2nd exercise

.

4.

1.

$$
\begin{aligned}
[X,Y](af+bg) &= X(Y(af+bg)) - Y(X((af+bg))) \\
&= X(Y^\mu\partial_\mu(af+bg)) - Y(X^\mu\partial_\mu(af+bg)) \\
&= X(Y^\mu a(\partial_\mu f) + Y^\mu b(\partial_\mu g)) - Y(X^\mu a(\partial_\mu f) + X^\mu b(\partial_\mu g)) \\
&= aX^v\partial_v(Y^\mu(\partial_\mu f)) - aY^v\partial_v(X^\mu(\partial_\mu f)) + bX^v\partial_v(Y^\mu(\partial_v g)) - \\
&\quad bY^v\partial_v(X^\mu(\partial_v g)) \\
&= a(X(Y(f)) - Y(X(f))) + b(X(Y(g)) - Y(X(g))) \\
&= a[X,Y]f + b[X,Y]g
\end{aligned}
$$

2.

$$
\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(Y^\mu\partial_\mu(fg)) - Y(X^\mu\partial_\mu(fg)) \\
&= X(Y^\mu(\partial_\mu f)g + Y^\mu(\partial_\mu g)f) - Y(X^\mu(\partial_\mu f)g + X^\mu(\partial_\mu g)f) \\
&= X^v\partial_v(Y^\mu(\partial_\mu f)g + Y^\mu(\partial_\mu g)f) - Y^v\partial_v(X^\mu(\partial_\mu f)g + X^\mu(\partial_\mu g)f) \\
&= X^v((\partial_v Y^\mu)((\partial_\mu f)g + (\partial_\mu g)f) + Y^\mu((\partial_v\partial_\mu f)g + (\partial_\mu f)(\partial_v g) + \\
&\quad (\partial_v\partial_\mu g)f + (\partial_\mu g)(\partial_v f))) - Y^v((\partial_u X^\mu)((\partial_\mu f)g + (\partial_\mu g)f) + \\
&\quad X^\mu((\partial_v\partial_\mu f)g + (\partial_\mu f)(\partial_v g) + (\partial_v\partial_\mu g)f + (\partial_\mu g)(\partial_v f))) \\
&= \begin{array}{c} X^v((\partial_v Y^\mu)((\partial_\mu f)g + (\partial_\mu g)f)) - Y^v(\partial_u X^\mu)((\partial_\mu f)g + (\partial_\mu g)f) \\ X^v Y^\mu((\partial_v\partial_\mu f)g + (\partial_\mu f)(\partial_v g) + (\partial_v\partial_\mu g)f + (\partial_\mu g)(\partial_v f)) \\ -Y^v X^\mu((\partial_v\partial_\mu f)g + (\partial_\mu f)(\partial_v g) + (\partial_v\partial_\mu g)f + (\partial_\mu g)(\partial_v f)) \end{array} \\
&= X^v((\partial_v Y^\mu)((\partial_\mu f)g + (\partial_\mu g)f)) - Y^v(\partial_u X^\mu)((\partial_\mu f)g + (\partial_\mu g)f) \\
&= \begin{array}{l} gX^v(\partial_v Y^\mu)(\partial_\mu f) - gY^v(\partial_\mu X^\mu)(\partial_\mu f) \\ +fX^v(\partial_v Y^\mu)(\partial_\mu g) - fY^v(\partial_\mu X^\mu)(\partial_\mu g) \end{array} \\
&= \begin{array}{l} g(X^v(\partial_v Y^\mu) - Y^v(\partial_\mu X^\mu))(\partial_\mu f) \\ +f(X^v(\partial_v Y^\mu) - Y^v(\partial_\mu X^\mu))(\partial_\mu f) \end{array} \\
&= g[X,Y]f + f[X,Y]g
\end{aligned}
$$

3.

$$
\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= X(Y^\mu\partial_\mu f) - Y(X^\mu\partial_\mu f) \\
&= X^\lambda\partial_\lambda(Y^\mu\partial_\mu f) - Y^\lambda\partial_\lambda(X^\mu\partial_\mu f) \\
&= X^\lambda(\partial_\lambda Y^\mu)(\partial_\mu f) + X^\lambda Y^\mu(\partial_\lambda\partial_\mu f) - Y^\lambda(\partial_\lambda X^\mu)(\partial_\mu f) - Y^\lambda X^\mu(\partial_\lambda\partial_\mu f) \\
&= X^\lambda(\partial_\lambda Y^\mu)(\partial_\mu f) - Y^\lambda(\partial_\lambda X^\mu)(\partial_\mu f) \\
[X,Y]^\mu\partial_\mu f &= (X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu)\partial_\mu f
\end{aligned}
$$

therefore $[X,Y]^\mu = X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu$

4.

$$
\begin{aligned}
\frac{\partial x^{\mu'}}{\partial x^\mu}[X,Y]^\mu &= \frac{\partial x^{\mu'}}{\partial x^\mu}(X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu) \\
&= X^\lambda\partial_\lambda\left(\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu\right) - X^\lambda Y^\mu\partial_\lambda\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) - Y^\lambda\partial_\lambda\left(\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu\right) + \\
&\quad Y^\lambda X^\mu\partial_\lambda\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= X^\lambda\partial_\lambda\bigg(\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu\bigg) \quad - \quad Y^\lambda\partial_\lambda\bigg(\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu\bigg) \quad + \quad Y^\lambda X^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\lambda\partial x^\mu} \quad - \\
&\quad X^\lambda Y^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\lambda\partial x^\mu} \\
&= X^\lambda\partial_\lambda\bigg(\frac{\partial x^{\mu'}}{\partial x^\mu}Y^\mu\bigg) - Y^\lambda\partial_\lambda\bigg(\frac{\partial x^{\mu'}}{\partial x^\mu}X^\mu\bigg) \\
&= X^\lambda\partial_\lambda Y^{\mu'} - Y^\lambda\partial_\lambda X^{\mu'} \\
&= X^\lambda\frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\partial_\lambda Y^{\mu'} - Y^\lambda\frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\partial_\lambda X^{\mu'} \\
&= X^{\lambda'}\partial_{\lambda'}Y^{\mu'} - Y^{\lambda'}\partial_{\lambda'}X^{\mu'} \\
&= [X,Y]^{\mu'}
\end{aligned}
$$

5.

$$
\begin{cases} X = \lambda\partial_1 + (\lambda+1)\partial_2 \\ Y = (\eta+1)\partial_1 + \eta\partial_2 \end{cases}
$$

require $X \neq \alpha Y$ namely,

$$
\text{set} \quad \begin{cases} \lambda = \alpha(\eta+1) \\ \lambda + 1 = \alpha\eta \end{cases} \Rightarrow \begin{cases} \lambda - \alpha = \alpha\eta \\ \lambda + 1 = \alpha\eta \end{cases}
$$

only $\alpha = -1$ the independence maybe not satisfy.

therefore require $\eta + 1 \neq -\lambda \;\Rightarrow\; \eta + \lambda + 1 \neq 0$

$$
\begin{aligned}
[X,Y]^1 &= (\lambda\partial_1 + (\lambda+1)\partial_2)(\eta+1) - ((\eta+1)\partial_1 + \eta\partial_2)\lambda \\
&= \lambda\partial_1(\eta+1) + (\lambda+1)\partial_2(\eta+1) - (\eta+1)\partial_1\lambda - \eta\partial_2\lambda \\
&= \lambda\partial_1\eta + \lambda\partial_2\eta + \partial_2\eta - \eta\partial_1\lambda - \partial_1\lambda - \eta\partial_2\lambda \neq 0
\end{aligned}
$$

$$
\begin{aligned}
[X,Y]^2 &= (\lambda\partial_1 + (\lambda+1)\partial_2)\eta - ((\eta+1)\partial_1 + \eta\partial_2)(\lambda+1) \\
&= [X,Y]^1 \neq 0
\end{aligned}
$$

at a point p

$$
\begin{aligned}
&\begin{cases} X_p &= \lambda_p\partial_1 + (\lambda_p+1)\partial_2 \\ Y_p &= (\eta_p+1)\partial_1 + \eta_p\partial_2 \end{cases} \\
\Rightarrow\; &\begin{cases} (\eta_p+1)X_p &= (\eta_p+1)\lambda_p\partial_1 + (\eta_p+1)(\lambda_p+1)\partial_2 \\ \lambda_p Y_p &= \lambda_p(\eta_p+1)\partial_1 + \eta_p\lambda_p\partial_2 \end{cases} \\
\Rightarrow\; &(\eta_p+1)X_p - \lambda_p Y_p = ((\eta_p+1)(\lambda_p+1) - \eta_p\lambda_p)\partial_2 \\
\Rightarrow\; &(\eta_p+1)X_p - \lambda_p Y_p = (\eta_p + \lambda_p + 1)\partial_2 \\
\Rightarrow\; &\partial_2 = \frac{\eta_p+1}{\eta_p+\lambda_p+1}X_p - \frac{\lambda_p}{\eta_p+\lambda_p+1}Y_p
\end{aligned}
$$

$$
\begin{aligned}
\lambda_p\partial_1 &= X_p - (\lambda_p+1)\partial_2 \\
&= X_p - \frac{(\eta_p+1)(\lambda_p+1)}{\eta_p+\lambda_p+1}X_p - \frac{\lambda_p(\lambda_p+1)}{\eta_p+\lambda_p+1}Y_p \\
\partial_1 &= \frac{\frac{(\eta_p+\lambda_p+1-(\eta_p+1)(\lambda_p+1))}{\lambda_p}}{\eta_p+\lambda_p+1}X_p - \frac{\lambda_p+1}{\eta_p+\lambda_p+1}Y_p \\
&= \frac{\frac{(\eta_p+\lambda_p+1-\eta_p\lambda_p-\eta_p-\lambda_p-1)}{\lambda_p}}{\eta_p+\lambda_p+1}X_p - \frac{\lambda_p+1}{\eta_p+\lambda_p+1}Y_p \\
&= \frac{\eta_p}{\eta_p+\lambda_p+1}X_p - \frac{\lambda_p+1}{\eta_p+\lambda_p+1}Y_p
\end{aligned}
$$

for arbitary vector $V=a\partial_1+b\partial_2$

$$\begin{aligned} V &= a\left(\frac{\eta_p}{\eta_p+\lambda_p+1}X_p-\frac{\lambda_p+1}{\eta_p+\lambda_p+1}Y_p\right)+b\left(\frac{\eta_p+1}{\eta_p+\lambda_p+1}X_p-\frac{\lambda_p}{\eta_p+\lambda_p+1}Y_p\right) \\ &= \frac{a\eta_p+b(\eta_p+1)}{\eta_p+\lambda_p+1}X_p-\frac{a(\lambda_p+1)+b\lambda_p}{\eta_p+\lambda_p+1}Y_p \\ &= \frac{(a+b)\eta_p+b}{\eta_p+\lambda_p+1}X_p-\frac{(a+b)\lambda_p+a}{\eta_p+\lambda_p+1}Y_p \end{aligned}$$

6.

(a). the curve $\vec{r}=(\cos\lambda,\sin\lambda,\lambda)$

it's wrong to treat $\vec{r}$ as a covariant vector

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases}$$

$$\begin{cases} \frac{\partial r}{\partial x}=\sin\theta\cos\phi \\ \frac{\partial r}{\partial y}=\sin\theta\sin\phi \\ \frac{\partial r}{\partial z}=-\cos\theta \end{cases} \begin{cases} \frac{\partial\theta}{\partial x} &= \frac{\cos\theta\cos\phi}{r} \\ \frac{\partial\theta}{\partial y} &= \frac{\cos\theta\sin\phi}{r} \\ \frac{\partial\theta}{\partial z} &= -\frac{\sin\theta}{r} \end{cases} \begin{cases} \frac{\partial\phi}{\partial x}=-\frac{\sin\phi}{r\sin\theta} \\ \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{r\sin\theta} \\ \frac{\partial\phi}{\partial z}=0 \end{cases}$$

therefore the curve

$$\begin{cases} r(\lambda) &= r_1\frac{1}{h_1}\frac{\partial r}{\partial x}+r_2\frac{1}{h_1}\frac{\partial r}{\partial y}+r_3\frac{1}{h_1}\frac{\partial r}{\partial z} \\ \theta(\lambda) &= r_1\frac{1}{h_2}\frac{\partial\theta}{\partial x}+r_2\frac{1}{h_2}\frac{\partial\theta}{\partial y}+r_3\frac{1}{h_2}\frac{\partial\theta}{\partial z} \\ \phi(\lambda) &= r_1\frac{1}{h_3}\frac{\partial\phi}{\partial x}+r_2\frac{1}{h_3}\frac{\partial\phi}{\partial y}+r_3\frac{1}{h_3}\frac{\partial\phi}{\partial z} \end{cases}$$

$$\begin{aligned} r &= \cos\lambda\sin\theta\cos\phi+\sin\lambda\sin\theta\sin\phi+\lambda\cos\theta \\ \theta &= \cos\lambda\cos\theta\cos\phi+\sin\lambda\cos\theta\sin\phi-\lambda\sin\theta \\ \phi &= -\cos\lambda\sin\phi+\sin\lambda\cos\phi \end{aligned}$$

(b). set tangent vector as symbol $\vec{v}$

in Cartesian.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\lambda} &= -\sin\lambda \\ \frac{\mathrm{d}y}{\mathrm{d}\lambda} &= \cos\lambda \\ \frac{\mathrm{d}z}{\mathrm{d}\lambda} &= 1 \end{aligned}$$

$\vec{v}=(-\sin\lambda,\cos\lambda,1)$

in spherical polar .

$$\begin{aligned} \vec{v} &= \sum_i f_i\vec{a}_i \\ &= \sum_i\sum_j x_i\left(\frac{1}{h_i}\frac{\partial a_i}{\partial x_j}\right)\left(h_i\frac{\partial x_i}{\partial a_i}\right)\vec{e}_i \\ &= (\ -\sin\lambda\ \ \cos\lambda\ \ 1\ )MM^{-1}\begin{pmatrix}\vec{e}_x\\ \vec{e}_y\\ \vec{e}_z\end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} -\sin\lambda & \cos\lambda & 1 \end{pmatrix} M \begin{pmatrix} \vec{a}_r \\ \vec{a}_\theta \\ \vec{a}_\phi \end{pmatrix} \\
&= \begin{array}{l} (-\sin\lambda\sin\theta\cos\phi + \cos\lambda\sin\theta\sin\phi + \cos\theta)\vec{a}_r \\ +(-\sin\lambda\cos\theta\cos\phi + \cos\lambda\cos\theta\sin\phi - 1\sin\theta)\vec{a}_\theta \\ +(\sin\lambda\sin\phi + \cos\lambda\cos\phi)\vec{a}_\phi \end{array}
\end{aligned}
$$

7.

$y=0$ then let $\phi=0$

(a)

$$
\begin{cases} x = \sinh\chi\sin\theta \\ z = \cosh\chi\cos\theta \end{cases}
$$

$$
\begin{aligned}
&\begin{cases} \mathrm{d}x = \sinh\chi\cos\theta\mathrm{d}\theta + \cosh\chi\sin\theta\mathrm{d}\chi \\ \mathrm{d}y = -\cosh\chi\sin\theta\mathrm{d}\theta + \sinh\chi\cos\theta\mathrm{d}\chi \end{cases} \\
\Rightarrow &\begin{cases} \sin\theta\cosh\chi\mathrm{d}x = \cosh\chi\sinh\chi\sin\theta\cos\theta\mathrm{d}\theta + \cosh^2\chi\sin^2\theta\mathrm{d}\chi \\ \cos\theta\sinh\chi\mathrm{d}y = -\cosh\chi\sinh\chi\cos\theta\sin\theta\mathrm{d}\theta + \sinh^2\chi\cos^2\theta\mathrm{d}\chi \end{cases} \\
\Rightarrow &(\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta)\mathrm{d}\chi = \sin\theta\cosh\chi\mathrm{d}x + \cos\theta\sinh\chi\mathrm{d}y \\
\Rightarrow &\mathrm{d}\chi = \frac{\sin\theta\cosh\chi\mathrm{d}x + \cos\theta\sinh\chi\mathrm{d}y}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow &\begin{cases} \cos\theta\sinh\chi\mathrm{d}x = \sinh^2\chi\cos^2\theta\mathrm{d}\theta + \cosh\chi\sinh\chi\sin\theta\cos\theta\mathrm{d}\chi \\ \sin\theta\cosh\chi\mathrm{d}y = -\cosh^2\chi\sin^2\theta\mathrm{d}\theta + \cosh\chi\sinh\chi\sin\theta\cos\theta\mathrm{d}\chi \end{cases} \\
\Rightarrow &(\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta)\mathrm{d}\theta = \cos\theta\sinh\chi\mathrm{d}x - \sin\theta\cosh\chi\mathrm{d}y \\
\Rightarrow &\mathrm{d}\theta = \frac{\cos\theta\sinh\chi\mathrm{d}x - \sin\theta\cosh\chi\mathrm{d}y}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta}
\end{aligned}
$$

therefore

$$
\begin{aligned}
\frac{\partial\chi}{\partial x} &= \frac{\sin\theta\cosh\chi}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta} \\
\frac{\partial\chi}{\partial y} &= \frac{\cos\theta\sinh\chi}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta} \\
\frac{\partial\theta}{\partial x} &= \frac{\cos\theta\sinh\chi}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta} \\
\frac{\partial\theta}{\partial y} &= \frac{-\sin\theta\cosh\chi}{\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\
&= (\sinh\chi\cos\theta\mathrm{d}\theta + \cosh\chi\sin\theta\mathrm{d}\chi)^2 + (-\cosh\chi\sin\theta\mathrm{d}\theta + \sinh\chi\cos\theta\mathrm{d}\chi)^2 \\
&= \sinh^2\chi\cos^2\theta\mathrm{d}\theta^2 + \cosh^2\chi\sin^2\theta\mathrm{d}\chi^2 + \sinh\chi\cos\theta\cosh\chi\sin\theta\mathrm{d}\theta\mathrm{d}\chi + \\
&\quad \cosh^2\chi\sin^2\theta\mathrm{d}\theta^2 + \sinh^2\chi\cos^2\theta\mathrm{d}\chi^2 - \cosh\chi\sin\theta\sinh\chi\cos\theta\mathrm{d}\theta\mathrm{d}\chi \\
&= (\sinh^2\chi\cos^2\theta + \cosh^2\chi\sin^2\theta)\mathrm{d}\theta^2 + (\cosh^2\chi\sin^2\theta + \sinh^2\chi\cos^2\theta)\mathrm{d}\chi^2
\end{aligned}
$$

when $\sinh\chi = \cosh\chi = r$ and $\mathrm{d}\chi = \frac{\mathrm{d}r}{r}$ they are the same.