3th exercise

9.

$$\begin{cases} x = r\sin\phi\sin\theta \\ y = r\cos\phi\sin\theta \\ z = r\cos\theta \end{cases}$$

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(\theta,\phi)}\right| &= \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix} = \begin{vmatrix} r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ r\cos\phi\cos\theta & -r\sin\phi\sin\theta \end{vmatrix} \\
&= -r^2\sin^2\phi\cos\theta\sin\theta - r^2\cos^2\phi\sin\theta\cos\theta \\
&= -r^2\cos\theta\sin\theta
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial(z,x)}{\partial(\theta,\phi)}\right| &= \begin{vmatrix} \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \end{vmatrix} = \begin{vmatrix} -r\sin\theta & 0 \\ r\sin\phi\cos\theta & r\cos\phi\sin\theta \end{vmatrix} \\
&= -r^2\cos\phi\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial(y,z)}{\partial(\theta,\phi)}\right| &= \begin{vmatrix} \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ -r\sin\theta & 0 \end{vmatrix} \\
&= -r^2\sin\phi\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
*F &= q\sin\theta \mathrm{d}\theta\wedge\mathrm{d}\phi \\
&= q\sin\theta\left|\frac{\partial(x^i,x^j)}{\partial(\theta,\phi)}\right|\mathrm{d}x^i\wedge\mathrm{d}x^j \\
&= q\sin\theta\left(\left|\frac{\partial(x,y)}{\partial(\theta,\phi)}\right|\mathrm{d}x\wedge\mathrm{d}y + \left|\frac{\partial(z,x)}{\partial(\theta,\phi)}\right|\mathrm{d}z\wedge\mathrm{d}x + \left|\frac{\partial(y,z)}{\partial(\theta,\phi)}\right|\mathrm{d}y\wedge\mathrm{d}z\right) \\
&= -q\sin\theta(r^2\cos\theta\sin\theta\mathrm{d}x\wedge\mathrm{d}y + r^2\cos\phi\sin^2\theta\mathrm{d}z\wedge\mathrm{d}x + r^2\sin\phi\sin^2\theta\mathrm{d}y\wedge\mathrm{d}z) \\
&= -qr^2\sin^2\theta(\cos\theta\mathrm{d}x\wedge\mathrm{d}y + \cos\phi\sin\theta\mathrm{d}z\wedge\mathrm{d}x + \sin\phi\sin\theta\mathrm{d}y\wedge\mathrm{d}z) \\
&= *F_{12}\mathrm{d}x\wedge\mathrm{d}y + *F_{31}\mathrm{d}z\wedge\mathrm{d}x + *F_{23}\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}$$

$$\begin{cases} (*F)_{12} &= -qr^2\sin^2\theta\cos\theta \\ (*F)_{23} &= -qr^2\sin^3\theta\sin\phi \\ (*F)_{31} &= -qr^2\sin^3\theta\cos\phi \\ \text{others} &= 0 \end{cases}$$

a. failed Orz

b.

$$\begin{aligned}
*(*F\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) &= \frac{1}{2!}\varepsilon^{\mu\nu}{}_{\sigma\rho}(*F)_{\mu\nu}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho \\
-F_{\sigma\rho}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho &= \varepsilon^{\mu\nu}{}_{\sigma\rho}(*F)_{\mu\nu}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho\ (\mu<\nu) \\
&= (\varepsilon^{12}{}_{\sigma\rho}(*F)_{12} + \varepsilon^{23}{}_{\sigma\rho}(*F)_{23} + \varepsilon^{13}{}_{\sigma\rho}(*F)_{13})\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho \\
F_{\sigma\rho}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho &= -(\varepsilon^{12}{}_{\sigma\rho}(*F)_{12} + \varepsilon^{23}{}_{\sigma\rho}(*F)_{23} + \varepsilon^{13}{}_{\sigma\rho}(*F)_{13})\mathrm{d}x^\sigma\wedge\mathrm{d}x^\rho
\end{aligned}$$

$$\begin{aligned}
F_{01} &= -(*F)_{23} = qr^2\sin^3\theta\sin\phi \\
F_{02} &= (*F)_{13} = qr^2\sin^3\theta\cos\phi \\
F_{03} &= -(*F)_{12} = qr^2\sin^2\theta\cos\theta \\
\text{others} &= 0
\end{aligned}$$

c. see above $c=1$

$$\begin{aligned}\vec{E} &= qr^2\sin^2\theta(\sin\theta\sin\phi\mathrm{d}x + \sin\theta\cos\phi\mathrm{d}y + \cos\theta\mathrm{d}z)\\ &= qr^2\sin^2\theta\mathrm{d}r\end{aligned}$$

$$\vec{B} = 0$$

d.

$$\begin{aligned}r\sin^2\theta &= r\left(1-\left(\frac{z}{r}\right)^2\right)\\ &= r-\frac{z^2}{r} = \sqrt{x^2+y^2+z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial{*F_{23}}}{\partial x} &= \frac{\partial(-qr^2\sin\phi\sin^3\theta)}{\partial x}\\ &= -q\frac{\partial(xr\sin^2\theta)}{\partial x}\\ &= -q - x\frac{\partial(r\sin^2\theta)}{\partial x}\\ &= -q - x\times\frac{x}{r} - x\times\frac{z^2}{r^2}\times\frac{x}{r}\\ &= -q - \frac{x^2}{r} - \frac{x^2z^2}{r^3}\end{aligned}$$

$$\begin{aligned}\frac{\partial{*F_{31}}}{\partial y} &= \frac{\partial(-qr^2\cos\phi\sin^3\theta)}{\partial y}\\ &= -q\frac{\partial(yr\sin^2\theta)}{\partial y}\\ &= -q - \frac{y^2}{r} - \frac{y^2z^2}{r^3}\end{aligned}$$

$$\begin{aligned}\frac{\partial{*F_{12}}}{\partial z} &= \frac{\partial(-qr^2\cos\theta\sin^2\theta)}{\partial z}\\ &= -q\frac{\partial(zr\sin^2\theta)}{\partial z}\\ &= -q - z\frac{\partial r\sin^2\theta}{\partial z}\\ &= -q - z\left(\frac{z}{r} + \frac{z^2}{r^2}\frac{z}{r} - \frac{2z}{r}\right)\\ &= -q + \frac{z^2}{r} - \frac{z^4}{r^3}\end{aligned}$$

$$\begin{aligned}\mathrm{d}\,(*F) &= \mathrm{d}(*F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)\\ &= \mathrm{d}(*F)_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= \partial_\sigma(*F)_{\mu\nu}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= \partial_\sigma(*F)_{\mu\nu}\mathrm{d}x^{[\sigma}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^{\nu]}\\ &= 4!\times\partial_{[\sigma}(*F)_{\mu\nu]}\mathrm{d}x^\sigma\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \text{ and } \sigma<\mu<\nu\\ &= 4!\times\frac{1}{3}(\partial_\sigma(*F)_{\mu\nu} + \partial_\mu(*F)_{\nu\sigma} + \partial_\nu(*F)_{\sigma\mu})\mathrm{d}x^\sigma\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= 8\times(\partial_1(*F)_{23} + \partial_2(*F)_{31} + \partial_3(*F)_{12})\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\ &= 8\times\left(-q-\frac{x^2}{r}-\frac{x^2z^2}{r^3} + -q - \frac{y^2}{r} - \frac{y^2z^2}{r^3} - q + \frac{z^2}{r} - \frac{z^4}{r^3}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&= 8\times\left(-3q-\frac{r^2}{r}+\frac{2z^2}{r}-\frac{z^2r^2}{r^3}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\
&= 8\times\left(-3q-r+\frac{z^2}{r}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
\int\mathrm{d}(*F) &= \int 8\times\left(-3q-r+\frac{z^2}{r}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\
&= 8\times\int\left(-3q-r+\frac{z^2}{r}\right)\mathrm{d}\tau\\
&= 8\times\int(-3q-r+r\cos^2\theta)r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\\
&= -24\times q\times\frac{4}{3}\pi R^3-8\times 4\pi\int_0^R r^3\mathrm{d}r+8\times 2\pi\int_0^R r^3\mathrm{d}r\int_0^\pi\cos^2\theta\sin\theta\mathrm{d}\theta\\
&= -32q\pi R^3-32\pi\times\frac{R^4}{4}+16\pi\times\frac{R^4}{4}\left(-\int_0^\pi\cos^2\theta\mathrm{d}\cos\theta\right)\\
&= -32q\pi R^3-32\pi\times\frac{R^4}{4}+16\pi\times\frac{R^4}{4}\left(\int_{-1}^1 t^2\mathrm{d}t\right)\\
&= 32\pi\left(\frac{1}{2}\times\frac{R^4}{4}\times\frac{2}{3}-qR^3-\frac{R^4}{4}\right)\\
&= -32\pi R^3\left(q+\frac{R}{6}\right)
\end{aligned}
$$

11.

.

$$
\begin{aligned}
S &= \int_\gamma A^{(3)}\\
F^{(4)} &= \mathrm{d}A^{(3)}
\end{aligned}
$$

$F_{[\mu\nu\sigma\rho]}=F_{\mu\nu\sigma\rho}\,;F_{(\mu\nu\sigma\rho)}=0\Rightarrow\mu<\nu<\sigma<\rho$ and $\mu,\nu,\sigma,\rho=0,1,...10$

a. $A^{(1)}\to$0-from particle $\Rightarrow A^{(3)}\to$2-from particle ?

b. $E_{\mu\nu\sigma}\ =\ F_{0\mu\nu\sigma}\Longrightarrow\#E_{\mu\nu\sigma}=3\times C_{10}^3=360$

according to $\nabla_\mu E_\mu=q$ definite charge $q=\nabla_\mu E_{\mu\nu\sigma}$ and is have 36 componets

c. $\left(*F^{(4)}\right)$ is 7-form the $\tilde{A}^{(6)}$ 6-form $\to$ 3-form particle

d. no idea ,but guess $A^{(3)}$

1.

a. $\nabla_\lambda\varepsilon_{\mu\nu\rho\sigma}=0$

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho\sigma}\nabla_\lambda\varepsilon_{\mu\nu\rho\sigma} &= \nabla_\lambda\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}-\varepsilon_{\mu\nu\rho\sigma}\nabla_\lambda\varepsilon^{\mu\nu\rho\sigma}\\
&= \nabla_\lambda 4!-\varepsilon^{\alpha\beta\gamma\eta}g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\eta}\nabla_\lambda\varepsilon^{\mu\nu\rho\sigma}\\
&= -\varepsilon^{\alpha\beta\gamma\eta}\nabla_\lambda g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\eta}\varepsilon^{\mu\nu\rho\sigma}\\
&= -\varepsilon^{\alpha\beta\gamma\eta}\nabla_\lambda\varepsilon_{\alpha\beta\gamma\eta}\\
2\times\varepsilon^{\mu\nu\rho\sigma}\nabla_\lambda\varepsilon_{\mu\nu\rho\sigma} &= 0
\end{aligned}
$$

b. $\nabla_\rho g^{\mu\nu}=0$

$$\begin{aligned}\nabla_\rho\delta_\mu^\nu &= \partial_\rho\delta_\mu^\nu+\Gamma_{\rho\lambda}^\nu\delta_\mu^\lambda-\Gamma_{\rho\mu}^\lambda\delta_\lambda^\nu\\ &= \Gamma_{\rho\mu}^\nu-\Gamma_{\rho\mu}^\nu\\ &= 0\end{aligned}$$

$$\begin{aligned}\nabla_\rho\delta_\mu^\nu &= \nabla_\rho(g^{\nu\sigma}g_{\sigma\mu})\\ 0 &= g^{\nu\sigma}\nabla_\rho g_{\sigma\mu}+g_{\sigma\mu}\nabla_\rho g^{\nu\sigma}\\ 0 &= g_{\sigma\mu}\nabla_\rho g^{\nu\sigma}\end{aligned}$$

therefore $\nabla_\rho g^{\nu\sigma}=0$

2. $\mu,\nu,\sigma,\rho...\in\{x,y,z\}$ $\mu',\nu',\sigma',\rho'...\in\{r,\theta,\phi\}$

$$\begin{cases}x=r\sin\theta\cos\phi\\ y=r\sin\theta\sin\phi\\ z=r\cos\theta\end{cases}$$

gradient.

$$\begin{aligned}\nabla_\mu\phi\hat{e}^\nu\delta_\nu^\mu &= \frac{\partial x^{\mu'}}{\partial x^\mu}\nabla_{\mu'}\phi\delta_\nu^\mu\frac{\partial x^\nu}{\partial x^{\nu'}}\hat{e}^{\nu'}\\ &= \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}\partial_{\mu'}\phi\delta_\nu^\mu\hat{e}^{\nu'}\\ &= \partial_{\mu'}\phi\hat{e}^{\mu'}\\ &= \partial_r\phi\mathrm{d}r+\partial_\theta\phi\mathrm{d}\theta+\partial_\phi\phi\mathrm{d}\phi\end{aligned}$$

div.

$$\begin{aligned}\nabla_\mu V^\mu &= \nabla_\mu\delta_\nu^\mu V^\nu\\ &= \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}\nabla_{\mu'}\delta_\nu^\mu V^{\nu'}\\ &= \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\nu'}}\nabla_{\mu'}V^{\nu'}\\ &= \delta_{\nu'}^{\mu'}\nabla_{\mu'}V^{\nu'}\\ &= \nabla_{\mu'}V^{\mu'}\\ &= \partial_r V^r+\Gamma_{r\mu'}^r V^{\mu'}+\partial_\theta V^\theta+\Gamma_{\theta\mu'}^\theta V^{\mu'}+\partial_\phi V^\phi+\Gamma_{\phi\mu'}^\phi V^{\mu'}\end{aligned}$$

reference:pro_3 $\Gamma_{\lambda\mu}^\lambda=\partial_\mu\ln|\sqrt{g_{\lambda\lambda}}|$ and $g_{rr}=1$; $g_{\theta\theta}=r^2$; $g_{\phi\phi}=r^2\sin^2\theta$

$$\begin{aligned}&\Gamma_{rr}^r=0\quad \Gamma_{\theta r}^\theta=\frac{1}{r}\quad \Gamma_{\phi r}^\phi=\frac{1}{r}\\ &\Gamma_{r\theta}^r=0\quad \Gamma_{\theta\theta}^\theta=0\quad \Gamma_{\phi\theta}^\phi=\frac{1}{\tan\theta}\\ &\Gamma_{r\phi}^r=0\quad \Gamma_{\theta\phi}^\theta=0\quad \Gamma_{\phi\phi}^\phi=0\end{aligned}$$

$$\nabla_{\mu'}V^{\mu'} = \partial_r V^r+\partial_\theta V^\theta+\partial_\phi V^\phi+\frac{2V^r}{r}+\frac{V^\theta}{\tan\theta}$$

curl. reference: $\Gamma^{\lambda}_{\mu\mu}=-\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$

$$
\begin{array}{lll}
\Gamma^r_{\phi\phi}=-\dfrac{1}{2}\partial_r r^2\sin^2\theta & \Gamma^\theta_{\phi\phi}=-\dfrac{1}{2r}\partial_\theta r^2\sin^2\theta & \\
\quad =-r\sin^2\theta & \quad =-r\sin\theta\cos\theta & \Gamma^\phi_{\mu\mu}=0 \\
\Gamma^r_{\theta\theta}=-\dfrac{1}{2}\partial_r r^2=-r & \Gamma^\theta_{rr}=0 &
\end{array}
$$

$$
\begin{aligned}
\varepsilon^{\mu\nu}{}_\rho\nabla_\mu w_\nu \mathrm{d}x^\sigma &= g_{\sigma\rho}\varepsilon^{\mu\nu\rho}\nabla_\mu w_\nu \mathrm{d}x^\sigma \\
&= \frac{g_{\sigma\rho}}{\sqrt{|g|}}\tilde{\varepsilon}^{\mu\nu\rho}\nabla_\mu w_\nu \mathrm{d}x^\sigma \\
&= \frac{g_{rr}\mathrm{d}r}{r^2\sin\theta}(\nabla_\theta w_\phi-\nabla_\phi w_\theta)+\frac{g_{\theta\theta}\mathrm{d}\theta}{r^2\sin\theta}(\nabla_\phi w_r-\nabla_r w_\phi) \\
&\quad +\frac{g_{\phi\phi}\mathrm{d}\phi}{r^2\sin\theta}(\nabla_\theta w_\phi-\nabla_\phi w_\theta) \\
&= \frac{\mathrm{d}r}{r^2\sin\theta}(\partial_\theta w_\phi-\partial_\phi w_\theta-\Gamma^\lambda_{\theta\phi}w_\lambda+\Gamma^\lambda_{\phi\theta}w_\lambda) \\
&\quad \frac{r^2\mathrm{d}\theta}{r^2\sin\theta}(\partial_\phi w_r-\partial_r w_\phi-\Gamma^\lambda_{\phi r}w_\lambda+\Gamma^\lambda_{r\phi}w_\lambda) \\
&\quad \frac{r^2\sin^2\theta\mathrm{d}\phi}{r^2\sin\theta}(\partial_\theta w_\phi-\partial_\phi w_\theta-\Gamma^\lambda_{\theta\phi}w_\lambda+\Gamma^\lambda_{\phi\theta}w_\lambda) \\
&= \frac{\mathrm{d}r}{r^2\sin\theta}(\partial_\theta w_\phi-\partial_\phi w_\theta) \\
&\quad \frac{r\mathrm{d}\theta}{r\sin\theta}(\partial_\phi w_r-\partial_r w_\phi) \\
&\quad \frac{r\sin\theta\mathrm{d}\phi}{r}(\partial_\theta w_\phi-\partial_\phi w_\theta)
\end{aligned}
$$

another version form wiki

$$
\mathrm{grad}\phi = \partial_r\phi+\frac{1}{r}\partial_\theta\phi+\frac{1}{r\sin\theta}\partial_\phi\phi
$$

$$
\begin{aligned}
\mathrm{div}\,\vec{V} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial(V_1r^2\sin\theta)}{\partial r}+\frac{\partial(V_2r\sin\theta)}{\partial\theta}+\frac{\partial(V_3r)}{\partial\phi}\right) \\
&= \frac{1}{r^2\sin\theta}(r^2\sin\theta\partial_rV_1+V_1\times 2\mathrm{r}\sin\theta+\mathrm{r}\sin\theta\partial_\theta V_2+V_2\times \mathrm{r}\cos\theta+r\partial_\phi V_3) \\
&= \partial_rV_1+\frac{2V_1}{r}+\frac{1}{r}\partial_\theta V_2+\frac{V_2}{r\tan\theta}+\frac{1}{r\sin\theta}\partial_\phi V_3 \\
&= \partial_rV_1+\frac{1}{r}\partial_\theta V_2+\frac{1}{r\sin\theta}\partial_\phi V_3+\frac{2V_1}{r}+\frac{V_2}{r\tan\theta}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{curl}\vec{V} &= \frac{1}{r^2\sin\theta}(\partial_\theta(V_3r\sin\theta)-\partial_\phi(V_2r))\vec{e_r} \\
&\quad \frac{r}{r^2\sin\theta}(\partial_\phi V_1-\partial_r(V_3r\sin\theta))\vec{e}_\theta \\
&\quad \frac{r\sin\theta}{r^2\sin\theta}(\partial_r(V_2r)-\partial_\theta V_1)\vec{e}_\phi \\
&= \frac{1}{r^2\sin\theta}(r\sin\theta\partial_\theta V_3+V_3r\cos\theta-r\partial_\phi V_2)\vec{e_r} \\
&\quad \frac{1}{r\sin\theta}(\partial_\phi V_1-r\sin\theta\partial_rV_3-V_3\sin\theta)\vec{e_\theta} \\
&\quad \frac{1}{r}(r\partial_rV_2+V_2-\partial_\theta V_1)\vec{e}_\phi
\end{aligned}
$$

then if transform below : $\nabla_\mu\phi$, $\nabla_\mu V^\mu$ and $\varepsilon^{\mu\nu}{}_\rho\nabla_\mu w_\nu\mathrm{d}x^\sigma$ are consisitent with what in the wiki obviouly!

$$
\begin{array}{rl}
w_r=V_1=V^r & \mathrm{d}r=\vec{e}_r=(\partial_r) \\
\dfrac{w_r}{r}=V_2=rV^\theta & r\mathrm{d}\theta=\vec{e}_\theta=\dfrac{(\partial_\theta)}{r} \\
\dfrac{w_r}{r\sin\theta}=V_3=r\sin\theta V^\phi & r\sin\theta\mathrm{d}\phi=\vec{e}_\phi=\dfrac{(\partial_\phi)}{r\sin\theta}
\end{array}
$$

therefore they are should not be the same,becuse they are one certain quality acting on different kinds of vectors.

ps: I just take a easy case of curl, for the curl of V^μ I haven't made it to transform.

3.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

$g_{\mu\nu} = g_{\mu\nu}\delta^\nu_\mu$

therefore

$$\begin{aligned} g^{ii}g_{ii} &= 1 \\ \Rightarrow g^{ii} &= (g_{ii})^{-1} \end{aligned}$$

-

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu}) \\ &= \frac{1}{2}g^{\lambda\sigma}\partial_\mu g_{\nu\sigma} + \frac{1}{2}g^{\lambda\sigma}\partial_\nu g_{\sigma\mu} \\ &= 0 \end{aligned}$$

-

$$\begin{aligned} \Gamma^\lambda_{\mu\mu} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\mu\sigma} + \partial_\mu g_{\sigma\mu} - \partial_\sigma g_{\mu\mu}) \\ &= \frac{1}{2}g^{\lambda\lambda}(-\partial_\sigma g_{\mu\mu}) \\ &= -\frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\mu\mu} \\ &= -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu} \end{aligned}$$

-

$$\begin{aligned} \Gamma^\lambda_{\mu\lambda} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \\ &= \frac{1}{2}g^{\lambda\lambda}(\partial_\mu g_{\lambda\lambda} + 0 - 0) \\ &= \frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\mu g_{\lambda\lambda} \\ &= \frac{1}{2}\partial_\mu \ln|g_{\lambda\lambda}| \\ &= \partial_\mu \ln|\sqrt{g_{\lambda\lambda}}| \end{aligned}$$

-

$$\begin{aligned} \Gamma^\lambda_{\lambda\lambda} &= \frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\lambda\lambda} \\ &= \partial_\lambda \ln|\sqrt{g_{\lambda\lambda}}| \end{aligned}$$

4.

a.

$$\begin{cases} x = uv\cos\phi \\ y = uv\sin\phi \\ z = \frac{1}{2}(u^2 - v^2) \end{cases}$$

$$
\begin{array}{lll}
\frac{\partial x}{\partial u}=\upsilon\cos\phi & \frac{\partial y}{\partial u}=\upsilon\sin\phi & \frac{\partial z}{\partial u}=u \\
\frac{\partial x}{\partial \upsilon}=u\cos\phi & \frac{\partial y}{\partial \upsilon}=u\sin\phi & \frac{\partial z}{\partial \upsilon}=-\upsilon \\
\frac{\partial x}{\partial \phi}=-u\upsilon\sin\phi & \frac{\partial y}{\partial \phi}=u\upsilon\cos\phi & \frac{\partial z}{\partial \phi}=0
\end{array}
$$

$$
\begin{cases}
\mathrm{d}x=\upsilon\cos\phi\mathrm{d}u+u\cos\phi\mathrm{d}\upsilon-u\upsilon\sin\phi\mathrm{d}\phi \\
\mathrm{d}y=\upsilon\sin\phi\mathrm{d}u+u\sin\phi\mathrm{d}\upsilon+u\upsilon\cos\phi\mathrm{d}\phi \\
\mathrm{d}z=2u\mathrm{d}u-2\upsilon\mathrm{d}\upsilon
\end{cases}
$$

$$
\begin{aligned}
\cos\phi\mathrm{d}x+\sin\phi\mathrm{d}y &= \upsilon(\cos^2\phi+\sin^2\phi)\mathrm{d}u+u(\cos^2\phi+\sin^2\phi)\mathrm{d}\upsilon \\
&= \upsilon\mathrm{d}u+u\mathrm{d}\upsilon
\end{aligned}
$$

$$
\begin{cases}
\upsilon\mathrm{d}u+u\mathrm{d}\upsilon &= \cos\phi\mathrm{d}x+\sin\phi\mathrm{d}y \\
u\mathrm{d}u-\upsilon\mathrm{d}\upsilon &= \frac{1}{2}\mathrm{d}z
\end{cases}
$$

$$
\Rightarrow \begin{cases}
(\upsilon^2+u^2)\mathrm{d}u &= \upsilon(\cos\phi\mathrm{d}x+\sin\phi\mathrm{d}y)+\frac{1}{2}u\mathrm{d}z \\
(u^2+\upsilon^2)\mathrm{d}\upsilon &= u(\cos\phi\mathrm{d}x+\sin\phi\mathrm{d}y)-\frac{1}{2}\upsilon\mathrm{d}z
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\mathrm{d}u &= \frac{\upsilon\cos\phi}{u^2+\upsilon^2}\mathrm{d}x+\frac{\upsilon\sin\phi}{u^2+\upsilon^2}\mathrm{d}y+\frac{u}{2(u^2+\upsilon^2)}\mathrm{d}z \\
\mathrm{d}\upsilon &= \frac{u\cos\phi}{u^2+\upsilon^2}\mathrm{d}x+\frac{u\sin\phi}{u^2+\upsilon^2}\mathrm{d}y-\frac{\upsilon}{2(u^2+\upsilon^2)}\mathrm{d}z
\end{cases}
$$

$$
\begin{aligned}
\mathrm{d}x &= \upsilon\cos\phi\left(\frac{\upsilon\cos\phi}{u^2+\upsilon^2}\mathrm{d}x+\frac{\upsilon\sin\phi}{u^2+\upsilon^2}\mathrm{d}y+\frac{u}{2(u^2+\upsilon^2)}\mathrm{d}z\right) \\
&\quad +u\cos\phi\left(\frac{u\cos\phi}{u^2+\upsilon^2}\mathrm{d}x+\frac{u\sin\phi}{u^2+\upsilon^2}\mathrm{d}y-\frac{\upsilon}{2(u^2+\upsilon^2)}\mathrm{d}z\right) \\
&\quad -u\upsilon\sin\phi\mathrm{d}\phi \\
&= \cos^2\phi\mathrm{d}x+\cos\phi\sin\phi\mathrm{d}y-u\upsilon\sin\phi\mathrm{d}\phi \\
u\upsilon\sin\phi\mathrm{d}\phi\mathrm{d}\phi &= -\sin^2\phi\mathrm{d}x+\cos\phi\sin\phi\mathrm{d}y \\
\mathrm{d}\phi &= -\frac{\sin\phi}{u\upsilon}\mathrm{d}x+\frac{\cos\phi}{u\upsilon}\mathrm{d}y
\end{aligned}
$$

then

$$
\begin{array}{lll}
\frac{\partial u}{\partial x}=\frac{\upsilon\cos\phi}{u^2+\upsilon^2} & \frac{\partial \upsilon}{\partial x}=\frac{u\cos\phi}{u^2+\upsilon^2} & \frac{\partial \phi}{\partial x}=-\frac{\sin\phi}{u\upsilon} \\
\frac{\partial u}{\partial y}=\frac{\upsilon\sin\phi}{u^2+\upsilon^2} & \frac{\partial \upsilon}{\partial y}=\frac{u\sin\phi}{u^2+\upsilon^2} & \frac{\partial \phi}{\partial y}=\frac{\sin\phi}{u\upsilon} \\
\frac{\partial u}{\partial z}=\frac{u}{2(u^2+\upsilon^2)} & \frac{\partial \upsilon}{\partial z}=-\frac{\upsilon}{2(u^2+\upsilon^2)} & \frac{\partial \phi}{\partial z}=0
\end{array}
$$

b.

$$
\begin{aligned}
\partial_u &= \frac{\partial x}{\partial u}\partial_x+\frac{\partial y}{\partial u}\partial_y+\frac{\partial z}{\partial u}\partial_z \\
&= \upsilon\cos\phi\partial_x+\upsilon\sin\phi\partial_y+u\partial_z
\end{aligned}
$$

$$
\begin{aligned}
\partial_\upsilon &= \frac{\partial x}{\partial \upsilon}\partial_x+\frac{\partial y}{\partial \upsilon}\partial_y+\frac{\partial z}{\partial \upsilon}\partial_z \\
&= u\cos\phi\partial_x+u\sin\phi\partial_y-\upsilon\partial_z
\end{aligned}
$$

$$
\begin{aligned}
\partial_\phi &= \frac{\partial x}{\partial \phi}\partial_x+\frac{\partial y}{\partial \phi}\partial_y+\frac{\partial z}{\partial \phi}\partial_z \\
&= -u\upsilon\sin\phi\partial_x+u\upsilon\cos\phi
\end{aligned}
$$

c.

$$
\begin{aligned}
g_{\mu'\nu'} &= \frac{\partial\xi^\mu}{\partial\xi^{\mu'}}\frac{\partial\xi^\nu}{\partial\xi^{\nu'}}g_{\mu\nu} \\
&= \frac{\partial\xi^\mu}{\partial\xi^{\mu'}}\frac{\partial\xi^\nu}{\partial\xi^{\nu'}}\delta_{\mu\nu}
\end{aligned}
$$

set:$\vec{e}_\mu = \left(\frac{\partial x}{\partial \xi^{\mu'}}, \frac{\partial y}{\partial \xi^{\mu'}}, \frac{\partial z}{\partial \xi^{\mu'}}\right)$

$$g_{\mu'\nu'} = \vec{e}_\mu \cdot \vec{e}_\nu$$

$$\begin{aligned}
g_{11} &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial u} \ldots \\
&= \upsilon^2 \cos^2\phi + \upsilon^2 \sin^2\phi + u^2 \\
&= \upsilon^2 + u^2
\end{aligned}$$

$$\begin{aligned}
g_{22} &= \frac{\partial x}{\partial \upsilon}\frac{\partial x}{\partial \upsilon} + \frac{\partial y}{\partial \upsilon}\frac{\partial y}{\partial \upsilon} + \frac{\partial z}{\partial \upsilon}\frac{\partial z}{\partial \upsilon} \\
&= u^2 \cos^2\phi + u^2 \sin^2\phi + \upsilon^2 \\
&= u^2 + \upsilon^2
\end{aligned}$$

$$\begin{aligned}
g_{33} &= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi} \\
&= (-u\upsilon \sin\phi)^2 + (u\upsilon \cos\phi)^2 \\
&= u^2 \upsilon^2
\end{aligned}$$

$$\begin{aligned}
g_{12} = g_{21} &= \vec{e}_1 \cdot \vec{e}_2 \\
&= \upsilon u \cos^2\phi + u\upsilon \sin^2\phi - u\upsilon \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_{13} = g_{31} &= \vec{e}_1 \cdot \vec{e}_3 \\
&= -u\upsilon^2 \cos\phi \sin\phi + u\upsilon^2 \sin\phi \cos\phi \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_{23} = g_{32} &= \vec{e}_2 \cdot \vec{e}_3 \\
&= -u^2 \upsilon \cos\phi \sin\phi + u^2 \upsilon \sin\phi \cos\phi \\
&= 0
\end{aligned}$$

$$g = \begin{pmatrix} u^2 + \upsilon^2 & & \\ & u^2 + \upsilon^2 & \\ & & u^2 \upsilon^2 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} \frac{1}{u^2 + \upsilon^2} & & \\ & \frac{1}{u^2 + \upsilon^2} & \\ & & \frac{1}{u^2 \upsilon^2} \end{pmatrix}$$

d.

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

$$\begin{aligned}
\Gamma^u_{\mu\nu} &= \frac{1}{2} g^{uu} (\partial_\mu g_{\nu u} + \partial_\nu g_{\mu u} - \partial_u g_{\mu\nu}) \\
\Gamma^u_{u\nu} &= \frac{1}{2} g^{uu} (\partial_u g_{\nu u} + \partial_\nu g_{uu} - \partial_u g_{u\nu}) \\
&= \frac{1}{2} g^{uu} \partial_\nu g_{uu} \\
&= \begin{cases} \frac{1}{2} g^{uu} \partial_u g_{uu} & \nu = u \\ \frac{1}{2} g^{uu} \partial_\upsilon g_{uu} & \nu = \upsilon \\ 0 & \nu = \phi \end{cases} = \begin{cases} \frac{u}{u^2 + \upsilon^2} & \nu = u \\ \frac{\upsilon}{u^2 + \upsilon^2} & \nu = \upsilon \\ 0 & \nu = \phi \end{cases} \\
\Gamma^u_{\upsilon\nu} &= \frac{1}{2} g^{uu} (\partial_\upsilon g_{\nu u} + \partial_\nu g_{\upsilon u} - \partial_u g_{\upsilon\nu})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}g^{uu}(\partial_\upsilon g_{\nu u} - \partial_u g_{\upsilon\nu}) \\
&= \begin{cases} \frac{1}{2}g^{uu}(\partial_\upsilon g_{\upsilon u} - \partial_u g_{\upsilon\upsilon}) & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} = \begin{cases} \frac{-\partial_u g_{\upsilon\upsilon}}{2(u^2+\upsilon^2)} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
&= \begin{cases} -\frac{u}{u^2+\upsilon^2} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
\Gamma^u_{\phi\phi} &= \frac{1}{2}g^{uu}(2\partial_\phi g_{\phi u} - \partial_u g_{\phi\phi}) \\
&= \frac{-u\upsilon^2}{u^2+\upsilon^2}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\upsilon_{\mu\nu} &= \frac{1}{2}g^{\upsilon\upsilon}(\partial_\mu g_{\nu\upsilon} + \partial_\nu g_{\mu\upsilon} - \partial_\upsilon g_{\mu\nu}) \\
\Gamma^\upsilon_{u\nu} &= \frac{1}{2}g^{\upsilon\upsilon}(\partial_u g_{\nu\upsilon} + \partial_\nu g_{u\upsilon} - \partial_\upsilon g_{u\nu}) \\
&= \frac{1}{2}g^{\upsilon\upsilon}(\partial_u g_{\nu\upsilon} - \partial_\upsilon g_{u\nu}) \\
&= \begin{cases} \frac{1}{2}g^{\upsilon\upsilon}(\partial_u g_{u\upsilon} - \partial_\upsilon g_{uu}) & \nu=u \\ \frac{1}{2}g^{\upsilon\upsilon}(\partial_u g_{\upsilon\upsilon} - \partial_\upsilon g_{u\upsilon}) & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} = \begin{cases} -\frac{1}{2}g^{\upsilon\upsilon}\partial_\upsilon g_{uu} & \nu=u \\ \frac{1}{2}g^{\upsilon\upsilon}\partial_u g_{\upsilon\upsilon} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
&= \begin{cases} -\frac{\upsilon}{u^2+\upsilon^2} & \nu=u \\ \frac{u}{u^2+\upsilon^2} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
\Gamma^\upsilon_{\upsilon\nu} &= \frac{1}{2}g^{\upsilon\upsilon}(\partial_\upsilon g_{\nu\upsilon} + \partial_\nu g_{\upsilon\upsilon} - \partial_\upsilon g_{\upsilon\nu}) \\
&= \frac{1}{2}g^{\upsilon\upsilon}\partial_\nu g_{\upsilon\upsilon} \\
&= \begin{cases} \frac{1}{2}g^{\upsilon\upsilon}\partial_\upsilon g_{\upsilon\upsilon} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} = \begin{cases} \frac{2\upsilon}{2(u^2+\upsilon^2)} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
&= \begin{cases} \frac{\upsilon}{u^2+\upsilon^2} & \nu=\upsilon \\ 0 & \nu=\phi \end{cases} \\
\Gamma^\upsilon_{\phi\phi} &= \frac{1}{2}g^{\upsilon\upsilon}(2\partial_\phi g_{\phi\upsilon} - \partial_\upsilon g_{\phi\phi}) \\
&= -\frac{2\upsilon u^2}{2(u^2+\upsilon^2)} \\
&= -\frac{\upsilon u^2}{u^2+\upsilon^2}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\phi_{\mu\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_\mu g_{\nu\phi} + \partial_\nu g_{\mu\phi} - \partial_\phi g_{\mu\nu}) \\
&= \frac{1}{2}g^{\phi\phi}(\partial_\mu g_{\nu\phi} + \partial_\nu g_{\mu\phi}) \\
\Gamma^\phi_{u\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_u g_{\nu\phi} + \partial_\nu g_{u\phi}) \\
&= \frac{1}{2}g^{\phi\phi}\partial_u g_{\nu\phi} \\
&= \begin{cases} \frac{1}{2}g^{\phi\phi}\partial_u g_{\phi\phi} & \nu=\phi \\ 0 & \text{others} \end{cases} = \begin{cases} \frac{2u\upsilon^2}{2u^2\upsilon^2} & \nu=\phi \\ 0 & \text{others} \end{cases} \\
&= \begin{cases} \frac{1}{u} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases} \\
\Gamma^\phi_{\upsilon\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_\upsilon g_{\nu\phi} + \partial_\nu g_{\upsilon\phi}) \\
&= \frac{1}{2}g^{\phi\phi}\partial_\upsilon g_{\nu\phi}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{cases} \frac{1}{2} g^{\phi\phi} \partial_\upsilon g_{\phi\phi} & \nu = \phi \\ 0 & \nu \neq \phi \end{cases} \\
&= \begin{cases} \frac{1}{\upsilon} & \nu = \phi \\ 0 & \nu \neq \phi \end{cases} \\
\Gamma^\phi_{\phi\phi} &= \frac{1}{2} g^{\phi\phi}(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi}) = 0
\end{aligned}
$$

e.

$$
\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \\
&= \partial_u V^u + \partial_\upsilon V^\upsilon + \partial_\phi V^\phi + \Gamma^u_{u\lambda} V^\lambda + \Gamma^\upsilon_{\upsilon\lambda} V^\lambda + \Gamma^\phi_{\phi\lambda} V^\lambda \\
&= \partial_u V^u + \partial_\upsilon V^\upsilon + \partial_\phi V^\phi \\
&\quad + \frac{u}{u^2+\upsilon^2} V^u + \frac{\upsilon}{u^2+\upsilon^2} V^\upsilon + \frac{u}{u^2+\upsilon^2} V^u + \frac{\upsilon}{u^2+\upsilon^2} V^\upsilon \\
&\quad + \frac{1}{u} V^u + \frac{1}{\upsilon} V^\upsilon \\
&= \partial_u V^u + \partial_\upsilon V^\upsilon + \partial_\phi V^\phi + \frac{2}{u^2+\upsilon^2}(V^u + V^\upsilon) + \frac{1}{u} V^u + \frac{1}{\upsilon} V^\upsilon
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu \nabla^\mu f &= \partial_u \nabla^u f + \partial_\upsilon \nabla^\upsilon f + \partial_\phi \nabla^\phi f + \frac{2}{u^2+\upsilon^2}(\nabla^u f + \nabla^\upsilon f) + \frac{1}{u} \nabla^u f + \frac{1}{\upsilon} \nabla^\upsilon f \\
& \partial_\mu \partial^\mu f + \partial_\upsilon \partial^\upsilon f + \partial_\phi \partial^\phi f + \frac{2}{u^2+\upsilon^2}(\partial^u f + \partial^\upsilon f) + \frac{1}{u} \partial^u f + \frac{1}{\upsilon} \partial^\upsilon f
\end{aligned}
$$

5. $g = \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}$; $g^{-1} = \begin{pmatrix} 1 & \\ & \frac{1}{\sin^2\theta} \end{pmatrix}$

$$
\begin{aligned}
\Gamma^\theta_{\theta\phi} &= \frac{1}{2} g^{\theta\nu}(\partial_\theta g_{\phi\nu} + \partial_\phi g_{\theta\nu} - \partial_\nu g_{\theta\phi}) \\
&= \frac{1}{2} g^{\theta\theta} \partial_\phi g_{\theta\theta} = 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\theta_{\phi\phi} &= \frac{1}{2} g^{\theta\nu}(2 \times \partial_\phi g_{\phi\nu} - \partial_\nu g_{\phi\phi}) \\
&= \frac{1}{2} g^{\theta\theta}(-\partial_\theta g_{\phi\phi}) \\
&= -\sin\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\phi_{\theta\theta} &= \frac{1}{2} g^{\phi\nu}(2 \times \partial_\theta g_{\theta\nu} - \partial_\nu g_{\theta\theta}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\phi_{\theta\phi} &= \frac{1}{2} g^{\phi\nu}(\partial_\theta g_{\phi\nu} + \partial_\phi g_{\theta\nu} - \partial_\nu g_{\theta\phi}) \\
&= \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = \frac{1}{2\sin\theta}\cos\theta \\
&= \frac{1}{\tan\theta}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\phi_{\phi\phi} &= \frac{1}{2} g^{\phi\nu}(2 \times \partial_\phi g_{\phi\nu} - \partial_\nu g_{\phi\phi}) \\
&= \frac{1}{2} g^{\phi\phi}(2 \times \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}) \\
&= 0
\end{aligned}
$$

a. $\begin{cases} \phi = \phi_0 \\ \theta = \lambda \end{cases}$

$$
\begin{aligned}
\frac{D}{\mathrm{d}\lambda} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} &= \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} + \Gamma^\theta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \\
&= \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} - 2 \times \sin\theta\cos\theta \frac{\mathrm{d}\phi_0}{\mathrm{d}\lambda} \frac{\mathrm{d}\phi_0}{\mathrm{d}\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}=0\\
\frac{D}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi_0}{\mathrm{d}\lambda} &= \frac{\mathrm{d}^2\phi_0}{\mathrm{d}\lambda^2}+\Gamma^{\phi}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\\
&= 0-2\times\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi_0}{\mathrm{d}\lambda}\\
&= 0
\end{aligned}
$$

$$
\begin{cases}\phi=\lambda\\ \theta=\theta_0\end{cases}
$$

$$
\begin{aligned}
\frac{D}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta_0}{\mathrm{d}\lambda} &= \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\lambda}+\Gamma^{\theta}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\\
&= 0+\frac{1}{\tan\theta}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\\
&= \frac{1}{\tan\theta}\\
\frac{D}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} &= \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}+\Gamma^{\phi}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\\
&= \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}+\frac{1}{\tan\theta}\frac{\mathrm{d}\theta_0}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0
\end{aligned}
$$

therefore when $\frac{1}{\tan\theta}=0\Rightarrow\theta=\frac{\pi}{2}$ the curve is geodesic.

b.

$$
\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\mu+\Gamma^{\mu}_{\nu\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}V^\sigma = 0
$$

then set when $\lambda=0$ $V^\mu=(1,0)$

$$
\begin{aligned}
&\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\theta+\Gamma^{\theta}_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}V^\phi = 0\\ \frac{\mathrm{d}}{\mathrm{d}\lambda}V^\phi+\Gamma^{\phi}_{\theta\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}V^\theta+\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}V^\phi\right) = 0\end{cases}\\
\Rightarrow&\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\theta-\sin\theta\cos\theta V^\phi = 0\\ \frac{\mathrm{d}}{\mathrm{d}\lambda}V^\phi+\frac{1}{\tan\theta}V^\theta = 0\end{cases}\\
\Rightarrow&\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\theta = 0\\ \frac{\mathrm{d}}{\mathrm{d}\lambda}V^\phi = 0\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}V^\theta-\sin\theta\sin\theta\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\phi &= \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}V^\theta-\sin\theta\cos\theta\left(-\frac{1}{\tan\theta}V^\theta\right)\\
&= V^{\theta\prime\prime}+\cos^2\theta V^\theta=0
\end{aligned}
$$

then

$$
\begin{aligned}
V^\theta &= A\cos(\cos\theta\lambda)+B\sin(\cos\theta\lambda)\\
V^\theta_0 &= A=1
\end{aligned}
$$

$$
\begin{aligned}
V^\phi &= \frac{1}{\sin\theta\cos\theta}\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\theta\\
&= -\frac{\cos\theta}{\sin\theta\cos\theta}\times\cos\theta\sin(\cos\theta\lambda)+\frac{B\cos\theta}{\sin\theta\cos\theta}\times\cos(\cos\theta\lambda)\\
&= -\frac{1}{\sin\theta}\sin(\cos\theta\lambda)+\frac{B}{\sin\theta}\cos(\cos\theta\lambda)\\
V^\phi_0 &= \frac{B}{\sin\theta}=0\\
&\Rightarrow B=0
\end{aligned}
$$

$$
\vec{V} = \left(\cos(\cos\theta\lambda),\cdot\right)
$$

the curve remain unchange

6.

$$\begin{aligned}
\mathrm{d}s^2 &= -(1+2\Phi)\mathrm{d}t^2+(1-2\Phi)\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\\
&= -\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2-2\Phi(\mathrm{d}t^2+\mathrm{d}r^2)
\end{aligned}$$

$$\begin{aligned}
g &= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} - 2\Phi\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\\
&= \begin{pmatrix} -1-2\Phi & & & \\ & 1-2\Phi & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\Gamma^t_{\mu\nu} &= \frac{1}{2}g^{tt}(\partial_\mu g_{\nu t}+\partial_\nu g_{\mu t}-\partial_t g_{\mu\nu})\\
&= \frac{1}{2}g^{tt}(\partial_\mu g_{\nu t}+\partial_\nu g_{\mu t})\\
\Gamma^t_{t\nu} &= \frac{1}{2}g^{tt}\partial_\nu g_{tt}\\
&= \begin{cases} -\frac{1}{2}\times\frac{2}{-1-2\Phi}\frac{\partial\Phi}{\partial r} & \nu=r\\ 0 & \nu\neq r\end{cases} = \begin{cases} \frac{1}{2(1+2\Phi)}\frac{\partial(1+2\Phi)}{\partial r} & \nu=r\\ 0 & \nu\neq r\end{cases}\\
&= \begin{cases} \frac{1}{2}\frac{\partial\ln(1+2\Phi)}{\partial r} & \nu=r\\ 0 & \nu\neq r\end{cases} = \begin{cases} \frac{\partial_r\Phi}{1+2\Phi} & \nu=r\\ 0 & \nu\neq r\end{cases}\\
\Gamma^t_{r\nu} &= \frac{1}{2}g^{tt}(\partial_r g_{\nu t}+\partial_\nu g_{rt})\\
&= \frac{1}{2}g^{tt}\partial_r g_{\nu t}\\
&= 0(\nu\neq t)\\
\Gamma^t_{\theta\nu} &= \frac{1}{2}g^{tt}(\partial_\theta g_{\nu t}+\partial_\nu g_{\theta t})=0\\
\Gamma^t_{\phi\nu} &= \frac{1}{2}g^{tt}(\partial_\phi g_{\nu t}+\partial_\nu g_{\phi t})=0
\end{aligned}$$

$$\begin{aligned}
\Gamma^r_{\mu\nu} &= \frac{1}{2}g^{rr}(\partial_\mu g_{\nu r}+\partial_\nu g_{\mu r}-\partial_r g_{\mu\nu})\\
\Gamma^r_{t\nu} &= \frac{1}{2}g^{rr}(\partial_t g_{\nu r}+\partial_\nu g_{tr}-\partial_r g_{t\nu})\\
&= -\frac{1}{2}g^{rr}\partial_r g_{t\nu}\\
&= \begin{cases} -\frac{1}{2}g^{rr}\partial_r g_{tt} & \nu=t\\ 0 & \nu\neq t\end{cases} \Rightarrow \begin{cases} -\frac{1}{2}\frac{-2}{1-2\Phi}\partial_r\Phi & \nu=t\\ 0 & \nu\neq t\end{cases}\\
&= \begin{cases} \frac{\partial_r\Phi}{1-2\Phi} & \nu=t\\ 0 & \nu\neq t\end{cases} = \begin{cases} -\frac{1}{2}\partial_r\ln(1-2\Phi) & \nu=t\\ 0 & \nu\neq t\end{cases}\\
\Gamma^r_{r\nu} &= \frac{1}{2}g^{rr}(\partial_r g_{\nu r}+\partial_\nu g_{rr}-\partial_r g_{r\nu})\\
&= \frac{1}{2}g^{rr}\partial_\nu g_{rr}\\
&= \begin{cases} \frac{1}{2}g^{rr}\partial_r g_{rr} & \nu=r\\ 0 & \nu\neq t,r\end{cases} = \begin{cases} \frac{1}{2}\frac{-2}{1-2\Phi}\partial_r\Phi\\ 0\end{cases}\\
&= \begin{cases} \frac{1}{2}\partial_r\ln(1-2\Phi) & \nu=r\\ 0 & \nu\neq t,r\end{cases}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{r}_{\theta\nu} &= \frac{1}{2}g^{rr}(\partial_\theta g_{\nu r}+\partial_\nu g_{\theta r}-\partial_r g_{\theta\nu})\\
&= -\frac{1}{2}g^{rr}\partial_r g_{\theta\nu}\\
&= \begin{cases} -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} & \nu=\theta \\ 0 & \nu=\phi \end{cases} = \begin{cases} -\frac{r}{1-2\Phi} & \nu=0 \\ 0 & \nu=\phi \end{cases}\\
\Gamma^{r}_{\phi\phi} &= \frac{1}{2}g^{\mathrm{rr}}(2\partial_\phi g_{\phi r}-\partial_r g_{\phi\phi}) = -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi}\\
&= -\frac{2r\sin^2\theta}{2(1-2\Phi)} = -\frac{r\sin^2\theta}{1-2\Phi}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\theta}_{\mu\nu} &= \frac{1}{2}g^{\theta\theta}(\partial_\mu g_{\nu\theta}+\partial_\nu g_{\mu\theta}-\partial_\theta g_{\mu\nu})\\
\Gamma^{\theta}_{t\nu} &= \frac{1}{2}g^{\theta\theta}(\partial_t g_{\nu\theta}+\partial_\nu g_{t\theta}-\partial_\theta g_{t\nu}) = 0\\
\Gamma^{\theta}_{r\nu} &= \frac{1}{2}g^{\theta\theta}(\partial_r g_{\nu\theta}+\partial_\nu g_{r\theta}-\partial_\theta g_{r\nu})\\
&= \frac{1}{2}g^{\theta\theta}\partial_r g_{\nu\theta}\\
&= \begin{cases} \frac{1}{2}g^{\theta\theta}\partial_r g_{\theta\theta} & \nu=\theta \\ 0 & \nu\neq\theta \end{cases} = \begin{cases} \frac{2r}{2r^2} \\ 0 \end{cases}\\
&= \begin{cases} \frac{1}{r} & \nu=\theta \\ 0 & \nu\neq\theta \end{cases}\\
\Gamma^{\theta}_{\theta\nu} &= \frac{1}{2}g^{\theta\theta}(\partial_\theta g_{\nu\theta}+\partial_\nu g_{\theta\theta}-\partial_\theta g_{\theta\nu})\\
&= \frac{1}{2}g^{\theta\theta}\partial_\nu g_{\theta\theta}\\
&= 0\,(\nu\neq t,r)\\
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}g^{\theta\theta}(2\partial_\phi g_{\phi\theta}-\partial_\theta g_{\phi\phi})\\
&= -\frac{2r^2\sin\theta\cos\theta}{2r^2} = -\sin\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\phi}_{\mu\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_\mu g_{\nu\phi}+\partial_\nu g_{\mu\phi}-\partial_\phi g_{\mu\nu})\\
\Gamma^{\phi}_{t\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_t g_{\nu\phi}+\partial_\nu g_{t\phi}-\partial_\phi g_{\mu\nu}) = 0\\
\Gamma^{\phi}_{r\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_r g_{\nu\phi}+\partial_\nu g_{r\phi}-\partial_\phi g_{r\nu})\\
&= \frac{1}{2}g^{\phi\phi}\partial_r g_{\nu\phi}\\
&= \begin{cases} \frac{1}{2}g^{\phi\phi}\partial_r g_{\phi\phi} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases} = \begin{cases} \frac{2r\sin^2\theta}{2r^2\sin^2\theta} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases}\\
&= \begin{cases} \frac{1}{r} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases}\\
\Gamma^{\phi}_{\theta\nu} &= \frac{1}{2}g^{\phi\phi}(\partial_\theta g_{\nu\phi}+\partial_\nu g_{\theta\phi}-\partial_\phi g_{\theta\nu})\\
&= \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\nu\phi}\\
&= \begin{cases} \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases} = \begin{cases} \frac{2r^2\sin\theta\cos\theta}{2r^2\sin^2\theta} \\ 0 \end{cases}\\
&= \begin{cases} \frac{1}{\tan\theta} & \nu=\phi \\ 0 & \nu\neq\phi \end{cases}\\
\Gamma^{\phi}_{\phi\phi} &= \frac{1}{2}g^{\phi\phi}(2\partial_\phi g_{\phi\phi}-\partial_\phi g_{\phi\phi}) = 0
\end{aligned}
$$

a. at first assume that $R,\theta,\phi=\text{constant}$

$x^\mu = (t, R, \theta, \phi)$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\sigma\rho} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} - (-\nabla\Phi) \ = \ 0$$

$$\left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \Gamma^t_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau^2} & = & 0 \\ \Gamma^r_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau^2} & = & \partial_r\Phi \\ \Gamma^\theta_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau^2} & = & 0 \\ \Gamma^\phi_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau^2} & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} & = & 0 \\ \Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 & = & \partial_r\Phi \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} & = & 0 \\ \frac{\partial_r\Phi}{1-2\Phi}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 & = & \partial_r\Phi \end{array}\right.$$

$$\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1-2\Phi}$$

therefore : $\ t \ = \ \tau\sqrt{1-2\Phi} + t_0 \ = \ \tau\sqrt{1+\frac{2GM}{R_i}} + t_0$

since $R_2 > R_1$,the t_1 on the tall building,move faster.

b. geodesic

There are $r = R(\text{constant})$ and $\theta = \frac{\pi}{2}$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \ = \ 0$$

$$\left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \Gamma^t_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} & = & 0 \\ \Gamma^r_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} & = & 0 \\ \Gamma^\theta_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} & = & 0 \\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda} + \Gamma^\phi_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} & = & 0 \\ \Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 & = & 0 \\ \Gamma^\theta_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 & = & 0 \\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} & = & 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} & = & 0 \\ \frac{\partial_r\Phi}{1-2\Phi}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - \frac{R\sin^2\frac{\pi}{2}}{1-2\Phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 & = & 0 \\ -\sin\frac{\pi}{2}\cos\frac{\pi}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 & = & 0 \\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} & = & 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcl} \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} & = & 0 \\ \frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2} & = & 0 \\ \partial_r\Phi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 & = & 0 \end{array}\right.$$

therefore

$$\begin{aligned} \frac{\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2}{\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2} &= \frac{\partial_r\Phi}{R} \\ \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 &= \frac{1}{R}\times\left(\frac{GM}{R^2}\right) \\ \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \sqrt{\frac{GM}{R^3}} \end{aligned}$$

the geodesic

$$\begin{cases} t &= \lambda + t_0 \\ r &= R \\ \theta &= \frac{\pi}{2} \\ \phi &= \sqrt{\frac{GM}{R^3}}\lambda + \phi_0 \end{cases}$$

c. two different curve.

form (b) satellite

$$\begin{aligned} \frac{\Delta\phi}{\Delta t} &= \sqrt{\frac{GM}{R^3}} \\ \frac{2\pi}{\Delta t} &= \sqrt{\frac{GM}{R^3}} \end{aligned}$$

$\Delta t = 2\pi\sqrt{\frac{R^3}{GM}}$

proper time:

$$\begin{aligned} \mathrm{d}\tau^2 &= -\frac{1}{c^2}\mathrm{d}s^2 \\ &= -\frac{1}{c^2}(-(c^2+2\Phi)\mathrm{d}t^2 + r^2\sin^2\theta\mathrm{d}\phi) \\ &= \left(1+\frac{2}{c^2}\Phi\right)(2\pi)^2\left(-\frac{R^2}{\Phi}\right) - \frac{R^2}{c^2}(2\pi)^2 \\ &= (2\pi)^2\left(-\frac{R^2}{\Phi} - \frac{2R^2}{c^2} - \frac{R^2}{c^2}\right) \\ &= (2\pi)^2\frac{R^2}{c^2}\left(-\frac{c^2}{\Phi} - 3\right) \\ &= (2\pi)^2\frac{R^2}{-\Phi}\left(1+\frac{3\Phi}{c^2}\right) \end{aligned}$$

$$\begin{aligned} \Delta t &= 2\pi\frac{R_2}{\sqrt{-\Phi}} = 2\pi \times 6.371 \times 10^6 m \times \sqrt{\frac{6.371 \times 10^6 m}{6.674 \times 10^{-11} m^3 \mathrm{kg}^{-1} s^{-2} \times 5.972 \times 10^{24}\mathrm{kg}}} \\ &= 5.061 \times 10^3 s \\ \Delta\tau &= 2\pi\frac{R_2}{\sqrt{-\Phi}}\sqrt{1+\frac{3\Phi}{c^2}} \\ &= \Delta t \times \sqrt{1 - \frac{3 \times 6.674 \times 10^{-11} m^3 \mathrm{kg}^{-1} s^{-2} \times 5.972 \times 10^{24}\mathrm{kg}}{6.371 \times 10^6 m \times (2.998 \times 10^8 m s^{-1})^2}} \\ &= \Delta t \times \sqrt{1 - 2.088 \times 10^{-9}} \\ &= 5.0609 \times 10^3 s \end{aligned}$$