Casimir operators

1. Enveloping algebra

1.1.

the Evneloping algebra of a lie algebra L —— $\mathrm{Env}(L)$ is the set of all possible linearly polynomials of generators in L

and I can't learn it strictly

2. Casimir operators

. Casimir operators $K_i$is a kind of Enveloping algebra. And with some more constrain

Definitiong.

- it could be expressed by the polynomial generator in L
- $[K_i, a_j] = 0$, a_j is arbitary generators of L
- $\#\{K_i\} =$ I can't learn it clear

example.

1. the casimir operators in quantum mechanic

   in the lie algebra, whose generators are $\hat{l}_x, \hat{l}_y, \hat{l}_z$.

   - the casimir operators is $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$
   - and the I'm sure $\left[\hat{l}^2, \hat{l}_i\right] = 0$
   - I don't know.

2. the casimir operators in Poincare group

   the operator is

$$\begin{cases} T^\mu = \partial_\mu = \frac{\partial}{\partial x^\mu} \\ T^{mn} = x^m \partial^n - x^n \partial^m \end{cases}$$

   - and frmm the class ,they needn't me to find the casimir operators [and I don't how to find it expect enumerating]

$$\begin{cases} K_1 = \nabla^2 \,\mathrm{or}\, \Delta = \partial_\mu \partial^\mu \\ K_2 = w^2 = w_\mu w^\mu \end{cases}$$

   and $w_\mu = \frac{1}{2} \varepsilon_{\mu v s \sigma} T^{vs} T^\sigma$

   - now compute the communicators

     a. $[\Delta, T^\mu]$

$$\begin{aligned} [\Delta, T^\mu] f(x) &= (\Delta T^\mu - T^\mu \Delta) f \\ &= (\partial_v \partial^v \partial^\mu - \partial^\mu \partial_v \partial^v) f \\ &= 0 \\ &\quad \text{of course} \end{aligned}$$

b. $[\Delta, T^{\mu\upsilon}]$

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\begin{aligned}
[\Delta, T^{\mu\upsilon}]f &= (\Delta T^{\mu\upsilon} - T^{\mu\upsilon}\Delta)f \\
&= \partial_s\partial^s(x^\mu\partial^\upsilon f - x^\upsilon\partial^\mu f) - (x^\mu\partial^\upsilon - x^\upsilon\partial^\mu)\partial_s\partial^s f \\
&= \partial_s\partial^s(x^\mu\partial^\upsilon f) - \partial_s\partial^s(x^\upsilon\partial^\mu f) - x^\mu\partial^\upsilon\partial_s\partial^s f + x^\upsilon\partial^\mu\partial_s\partial^s f \\
&= \partial^s(\partial_s(x^\mu\partial^\upsilon f)) - \partial^s(\partial_s(x^\upsilon\partial^\mu f)) - x^\mu\partial^\upsilon\partial_s\partial^s f + x^\upsilon\partial^\mu\partial_s\partial^s f \\
&= \partial^s(\delta_s^\mu\partial^\upsilon f) + \partial^s(x^\mu\partial_s\partial^\upsilon f) - \partial^s(\delta_s^\upsilon\partial^\mu f) - \partial^s(x^\upsilon\partial_s\partial^\mu f) - \\
&\quad x^\mu\partial^\upsilon\partial_s\partial^s f + x^\upsilon\partial^\mu\partial_s\partial^s f \\
&= \partial^\mu\not\partial^\upsilon f \quad - \quad \partial^u\not\partial^\mu f \quad + \quad (\partial^s x^\mu)\partial_s\partial^\upsilon f \ + \\
&\quad x^\mu\partial^s\partial_s\partial^\upsilon f \not- x^\mu\partial^s\partial_s\partial^\upsilon f \quad - \quad (\partial^s x^\upsilon)\partial_s\partial^\mu f \quad - \\
&\quad x^\upsilon\partial^s\partial_s\partial^\mu f \not+ x^\upsilon\partial^s\partial_s\partial^\mu f \\
&= \partial^s\eta^{\mu k}x_k\partial_s\partial^\upsilon f - \partial^s\eta^{\upsilon k}x_k\partial_s\partial^\mu f \\
&= \eta^{\mu k}\delta_k^s\partial_s\partial^\upsilon f - \eta^{\upsilon k}\delta_k^s\partial_s\partial^\mu f \\
&= \eta^{\mu s}\partial_s\partial^\upsilon f - \eta^{\upsilon s}\partial_s\partial^\mu f \\
&= \partial^\mu\partial^\upsilon f - \partial^\upsilon\partial^\mu f \\
&= 0
\end{aligned}
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c. $[w^2, T^\mu]$

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\begin{aligned}
w^2 &= w_\mu w^\mu \\
&= \frac{1}{2}\varepsilon_{\mu\upsilon\rho\sigma}T^{\upsilon\rho}T^\sigma\frac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}T_{\alpha\beta}T_\gamma \\
&= \frac{1}{4}\varepsilon_{\mu\upsilon\rho\sigma}\varepsilon^{\mu\alpha\beta\gamma}T^{\upsilon\rho}T^\sigma T_{\alpha\beta}T_\gamma \\
&= \frac{1}{4}\big(\delta_\upsilon^\alpha\delta_\rho^\beta\delta_\sigma^\gamma \ - \ \delta_\upsilon^\alpha\delta_\sigma^\beta\delta_\rho^\gamma \ + \ \delta_\sigma^\alpha\delta_\upsilon^\beta\delta_\rho^\gamma \ - \ \delta_\rho^\alpha\delta_\upsilon^\beta\delta_\sigma^\gamma \ + \ \delta_\rho^\alpha\delta_\sigma^\beta\delta_\upsilon^\gamma \ - \\
&\quad \delta_\sigma^\alpha\delta_\rho^\beta\delta_\upsilon^\gamma\big)T^{\upsilon\rho}T^\sigma T_{\alpha\beta}T_\gamma \\
&= \frac{1}{4}(T^{\upsilon\rho}T^\sigma T_{\upsilon\rho}T_\sigma - T^{\upsilon\rho}T^\sigma T_{\upsilon\sigma}T_\rho + T^{\upsilon\rho}T^\sigma T_{\sigma\upsilon}T_\rho - T^{\upsilon\rho}T^\sigma T_{\rho\upsilon}T_\sigma + \\
&\quad T^{\upsilon\rho}T^\sigma T_{\rho\sigma}T_\upsilon - T^{\upsilon\rho}T^\sigma T_{\sigma\rho}T_\upsilon) \\
&= \frac{1}{4}(2T^{\upsilon\rho}T^\sigma T_{\upsilon\rho}T_\sigma + 2T^{\upsilon\rho}T^\sigma T_{\sigma\upsilon}T_\rho + 2T^{\upsilon\rho}T^\sigma T_{\rho\sigma}T_\upsilon) \\
&= \frac{1}{2}(T^{\upsilon\rho}T^\sigma T_{\upsilon\rho}T_\sigma + T^{\upsilon\rho}T^\sigma T_{\sigma\upsilon}T_\rho + T^{\upsilon\rho}T^\sigma T_{\rho\sigma}T_\upsilon)
\end{aligned}
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\begin{aligned}
T^{\upsilon\rho}T^\sigma T_{\upsilon\rho}T_\sigma &= T^{\upsilon\rho}T^\sigma[T_{\upsilon\rho}, T_\sigma] + T^{\upsilon\rho}T^\sigma T_\sigma T_{\upsilon\rho} \\
&= T^{\upsilon\rho}T^\sigma(\eta_{\rho\sigma}T_\upsilon - \eta_{\upsilon\sigma}T_\rho) + T^{\upsilon\rho}\Delta T_{\upsilon\rho} \\
&= \eta_{\rho\sigma}T^{\upsilon\rho}T^\sigma T_\upsilon - \eta_{\upsilon\sigma}T^{\upsilon\rho}T^\sigma T_\rho + \Delta T^{\upsilon\rho}T_{\upsilon\rho} \\
&= T^{\upsilon\rho}(T_\rho T_\upsilon - T_\upsilon T_\rho) + \Delta T^{\upsilon\rho}T_{\upsilon\rho} \\
&= \Delta(T^{\upsilon\rho})^2
\end{aligned}
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\begin{aligned}
[T_{\upsilon\rho}, T_\sigma] &= T_{\upsilon\rho}T_\sigma - T_\sigma T_{\upsilon\rho} \\
&= \eta_{\upsilon k}\eta_{\rho l}T^{kl}\eta_{j\sigma}T^j - \eta_{j\sigma}T^j\eta_{\upsilon k}\eta_{\rho l}T^{kl} \\
&= \eta_{\upsilon k}\eta_{\rho l}\eta_{j\sigma}[T^{kl}, T^j] \\
&= \eta_{\upsilon k}\eta_{\rho l}\eta_{j\sigma}(\eta^{lj}T^k - \eta^{kj}T^l) \\
&= \eta_{\upsilon k}\eta_{\rho l}(\delta_\sigma^l T^k - \delta_\sigma^k T^l) \\
&= \eta_{\upsilon k}\eta_{\rho\sigma}T^k - \eta_{\upsilon\sigma}\eta_{\rho l}T^l \\
&= \eta_{\rho\sigma}T_\upsilon - \eta_{\upsilon\sigma}T_\rho
\end{aligned}
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$$\begin{aligned}T^{\upsilon\rho}T^{\sigma}T_{\sigma\upsilon}T_{\rho} &= T^{\upsilon\rho}[T^{\sigma},T_{\sigma\upsilon}]T_{\rho}+T^{\upsilon\rho}T_{\sigma\upsilon}T^{\sigma}T_{\rho}\\ &= T^{\upsilon\rho}T_{\sigma\upsilon}T^{\sigma}T_{\rho}\end{aligned}$$

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$$\begin{aligned}[T^{\sigma},T_{\sigma\upsilon}] &= -[T_{\sigma\upsilon},T^{\sigma}]\\ &= -(T_{\sigma\upsilon}T^{\sigma}-T^{\sigma}T_{\sigma\upsilon})\\ &= -(T_{\sigma\upsilon}\eta^{\sigma k}T_k-\eta^{\sigma k}T_kT_{\sigma\upsilon})\\ &= -\eta^{\sigma k}[T_{\sigma\upsilon},T_k]\\ &= -\eta^{\sigma k}(\eta_{\upsilon k}T_{\sigma}-\eta_{\sigma k}T_{\upsilon})\\ &= -(\delta^{\sigma}_{\upsilon}T_{\sigma}-\delta^{\sigma}_{\sigma}T_{\upsilon})\\ &= 0\end{aligned}$$

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$$\begin{aligned}T^{\upsilon\rho}T^{\sigma}T_{\rho\sigma}T_{\upsilon} &= -T^{\upsilon\rho}T^{\sigma}T_{\sigma\rho}T_{\upsilon}\\ &= -T^{\upsilon\rho}T_{\sigma\rho}T^{\sigma}T_{\upsilon}\end{aligned}$$

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$$\begin{aligned}T^{\upsilon\rho}T^{\sigma}T_{\sigma\upsilon}T_{\rho}+T^{\upsilon\rho}T^{\sigma}T_{\rho\sigma}T_{\upsilon} &= T^{\upsilon\rho}T_{\sigma\upsilon}T_{\rho}T^{\sigma}-T^{\upsilon\rho}T_{\sigma\rho}T_{\upsilon}T^{\sigma}\\ &= T^{\upsilon\rho}T_{\sigma\upsilon}T^{\sigma}T_{\rho}-T^{\rho\upsilon}T_{\sigma\upsilon}T^{\sigma}T_{\rho}\\ &= 2T^{\upsilon\rho}T_{\sigma\upsilon}T^{\sigma}T_{\rho}\end{aligned}$$

then

$$w^2=\frac{1}{2}\Delta T^{\upsilon\rho}T_{\upsilon\rho}+T^{\rho\upsilon}T_{\upsilon\sigma}T^{\sigma}T_{\rho}$$

$$\begin{aligned}[\Delta(T^{\upsilon\rho})^2,T^{\mu}] &= \Delta(T^{\upsilon\rho})^2T^{\mu}-T^{\mu}\Delta(T^{\upsilon\rho})^2\\ &= \Delta(T^{\upsilon\rho}T_{\upsilon\rho}T^{\mu}-T^{\mu}T^{\upsilon\rho}T_{\upsilon\rho})\\ &= \Delta(T^{\upsilon\rho}[T_{\upsilon\rho},T^{\mu}]+T^{\upsilon\rho}T^{\mu}T_{\upsilon\rho}-T^{\mu}T^{\upsilon\rho}T_{\upsilon\rho})\\ &= \Delta(T^{\upsilon\rho}(\delta^{\mu}_{\rho}T_{\upsilon}-\delta^{\mu}_{\upsilon}T_{\rho})+[T^{\upsilon\rho},T^{\mu}]T_{\upsilon\rho}+T^{\mu}T^{\upsilon\rho}T_{\upsilon\rho}-\\ &\quad T^{\mu}T^{\upsilon\rho}T_{\upsilon\rho})\\ &= \Delta(T^{\upsilon\mu}T_{\upsilon}-T^{\mu\rho}T_{\rho}+\eta^{\rho\mu}T^{\upsilon}T_{\upsilon\rho}-\eta^{\upsilon\mu}T^{\rho}T_{\upsilon\rho})\\ &= \Delta(T^{\upsilon\mu}T_{\upsilon}-T^{\mu\rho}T_{\rho}+\eta^{\rho\mu}\eta^{\upsilon k}T_k\eta_{\upsilon i}\eta_{\rho j}T^{ij}-\\ &\quad \eta^{\upsilon\mu}\eta^{\rho k}T_k\eta_{\upsilon i}\eta_{\rho j}T^{ij})\\ &= \Delta(T^{\upsilon\mu}T_{\upsilon}-T^{\mu\rho}T_{\rho}+\delta^{\mu}_{j}\delta^{k}_{i}T_kT^{ij}-\delta^{\mu}_{i}\delta^{k}_{j}T_kT^{ij})\\ &= \Delta(T^{\upsilon\mu}T_{\upsilon}-T^{\mu\rho}T_{\rho}+T_kT^{k\mu}-T_kT^{\mu k})\\ &= \Delta(2T^{\mu v}T_v-2T_vT^{\mu v})\\ &= 2\Delta[T^{\mu v},T_v]\\ &= 0\end{aligned}$$

$$\begin{aligned}[T^{\upsilon\rho}T_{\sigma\upsilon}T^{\sigma}T_{\rho},T^{\mu}] &= (T^{\upsilon\rho}T_{\sigma\upsilon}T^{\mu}-T^{\mu}T^{\upsilon\rho}T_{\sigma\upsilon})T^{\sigma}T_{\rho}\\ &= (T^{\upsilon\rho}[T_{\sigma\upsilon},T^{\mu}]+T^{\upsilon\rho}T^{\mu}T_{\sigma\upsilon}-T^{\mu}T^{\upsilon\rho}T_{\sigma\upsilon})T^{\sigma}T_{\rho}\\ &= (T^{\upsilon\rho}(\delta^{\mu}_{\upsilon}T_{\sigma}-\delta^{\mu}_{\sigma}T_{\upsilon})+[T^{\upsilon\rho},T^{\mu}]T_{\sigma\upsilon}+T^{\mu}T^{\upsilon\rho}T_{\sigma\upsilon}-\\ &\quad T^{\mu}T^{\upsilon\rho}T_{\sigma\upsilon})T^{\sigma}T_{\rho}\\ &= (T^{\mu\rho}T_{\sigma}-\delta^{\mu}_{\sigma}T^{\upsilon\rho}T_{\upsilon}+\eta^{\rho\mu}T^{\upsilon}T_{\sigma\upsilon}-\eta^{\upsilon\mu}T^{\rho}T_{\sigma\upsilon})T^{\sigma}T_{\rho}\end{aligned}$$

$$
\begin{aligned}
&= (T^{\mu\rho}T_\sigma - \delta^\mu_\sigma T^{\upsilon\rho}T_\upsilon + \eta^{\rho\mu}\eta^{\upsilon k}T_k\eta_{\sigma i}\eta_{\upsilon j}T^{ij} - \\
&\quad \eta^{\upsilon\mu}\eta^{\rho k}T_k\eta_{\sigma i}\eta_{\upsilon j}T^{ij})T^\sigma T_\rho \\
&= (T^{\mu\rho}T_\sigma - \delta^\mu_\sigma T^{\upsilon\rho}T_\upsilon + \delta^k_j\eta^{\rho\mu}\eta_{\sigma i}T_kT^{ij} - \\
&\quad \delta^\mu_j\eta^{\rho k}\eta_{\sigma i}T_kT^{ij})T^\sigma T_\rho \\
&= (T^{\mu\rho}T_\sigma - \delta^\mu_\sigma T^{\upsilon\rho}T_\upsilon + \eta^{\rho\mu}\eta_{\sigma i}T_kT^{ik} - \\
&\quad \eta^{\rho k}\eta_{\sigma i}T_kT^{i\mu})T^\sigma T_\rho \\
&\neq 0?
\end{aligned}
$$