Lorenz transformation

1.Introduction to transformation

1. the derivation

   Hypothesis: the invariance of interval($\mathrm{d}s$) in 4D space-time under the transformation

$$\mathrm{d}s^2 = (c\mathrm{d}t)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

   Symbol clear up:

$$\begin{cases} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$$

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu = \mathrm{d}x^\mu \eta_{\mu\nu} \mathrm{d}x^\nu$$

   easy to see here $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

   a. in a simple case

      consider:

$$c^2\mathrm{d}t^2 - \mathrm{d}x^2 = c^2\mathrm{d}t'^2 - \mathrm{d}x'^2$$

      find a transformation feed it.

      assumption:

$$\begin{cases} \mathrm{d}x = \Lambda^1{}_0 \mathrm{d}ct' + \Lambda^1{}_1 \mathrm{d}x' \\ \mathrm{d}ct = \Lambda^0{}_0 \mathrm{d}ct' + \Lambda^0{}_1 \mathrm{d}x' \end{cases}$$

      then

$$\begin{aligned} (\Lambda^0_0 \mathrm{d}ct' + \Lambda^0_1 \mathrm{d}x')^2 - (\Lambda^1_0 \mathrm{d}ct' + \Lambda^1_1 \mathrm{d}x')^2 &= \mathrm{d}ct'^2 - \mathrm{d}x'^2 \\ \left((\Lambda^0_0)^2 \mathrm{d}ct'^2 + (\Lambda^0_1)^2 \mathrm{d}x'^2 + 2\Lambda^0_0 \Lambda^0_1 \mathrm{d}ct' \mathrm{d}x'\right) - ((\Lambda^1_0)^2 \mathrm{d}ct' + (\Lambda^1_1)^2 \mathrm{d}x' + \\ 2\Lambda^1_0 \Lambda^1_1 \mathrm{d}ct' \mathrm{d}x') &= \mathrm{d}ct'^2 - \mathrm{d}x'^2 \end{aligned}$$

$$\begin{cases} (\Lambda^0_0)^2 - (\Lambda^1_0)^2 = 1 \\ (\Lambda^0_1)^2 - (\Lambda^1_1)^2 = -1 \\ \Lambda^0_0 \Lambda^0_1 - \Lambda^1_0 \Lambda^1_1 = 0 \end{cases}$$

I haven't solve the equation groups(could solve). just know there are one free degree.

and

$$\begin{cases} a^2-c^2=1 \\ b^2-d^2=-1 \\ ab-cd=0 \end{cases}$$

more clear

set $\frac{a}{c}=\frac{d}{b}=u$ then $a=cu$; $d=bu$

$$\begin{cases} c^2u^2-c^2=1 \\ b^2-b^2u^2=-1 \end{cases} \Rightarrow \begin{cases} c^2=\frac{1}{u^2-1} \\ b^2=\frac{1}{u^2-1} \end{cases}$$
$$\Rightarrow c^2=b^2$$

and

$$\begin{cases} a=cu \\ d=bu \end{cases} \Rightarrow \begin{cases} a=\gamma_1\frac{u}{\sqrt{u^2-1}} \\ d=\gamma_2\frac{u}{\sqrt{u^2-1}} \end{cases}$$
$$(\gamma_i^2=1)$$

in the special case that $\mathrm{d}x'=0$ then

$$\begin{cases} \mathrm{d}x=\Lambda^1{}_0\mathrm{d}ct' \\ \mathrm{d}ct=\Lambda^0{}_0\mathrm{d}ct' \end{cases} \Rightarrow \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\Lambda^1_0}{\Lambda^0_0}$$
$$\frac{v}{c}=\frac{1}{u}$$

now feel confused at the meaning of v

and then

$$\begin{pmatrix} \gamma_1\frac{u}{\sqrt{u^2-1}} & \gamma_2\frac{1}{\sqrt{u^2-1}} \\ \gamma_1\frac{1}{\sqrt{u^2-1}} & \gamma_2\frac{u}{\sqrt{u^2-1}} \end{pmatrix} = \begin{pmatrix} \gamma_1\frac{\frac{c}{v}}{\sqrt{(\frac{c}{v})^2-1}} & \gamma_2\frac{1}{\sqrt{(\frac{c}{v})^2-1}} \\ \gamma_1\frac{1}{\sqrt{(\frac{c}{v})^2-1}} & \gamma_2\frac{\frac{c}{v}}{\sqrt{(\frac{c}{v})^2-1}} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_1\frac{1}{\sqrt{1-(\frac{v}{c})^2}} & \gamma_2\frac{\frac{v}{c}}{\sqrt{1-(\frac{v}{c})^2}} \\ \gamma_1\frac{\frac{v}{c}}{\sqrt{1-(\frac{v}{c})^2}} & \gamma_2\frac{1}{\sqrt{1-(\frac{v}{c})^2}} \end{pmatrix}$$

I don't know the reason that $\gamma_i=1$ in the end

and the determination

$$|\Lambda| = \gamma_1\gamma_2$$

b. in the general case

$$\mathrm{d}(ct)^2-\mathrm{d}x^2-\mathrm{d}y^2-\mathrm{d}z^2 \;=\; \mathrm{d}(ct')^2-\mathrm{d}x'^2-\mathrm{d}y'^2-\mathrm{d}z'^2$$

then similarly

$$\mathrm{d}x^i=\Lambda^i{}_j\mathrm{d}x'^j$$

$$\begin{aligned}
\mathrm{d}(ct)^2 &= \left(\Lambda^0{}_j\mathrm{d}x'^j\right)^2\\
&= \Lambda^0{}_j\Lambda^0{}_i\mathrm{d}x'^j\mathrm{d}x'^i\\
\mathrm{d}x^2 &= \Lambda^1{}_j\Lambda^1{}_i\mathrm{d}x'^j\mathrm{d}x'^i\\
\mathrm{d}y^2 &= \Lambda^2{}_j\Lambda^2{}_i\mathrm{d}x'^j\mathrm{d}x'^i\\
\mathrm{d}z^2 &= \Lambda^3{}_j\Lambda^3{}_i\mathrm{d}x'^j\mathrm{d}x'^i
\end{aligned}$$

therefore

$$\Lambda^0{}_j\Lambda^0{}_i\mathrm{d}x'^j\mathrm{d}x'^i - \Lambda^1{}_j\Lambda^1{}_i\mathrm{d}x'^j\mathrm{d}x'^i - \Lambda^2{}_j\Lambda^2{}_i\mathrm{d}x'^j\mathrm{d}x'^i - \Lambda^3{}_j\Lambda^3{}_i\mathrm{d}x'^j\mathrm{d}x'^i = \mathrm{d}x'^0\mathrm{d}x'^0-\mathrm{d}x'^m\mathrm{d}x'^m$$

$$\begin{cases}
\Lambda^0{}_0\Lambda^0{}_0-\Lambda^1{}_0\Lambda^1{}_0-\Lambda^2{}_0\Lambda^2{}_0-\Lambda^3{}_0\Lambda^3{}_0=1\\
\Lambda^0{}_1\Lambda^0{}_1-\Lambda^1{}_1\Lambda^1{}_1-\Lambda^2{}_1\Lambda^2{}_1-\Lambda^3{}_1\Lambda^3{}_1=-1\\
\Lambda^0{}_2\Lambda^0{}_2-\Lambda^1{}_2\Lambda^1{}_2-\Lambda^2{}_2\Lambda^2{}_2-\Lambda^3{}_2\Lambda^3{}_2=-1\\
\Lambda^0{}_3\Lambda^0{}_3-\Lambda^1{}_3\Lambda^1{}_3-\Lambda^2{}_3\Lambda^2{}_3-\Lambda^3{}_3\Lambda^3{}_3=-1\\
\ldots=0\\
\ldots=0\\
\ldots\text{orz}
\end{cases}$$

i. the other angle

if the quaity v really mean the the speed of coor-O' related to coor-O

then set $\gamma_i=\frac{1}{\sqrt{1-\frac{v_i^2}{c}}}$; $\beta_i=\frac{v_i}{c}$,

$$\begin{aligned}
\|\Lambda\| &= \|\Lambda_x\|\|\Lambda_y\|\|\Lambda_z\|\\
&= \begin{pmatrix}\gamma_x & \beta_x\gamma_x & & \\ \beta_x\gamma_x & \gamma_x & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}\gamma_y & & \beta_y\gamma_y & \\ & 1 & & \\ \beta_y\gamma_y & & \gamma_y & \\ & & & 1\end{pmatrix}\begin{pmatrix}\gamma_z & & & \beta_z\gamma_z\\ & 1 & & \\ & & 1 & \\ \beta_z\gamma_z & & & \gamma_z\end{pmatrix}\\
&= \begin{pmatrix}\gamma_x\gamma_y & \beta_x\gamma_x & \beta_y\gamma_y\gamma_x & 0\\ \beta_x\gamma_x\gamma_y & \gamma_x & \beta_x\beta_y\gamma_x\gamma_y & 0\\ \beta_y\gamma_y & 0 & \gamma_y & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_z & & & \beta_z\gamma_z\\ & 1 & & \\ & & 1 & \\ \beta_z\gamma_z & & & \gamma_z\end{pmatrix}\\
&= \begin{pmatrix}\gamma_x\gamma_y\gamma_z & \beta_x\gamma_x & \beta_y\gamma_y\gamma_x & \beta_z\gamma_x\gamma_y\gamma_z\\ \beta_x\gamma_x\gamma_y\gamma_z & \gamma_x & \beta_x\beta_y\gamma_x\gamma_y & \beta_x\beta_z\gamma_x\gamma_y\gamma_z\\ \beta_y\gamma_y\gamma_z & 0 & \gamma_y & \beta_y\beta_z\gamma_y\gamma_z\\ 0 & 0 & 0 & \gamma_z\end{pmatrix}
\end{aligned}$$

it must be wrong

ii. the other angle too (wiki:transformation boost)

$$\begin{pmatrix} ct' \\ x' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_x & -\beta_x\gamma_x & & \\ -\beta_x\gamma_x & \gamma_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ & R \end{pmatrix}\begin{pmatrix} t' \\ \vec{r}\,' \end{pmatrix} = \begin{pmatrix} 1 & \\ & R \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & R \end{pmatrix}^{-1}\begin{pmatrix} 1 & \\ & R \end{pmatrix}\begin{pmatrix} t \\ \vec{r} \end{pmatrix}$$

therefore

$$\begin{aligned}
\text{matrix} &= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma & 0 & 0 \end{pmatrix} \\ R\begin{pmatrix} -\beta\gamma \\ 0 \\ 0 \end{pmatrix} & R\begin{pmatrix} \gamma & & \\ & 1 & \\ & & 1 \end{pmatrix} \end{pmatrix}\begin{pmatrix} 1 & \\ & R^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma & 0 & 0 \end{pmatrix}R^{-1} \\ R\begin{pmatrix} -\beta\gamma \\ 0 \\ 0 \end{pmatrix} & R\begin{pmatrix} \gamma & & \\ & 1 & \\ & & 1 \end{pmatrix}R^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma & 0 & 0 \end{pmatrix}R^{-1} \\ R\begin{pmatrix} -\beta\gamma \\ 0 \\ 0 \end{pmatrix} & R\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}+\begin{pmatrix} \gamma-1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\right)R^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma & 0 & 0 \end{pmatrix}R^{-1} \\ R\begin{pmatrix} -\beta\gamma \\ 0 \\ 0 \end{pmatrix} & \mathbb{I}+R\begin{pmatrix} \gamma-1 & & \\ & 0 & \\ & & 0 \end{pmatrix}R^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma R^{-1}_{11} & -\beta\gamma R^{-1}_{12} & -\beta\gamma R^{-1}_{21} \end{pmatrix} \\ \begin{pmatrix} -\beta\gamma R_{11} \\ -\beta\gamma R_{21} \\ -\beta\gamma R31 \end{pmatrix} & \mathbb{I}+(\gamma-1)\begin{pmatrix} R_{11} & & \\ R_{21} & 0 & \\ R_{31} & & 0 \end{pmatrix}R^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma & \begin{pmatrix} -\beta\gamma R^{-1}_{11} & -\beta\gamma R^{-1}_{12} & -\beta\gamma R^{-1}_{21} \end{pmatrix} \\ \begin{pmatrix} -\beta\gamma R_{11} \\ -\beta\gamma R_{21} \\ -\beta\gamma R31 \end{pmatrix} & \mathbb{I}+(\gamma-1)\begin{pmatrix} R_{11}R^{-1}_{11} & R_{11}R^{-1}_{12} & R_{11}R^{-1}_{13} \\ R_{21}R^{-1}_{11} & R_{21}R^{-1}_{12} & R_{31}R_{13} \\ R_{31}R^{-1}_{11} & R_{21}R^{-1}_{12} & R_{31}R_{13} \end{pmatrix} \end{pmatrix}
\end{aligned}$$

becuase of I know the answer in wiki.

then

$$R_{i1} = R^{-1}_{1i} = \frac{\beta_i}{\beta} = \frac{v_i}{v} = \frac{x^i}{\sqrt{x^j x^j}}$$

oh! I forget it because of

$$R\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

then

$$\vec{e}_i = \sum_j R_{ij}\vec{a}_j$$

therefore

$$R_{ij} = \vec{e}_i \cdot \vec{a}_j$$

under the representation formal.

$$R = \begin{pmatrix} \vec{e}_1\cdot\vec{a}_1 & \vec{e}_1\cdot\vec{a}_2 & \vec{e}_1\cdot\vec{a}_3 \\ \vec{e}_2\cdot\vec{a}_1 & \vec{e}_2\cdot\vec{a}_2 & \vec{e}_2\cdot\vec{a}_3 \\ \vec{e}_3\cdot\vec{a}_1 & \vec{e}_3\cdot\vec{a}_2 & \vec{e}_3\cdot\vec{a}_3 \end{pmatrix}$$

uesless seems like orz.

and how to get the R seems wired

1. reference : textbook of classical-mechanics Euler-angel

$$R=\begin{pmatrix} \cos\psi\cos\phi-\sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi+\sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi-\cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi+\cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$\cos\psi\cos\phi-\sin\psi\cos\theta\sin\phi = \mathrm{orz}$$

2. the form of lorenz transformation

   there three kinds of

   a. boost

      PS: $\tan h\phi_i = \frac{v_i}{c}$

$$\begin{pmatrix} \cos h\phi_1 & -\sin h\phi_1 & & \\ -\sin h\phi_1 & \cos h\phi_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos h\phi_2 & & -\sin h\phi_2 & \\ & 1 & & \\ -\sin h\phi_2 & & \cos h\phi_2 & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos h\phi_3 & & & -\sin h\phi_3 \\ & 1 & & \\ & & 1 & \\ -\sin h\phi_3 & & & \cos h\phi_3 \end{pmatrix}$$

   b. spatial transformation

$$\begin{pmatrix} 1 & & & \\ & \cos\varphi & -\sin\varphi & \\ & \sin\varphi & \cos\varphi & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & & -\sin\theta \\ & & 1 & \\ & \sin\theta & & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\psi & -\sin\psi \\ & & \sin\psi & \cos\psi \end{pmatrix}$$

tell nothing about the (x, y, z)

c. translation

it seems to be more different: we can write down it as the matrix representaion.

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}$$