5th exercise

1. treat the whole process as a completely inelastic collision under the gravity of Schwarzschild metric.

the outcome (depend on the mass of monkey and coconut and the initial orbital radius)
maybe i. drop into black hole. ii.moving along other circular geodesic orbit. iii. leave off the black hole.

3.

a. according to the equation: in Schwarzschild metric.

$$\begin{aligned} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)\epsilon &= E^2 \\ \Rightarrow \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= E^2 + \left(\frac{2GM}{r} - 1\right)\epsilon \end{aligned}$$

since here inside the blackhole, namely $r < 2GM \ \Rightarrow \frac{2GM}{r} - 1 > 0$

therefore

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| = \sqrt{E^2 + \left(\frac{2GM}{r} - 1\right)\epsilon}$$

for timelike particle $\epsilon = 1$ then could see

$$\begin{aligned} \left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| &= \sqrt{E^2 + \left(\frac{2GM}{r} - 1\right)\epsilon} \\ &= \sqrt{E^2 + \left(\frac{2GM}{r} - 1\right)} \geqslant \sqrt{\frac{2GM}{r} - 1} \end{aligned}$$

b.

$$\mathrm{d}\tau \leqslant \frac{\mathrm{d}r}{\sqrt{\frac{2GM}{r} - 1}}$$

to calculate the maxium of τ then

$$\begin{aligned} \Delta\tau &= \int_{2GM}^{0} \frac{-\mathrm{d}r}{\sqrt{\frac{2GM}{r} - 1}} \\ \xlongequal[r = \frac{2GM}{u^2+1}]{u = \sqrt{\frac{2GM}{r} - 1}} &= -\int_0^{\infty} \frac{2GM}{u} \times \left(-\frac{2u}{(u^2+1)^2}\right)\mathrm{d}u \\ &= \\ &= 4GM \int_0^{\infty} \frac{\mathrm{d}u}{(u^2+1)^2} \\ &= 4GM \times \frac{\pi}{4} \\ &= \pi GM \end{aligned}$$

c. solar mass: $M = 1.989 \times 10^{35}\text{kg}$,$G = 6.674 \times 10^{-11}\, m^3\, \text{kg}^{-1}\, s^{-2}$

$$\begin{aligned}
(\Delta\tau)s &= \frac{\pi G M}{c^3} = \frac{3.1415 \times 1.989 \times 10^{35}\text{kg} \times 6.674 \times 10^{-11}\, m^3\, \text{kg}^{-1}\, s^{-2}}{(2.998 \times 10^8 m\, s^{-1})^3} \\
&= 1.548 s
\end{aligned}$$

b′. below may be not right

$$\mathrm{d}\tau = \frac{-\mathrm{d}r}{\sqrt{E^2 + \left(\frac{2GM}{r} - 1\right)}} = \frac{-\mathrm{d}r}{\sqrt{\frac{2GM}{r} + E^2 - 1}}$$

$$\begin{aligned}
\Delta\tau &= -\int_{2GM}^{0} \frac{\mathrm{d}r}{\sqrt{\frac{2GM}{r} + E^2 - 1}} \\
&\xlongequal[r=\frac{2GM}{u^2+1-E^2}]{u=\sqrt{\frac{2GM}{r}+E^2-1}} \int_E^\infty \frac{2GM}{u} \times (-1) \frac{2u\mathrm{d}u}{(u^2+1-E^2)^2} \\
&= 4GM \int_E^\infty \frac{\mathrm{d}u}{(u^2+1-E^2)^2} \\
&\xlongequal[u=\sqrt{\upsilon^2+E^2}]{\upsilon^2=u^2-E^2} -4GM \int_0^\infty \frac{1}{(\upsilon^2+1)^2} \times \frac{\upsilon}{\sqrt{\upsilon^2+E^2}} \mathrm{d}\upsilon \\
&\leqslant 4GM \int_0^\infty \frac{\mathrm{d}u}{(u^2+1)^2} = 4GM \times \frac{\pi}{4} = \pi G M
\end{aligned}$$

4.

the equation :

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} &= 0 \\
g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} + \Lambda g^{\mu\nu} g_{\mu\nu} &= 0 \\
R - 2R + 4\Lambda &= 0 \\
R &= 4\Lambda
\end{aligned}$$

then

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2} \times 4\Lambda g_{\mu\nu} + \Lambda g_{\mu\nu} &= 0 \\
R_{\mu\nu} - \Lambda g_{\mu\nu} &= 0
\end{aligned}$$

the general spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\alpha(r,t)}\mathrm{d}t + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\phi^2)$$

using the Ricci tensor in textbook and add a term $-\Lambda g_{\mu\nu}$

$$\begin{aligned}
R_{tt} &= (\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta) + e^{2(\alpha-\beta)}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right) \\
R_{rr} &= -\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right) + e^{2(\beta-\alpha)}(\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta) \\
R_{tr} &= \frac{2}{r}\partial_t\beta
\end{aligned}$$

$$
\begin{aligned}
R_{\theta\theta} &= e^{-2\beta}(r(\partial_r\beta-\partial_r\alpha)-1)+1\\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta
\end{aligned}
$$

the equation $R_{\mu\nu}-\Lambda g_{\mu\nu}=0$

$$
\left\{
\begin{aligned}
(\partial_t^2\beta+(\partial_t\beta)^2-\partial_t\alpha\partial_t\beta)+e^{2(\alpha-\beta)}\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha\partial_r\beta+\tfrac{2}{r}\partial_r\alpha\big)-\Lambda e^{2\alpha} &= 0\\
-\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha\partial_r\beta-\tfrac{2}{r}\partial_r\beta\big)+e^{2(\beta-\alpha)}(\partial_t^2\beta+(\partial_t\beta)^2-\partial_t\alpha\partial_t\beta)+\Lambda e^{2\beta} &= 0\\
\tfrac{2}{r}\partial_t\beta &= 0\\
e^{-2\beta}(r(\partial_r\beta-\partial_r\alpha)-1)+1+\Lambda r^2 &= 0\\
(e^{-2\beta}(r(\partial_r\beta-\partial_r\alpha)-1)+1)\sin^2\theta+\Lambda r^2\sin^2\theta &= 0
\end{aligned}
\right.
$$

as $\partial_t\beta=0$

$$
\Rightarrow\left\{
\begin{aligned}
\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha\partial_r\beta+\tfrac{2}{r}\partial_r\alpha\big)-\Lambda e^{2\beta} &= 0\\
-\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha\partial_r\beta-\tfrac{2}{r}\partial_r\beta\big)+\Lambda e^{2\beta} &= 0\\
e^{-2\beta}(r(\partial_r\beta-\partial_r\alpha)-1)+1+\Lambda r^2 &= 0
\end{aligned}
\right.
$$

$\Rightarrow -e^{-2\beta}r\partial_r\partial_t\alpha=0 \;\Rightarrow\;$ set $\alpha(t,r)=\alpha_0(t)+\alpha_1(r)$, and could see the equation below have no relation with $\alpha_0(t)$,that's to say for arbitary $\alpha_0(t)$ the equations will all satisfy.

therefore set $\alpha_0=0$ then $\alpha=\alpha_1$

$\Rightarrow\; \frac{4}{r}\partial_r(\alpha+\beta)=0\Rightarrow\alpha+\beta=c(t)$ could set $c(t)=0$ here, therefore $\alpha=-\beta$

$$
\Rightarrow\left\{
\begin{aligned}
\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha(-\partial_r\alpha)+\tfrac{2}{r}\partial_r\alpha\big)-\Lambda e^{-2\alpha} &= 0\\
-\big(\partial_r^2\alpha+(\partial_r\alpha)^2-\partial_r\alpha(-\partial_r\alpha)+\tfrac{2}{r}\partial_r\alpha\big)+\Lambda e^{-2\alpha} &= 0\\
e^{2\alpha}(r(-\partial_r\alpha-\partial_r\alpha)-1)+1+\Lambda r^2 &= 0
\end{aligned}
\right.
$$

$$
\Rightarrow\left\{
\begin{aligned}
\big(\partial_r^2\alpha+2(\partial_r\alpha)^2+\tfrac{2}{r}\partial_r\alpha\big)-\Lambda e^{-2\alpha} &= 0\\
e^{2\alpha}(-2r\partial_r\alpha-1)+1+\Lambda r^2 &= 0
\end{aligned}
\right.
$$

$$
\Rightarrow\left\{
\begin{aligned}
\big(\partial_r^2\alpha+2(\partial_r\alpha)^2+\tfrac{2}{r}\partial_r\alpha\big)-\Lambda e^{-2\alpha} &= 0\\
\partial_r\alpha+\tfrac{1}{2r}+\tfrac{1}{2r}(1+\Lambda r^2)e^{-2\alpha} &= 0
\end{aligned}
\right.
$$

$$
\begin{aligned}
(2r\partial_r\alpha+1)e^{2\alpha} &= 1+\Lambda r^2\\
\partial_r(re^{2\alpha}) &= 1+\Lambda r^2\\
\frac{\mathrm{d}(re^{2\alpha})}{\mathrm{d}r} &= 1+\Lambda r^2\\
re^{2\alpha} &= \int \mathrm{d}r\{1+\Lambda r^2\}\\
&= r+\frac{\Lambda}{3}r^3+C\\
e^{2\alpha} &= 1+\frac{\Lambda}{3}r^2+\frac{C}{r}
\end{aligned}
$$

since $\Lambda\to0$ the result will be the same as Schwarzschild solution. therefore $C=-2GM$

the metric :

$$
\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\mathrm{d}t^2+\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r+r^2\mathrm{d}\Omega^2
$$

b.

here choose $\theta = \frac{\pi}{2}$,too

$$\begin{aligned}
\epsilon &= -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \\
&= \left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2-\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2-r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\left(r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2+\epsilon\right) &= \left(\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \\
\left(\frac{\mathrm{d}r}{\mathrm{d}r}\right)^2+\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\left(\frac{L^2}{r^2}+\epsilon\right) &= E^2
\end{aligned}$$

for massive particle $\epsilon = 1$, $V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}+\frac{\Lambda}{3}r^2\right)\left(\frac{L^2}{r^2}+1\right)$

therefore the radical geodesic equation of motion:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}r}\right)^2+2V(r) = E^2$$

5.

as $g_{\mu\nu}U^\mu U^\nu = -1$,$U_\mu = (U_0, 0, 0, 0)$

$$\begin{aligned}
g^{\mu\nu}U_\mu U_\nu &= -\left(1-\frac{2GM}{r_*}\right)^{-1}U_0^2 = -1 \\
\Rightarrow\ U_0^2 &= \left(1-\frac{2GM}{r_*}\right) \\
\Rightarrow\ U_0 &= -\sqrt{1-\frac{2GM}{r_*}}
\end{aligned}$$

a. bacon move along the geodesic, and angular momentum $L = 0$

$$\left\{\begin{aligned}
\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} &= E \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2+\left(1-\frac{2GM}{r}\right) &= E^2
\end{aligned}\right. \quad \Rightarrow \frac{\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}{\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2} = \frac{E^2-\left(1-\frac{2GM}{r}\right)}{\frac{E^2}{\left(1-\frac{2GM}{r}\right)^2}}$$

$$\Rightarrow \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(1-\frac{1}{E^2}\left(1-\frac{2GM}{r}\right)\right)\left(1-\frac{2GM}{r}\right)^2$$

$$\Rightarrow \left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = \sqrt{\left(1-\frac{1}{E^2}\left(1-\frac{2GM}{r}\right)\right)\left(1-\frac{2GM}{r}\right)^2}$$

b.

$$\begin{aligned}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= E^2-\left(1-\frac{2GM}{r}\right)\\ \left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| &= \sqrt{E^2-\left(1-\frac{2GM}{r}\right)}\end{aligned}$$

the comoving cooridinate observe speed

$$\frac{\mathrm{d}r_*}{\mathrm{d}t} = \left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| = \sqrt{E^2-\left(1-\frac{2GM}{r}\right)}$$

when $r=2GM$,the $\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right|=E$

c.

bacon emit a photo ,moving along the geodesic.

observers

$$\begin{aligned}p^\mu U_\mu &= -E_{\text{photo}}\\ \frac{\mathrm{d}t}{\mathrm{d}\tau}U_t &= -\frac{2\pi}{\lambda}\\ \frac{\frac{2\pi}{\lambda_{\text{em}}}}{\left(1-\frac{2GM}{r_{\text{em}}}\right)}\times\left(-\sqrt{\left(1-\frac{2GM}{r_*}\right)}\right) &= -\frac{2\pi}{\lambda_{\text{ob}}}\\ \lambda_{\text{ob}} &= \lambda_{\text{em}}\left(\frac{1-\frac{2GM}{r_{\text{em}}}}{\sqrt{1-\frac{2GM}{r_*}}}\right)\end{aligned}$$

d. for null curve

$$\left\{\begin{aligned}\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} &= E\\ \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= E^2\end{aligned}\right. \Rightarrow \frac{\left(1-\frac{2GM}{r}\right)^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2}{\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2}=1$$

$$\Rightarrow \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \left(\frac{1}{1-\frac{2GM}{r}}\right)^2$$

as $r\rightarrow r_*$ and $r_*>r$ therefore:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}r} &= \frac{1}{1-\frac{2GM}{r}}\\ \mathrm{d}t &= \frac{1}{1-\frac{2GM}{r}}\mathrm{d}r\\ \int_0^{t_{\text{ob}}}\mathrm{d}t &= \int_{r_{\text{em}}}^{r_*}\frac{\mathrm{d}r}{1-\frac{2GM}{r}}\\ t_{\text{ob}} &= \int_{r_{\text{em}}}^{r_*}\frac{r\mathrm{d}r}{r-2GM}\\ &= \int_{r_{\text{em}}}^{r_*}\frac{(r-2GM)+2GM}{r-2GM}\mathrm{d}r\\ &= \int_{r_{\text{em}}}^{r_*}1+\frac{2GM\mathrm{d}r}{r-2GM}\\ t_{\text{ob}} &= r_*-r_{\text{em}}+2GM\ln\left(\frac{r_*-2GM}{r_{\text{em}}-2GM}\right)\end{aligned}$$

e. using the result (d)

$$e^{\frac{t_{\rm ob}-r_*+r_{\rm em}}{2GM}} = \frac{r_*-2GM}{r_{\rm em}-2GM}$$

$$\begin{aligned}
\frac{\lambda_{\rm ob}}{\lambda_{\rm em}} &= \frac{1-\frac{2GM}{r}}{\sqrt{1-\frac{2GM}{r_*}}} = \frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r_*}} \times \sqrt{1-\frac{2GM}{r_*}} \\
&= \frac{r-2GM}{r_*-2GM} \times \frac{r_*}{r}\sqrt{1-\frac{GM}{r_*}} \\
&= e^{-\frac{t_{\rm ob}}{2GM}} \times \frac{e^{\frac{r-r_*}{2GM}}}{r} \times r_*\sqrt{1-\frac{GM}{r_*}} \\
&\xlongequal{r\to 2GM} e^{-\frac{t_{\rm ob}}{2GM}} \frac{e^{-\frac{r_*}{2GM}+1}}{2GM} r_*\sqrt{1-\frac{GM}{r_*}}
\end{aligned}$$

when $r \to 2GM \Rightarrow \frac{\lambda_{\rm ob}}{\lambda_{\rm em}} \to 0$, could see that require $t_{\rm ob} \to \infty$ consisting with imagination,therefore the aproximation is not bad.

$$\begin{aligned}
\frac{\lambda_{\rm ob}}{\lambda_{\rm em}} &\propto e^{-\frac{t_{\rm ob}}{2GM}} \\
&\Rightarrow T = -2GM
\end{aligned}$$