MAT120: Integral Calculus and Differential Equations
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Section - 07
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1 Evaluate the following integrals by Integration by Parts

1.1 $\int x(tan^{-1}x)dx$

Solution

Let, u = x

$$v = tan^{-1}x$$

du = x dx

$$dv = \frac{1}{x^2+1}dx$$

$u = \frac{x^2}{2}$

According to integration by parts formula, we know that,
$\int u\ v\ dx = u \int vdx - \int \frac{du}{dx}(\int vdx)dx$

$\Rightarrow \int u\ v\ dx = \frac{x^2}{2} \int tan^{-1}x - \int \frac{x^2}{2(x^2+1)}dx$

$\Rightarrow \int u\ v\ dx = \frac{x^2}{2} \int tan^{-1}x - \frac{1}{2} \int (1 - \frac{x^2}{2(x^2+1)})dx$

$\Rightarrow \int u\ v\ dx = \frac{x^2}{2} \int tan^{-1}x - \frac{1}{2} \int 1\ dx + \frac{1}{2} \int (\frac{1}{x^2+1})dx$

$\Rightarrow \int u\ v\ dx = \frac{x^2}{2} \int tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\ tan^{-1}\ (x) + c \qquad [\because \frac{1}{x^2+1}] = tan^{-1}\ (x)]$

$\therefore\ \int u\ v\ dx = \frac{1}{2}(x^2+1)tan^{-1}(x) - x + c \qquad [_{answer}]$

1.2 $\int \sqrt{x}\, ln \sqrt{x}\, dx$

$= \int \sqrt{x}\, ln(x^{\frac{1}{2}})\, dx$
$= \frac{1}{2} \int \sqrt{x}\, ln(x)\, dx$

Let,

$u = ln(x)$ $\qquad\qquad$ $v = \sqrt{x}$

$du = \frac{1}{x}\, dx$ $\qquad\qquad$ $v = \frac{2x^{\frac{3}{2}}}{3}\, dx$

According to integration by parts formula, we know that,
$\int u\, v\, dx = u \int v dx - \int \frac{du}{dx} (\int v dx) dx$

$$\int \sqrt{x}\, ln \sqrt{x} \;= ln(x)\, \frac{2}{3}\, (x^{\frac{3}{2}})\, \frac{1}{3}\; \int \sqrt{x}\, dx$$

$$\Rightarrow \int \sqrt{x}\, ln \sqrt{x} \;= \frac{1}{3}\, x^{\frac{3}{2}}\, ln(x) - \frac{2}{9}\, x^{\frac{3}{2}} + c$$

$$\Rightarrow \int \sqrt{x}\, ln \sqrt{x} \;= \frac{1}{9}\, x^{\frac{3}{2}}\, (3\, ln(x) - 2) \; + c$$ [answer]

2 Use reduction formula to evaluate: $\int_0^{\frac{\pi}{2}} cos^6(x)\, dx$

$According\, to\, reduction\; formula\; we\; know\; that,$

$$\int cos^n(x)\, dx = \frac{1}{n}\, cos^{n-1}(x) sin(x) \; + \frac{n-1}{n} \int cos^{n-2}(x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x) dx = \frac{1}{6} \int_0^{\frac{\pi}{2}} cos^{6-1}(x)\; sinx \; + \frac{6-1}{6} \int_0^{\frac{\pi}{2}} cos^{6-2}(x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = [\frac{1}{6} cos^5 x\; sinx]_0^{\frac{\pi}{2}} \; + \frac{5}{6} \int_0^{\frac{\pi}{2}} cos^4 x\, dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = 0 + \frac{5}{6} \int_0^{\frac{\pi}{2}} cos^4(x) dx \qquad [\because \frac{1}{6} cos^5(\frac{\pi}{2})\; sin(\frac{\pi}{2}) \; - \frac{1}{6} cos^5(0) sin(0) = 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = \frac{5}{6} (\frac{1}{4} cos^{4-1} x\; sinx + \frac{4-1}{4} \int_{\frac{\pi}{2}}^0 cos^{4-2} x\, dx) \qquad [\because\; applying\, reduction\, method]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = \left[\frac{5}{24} cos^3 x\; sinx\right]_0^{\frac{\pi}{2}} \; + \frac{5}{8} \int_0^{\frac{\pi}{2}} cos^2 x\, dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = 0 + \frac{5}{8} \int_0^{\frac{\pi}{2}} cos^2(x)\, dx \qquad \left[\because \frac{5}{24} cos^3(\frac{\pi}{2})\; sin(\frac{\pi}{2}) - \frac{5}{24} cos^3(0) sin(0) = 0\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\, dx = \frac{5}{8} \int_0^{\frac{\pi}{2}} cos(2x) + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\, dx$$

$$
\begin{aligned}
&let, \\
&\quad u = 2x \\
&\because du = 2\,dx \\
&So,\ we\ get\ a\ new\ lower\ bound\ from\ here, \\
&\quad u = 2 \times 0 \\
&\quad\ \ = 0 \\
&and\ in\ upper\ bound \\
&\quad u = 2 \times \frac{\pi}{2} \\
&\quad\ \ = \pi \\
&So, \\
&\quad \frac{5}{32}\int_0^{\pi} cos(u)du + \frac{5}{16}\int_0^{\frac{\pi}{2}} 1dx
\end{aligned}
$$

$Applying\ fundamental\ theorem\ of\ calculus$

$$
\left[\frac{5sin(u)}{32}\right]_0^{\pi} = \frac{5sin(\pi)}{32} - \frac{5sin(0)}{32} \qquad \left[\therefore \int cos(u)\ =\ sin(u)\right]
$$

$$
\left[\frac{5sin(u)}{32}\right]_0^{\pi} = 0
$$

$So,\ we\ get$

$$
\begin{aligned}
&\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\,dx = \frac{5}{16}\int_0^{\frac{\pi}{2}}\ 1\,dx \\
&\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\,dx = \left[\frac{5x}{16}\right]_0^{\frac{\pi}{2}} \\
&\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\,dx = \frac{5\pi}{32} - \frac{5 \times 0}{32} \\
&\Rightarrow \int_0^{\frac{\pi}{2}} cos^6(x)\,dx = \frac{5\pi}{32} \qquad [answer]
\end{aligned}
$$

3 Evaluate the integral using appropriate substitution: $\int \frac{cos4\theta}{1+2sin\ 4\theta} d\theta$

$$let,$$

$$u = 4\theta$$

$$du = 4\ d\theta$$

$$So,\ we\ get$$

$$\frac{1}{4}\int \frac{cos(u)}{1+2sin(u)}du$$

$$y = 2sin(u) + 1$$

$$dy = 2cos(u)du$$

$$So, we\ get$$

$$\frac{1}{8}\int \frac{1}{y}\ dy$$

$$= \frac{1}{8}\ \times\ ln(y) + c$$

$$= \frac{1}{8}\ ln(2sin(u) + 1) + c$$

$$= \frac{1}{8}\ ln(2sin(4\theta) + 1) + c \qquad [answer]$$

4 Use Gamma Function to evaluate $\int_0^{\infty} x^5\ e^{-x^2}\ dx$

$$
\begin{aligned}
&let, \\
&x^2 = u \\
\Rightarrow\ & \frac{d}{dx}(x^2) = \frac{d}{dx}(u) \\
\Rightarrow\ & 2x = \frac{du}{dx} \\
\Rightarrow\ & x\ dx = \frac{du}{2} \\
\therefore\ & dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}
\end{aligned}
$$

$So,$

$$
\begin{aligned}
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \int_0^{\infty} (\sqrt{u}\,)^5\ \cdot e^{-u} \cdot \frac{du}{2\sqrt{u}} \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \int_0^{\infty} u^{\frac{5}{2}} \cdot e^{-u} \cdot \frac{du}{2\ u^{\frac{1}{2}}} \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \frac{1}{2}\int_0^{\infty} e^{-u} \cdot u^{\frac{5-1}{2}} du \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \frac{1}{2}\int_0^{\infty} e^{-u} \cdot u^2\ du \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \frac{1}{2}\int_0^{\infty} e^{-u} \cdot u^{3-1} du \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \frac{1}{2} \times \Gamma 3 \qquad \left[\therefore \int_0^{\infty}\ e^{-u}\ u^{n-1}\ du\ = \Gamma n\right] \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= \frac{1}{2} \times 2! \\
\int_0^{\infty} x^5\ e^{-x^2}\ dx &= 1 \qquad [answer]
\end{aligned}
$$

5 Prove the Fundamental theorem of Calculus (It cannot be an exact copy from any source. You can use references but it should be properly cited. Also, you cannot copy paste directly. In that case you won't receive any marks).

According to Fundamental Theorem of Calculus we know that, if there is a graph which occupies a place say, A under the graph of f which is continuous [a,b] and F is an anti derivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$\textit{Let, we are given an function } f(t)$$
$$\textit{define the function } F(x) -$$
$$F(x) = \int_a^x f(t)dt$$

$$\textit{Say, } x_1 \textit{ and } \Delta x \textit{ are two values between } [a+b]$$
$$\textit{So, we get,}$$
$$F(x_1) = \int_a^{x_1} f(t)\ dt;$$
$$\textit{and}$$
$$F(x_1+\Delta x) = \int_a^{x_1+\Delta x} f(t)dt$$

$$\textit{If we substract, then we get,}$$
$$\int_a^{x_1+\Delta x} f(t)dt - \int_a^{x_1} f(t)dt = \int_{x_1}^{x_1+\Delta x} f(t)dt$$
$$F(x_1+\Delta x) - F(x_1) = \int_{x_1}^{x_1+\Delta x} f(t)dt$$

$$\textit{According to mean value theorem,}$$
$$\textit{A real number exists between } r \in [x_1, x_1+\Delta]$$
$$\int_{x_1}^{x_1+\Delta x} f(t)dt = f(r)\cdot\Delta x$$
$$F(x_1+\Delta x) - F(x_1) = f(r)\cdot\Delta x$$
$$\frac{F(x_1+\Delta x) - F(x_1)}{\Delta x} = f(r)\cdot\Delta x$$
$$\lim_{\Delta x\to 0}\frac{F(x_1+\Delta x) - F(x_1)}{\Delta x} = \lim_{\Delta x\to 0} f(r)$$
$$F'(x_1) = \lim_{\Delta x\to 0} f(r) \qquad \left[\therefore \lim_{\Delta x\to 0}\frac{F(x_1+\Delta x) - F(x_1)}{\Delta x} = F'(x_1)\right]$$

$$
\begin{aligned}
So, \lim_{\Delta x \to 0} x_1 &= x_1 \\
and \lim_{\Delta x \to 0} x_1 + \Delta x &= x_1 \\
\therefore \lim_{\Delta x \to 0} r &= x_1
\end{aligned}
$$

$$We\ can\ now\ say\ that,\ \ f\ is\ continuous\ at\ r,$$

$$\therefore F'(x_1) = f(x_1)$$

[proved]

References

[1] Howard A.,Iril B., Stephen D., Calculus. 10th ed.,John Willey and Sons, United States of America, 2012.

[2] Spivak, Michael (1980), Calculus (2nd ed.), Houston, Texas: Publish or Perish Inc.

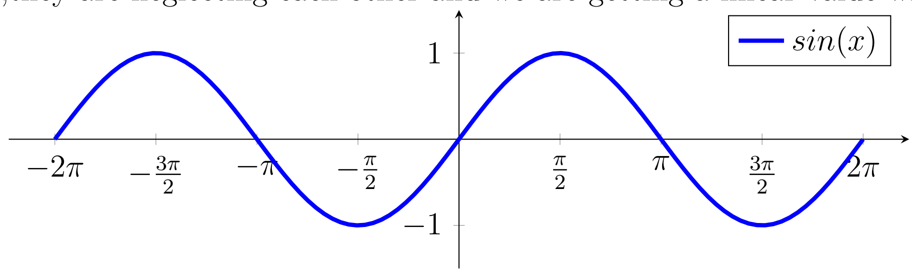
[3] Wikipedia : Fundamental Theorem of Calculus.
https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

6 You are asked to integrate the following definite integrals, $\int_0^{2\pi} sinx\ dx$

$$(a)$$

$$
\begin{aligned}
\int_0^{2\pi} sinx\ dx &= [-cos\ x]_0^{2\pi} \\
&= -\ [cos\ x]_0^{2\pi} \\
&= -[cos 2\pi\ -\ cos(0)] \\
&= -(1-1) \\
&= 0
\end{aligned}
$$

According to Fundamental Theorem of Calculus we know that, we will get an area after integration and area has a space which is obviously greater than zero.
But, here we are getting zero, because the value of $cos2\pi$ is positive and the value of cos(0) is negative. So,they are neglecting each other and we are getting a linear value which is zero.



(b)

$$\int_0^{2\pi} |sinx|\ dx = \int_0^{\pi} + sinx\, dx\ +\ \int_{\pi}^{2\pi} - sinx\, dx \qquad\qquad [plotting\ the\ value\ in\ graph]$$

$$= \int_0^{\pi}\ sinx\, dx\ -\int_{\pi}^{2\pi} + sinx\, dx$$

$$= [-cosx]_0^{\pi} - [-cosx]_{\pi}^{2\pi}$$

$$= -\left[cosx\right]_0^{\pi} - [-cosx]_{\pi}^{2\pi}$$

$$= -\left[cos\pi\ -\ cos0\right] - [-cos2\pi\ - cos\pi]$$

$$= -\left[-1-1\right] + [1-(-1)]$$

$$= 2\ + 2$$

$$= 4$$

$[answer]$