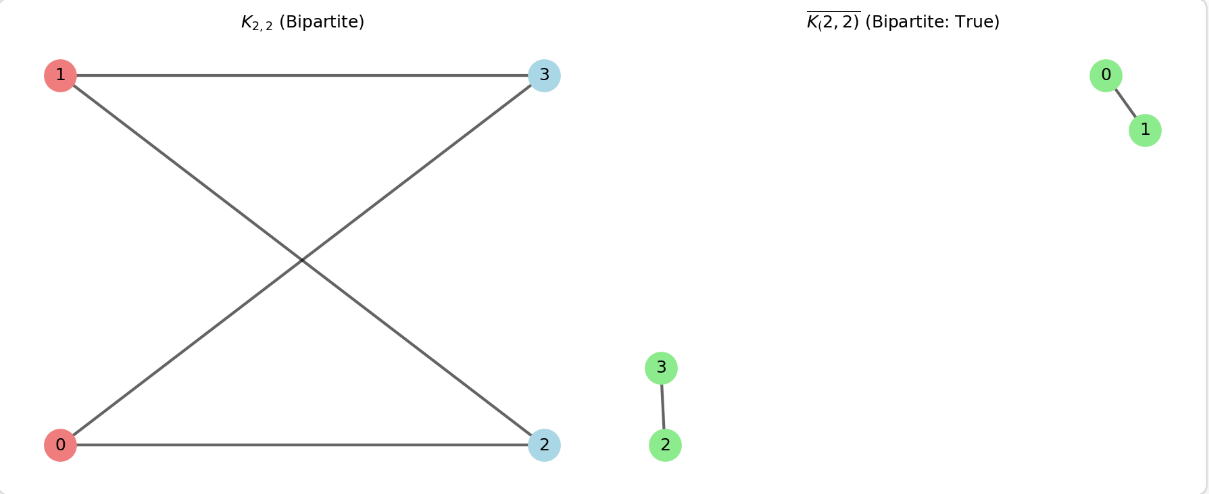
```
Example G1 * H1:
G: n1=2, m1=1; H: n2=3, m2=3
G * H: |V|=6, |E|=9
Formula n1*m2 + n2*m1 = 9
Matches formula? True

Example G1 * H2:
G: n1=2, m1=1; H: n2=4, m2=3
G * H: |V|=8, |E|=10
Formula n1*m2 + n2*m1 = 10
Matches formula? True

Example G2 * H1:
G: n1=3, m1=2; H: n2=3, m2=3
G * H: |V|=9, |E|=15
```

```
Formula n1*m2 + n2*m1 = 15
Matches formula? True

Example G2 * H2:
G: n1=3, m1=2; H: n2=4, m2=3
G * H: |V|=12, |E|=17
Formula n1*m2 + n2*m1 = 17
Matches formula? True

Example G3 * H1:
G: n1=4, m1=4; H: n2=3, m2=3
G * H: |V|=12, |E|=24
Formula n1*m2 + n2*m1 = 24
Matches formula? True

Example G3 * H2:
G: n1=4, m1=4; H: n2=4, m2=3
G * H: |V|=16, |E|=28
Formula n1*m2 + n2*m1 = 28
Matches formula? True
```

Question 7: Bipartite Graphs with Bipartite Complements

(a) Find a bipartite graph whose complement is also bipartite

I tried $K_{2,2}$: it's bipartite, and its complement is just two disjoint edges, so the complement is bipartite too. I drew both below.



(a') Experiment with more examples

I ran a few more bipartite graphs and checked whether their complements are bipartite.

```
K_{2,2}:
  Original bipartite: True
  Complement bipartite: True
  Original: 4 vertices, 4 edges
  Complement: 4 vertices, 2 edges

K_{3,3}:
  Original bipartite: True
  Complement bipartite: False
  Original: 6 vertices, 9 edges
  Complement: 6 vertices, 6 edges

Empty bipartite (3,3):
  Original bipartite: True
  Complement bipartite: False
  Original: 6 vertices, 9 edges
  Complement: 6 vertices, 6 edges
```

(c) Proposition and proof

Proposition: A bipartite graph G has a bipartite complement iff $G = K_{2,2}$ or G is empty (with $n \leq 2$).

Proof: ($\Rightarrow$) If $\overline{G}$ is bipartite it has no odd cycles. If G has a part of size ≥ 3, then in $\overline{G}$ that part gets edges inside it and we get a triangle, so $\overline{G}$ isn't bipartite. So both parts have size ≤ 2. For $K_{2,2}$, $\overline{G}$ is two disjoint edges (bipartite). For bigger $K_{m,n}$ the complement has cliques, so not bipartite.

($\Leftarrow$) $K_{2,2}$: complement is two disjoint edges. Empty graph: complement is K_n, bipartite only when $n \leq 2$. $\square$