Assignment # 2 Solution

#1

$x_0 = 0 \Rightarrow f[x_0] = 0$]$\Rightarrow f[x_0, x_1] = \frac{2}{\pi}$]$\Rightarrow f[x_0, x_1, x_2] = -\frac{4}{\pi^2}$]$\Rightarrow f[x_0, x_1, x_2, x_3] = \frac{8}{3\pi^3}$

$x_1 = \pi/2 \Rightarrow f[x_1] = 1$]$\Rightarrow f[x_1, x_2] = -\frac{2}{\pi}$]$\Rightarrow f[x_1, x_2, x_3] = 0$

$x_2 = \pi \Rightarrow f[x_2] = 0$]$\Rightarrow f[x_2, x_3] = -\frac{2}{\pi}$

$x_3 = 3\pi/2 \Rightarrow f[x_3] = -1$

(1) $a_0 = 0$; $a_1 = \frac{2}{\pi}$ and $a_2 = -\frac{4}{\pi^2}$.

(2) $P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$$= 0 + \frac{2}{\pi}(x - 0) + \left(-\frac{4}{\pi^2}\right)(x - 0)(x - \pi/2) = \frac{2}{\pi}x - \frac{4}{\pi^2}\left(x^2 - \frac{\pi}{2}x\right)$$

$$\Rightarrow P_2(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x.$$

(3) Here: $a_3 = \frac{8}{3\pi^3}$.

So, $P_3(x) = P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2)$

$$= -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x + \frac{8}{3\pi^3}x(x - \pi/2)(x - \pi)$$

$$= -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x + \frac{8}{3\pi^3}\left(x^3 - \frac{3\pi}{2}x^2 + \frac{\pi^2}{2}x\right)$$

$$= \frac{8}{3\pi^3}x^3 + \left(-\frac{4}{\pi^2} - \frac{4}{\pi^2}\right)x^2 + \left(\frac{4}{\pi} + \frac{4}{3\pi}\right)x$$

$$\Rightarrow P_3(x) = \frac{8}{3\pi^3}x^3 - \frac{8}{\pi^2}x^2 + \frac{16}{3\pi}x.$$

(4) $|f(x) - P_3(x)| = \left|\frac{1}{4!}\sin''''(\xi)(x - x_0)(x - x_1)(x - x_2)(x - x_3)\right|$

$\therefore w(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) = x(x - \pi/2)(x - \pi)(x - 3\pi/2)$.

(5) $|f(x) - P_3(x)| \le \frac{1}{4!}|\sin''''(\xi)|_{max}\,|w(x)|_{max}$

Now, for $I = [0, 3\pi/2] \Rightarrow |\sin''''(\xi)| = 1$

Now, for $w(x) = x^4 - 3\pi x^3 + \frac{11}{4}\pi^2x^2 - \frac{3}{4}\pi^3 x \Rightarrow w'(x) = 4x^3 - 9\pi x^2 + \frac{11}{2}\pi^2 x - \frac{3}{4}\pi^3$.

Hence: $w'(x) = 0 \Rightarrow x = \frac{3\pi}{4}$ and $(3 \pm \sqrt{5})\frac{\pi}{4}$

And $\left|w\left(\frac{3\pi}{4}\right)\right| = \frac{9}{256}\pi^4$ and $\left|w\left((3 \pm \sqrt{5})\pi/4\right)\right| = \frac{\pi^4}{16} = \frac{16\pi^4}{256}$

Therefore: $|f(x) - P_3(x)| \le \frac{1}{4!}(1)\left(\frac{\pi^4}{16}\right) = 0.25367$.