3rd-exercise-2

8.

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi) \\
g &= \begin{pmatrix} 1 & & \\ & \sin^2\psi & \\ & & \sin^2\psi\sin^2\theta \end{pmatrix} \\
g^{-1} &= \begin{pmatrix} 1 & & \\ & \frac{1}{\sin^2\psi} & \\ & & \frac{1}{\sin^2\psi\sin^2\theta} \end{pmatrix}
\end{aligned}
$$

a.

$$
[m_n] = \begin{pmatrix} 0 & 0 & 0 \\ 2\sin\psi\cos\psi & 0 & 0 \\ 2\sin\psi\cos\psi\sin^2\theta & 2\sin^2\psi\sin\theta\cos\theta & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\Gamma^0_{11} &= -\frac{1}{2}g^{\psi\psi} \times 2\sin\psi\cos\psi = -\sin\psi\cos\psi \\
\Gamma^1_{10} &= \frac{1}{2}g^{\theta\theta} \times 2\sin\psi\cos\psi = \frac{1}{\tan\psi} \\
\Gamma^0_{22} &= -\frac{1}{2}g^{\psi\psi} \times 2\sin\psi\cos\psi\sin^2\theta = -\sin\psi\cos\psi\sin^2\theta \\
\Gamma^2_{20} &= \frac{1}{2}g^{\phi\phi} \times 2\sin\psi\cos\psi\sin^2\theta = \frac{1}{\tan\psi} \\
\Gamma^1_{22} &= -\frac{1}{2}g^{\theta\theta} \times 2\sin^2\psi\sin\theta\cos\theta = -\sin\theta\cos\theta \\
\Gamma^2_{21} &= \frac{1}{2}g^{\phi\phi} \times 2\sin^2\psi\sin\theta\cos\theta = \frac{1}{\tan\theta}
\end{aligned}
$$

b.

$$
\begin{aligned}
R^1{}_{212} &= \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12} \\
&= \partial_1(-\sin\theta\cos\theta) + \Gamma^1_{10}\Gamma^0_{22} - \Gamma^1_{22}\Gamma^2_{21} \\
&= -\cos^2\theta + \sin^2\theta + \frac{1}{\tan\psi} \times (-\sin\psi\cos\psi\sin^2\theta) + \sin\theta\cos\theta \times \frac{1}{\tan\theta} \\
&= \sin^2\theta - \cos^2\psi\sin^2\theta = \sin^2\psi\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R^2{}_{112} &= \partial_1\Gamma^2_{21} - \partial_2\Gamma^2_{11} + \Gamma^2_{1\lambda}\Gamma^\lambda_{21} - \Gamma^2_{2\lambda}\Gamma^\lambda_{11} \\
&= \partial_1\frac{1}{\tan\theta} + \Gamma^2_{21}\Gamma^2_{21} - \Gamma^2_{20}\Gamma^0_{11} \\
&= \frac{-1}{\sin^2\theta} + \frac{1}{\tan\theta} \times \frac{1}{\tan\theta} + \frac{1}{\tan\psi} \times \sin\psi\cos\psi \\
&= -1 + \cos^2\psi = -\sin^2\psi
\end{aligned}
$$

$$
\begin{aligned}
R^0{}_{202} &= \partial_0\Gamma^0_{22} - \partial_2\Gamma^0_{02} + \Gamma^0_{0\lambda}\Gamma^\lambda_{22} - \Gamma^0_{2\lambda}\Gamma^\lambda_{02} \\
&= \partial_0(-\sin\psi\cos\psi\sin^2\theta) - \Gamma^0_{22}\Gamma^2_{20} \\
&= -\sin^2\theta(\cos^2\psi - \sin^2\psi) + \sin\psi\cos\psi\sin^2\theta \times \frac{1}{\tan\psi}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sin^2\theta\cos^2\psi + \sin^2\theta\sin^2\psi + \cos^2\psi\sin^2\theta \\
&= \sin^2\theta\cos^2\psi
\end{aligned}
$$

$$
\begin{aligned}
R^0{}_{101} &= \partial_0\Gamma^0_{11} - \partial_1\Gamma^0_{01} + \Gamma^0_{0\lambda}\Gamma^\lambda_{11} - \Gamma^0_{1\lambda}\Gamma^\lambda_{01} \\
&= \partial_0(-\sin\psi\cos\psi) + \Gamma^0_{11}\Gamma^1_{10} \\
&= -\cos^2\psi + \sin^2\psi - \sin\psi\cos\psi \times \frac{1}{\tan\psi} \\
&= \sin^2\psi
\end{aligned}
$$

$$
\begin{aligned}
R^1{}_{001} &= \partial_0\Gamma^1_{10} - \partial_1\Gamma^1_{00} + \Gamma^1_{0\lambda}\Gamma^\lambda_{10} - \Gamma^1_{1\lambda}\Gamma^\lambda_{00} \\
&= \partial_0\frac{1}{\tan\psi} + \Gamma^1_{10}\Gamma^1_{10} \\
&= \frac{-1}{\sin^2\psi} + \frac{\cos^2\psi}{\sin^2\psi} = -1
\end{aligned}
$$

$$
\begin{aligned}
R^2{}_{002} &= \partial_0\Gamma^2_{20} - \partial_2\Gamma^2_{00} + \Gamma^2_{0\lambda}\Gamma^\lambda_{20} - \Gamma^2_{2\lambda}\Gamma^\lambda_{00} \\
&= \partial_0\frac{1}{\tan\psi} + \Gamma^2_{20}\Gamma^2_{20} \\
&= \frac{-1}{\sin^2\psi} + \frac{\cos^2\psi}{\sin^2\psi} = -1
\end{aligned}
$$

$$
\begin{aligned}
R^0{}_{101} &= \sin^2\psi & R^1{}_{001} &= -1 \\
R^0{}_{202} &= \sin^2\psi\sin^2\theta & R^2{}_{002} &= -1 \\
R^1{}_{212} &= \sin^2\psi\sin^2\theta & R^2{}_{112} &= -\sin^2\psi
\end{aligned}
$$

$$
\begin{aligned}
R_{00} &= R^1{}_{010} + R^2{}_{020} = 1 + 1 = 2 \\
R_{11} &= R^0{}_{101} + R^2{}_{121} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{22} &= R^0{}_{202} + R^1{}_{212} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R &= g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} \\
&= 2 + \frac{1}{\sin^2\psi} \times 2\sin^2\psi + \frac{1}{\sin^2\psi\sin^2\theta} \times 2\sin^2\psi\sin^2\theta \\
&= 6
\end{aligned}
$$

c.

$$
\begin{aligned}
R_{\rho\sigma\mu\nu} &= \frac{6}{3(3-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \\
&= g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}
\end{aligned}
$$

$$
\begin{aligned}
R_{0101} &= g_{0\lambda}R^\lambda{}_{101} = R^0{}_{101} = \sin^2\psi \\
R_{0102} &= g_{0\lambda}R^\lambda{}_{102} = R^0{}_{102} = 0 \\
R_{0112} &= g_{0\lambda}R^\lambda{}_{112} = 0 \\
R_{0202} &= g_{0\lambda}R^\lambda{}_{202} = R^0{}_{202} = \sin^2\psi\sin^2\theta \\
R_{0212} &= g_{0\lambda}R^\lambda{}_{212} = 0 \\
R_{1212} &= g_{1\lambda}R^\lambda{}_{212} = \sin^2\psi \times \sin^2\psi\sin^2\theta = \sin^4\psi\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R_{0101} &= g_{00}g_{11} - g_{01}g_{10} = \sin^2\psi \\
R_{0102} &= g_{00}g_{12} - g_{02}g_{10} = 0
\end{aligned}
$$

$$
\begin{aligned}
R_{0112} &= g_{01}g_{12} - g_{02}g_{11} = 0 \\
R_{0202} &= g_{00}g_{22} - g_{02}g_{20} = \sin^2\psi\sin^2\theta \\
R_{0212} &= g_{01}g_{22} - g_{02}g_{21} = 0 \\
R_{1212} &= g_{11}g_{22} - g_{12}g_{21} = \sin^4\psi\sin^2\theta
\end{aligned}
$$

therefore it's maximally symmetric space

11.

$$
\mathrm{d}s^2 = f^2(r)\mathrm{d}r^2 + r^2 f^2(r)\mathrm{d}\theta^2
$$

$$
[m_n] = \begin{pmatrix} 2ff' & 0 \\ 2r^2 ff' + 2rf^2 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2f^2} \times 2ff' = \frac{f'}{f} \\
\Gamma^1_{22} &= -\frac{1}{2f^2} \times (2r^2 ff' + 2rf^2) = -\left(\frac{r^2 f'}{f} + r\right) = -r^2 h \\
\Gamma^2_{21} &= \frac{1}{2r^2 f^2} \times (2r^2 ff' + 2rf^2) = \frac{f'}{f} + \frac{1}{r} = h
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &= R^2{}_{121} = \partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{21} + \Gamma^2_{2\lambda}\Gamma^\lambda_{11} - \Gamma^2_{1\lambda}\Gamma^\lambda_{21} \\
&= -\partial_1\Gamma^2_{21} + \Gamma^2_{21}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^2_{21} \\
&= -\partial_1\left(\frac{f'}{f} + \frac{1}{r}\right) + \left(\frac{f'}{f} + \frac{1}{r}\right)\frac{f'}{f} - \left(\frac{f'}{f} + \frac{1}{r}\right)^2 \\
R_{22} &= R^1{}_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12} \\
&= \partial_1\left(-\left(\frac{r^2 f'}{f} + r\right)\right) - \left(\frac{r^2 f'}{f} + r\right)\frac{f'}{f} + \left(\frac{r^2 f'}{f} + r\right)\left(\frac{f'}{f} + \frac{1}{r}\right) \\
&= -\partial_1 r^2\left(\frac{f'}{f} + \frac{1}{r}\right) - r^2\left(\frac{f'}{f} + \frac{1}{r}\right)\frac{f'}{f} + r^2\left(\frac{f'}{f} + \frac{1}{r}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
R_{11} = R^2{}_{121} &= \partial_2\Gamma^2_{11} - \partial_1\Gamma^2_{21} + \Gamma^2_{2\lambda}\Gamma^\lambda_{11} - \Gamma^2_{1\lambda}\Gamma^\lambda_{21} \\
&= -\partial_r h + \Gamma^2_{21}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^2_{21} \\
&= -h' + h\frac{f'}{f} - h^2 \\
R_{22} = R^1{}_{212} &= \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12} \\
&= \partial_r(-r^2 h) + \Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{21} \\
&= -\partial_r(r^2 h) - r^2 h\frac{f'}{f} + r^2 h^2
\end{aligned}
$$

$$
\begin{aligned}
R &= g^{11}R_{11} + g^{22}R_{22} \\
&= \frac{1}{f^2}\left(-h' + h\frac{f'}{f} - h^2\right) + \frac{1}{r^2 f^2}\left(-\partial_r(r^2 h) - r^2 h\frac{f'}{f} + r^2 h^2\right) \\
&= -\frac{h'}{f^2} - \frac{1}{r^2 f^2}\partial_r(r^2 h) \\
&= -\frac{h'}{f^2} - \frac{1}{r^2 f^2}(r^2 h' + 2rh) \\
&= -\frac{2h'}{f^2} - \frac{2rh}{r^2 f^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2}{rf^2}(rh'+h) \\
&= -\frac{2}{rf^2}\left(r\left(\frac{f'}{f}+\frac{1}{r}\right)'+\left(\frac{f'}{f}+\frac{1}{r}\right)\right) \\
&= -\frac{2}{rf^2}\left(r\left(\frac{f''}{f}-\frac{f'^2}{f^2}-\frac{1}{r^2}\right)+\frac{f'}{f}+\frac{1}{r}\right) \\
&= -\frac{2}{rf^2}\left(r\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)+\frac{f'}{f}\right)
\end{aligned}
$$

the differential equation

$$
\begin{aligned}
R &= -\frac{2}{a^2} \\
-\frac{2}{rf^2}\left(r\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)+\frac{f'}{f}\right) &= -\frac{2}{a^2} \\
\frac{1}{rf^2}\left(r\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)+\frac{f'}{f}\right) &= \frac{1}{a^2} \\
\frac{1}{rf^2}\left(r\left(\frac{f'}{f}\right)'+\frac{f'}{f}\right) &= \frac{1}{a^2}
\end{aligned}
$$

I'm not successed in solving the equation. orz

I got the answer form Mobius transformation
Refence: The road to reality[Penrose]chinese version P101

$$
\begin{aligned}
x+iy &= \frac{z-1}{iz+i}=\frac{u+iv-1}{i(u+iv)+i}=\frac{u-1+iv}{-v+(u+1)i} \\
&= \frac{(u-1+iv)(v+(u+1)i)}{(-v+(u+1)i)(v+(u+1)i)} \\
&= \frac{(u-1)v-v(u+1)+i(v^2+u^2-1)}{-v^2-(u+1)^2} \\
&= \frac{2v+i(1-v^2-u^2)}{v^2+(u+1)^2} \\
\Rightarrow & \\
x &= \frac{2v}{v^2+(u+1)^2} \\
y &= \frac{1-v^2-u^2}{v^2+(u+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial u} &= -\frac{4v(u+1)}{(v^2+(u+1)^2)^2} \\
\frac{\partial x}{\partial v} &= \frac{2}{v^2+(u+1)^2}-\frac{2v\times 2v}{(v^2+(u+1)^2)^2} \\
&= \frac{2(v^2+(u+1)^2)-4v^2}{(v^2+(u+1)^2)^2}=\frac{2((u+1)^2-v^2)}{(v^2+(u+1)^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial y}{\partial u} &= \frac{-2u}{v^2+(u+1)^2}-\frac{(1-v^2-u^2)\times 2(u+1)}{(v^2+(u+1)^2)^2} \\
&= \frac{-2u(v^2+(u+1)^2)-2(u+1)(1-v^2-u^2)}{(v^2+(u+1)^2)^2} \\
&= \frac{2(-uv^2-u(u+1)^2-(u+1)+u^2(u+1)+v^2(u+1))}{(v^2+(u+1)^2)^2} \\
&= \frac{2(v^2-(u+1)(u(u+1)+1-u^2))}{(v^2+(u+1)^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2(\upsilon^2-(u+1)^2)}{(\upsilon^2+(u+1)^2)^2}\\
\frac{\partial y}{\partial \upsilon} &= \frac{-2\upsilon}{\upsilon^2+(u+1)^2}-\frac{(1-\upsilon^2-u^2)\times 2\upsilon}{(\upsilon^2+(u+1)^2)^2}\\
&= \frac{2(-\upsilon(\upsilon^2+(u+1)^2)-(\upsilon-\upsilon^3-u^2\upsilon))}{(\upsilon^2+(u+1)^2)^2}\\
&= \frac{2(-\upsilon(u+1)^2-\upsilon+u^2\upsilon)}{(\upsilon^2+(u+1)^2)^2}\\
&= \frac{2(-\upsilon(u^2+2u+1)-\upsilon+u^2\upsilon)}{(\upsilon^2+(u+1)^2)^2}=\frac{2(-2u\upsilon-2\upsilon)}{(\upsilon^2+(u+1)^2)^2}\\
&= \frac{-4\upsilon(u+1)}{(\upsilon^2+(u+1)^2)^2}
\end{aligned}
$$

set $\rho=\upsilon^2+(u+1)^2$ $\frac{\partial(x,y)}{\partial(u,\upsilon)}=\begin{pmatrix} \frac{c}{\rho^2} & \frac{b}{\rho^2}\\ -\frac{b}{\rho^2} & \frac{c}{\rho^2}\end{pmatrix}$

put into the metric $\mathrm{d}s^2 = \frac{a^2}{y^2}\mathrm{d}x^2+\frac{a^2}{y^2}\mathrm{d}y^2$

$b^2+c^2=(4\upsilon(u+1))^2+(2(\upsilon^2-(u+1)^2))^2=4(4\upsilon^2(u+1)^2+(\upsilon^2-(u+1)^2)^2)=4(\upsilon^2+(u+1)^2)^2$

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{a^2}{\left(\frac{1-\upsilon^2-u^2}{\upsilon^2+(u+1)^2}\right)^2}((\partial_u x\mathrm{d}u+\partial_\upsilon x\mathrm{d}\upsilon)^2+(\partial_u y\mathrm{d}u+\partial_\upsilon y\mathrm{d}\upsilon)^2)\\
&= \frac{a^2\rho^2}{(1-\upsilon^2-u^2)^2}\left(\left(\frac{c}{\rho^2}\mathrm{d}u+\frac{b}{\rho^2}\mathrm{d}\upsilon\right)^2+\left(-\frac{b}{\rho^2}\mathrm{d}u+\frac{c}{\rho^2}\mathrm{d}\upsilon\right)^2\right)\\
&= \frac{a^2}{\rho^2(1-\upsilon^2-u^2)^2}(c^2\mathrm{d}u^2+b^2\mathrm{d}\upsilon+2cb\mathrm{d}\upsilon\mathrm{d}u+b^2\mathrm{d}u+c^2\mathrm{d}\upsilon-2bc\mathrm{d}u\mathrm{d}\upsilon)\\
&= \frac{a^2}{\rho^2(1-\upsilon^2-u^2)^2}((c^2+b^2)\mathrm{d}u^2+(c^2+b^2)\mathrm{d}\upsilon^2)\\
&= \frac{a^2}{\rho^2(1-\upsilon^2-u^2)^2}(4\rho^2\mathrm{d}u^2+4\rho^2\mathrm{d}\upsilon^2)\\
&= \frac{4a^2}{(1-r^2)^2}(\mathrm{d}u^2+\mathrm{d}\upsilon^2)\\
&= \frac{4a^2}{(1-r^2)^2}(\mathrm{d}r^2+r^2\mathrm{d}\theta^2)
\end{aligned}
$$

check the diffrential equation: solution $f(r)=\frac{2a}{1-r^2}$ $f'=-\frac{2a}{(1-r^2)^2}\times(-2r)=\frac{4ar}{(1-r^2)^2}$

$$
\begin{aligned}
\frac{1}{rf^2}\left(r\left(\frac{f'}{f}\right)'+\frac{f'}{f}\right) &= \frac{1}{r\left(\frac{2a}{1-r^2}\right)^2}\left(r\left(\frac{\frac{4ar}{(1-r^2)^2}}{\frac{2a}{1-r^2}}\right)+\frac{\frac{4ar}{(1-r^2)^2}}{\frac{2a}{1-r^2}}\right)\\
&= \frac{(1-r^2)^2}{4a^2r}\left(r\left(\frac{2r}{1-r^2}\right)'+\frac{2r}{1-r^2}\right)\\
&= \frac{(1-r^2)^2}{4a^2r}\left(r\left(\frac{2(1-r^2)-2r\times(-2r)}{(1-r^2)^2}\right)+\frac{2r}{1-r^2}\right)\\
&= \frac{(1-r^2)^2}{4a^2r}\left(r\left(\frac{2-2r^2+4r^2}{(1-r^2)^2}\right)+\frac{2r}{1-r^2}\right)\\
&= \frac{(1-r^2)^2}{4a^2}\left(\frac{2+2r^2}{(1-r^2)^2}+\frac{2(1-r^2)}{(1-r^2)^2}\right)\\
&= \frac{(1-r^2)^2}{4a^2}\left(\frac{4}{(1-r^2)^2}\right)
\end{aligned}
$$

$$= \frac{1}{a^2}$$

therefore when require $y>0 \Rightarrow \ y \ = \ \frac{1-v^2-u^2}{v^2+(u+1)^2} > 0$ then $1-v^2-u^2>0 \Leftrightarrow r^2=u^2+v^2<1$

namely the range of r is $(0,1)$

13.

killing equation upper indice version

$$(g_{\sigma\mu}\partial_\nu + g_{\sigma\nu}\partial_\mu)K^\sigma + (g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda} + g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda})K^\lambda \ = \ 0$$

a.

$$(\eta_{\sigma\mu}\partial_\nu + \eta_{\sigma\nu}\partial_\mu)K^\sigma \ = \ 0$$

$$\left\{\begin{array}{rcl} 2\eta_{\sigma t}\partial_t K^\sigma &=& 0 \\ (\eta_{\sigma t}\partial_x + \eta_{\sigma x}\partial_t)K^\sigma &=& 0 \\ (\eta_{\sigma t}\partial_y + \eta_{\sigma y}\partial_t)K^\sigma &=& 0 \\ (\eta_{\sigma t}\partial_z + \eta_{\sigma z}\partial_t)K^\sigma &=& 0 \\ 2\eta_{\sigma x}\partial_x K^\sigma &=& 0 \\ (\eta_{\sigma y}\partial_x + \eta_{\sigma x}\partial_y)K^\sigma &=& 0 \\ (\eta_{\sigma x}\partial_z + \eta_{\sigma z}\partial_x)K^\sigma &=& 0 \\ 2\eta_{\sigma y}\partial_y K^\sigma &=& 0 \\ (\eta_{\sigma y}\partial_z + \eta_{\sigma z}\partial_y)K^\sigma &=& 0 \\ 2\eta_{\sigma z}\partial_z K^\sigma &=& 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} -\partial_t K^t &=& 0 \\ -\partial_x K^t + \partial_t K^x &=& 0 \\ -\partial_y K^t + \partial_t K^y &=& 0 \\ -\partial_z K^t + \partial_t K^z &=& 0 \\ \partial_x K^x &=& 0 \\ \partial_x K^y + \partial_y K^x &=& 0 \\ \partial_z K^x + \partial_x K^z &=& 0 \\ \partial_y K^y &=& 0 \\ \partial_z K^y + \partial_y K^z &=& 0 \\ \partial_z K^z &=& 0 \end{array}\right.$$

claim the solution as an order one expansion $K^\mu = a^\mu{}_\nu x^\nu + c^\mu$

$$\left\{\begin{array}{rcl} a^0{}_\nu\partial_0 x^\nu &=& 0 \\ -a^0{}_\nu\partial_1 x^\nu + a^1{}_\nu\partial_0 x^\nu &=& 0 \\ -a^0{}_\nu\partial_2 x^\nu + a^2{}_\nu\partial_0 x^\nu &=& 0 \\ -a^0{}_\nu\partial_3 x^\nu + a^3{}_\nu\partial_0 x^\nu &=& 0 \\ a^1{}_\nu\partial_1 x^\nu &=& 0 \\ a^2{}_\nu\partial_1 x^\nu + a^1{}_\nu\partial_2 x^\nu &=& 0 \\ a^1{}_\nu\partial_3 x^\nu + a^3{}_\nu\partial_1 x^\nu &=& 0 \\ a^2{}_\nu\partial_2 x^\nu &=& 0 \\ a^2{}_\nu\partial_3 x^\nu + a^3{}_\nu\partial_2 x^\nu &=& 0 \\ a^3{}_\nu\partial_3 x^\nu &=& 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} a^0{}_0 &=& 0 \\ -a^0{}_1 + a^1{}_0 &=& 0 \\ -a^0{}_2 + a^2{}_0 &=& 0 \\ -a^0{}_3 + a^3{}_0 &=& 0 \\ a^1{}_1 &=& 0 \\ a^2{}_1 + a^1{}_2 &=& 0 \\ a^1{}_3 + a^3{}_1 &=& 0 \\ a^2{}_2 &=& 0 \\ a^2{}_3 + a^3{}_2 &=& 0 \\ a^3{}_3 &=& 0 \end{array}\right.$$

therefore

$$\begin{array}{rcl} K^\mu &=& a^\mu{}_\nu x^\nu + c^\mu \\ \begin{pmatrix} K^1 \\ K^2 \\ K^3 \\ K^4 \\ 1 \end{pmatrix} &=& \begin{pmatrix} 0 & a^0{}_1 & a^0{}_2 & a^0{}_3 & c^1 \\ a^0{}_1 & 0 & a^1{}_2 & a^1{}_3 & c^2 \\ a^0{}_2 & -a^1{}_2 & 0 & a^2{}_3 & c^3 \\ a^0{}_3 & -a^1{}_3 & -a^2{}_3 & 0 & c^4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ 1 \end{pmatrix} \end{array}$$

could see explictly ,there are 10 independent killing vector.cause of 10 free paramenters.

b.

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v - \mathrm{d}v\mathrm{d}u + a^2(u)\mathrm{d}x^2 + b^2(u)\mathrm{d}y^2$$

$$g = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & a^2 & \\ & & & b^2 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & \frac{1}{a^2} & \\ & & & \frac{1}{b^2} \end{pmatrix}$$

$$\begin{aligned}
\Gamma^u_{\mu\nu} &= \frac{1}{2}g^{uv}(\partial_\mu g_{\nu v} + \partial_\nu g_{\mu v} - \partial_v g_{\mu\nu}) = 0 \\
\Gamma^v_{\mu\nu} &= \frac{1}{2}g^{vu}(\partial_\mu g_{\nu u} + \partial_\nu g_{\mu u} - \partial_u g_{\mu\nu}) \\
&= \frac{1}{2}\partial_u g_{\mu\nu} \\
&= \begin{cases} \Gamma^v_{xx} = a a' \\ \Gamma^v_{yy} = b b' \\ 0 & \text{others} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\Gamma^x_{\mu\nu} &= \frac{1}{2}g^{xx}(\partial_\mu g_{\nu x} + \partial_\nu g_{\mu x} - \partial_x g_{\mu\nu}) \\
&= \frac{1}{2a^2}(\partial_\mu g_{\nu x} + \partial_\nu g_{\mu x}) \\
&= \begin{cases} \Gamma^x_{xu} = \frac{1}{2a^2}\partial_u g_{xx} \\ 0 & \text{others} \end{cases} = \begin{cases} \Gamma^x_{ux} = \frac{a'}{a} \\ 0 & \text{others} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\Gamma^y_{\mu\nu} &= \frac{1}{2}g^{yy}(\partial_\mu g_{\nu y} + \partial_\nu g_{\mu y} - \partial_y g_{\mu\nu}) \\
&= \frac{1}{2b^2}(\partial_\mu g_{\nu y} + \partial_\nu g_{\mu y}) \\
&= \begin{cases} \Gamma^y_{yu} = \frac{1}{2b^2}\partial_u g_{yy} \\ 0 & \text{others} \end{cases} = \begin{cases} \Gamma^y_{yu} = \frac{b'}{b} \\ 0 & \text{others} \end{cases}
\end{aligned}$$

$$(g_{\sigma\mu}\partial_\nu + g_{\sigma\nu}\partial_\mu)K^\sigma + (g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda} + g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda})K^\lambda = 0$$

$$\begin{cases}
-2g_{\sigma u}\,\partial_u K^\sigma + (g_{\sigma u}\Gamma^\sigma_{u\lambda} + g_{\sigma u}\Gamma^\sigma_{u\lambda})K^\lambda = 0 \\
-2g_{\sigma v}\,\partial_v K^\sigma + (g_{\sigma v}\Gamma^\sigma_{v\lambda} + g_{\sigma v}\Gamma^\sigma_{v\lambda})K^\lambda = 0 \\
-2g_{\sigma x}\,\partial_x K^\sigma + (g_{\sigma x}\Gamma^\sigma_{x\lambda} + g_{\sigma x}\Gamma^\sigma_{x\lambda})K^\lambda = 0 \\
-2g_{\sigma y}\,\partial_y K^\sigma + (g_{\sigma y}\Gamma^\sigma_{y\lambda} + g_{\sigma y}\Gamma^\sigma_{y\lambda})K^\lambda = 0 \\
(g_{\sigma u}\partial_v + g_{\sigma v}\partial_u)K^\sigma + (g_{\sigma u}\Gamma^\sigma_{v\lambda} + g_{\sigma v}\Gamma^\sigma_{u\lambda})K^\lambda = 0 \\
(g_{\sigma u}\partial_x + g_{\sigma x}\partial_u)K^\sigma + (g_{\sigma u}\Gamma^\sigma_{x\lambda} + g_{\sigma x}\Gamma^\sigma_{u\lambda})K^\lambda = 0 \\
(g_{\sigma u}\partial_y + g_{\sigma y}\partial_u)K^\sigma + (g_{\sigma u}\Gamma^\sigma_{y\lambda} + g_{\sigma y}\Gamma^\sigma_{u\lambda})K^\lambda = 0 \\
(g_{\sigma v}\partial_x + g_{\sigma x}\partial_v)K^\sigma + (g_{\sigma v}\Gamma^\sigma_{x\lambda} + g_{\sigma x}\Gamma^\sigma_{v\lambda})K^\lambda = 0 \\
(g_{\sigma v}\partial_y + g_{\sigma y}\partial_v)K^\sigma + (g_{\sigma v}\Gamma^\sigma_{y\lambda} + g_{\sigma y}\Gamma^\sigma_{v\lambda})K^\lambda = 0 \\
(g_{\sigma x}\partial_y + g_{\sigma y}\partial_x)K^\sigma + (g_{\sigma x}\Gamma^\sigma_{y\lambda} + g_{\sigma y}\Gamma^\sigma_{x\lambda})K^\lambda = 0
\end{cases}$$

$$
\Rightarrow \left\{ \begin{array}{rcl}
2\partial_u K^v &=& 0 \\
2\partial_v K^u &=& 0 \\
-2a^2 \partial_x K^x + 2a^2 \frac{a'}{a} K^u &=& 0 \\
-2b^2 \partial_y K^y + 2b^2 \frac{b'}{b} K^u &=& 0 \\
-\partial_v K^v - \partial_u K^u &=& 0 \\
-\partial_x K^v + a^2 \partial_u K^x - a a' K^x + a^2 \frac{a'}{a} K^x &=& 0 \\
-\partial_y K^v + b^2 \partial_u K^y - b b' K^y + b^2 \frac{b'}{b} K^y &=& 0 \\
-\partial_x K^u + a^2 \partial_v K^x &=& 0 \\
-\partial_y K^u + b^2 \partial_v K^y &=& 0 \\
a^2 \partial_y K^x + b^2 \partial_x K^y &=& 0
\end{array} \right.
$$

$$
\Rightarrow \left\{ \begin{array}{rcl}
\partial_u K^v &=& 0 \\
\partial_v K^u &=& 0 \\
-a \partial_x K^x + a' K^u &=& 0 \\
-b \partial_y K^y + b' K^u &=& 0 \\
-\partial_v K^v - \partial_u K^u &=& 0 \\
-\partial_x K^v + a^2 \partial_u K^x &=& 0 \\
-\partial_y K^v + b^2 \partial_u K^y &=& 0 \\
-\partial_x K^u + a^2 \partial_v K^x &=& 0 \\
-\partial_y K^u + b^2 \partial_v K^y &=& 0 \\
a^2 \partial_y K^x + b^2 \partial_x K^y &=& 0
\end{array} \right.
\Rightarrow \left\{ \begin{array}{rcl}
\partial_u K^v &=& 0 \\
0 &=& 0 \\
\partial_x K^x &=& 0 \\
\partial_y K^y &=& 0 \\
\partial_v K^v &=& 0 \\
-\partial_x K^v + a^2 \partial_u K^x &=& 0 \\
-\partial_y K^v + b^2 \partial_u K^y &=& 0 \\
\partial_v K^x &=& 0 \\
\partial_v K^y &=& 0 \\
a^2 \partial_y K^x + b^2 \partial_x K^y &=& 0
\end{array} \right.
$$

$$
\Rightarrow \left\{ \begin{array}{rcl}
\partial_u K^v &=& 0 \\
\partial_v K^v &=& 0 \\
\partial_v K^x &=& 0 \\
\partial_x K^x &=& 0 \\
\partial_v K^y &=& 0 \\
\partial_y K^y &=& 0 \\
-\partial_x K^v + a^2 \partial_u K^x &=& 0 \\
-\partial_y K^v + b^2 \partial_u K^y &=& 0 \\
a^2 \partial_y K^x + b^2 \partial_x K^y &=& 0
\end{array} \right.
$$

guess
$$
\left\{ \begin{array}{rcl}
K^v(x, y) &=& c_1 x + c_2 y + c_3 \\
K^x(u, y) &=& c_4 \int_0^u \frac{1}{a^2} \mathrm{d}t + c_5 \\
K^y(u, x) &=& c_6 \int_0^u \frac{1}{b^2} \mathrm{d}t + c_7
\end{array} \right.
$$

$$
\left\{ \begin{array}{rcl}
\partial_u K^v &=& 0 \\
\partial_v K^v &=& 0 \\
\partial_v K^x &=& 0 \\
\partial_x K^x &=& 0 \\
\partial_v K^y &=& 0 \\
\partial_y K^y &=& 0 \\
-\partial_x K^v + a^2 \partial_u K^x &=& 0 \\
-\partial_y K^v + b^2 \partial_u K^y &=& 0 \\
a^2 \partial_y K^x + b^2 \partial_x K^y &=& 0
\end{array} \right.
\Rightarrow \left\{ \begin{array}{rcl}
0 &=& 0 \\
0 &=& 0 \\
0 &=& 0 \\
0 &=& 0 \\
0 &=& 0 \\
0 &=& 0 \\
-c_1 + c_4 a^2 \times \frac{1}{a^2} &=& 0 \\
-c_2 + c_6 b^2 \times \frac{1}{b^2} &=& 0 \\
0 &=& 0
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
c_1 = c_4 \\
c_2 = c_6
\end{array} \right.
$$

then

$$\begin{pmatrix} K^u \\ K^\nu \\ K^x \\ K^y \end{pmatrix} = \begin{pmatrix} 0 \\ c_1x+c_2y+c_3 \\ c_1\int_0^u \frac{1}{a^2}\mathrm{d}t+c_5 \\ c_2\int_0^u \frac{1}{b^2}\mathrm{d}t+c_7 \end{pmatrix}$$

could see there are 5 free parameters ,namely 5 five independent killing vector.

could see form the metric $K^u=0$

guess evidence here:

$$\left\{\begin{array}{rcl} \partial_u K^v &=& 0 \\ -\partial_x K^v + a^2\partial_u K^x &=& 0 \\ \partial_v K^x &=& 0 \\ \partial_x K^x &=& 0 \end{array}\right. \Rightarrow \partial_u(a^2\partial_u K^x)=0$$

$$\Rightarrow a^2\partial_u K^x = C_1(y)$$

$$\Rightarrow K^x = C_1(y)\int_0^u \frac{1}{a^2(t)}\mathrm{d}t + C_2(y)$$

similarly

$$K^y = C_3(x)\int_0^u \frac{1}{b^2(t)}\mathrm{d}t + C_4(x)$$

16.

a.

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}^2\psi + \sin^2\psi\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\mathrm{d}\phi^2 \\ &= \mathrm{d}^2\psi + (\sin\psi\mathrm{d}\theta)(\sin\psi\mathrm{d}\theta) + (\sin\psi\sin\theta\mathrm{d}\phi)(\sin\psi\sin\theta\mathrm{d}\phi) \\ &= \hat{\theta}^1\otimes\hat{\theta}^1 + \hat{\theta}^2\otimes\hat{\theta}^2 + \hat{\theta}^3\otimes\hat{\theta}^3 \end{aligned}$$

therefore

$$[e_a^\mu] = \begin{pmatrix} 1 & & \\ & \frac{1}{\sin\psi} & \\ & & \frac{1}{\sin\psi\sin\theta} \end{pmatrix}$$

$$[e_\mu^a] = \begin{pmatrix} 1 & & \\ & \sin\psi & \\ & & \sin\psi\sin\theta \end{pmatrix}$$

$$e^0 = \mathrm{d}\psi \quad e^1 = \sin\psi\mathrm{d}\theta \quad e^2 = \sin\psi\sin\theta\mathrm{d}\phi$$

b.

$$\begin{aligned} \mathrm{d}e^0 &= 0 \\ \mathrm{d}e^1 &= \partial_\nu\sin\psi\mathrm{d}x^\nu\wedge\mathrm{d}\theta = \cos\psi\mathrm{d}\psi\wedge\mathrm{d}\theta \\ \mathrm{d}e^2 &= \partial_\nu\sin\psi\sin\theta\mathrm{d}x^\nu\wedge\mathrm{d}\phi = \cos\psi\sin\theta\mathrm{d}\psi\wedge\mathrm{d}\phi + \sin\psi\cos\theta\mathrm{d}\theta\wedge\mathrm{d}\phi \end{aligned}$$

$$\begin{aligned} w^0{}_b\wedge e^b &= -\mathrm{d}e^0 \\ w^0{}_1\wedge e^1 + w^0{}_2\wedge e^2 &= 0 \\ w^0{}_1\wedge\sin\psi\mathrm{d}\theta + w^0{}_2\wedge\sin\psi\sin\theta\mathrm{d}\phi &= 0 \end{aligned}$$

$$\begin{aligned} \sin\psi w^0{}_1 \wedge \mathrm{d}\theta + \sin\psi\sin\theta w^0{}_2 \wedge \mathrm{d}\phi &= 0 \\ w^0{}_1 \wedge \mathrm{d}\theta + \sin\theta w^0{}_2 \wedge \mathrm{d}\phi &= 0 \end{aligned}$$

$$\begin{aligned} w^1{}_b \wedge e^b &= -\mathrm{d}e^1 \\ w^1{}_0 \wedge e^0 + w^1{}_2 \wedge e^2 &= -(\cos\psi \mathrm{d}\psi \wedge \mathrm{d}\theta) \\ w^1{}_0 \wedge \mathrm{d}\psi + w^1{}_2 \wedge \sin\psi\sin\theta\mathrm{d}\phi &= -\cos\psi\mathrm{d}\psi \wedge \mathrm{d}\theta \end{aligned}$$

$$\begin{aligned} w^2{}_b \wedge e^b &= -\mathrm{d}e^2 \\ w^2{}_0 \wedge e^0 + w^2{}_1 \wedge e^1 &= -\cos\psi\sin\theta\mathrm{d}\psi \wedge \mathrm{d}\phi \\ w^2{}_0 \wedge \mathrm{d}\psi + \sin\psi w^2{}_1 \wedge \mathrm{d}\theta &= -\cos\psi\sin\theta\mathrm{d}\psi \wedge \mathrm{d}\phi - \sin\psi\cos\theta\mathrm{d}\theta \wedge \mathrm{d}\phi \end{aligned}$$

$$\left\{ \begin{array}{lcl} w^0{}_1 &=& \alpha\mathrm{d}\theta \\ w^0{}_2 &=& \beta\mathrm{d}\phi \\ w^1{}_0 &=& \cos\psi\mathrm{d}\theta \\ w^1{}_2 &=& \gamma\mathrm{d}\phi \\ w^2{}_0 &=& \cos\psi\sin\theta\mathrm{d}\phi \\ w^2{}_1 &=& \cos\theta\mathrm{d}\phi \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} w^0{}_1 &=& -\cos\psi\mathrm{d}\theta \\ w^0{}_2 &=& -\cos\psi\sin\theta\mathrm{d}\phi \\ w^1{}_2 &=& -\cos\theta\mathrm{d}\phi \end{array} \right.$$

$$\begin{aligned} R^0{}_0 &= \mathrm{d}w^0{}_0 + w^0{}_b \wedge w^b{}_0 = 0 \\ R^0{}_1 &= \mathrm{d}w^0{}_1 + w^0{}_b \wedge w^b{}_1 \\ &= \sin\psi\mathrm{d}\psi \wedge \mathrm{d}\theta + w^0{}_2 \wedge w^2{}_1 \\ &= \sin\psi\mathrm{d}\psi \wedge \mathrm{d}\theta - \cos\psi\sin\theta\cos\theta\mathrm{d}\phi \wedge \mathrm{d}\phi \\ &= \sin\psi\mathrm{d}\psi \wedge \mathrm{d}\theta \end{aligned}$$

$$\begin{aligned} R^0{}_2 &= \mathrm{d}w^0{}_2 + w^0{}_b \wedge w^b{}_2 \\ &= -\partial_\nu(\cos\psi\sin\theta)\mathrm{d}x^\nu \wedge \mathrm{d}\phi + w^0{}_1 \wedge w^1{}_2 \\ &= \sin\psi\sin\theta\mathrm{d}\psi \wedge \mathrm{d}\phi - \cos\psi\cos\theta\mathrm{d}\theta \wedge \mathrm{d}\phi + (-\cos\psi\mathrm{d}\theta) \wedge (-\cos\theta\mathrm{d}\phi) \\ &= \sin\psi\sin\theta\mathrm{d}\psi \wedge \mathrm{d}\phi \end{aligned}$$

$$\begin{aligned} R^1{}_1 &= 0 \\ R^1{}_2 &= \mathrm{d}w^1{}_2 + w^1{}_b \wedge w^b{}_2 \\ &= -\mathrm{d}\cos\theta \wedge \mathrm{d}\phi + w^1{}_0 \wedge w^0{}_2 \\ &= \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi + \cos\psi\mathrm{d}\theta \wedge (-\cos\psi\sin\theta\mathrm{d}\phi) \\ &= (\sin\theta - \cos^2\psi\sin\theta)\mathrm{d}\theta \wedge \mathrm{d}\phi \\ &= \sin\theta\sin^2\psi\mathrm{d}\theta \wedge \mathrm{d}\phi \end{aligned}$$

$$R^2{}_2 = 0$$

$$\begin{aligned} R^\sigma{}_\rho &= e^a R^b{}_a e_b \\ &= e_b R^b{}_a e^a \\ & \begin{pmatrix} 1 & & \\ & \frac{1}{\sin\psi} & \\ & & \frac{1}{\sin\psi\sin\theta} \end{pmatrix} \begin{pmatrix} & R^0{}_1 & R^0{}_2 \\ -R^0{}_1 & & R^1{}_2 \\ -R^0{}_2 & -R^1{}_2 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & \sin\psi & \\ & & \sin\psi\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ & \frac{1}{\sin\psi} & \\ & & \frac{1}{\sin\psi\sin\theta} \end{pmatrix} \begin{pmatrix} & \sin\psi R^0{}_1 & \sin\psi\sin\theta R^0{}_2 \\ -R^0{}_1 & & \sin\psi\sin\theta R^1{}_2 \\ -R^0{}_2 & -\sin\psi R^1{}_2 & \end{pmatrix} \\ &= \begin{pmatrix} & \sin\psi R^0{}_1 & \sin\psi\sin\theta R^0{}_2 \\ -\frac{1}{\sin\psi} R^0{}_1 & & \frac{1}{\sin\psi} \times \sin\psi\sin\theta R^1{}_2 \\ -\frac{1}{\sin\psi\sin\theta} R^0{}_2 & -\frac{1}{\sin\psi\sin\theta} \times \sin\psi R^1{}_2 & \end{pmatrix} \end{aligned}$$

$$
= \begin{pmatrix} & \sin\psi R^0{}_1 & \sin\psi\sin\theta R^0{}_2 \\ -\frac{1}{\sin\psi}R^0{}_1 & & \sin\theta R^1{}_2 \\ -\frac{1}{\sin\psi\sin\theta}R^0{}_2 & -\frac{1}{\sin\theta}R^1{}_2 & \end{pmatrix}
$$

$$
= \begin{pmatrix} & \sin^2\psi \mathrm{d}\psi\wedge\mathrm{d}\theta & \sin^2\psi\sin^2\theta\mathrm{d}\psi\wedge\mathrm{d}\phi \\ -\mathrm{d}\psi\wedge\mathrm{d}\theta & & \sin^2\psi\sin^2\theta\mathrm{d}\theta\wedge\mathrm{d}\phi \\ -\mathrm{d}\psi\wedge\mathrm{d}\phi & -\sin^2\psi\mathrm{d}\theta\wedge\mathrm{d}\phi & \end{pmatrix}
$$

$$
\begin{array}{rl}
R^0{}_{101} = \sin^2\psi & R^1{}_{001} = -1 \\
R^0{}_{202} = \sin^2\psi\sin^2\theta & R^2{}_{002} = -1 \\
R^1{}_{212} = \sin^2\psi\sin^2\theta & R^2{}_{112} = -\sin^2\psi
\end{array}
$$

$$
\begin{aligned}
R_{00} &= R^1{}_{010} + R^2{}_{020} = 1 + 1 = 2 \\
R_{11} &= R^0{}_{101} + R^2{}_{121} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{22} &= R^0{}_{202} + R^1{}_{212} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta
\end{aligned}
$$