4th exercise

1.

a.

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right)$$

$$\begin{aligned}
S_M &= \int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right\} \\
&= \int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{4}g^{\mu\sigma}g^{\nu\sigma}F_{\sigma\rho}F_{\mu\nu} + g^{\mu\sigma}A_\mu J_\sigma\right\}
\end{aligned}$$

$$\begin{aligned}
\delta S_M &= \int \mathrm{d}^4x\delta\sqrt{-g}\left\{-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + g^{\mu\sigma}A_\mu J_\sigma\right\} \\
&\quad+\int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{4}\delta(g^{\mu\sigma}g^{\nu\rho})F_{\sigma\rho}F_{\mu\nu} + \delta g^{\mu\sigma}A_\mu J_\sigma\right\} \\
&= \int \mathrm{d}^4x\frac{-g}{2\sqrt{-g}}g^{\kappa\lambda}\delta g_{\kappa\lambda}\left\{-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + g^{\mu\sigma}A_\mu J_\sigma\right\} \\
&\quad+\int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{4}\delta g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} - \frac{1}{4}g^{\mu\sigma}\delta g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + \delta g^{\mu\sigma}A_\mu J_\sigma\right\} \\
&= \int \mathrm{d}^4x\left(-\frac{1}{2}\right)g_{\kappa\lambda}\delta g^{\kappa\lambda}\left\{-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + g^{\mu\sigma}A_\mu J_\sigma\right\} \\
&\quad+\int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\sigma}\left\{-\frac{1}{2}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + A_\mu J_\sigma\right\} \\
&= -\int \mathrm{d}^4x\frac{1}{2}\sqrt{-g}\delta g^{\kappa\lambda}\left\{-\frac{1}{4}g_{\kappa\lambda}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + g_{\kappa\lambda}g^{\mu\sigma}A_\mu J_\sigma + g^{\nu\rho}F_{\lambda\rho}F_{\kappa\nu} - 2A_\kappa J_\lambda\right\}
\end{aligned}$$

therefore

$$\begin{aligned}
T_{\kappa\lambda} &= -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\kappa\lambda}} \\
&= -\frac{1}{4}g_{\kappa\lambda}g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}F_{\mu\nu} + g_{\kappa\lambda}g^{\mu\sigma}A_\mu J_\sigma + g^{\nu\rho}F_{\lambda\rho}F_{\kappa\nu} - 2A_\kappa J_\lambda
\end{aligned}$$

b.

$$S[g^{\mu\nu}, \partial_\mu A_\nu] = \int \mathrm{d}^4\sqrt{-g}\mathcal{L}'(g^{\mu\nu}, \partial_\mu A_\nu)$$

$$\begin{aligned}
\delta S' &= \delta\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}' \\
&= \int \mathrm{d}^4x\left\{\left(\mathcal{L}'\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} + \sqrt{-g}\frac{\partial\mathcal{L}'}{\partial g^{\mu\nu}}\right)\delta g^{\mu\nu} + \sqrt{-g}\frac{\partial\mathcal{L}'}{\partial(\nabla_\mu A_\nu)}\delta(\nabla_\mu A_\nu)\right\} \\
&= \int \mathrm{d}^4x\left\{\left(\mathcal{L}'\frac{-g}{2\sqrt{-g}}\times(-g_{\mu\nu}) + \sqrt{-g}\frac{\partial\mathcal{L}'}{\partial g^{\mu\nu}}\right)\delta g^{\mu\nu} + \sqrt{-g}\frac{\partial\mathcal{L}'}{\partial(\nabla_\mu A_\nu)}\nabla_\mu\delta A_\nu\right\} \\
&= \int \mathrm{d}^4x\sqrt{-g}\left\{\left(\frac{\partial\mathcal{L}'}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}'g_{\mu\nu}\right)\delta g^{\mu\nu} + \nabla_\mu\left(\frac{\partial\mathcal{L}'}{\partial(\nabla_\mu A_\nu)}\delta A_\nu\right) - \nabla_\mu\frac{\partial\mathcal{L}'}{\delta(\nabla_\mu A_\nu)}\delta A_\nu\right\} \\
&= \int \mathrm{d}^4x\sqrt{-g}\left\{\left(\frac{\partial\mathcal{L}'}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}'g_{\mu\nu}\right)\delta g^{\mu\nu} - \nabla_\mu\frac{\partial\mathcal{L}'}{\partial(\nabla_\mu A_\nu)}\delta A_\nu\right\}
\end{aligned}$$

therefore

$$\begin{aligned}
T'_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S'}{\delta g^{\mu\nu}} \\
&= \mathcal{L}' g_{\mu\nu} - 2\frac{\partial \mathcal{L}'}{\partial g^{\mu\nu}} \\
&= \beta R^{\lambda\kappa} g^{\rho\sigma} F_{\lambda\rho} F_{\kappa\sigma} g_{\mu\nu} - \beta F_{\lambda\rho} F_{\kappa\sigma} \frac{\partial R^{\lambda\kappa}}{\partial g^{\mu\nu}} g^{\rho\sigma} - \beta F_{\lambda\mu} F_{\kappa\nu} R^{\lambda\kappa} \\
&= \beta g^{\sigma\rho} F_{\lambda\rho} F_{\kappa\sigma}\left(R^{\lambda\kappa} g_{\mu\nu} - \frac{\partial R^{\lambda\kappa}}{\partial g^{\mu\nu}}\right) - \beta F_{\lambda\mu} F_{\kappa\nu} R^{\lambda\kappa}
\end{aligned}$$

Einstein eqation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} + T'_{\mu\nu}$$

Maxwell equation

$$\begin{aligned}
-\nabla_\mu \frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)} + J^\nu + \nabla_\mu F^{\mu\nu} &= 0 \\
\nabla_\mu F^{\nu\mu} + 2\beta \nabla_\mu((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma}) &= J^\nu
\end{aligned}$$

part 1. $\frac{\partial \mathcal{L}'}{\partial g^{\mu\nu}}$

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial g^{\mu\nu}} &= \frac{\partial}{\partial g^{\mu\nu}}(\beta R^{\lambda\kappa} g^{\rho\sigma} F_{\lambda\rho} F_{\kappa\sigma}) \\
&= \beta F_{\lambda\rho} F_{\kappa\sigma}\left(\frac{\partial R^{\lambda\kappa}}{\partial g^{\mu\nu}} g^{\rho\sigma} + R^{\lambda\kappa}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}}\right) \\
&= \beta F_{\lambda\rho} F_{\kappa\sigma}\left(\frac{\partial R^{\lambda\kappa}}{\partial g^{\mu\nu}} g^{\rho\sigma} + \frac{1}{2} R^{\lambda\kappa}(\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu)\right) \\
&= \beta F_{\lambda\rho} F_{\kappa\sigma} \frac{\partial R^{\lambda\kappa}}{\partial g^{\mu\nu}} g^{\rho\sigma} + \beta F_{\lambda\mu} F_{\kappa\nu} R^{\lambda\kappa}
\end{aligned}$$

part 2. $\nabla_\mu \frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)}$

$$\begin{aligned}
\nabla_\mu \frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)} &= \beta g^{\rho\sigma} \nabla_\mu\left(R^{\lambda\kappa} \frac{\partial}{\partial(\nabla_\mu A_\nu)} F_{\lambda\rho} F_{\kappa\sigma}\right) \\
&= \beta g^{\rho\sigma} \nabla_\mu\left(R^{\lambda\kappa} \frac{\partial}{\partial(\nabla_\mu A_\nu)} F_{\lambda\rho} F_{\kappa\sigma}\right) \\
&= \beta g^{\rho\sigma} \nabla_\mu\left(R^{\lambda\kappa}\left(F_{\kappa\sigma} \frac{\partial F_{\lambda\rho}}{\partial(\nabla_\mu A_\nu)} + F_{\lambda\rho} \frac{\partial F_{\kappa\sigma}}{\partial(\nabla_\mu A_\nu)}\right)\right) \\
&= \beta g^{\rho\sigma} \nabla_\mu (R^{\lambda\kappa} F_{\kappa\sigma}(\delta^\mu_\lambda \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\lambda) + R^{\lambda\kappa} F_{\lambda\rho}(\delta^\mu_\kappa \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\kappa)) \\
&= \beta \nabla_\mu (g^{\nu\sigma} R^{\mu\kappa} F_{\kappa\sigma} - g^{\mu\sigma} R^{\nu\kappa} F_{\kappa\sigma} + g^{\nu\rho} R^{\lambda\mu} F_{\lambda\rho} - g^{\mu\rho} R^{\lambda\nu} F_{\lambda\rho}) \\
&= 2\beta \nabla_\mu (g^{\nu\sigma} R^{\mu\kappa} F_{\kappa\sigma} - g^{\mu\sigma} R^{\nu\kappa} F_{\kappa\sigma}) \\
&= 2\beta \nabla_\mu ((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma})
\end{aligned}$$

the current

$$J^\nu = \nabla_\mu F^{\nu\mu} + 2\beta \nabla_\mu ((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma})$$

$$\begin{aligned}
\nabla_\nu J^\nu &= \nabla_\nu \nabla_\mu F^{\nu\mu} + 2\beta \nabla_\nu \nabla_\mu ((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma}) \\
&= 0 \times 2\beta \times 0 \\
&= 0
\end{aligned}$$

part 1. $\nabla_\nu \nabla_\mu F^{\nu\mu}$

$$\begin{aligned}
\nabla_\nu \nabla_\mu F^{\nu\mu} &= \nabla_\nu \nabla_\mu \nabla^\nu A^\mu - \nabla_\nu \nabla_\mu \nabla^\mu A^\nu \\
&= \nabla_\nu \nabla_\mu \nabla^\nu A^\mu - \nabla_\mu \nabla_\nu \nabla^\nu A^\mu \\
&= [\nabla_\nu, \nabla_\mu] \nabla^\nu A^\mu \\
&= R^\nu{}_{\rho\nu\mu} \nabla^\rho A^\mu + R^\mu{}_{\rho\nu\mu} \nabla^\nu A^\rho \\
&= R_{\rho\mu} \nabla^\rho A^\mu - R_{\rho\nu} \nabla^\nu A^\rho \\
&= R_{\rho\mu} \nabla^\rho A^\mu - R_{\nu\rho} \nabla^\nu A^\rho \\
&= 0
\end{aligned}$$

part 2. $\nabla_\nu \nabla_\mu ((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma})$

$$\begin{aligned}
& \nabla_\nu \nabla_\mu ((g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho}) F_{\rho\sigma}) \\
= & \nabla_\nu \nabla_\mu (g^{\nu\sigma} R^{\mu\rho} F_{\rho\sigma}) - \nabla_\nu \nabla_\mu (g^{\mu\sigma} R^{\nu\rho} F_{\rho\sigma}) \\
= & \nabla_\nu \nabla_\mu (g^{\nu\sigma} R^{\mu\rho} F_{\rho\sigma}) - \nabla_\mu \nabla_\nu (g^{\nu\sigma} R^{\mu\rho} F_{\rho\sigma}) \\
= & [\nabla_\nu, \nabla_\mu] (g^{\nu\sigma} R^{\mu\rho} F_{\rho\sigma}) \\
= & R^\nu{}_{\kappa\nu\mu} g^{\kappa\sigma} R^{\mu\rho} F_{\rho\sigma} + R^\mu{}_{\kappa\nu\mu} g^{\nu\sigma} R^{\kappa\rho} F_{\rho\sigma} \\
= & R_{\kappa\mu} g^{\kappa\sigma} R^{\mu\rho} F_{\rho\sigma} - R_{\kappa\nu} g^{\nu\sigma} R^{\kappa\rho} F_{\rho\sigma} \\
= & g^{\nu\sigma} R_{\nu\mu} R^{\mu\rho} F_{\rho\sigma} - g^{\nu\sigma} R_{\mu\nu} R^{\mu\rho} F_{\rho\sigma} \\
= & 0
\end{aligned}$$

the current is consered.

2.

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

varying the action

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4 x \{\delta \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\, \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}\} \\
&= \int \mathrm{d}^4 x \sqrt{-g} \left\{ \frac{1}{\sqrt{-g}} \times \frac{-g}{2\sqrt{-g}} (-g_{\mu\nu} \delta g^{\mu\nu}) R + \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right\} \\
&= \int \mathrm{d}^4 x \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} R + \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right\} \\
&= \int \mathrm{d}^4 x \sqrt{-g} \left\{ \delta g^{\mu\nu} \left(-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \right) + g^{\mu\nu} \delta R_{\mu\nu} \right\}
\end{aligned}$$

we known $-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} = 0$ then, that's Einstein equation.

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4 x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} \\
&= \int \mathrm{d}^4 x \sqrt{-g}\, g^{\mu\nu} \{\nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu})\} \\
&= \int \mathrm{d}^4 x \sqrt{-g} (g^{\mu\nu} \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - g^{\mu\nu} \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu})) \\
&= \int \mathrm{d}^4 x \sqrt{-g} (\nabla_\lambda (g^{\mu\nu} \delta \Gamma^\lambda_{\nu\mu}) - \delta \Gamma^\lambda_{\nu\mu} \nabla_\lambda g^{\mu\nu} - \nabla_\nu (g^{\mu\nu} (\delta \Gamma^\lambda_{\lambda\mu})) + \delta \Gamma^\lambda_{\lambda\mu} \nabla_\nu g^{\mu\nu}) \\
&= \int \mathrm{d}^4 x \sqrt{-g} \nabla_\sigma (g^{\mu\nu} \delta \Gamma^\sigma_{\nu\mu} - g^{\mu\sigma} \delta \Gamma^\lambda_{\lambda\mu}) + \int \mathrm{d}^4 x \sqrt{-g} \{\delta \Gamma^\lambda_{\lambda\mu} \nabla_\nu g^{\mu\nu} - \delta \Gamma^\lambda_{\nu\mu} \nabla_\lambda g^{\mu\nu}\}
\end{aligned}$$

$$
\begin{aligned}
&= \int \mathrm{d}^4x\sqrt{-g}\{\delta\Gamma^{\lambda}_{\lambda\mu}\nabla_\nu g^{\mu\nu} - \delta\Gamma^{\lambda}_{\nu\mu}\nabla_\lambda g^{\mu\nu}\} \\
&= \int \mathrm{d}^4x\sqrt{-g}\{\delta\Gamma^{\lambda}_{\sigma\mu}\delta^{\sigma}_{\lambda}\nabla_\nu g^{\mu\nu} - \delta\Gamma^{\lambda}_{\sigma\mu}\nabla_\lambda g^{\mu\sigma}\} \\
&= \int \mathrm{d}^4x\sqrt{-g}\delta\Gamma^{\lambda}_{\sigma\mu}\{\delta^{\sigma}_{\lambda}\nabla_\nu g^{\mu\nu} - \nabla_\lambda g^{\mu\sigma}\}
\end{aligned}
$$

therefore

$$
\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Gamma^{\lambda}_{\sigma\mu}} = \delta^{\sigma}_{\lambda}\nabla_\nu g^{\mu\nu} - \nabla_\lambda g^{\mu\sigma} = 0
$$

if $\sigma = \lambda$ then $\Rightarrow \nabla_\nu g^{\mu\nu} - \nabla_\sigma g^{\mu\sigma} \equiv 0$

if $\sigma \neq \lambda$ then $\Rightarrow -\nabla_\lambda g^{\mu\sigma} = 0$

and λ still could be the same with the μ ,So we get $\nabla_\lambda g^{\mu\sigma} = 0$ without $\sigma \neq \lambda$

that is metric compatible.