The properties of the generators of Poincare transformation

1. the generators as operator

$$\begin{cases} T_{mn} = x_m\partial_n - x_n\partial_m \\ T_\mu = \partial_\mu \end{cases}$$

similarly

$$\begin{cases} T^{mn} = x^m\partial^n - x^n\partial^m \\ T^\mu = \partial^\mu \end{cases}$$

they are form a lie algebra as the generator with respect to Poincare group

2. check the communicate relation they have

   a. $[T^\mu, T^v]$

$$[T^\mu, T^v]f(x)$$

PS: here $f(x) \Leftrightarrow f(x_0, x_1, x_2, x_3)$

$$\begin{aligned} [T^\mu, T^v]f(x) &= (\partial^\mu\partial^v - \partial^v\partial^\mu)f \\ &= \partial^\mu\partial^v f - \partial^v\partial^\mu f \\ &= 0 \\ &\quad \text{of course} \end{aligned}$$

   b. $[T^{mn}, T^v]$

$$\begin{aligned} [T^{mn}, T^v]f &= (T^{mn}T^v - T^vT^{mn})f \\ &= ((x^m\partial^n - x^n\partial^m)\partial^v - \partial^v(x^m\partial^n - x^n\partial^m))f \\ &= (x^m\partial^n - x^n\partial^m)\partial^v f - \partial^v(x^m\partial^n - x^n\partial^m)f \\ &= x^m\partial^n\partial^v f - x^n\partial^m\partial^v f - \partial^v(x^m\partial^n f) + \partial^v(x^n\partial^m f) \\ &= x^m\partial^n\partial^v f - x^n\partial^m\partial^v f - (\partial^v x^m)\partial^n f - x^m\partial^v\partial^n f + (\partial^v x^n)\partial^m f + \\ &\quad x^n\partial^v\partial^m f \\ &= (\partial^v x^n)\partial^m f - (\partial^v x^m)\partial^n f \\ &= ((\partial^v\eta^{nk}x_k)\partial^m - (\partial^v\eta^{mk}x_k)\partial^n)f \\ &= (\eta^{nk}(\partial^v x_k)\partial^m - \eta^{mk}(\partial^v x_k)\partial^n)f \\ &= (\eta^{nk}\delta^v_k\partial^m - \eta^{mk}\delta^v_k\partial^n)f \\ &= (\eta^{nv}\partial^m - \eta^{mv}\partial^n)f \\ &= (\eta^{nv}T^m - \eta^{mv}T^n)f \end{aligned}$$

$$[T^{mn}, T^v] = \eta^{nv}T^m - \eta^{mv}T^n$$

   c. $[T^{mn}, T^{\mu v}]$

$$\begin{aligned} [T^{mn}, T^{\mu v}]f &= ((x^m\partial^n - x^n\partial^m)(x^\mu\partial^v - x^v\partial^\mu) - (x^\mu\partial^v - x^v\partial^\mu)(x^m\partial^n - x^n\partial^m))f \\ &= (x^m\partial^n - x^n\partial^m)(x^\mu\partial^v f - x^v\partial^\mu f) - (x^\mu\partial^v - x^v\partial^\mu)(x^m\partial^n f - x^n\partial^m f) \\ &= (x^m\partial^n - x^n\partial^m)x^\mu\partial^v f - (x^m\partial^n - x^n\partial^m)x^v\partial^\mu f - (x^\mu\partial^v - \\ &\quad x^v\partial^\mu)x^m\partial^n f + (x^\mu\partial^v - x^v\partial^\mu)x^n\partial^m f \\ &= x^m\partial^n(x^\mu\partial^v f) - x^n\partial^m(x^\mu\partial^v f) - x^m\partial^n(x^v\partial^\mu f) + x^n\partial^m(x^v\partial^\mu f) - \\ &\quad x^\mu\partial^v(x^m\partial^n f) + x^v\partial^\mu(x^m\partial^n f) + x^\mu\partial^v(x^n\partial^m f) - x^v\partial^\mu(x^n\partial^m f) \end{aligned}$$

$$
\begin{aligned}
& x^m(\partial^n x^\mu)\partial^v f + x^m x^\mu \partial^n \partial^v f \\
& -x^n(\partial^m x^\mu)\partial^v f - x^n x^\mu \partial^m \partial^v f \\
& -x^m(\partial^n x^v)\partial^\mu f - x^m x^v \partial^n \partial^\mu f \\
= \; & x^n(\partial^m x^v)\partial^\mu f + x^n x^v \partial^m \partial^\mu f \\
& -x^\mu(\partial^v x^m)\partial^n f - x^\mu x^m \partial^v \partial^n f \\
& x^v(\partial^\mu x^m)\partial^n f + x^v x^m \partial^\mu \partial^n f \\
& x^\mu(\partial^v x^n)\partial^m f + x^\mu x^n \partial^v \partial^m f \\
& -x^v(\partial^\mu x^n)\partial^m f - x^v x^n \partial^\mu \partial^m f \\
= \; & (x^m \eta^{\mu n}\partial^v - x^n \eta^{\mu m}\partial^v - x^m \eta^{vn}\partial^\mu + x^n \eta^{vm}\partial^\mu - x^\mu \eta^{mv}\partial^n + \\
& x^v \eta^{m\mu}\partial^n + x^\mu \eta^{nv}\partial^m - x^v \eta^{n\mu}\partial^m) f \\
= \; & (\eta^{\mu n}(x^m\partial^v - x^v\partial^m) + \eta^{vm}(x^n\partial^\mu - x^\mu\partial^n) + \eta^{m\mu}(x^v\partial^n - x^n\partial^v) + \\
& \eta^{nv}(x^\mu\partial^m - x^m\partial^\mu)) f \\
= \; & (\eta^{\mu n}T^{mv} + \eta^{vm}T^{n\mu} + \eta^{m\mu}T^{vn} + \eta^{nv}T^{\mu m}) f
\end{aligned}
$$

then

$$
[T^{mn}, T^{\mu v}] = \eta^{\mu n}T^{mv} + \eta^{vm}T^{n\mu} + \eta^{m\mu}T^{vn} + \eta^{nv}T^{\mu m}
$$