MAT120: Integral Calculus and Differential Equations
BRAC University

Syed Zuhair Hossain
St. ID - 19101573
Section - 07
Set-Q
Assignment - 04

September 9, 2020

1 Evaluate the integral

$\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2+y^2}\, dx\ dy\ dz$

Solution :

$$\begin{aligned} &Let, \\ u &= \frac{x}{y} \qquad\qquad \Rightarrow x^2 = y^2u^2 \\ du &= \frac{1}{y}dx \\ \therefore dx &= y\ du \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{y}{x^2+y^2}\ dx &= \int \frac{y^2}{y^2u^2+y^2}du \\ &= \int \frac{y^2}{u^2+1}du \\ &= \int \frac{1}{u^2+1}du \\ &= tan^{-1}(u) \\ &= tan^{-1}\frac{x}{y} \end{aligned}$$

$$\int_1^2 \int_2^3 (\frac{\pi}{3}) dy dx$$

$$= \int_1^2 \frac{\pi}{3} \times [y]_z^2 \, dz$$

$$= \int_1^2 \frac{\pi}{3}(2 - z) dz$$

$$= \int_1^2 \frac{2\pi}{3} dz - \int_1^2 \frac{\pi}{3} z dz$$

$$= \frac{2\pi}{3} \int_1^2 dz - \frac{\pi}{3} \int_1^2 z dz$$

$$= \frac{2\pi}{3} \int_1^2 dz - \left[\frac{\pi}{3} \times \frac{z^2}{2}\right]_1^2$$

$$= \frac{2\pi}{3} \int_1^2 dz + (-\frac{4\pi}{6} + \frac{1 \cdot \pi}{6})$$

$$= \frac{2\pi}{3} \int_1^2 dz - \frac{\pi}{2}$$

$$= \frac{2\pi \times 2}{3} - \frac{2\pi \cdot 1}{3} - \frac{\pi}{2}$$

$$= \frac{2\pi}{3} - \frac{\pi}{2}$$

$$= \frac{4\pi - 3\pi}{6}$$

$$= \frac{\pi}{6} \qquad [Answer]$$

2 Solve

$$(x+1)\frac{dy}{dx} + y = lnx, y(1) = 10$$

Converting into standard form:

$$\frac{dy}{dx} + \left(\frac{1}{x+1}\right) y = \frac{ln(x)}{(x+1)} \qquad [\textit{dividing by (x+1) in both side}]$$

according to the formula

$$y' + \rho(t)y = g(t)$$

$$\therefore \rho(x) = e^{\int \frac{1}{x+1} dx}$$

$$= e^{ln(x+1)}$$

$$= (x+1)$$

$$\therefore (x+1)y = \int ln(x)dx$$

$$\Rightarrow (x+1)y = xln(x) - x + c$$

$$\therefore y = \frac{x(ln(x) - x + c)}{x+1}$$

apply the given condition,

$$y(1) = 10 \qquad x = 1$$

$$\Rightarrow \frac{1 \times ln(1) - 1 + c}{1+1} = 10$$

$$\Rightarrow \frac{0 - 1 + c}{2} = 0$$

$$c = 21$$

$$\therefore y = \frac{xln(x) - x + 21}{(x+1)} \qquad \textbf{[Answer]}$$

3 Evaluate $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} dydx$

$$
\begin{aligned}
&= \int_1^4 \frac{3}{2} \left[e^{\frac{y}{\sqrt{x}}} \times \sqrt{x} \right]_0^{\sqrt{x}} dx \\
&= \int_1^4 \frac{3}{2} \left[e^1 \cdot \sqrt{x} - e^0 \cdot \sqrt{x} \right] dx \\
&= \int_1^4 \frac{3}{2} (e\sqrt{x} - \sqrt{x}) dx \\
&= \int_1^4 \frac{3}{2} \times \sqrt{x} \times (e-1) dx \\
&= \frac{3}{2} \int_1^4 \sqrt{x}(e-1) dx \\
&= \frac{3}{2} \times (e-1) \int_1^4 \sqrt{x} dx \\
&= \frac{3}{2}(e-1)[\frac{2x^{\frac{3}{2}}}{3}]_1^4 \\
&= (e-1)[x^{\frac{3}{2}}]_1^4 \\
&= (e-1)[4^{\frac{3}{2}} - 1] \\
&= 7(e-1) \\
&\approx 12.0279728 \qquad \textbf{[Answer]}
\end{aligned}
$$

4 Solve the differential equation using variables separable method : $\mathbf{x}^2\frac{dy}{dx} = y - xy; y(-1) = -1.$

$$\Rightarrow x^2\frac{dx}{dy} = y(1-x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1-x}{x*2} = \frac{1}{x^2} - \frac{1}{x}$$
$$\Rightarrow \int y^{-1}dy = \int \frac{(1-x)}{x^2}dx$$

Now, integrating both sides,

$$\int \frac{dy}{y} = \int \frac{1}{x^2}dx - \int \frac{1}{x}dx$$
$$\Rightarrow ln(y) = -\frac{1}{x} - ln(x) + c$$

Given that,

$$y = -1 \qquad x = -1$$
$$\therefore ln|-1| = \frac{-1}{-1} - ln|-1| + c$$
$$(1) = 1 - ln(1) + c$$
$$\therefore c = 0$$

$$\therefore logy = \frac{1}{x} - log(x) - 1$$
$$\Rightarrow log(y) + log(x) = -\frac{1}{x} - 1$$
$$\Rightarrow log(yx) = -\frac{1}{x} - 1$$
$$\Rightarrow yx = e^{-\frac{1}{x}-1}$$
$$\Rightarrow yx = e^{\frac{-1}{x}}e^{-1}$$
$$\Rightarrow y = \frac{e^{\frac{-1}{x}}}{e^x} \qquad \textbf{[Answer]}$$

5 Evaluate the integral: $\int_0^{\frac{\pi}{4}} \int_0^1 \int_0^{x^2} xcosy\ dz\ dx\ dy$

$$= \int_0^{\frac{\pi}{4}} \int_0^1 [xzcosy]_0^{x^2} dxdy$$
$$= \int_0^{\frac{\pi}{4}} \int_0^1 (x \cdot x^2 cosy - x \cdot 0 \cdot cosy) dxdy$$
$$= \int_0^{\frac{\pi}{4}} \int_0^1 x^3 cosydxdy$$
$$= \int_0^{\frac{\pi}{4}} cosy \int_0^1 x^3 dxdy$$
$$= \int_0^{\frac{\pi}{4}} \left[\frac{x^4}{4} cos(y)\right]_0^1 dy$$
$$= \int_0^{\frac{\pi}{4}} \left[\frac{x^4}{4} cos(y) - \frac{0}{4} cos(y)\right] dy$$
$$= \int_0^{\frac{\pi}{4}} \frac{cosy}{4} dy$$
$$= \frac{1}{4} \int_0^{\frac{\pi}{4}} cos\ y\ dy$$
$$= \left[\frac{1}{4} \times siny\right]_0^{\frac{\pi}{4}}$$
$$= \left[\frac{1}{4} \times siny\right]_0^{\frac{\pi}{4}}$$
$$= (\frac{1}{4} sin\frac{\pi}{4} - \frac{1}{4} sin(0))$$
$$= \frac{1}{4} \times \frac{1}{\sqrt{2}} - 0$$
$$= \frac{1}{4\sqrt{2}}$$

[Answer]

6 Solve the system for x and y in terms of u and v then find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

u = x-y; v = 2x+y

$$Here,$$
$$u = x - y............(1)$$
$$v = 2x + y...........(2)$$

adding equation (1) and (2) we get,

$$u + v = 3x$$
$$x = \frac{u+v}{3}$$
$$x = \frac{u}{3} + \frac{v}{3}$$

Again, extracting equation (2) from (1) we get,

$$v - u = 2y - x$$
$$2y = v - u + x$$
$$2y = v - u + \frac{u}{3} + \frac{v}{3}$$
$$y = \frac{3v - 3u + u + v}{3 \times 2}$$
$$y = \frac{4v - 2u}{6}$$
$$y = \frac{2v}{3} - \frac{1 \cdot u}{3}$$

$$\frac{\partial x}{\partial u} = \frac{1}{3} \qquad \frac{\partial y}{\partial u} = -\frac{2}{3}$$

$$\frac{\partial x}{\partial v} = \frac{1}{3} \qquad \frac{\partial y}{\partial u} = \frac{1}{3}$$

Jacobian,J $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3} \times \dfrac{1}{3} - \left(-\dfrac{2}{3}\right) \times \dfrac{1}{3} = \dfrac{1}{9} + \dfrac{2}{9} = \dfrac{1}{3}$

[Answer]