MAT120 : Final Exam #Set-09

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September 24, 2020

Answer to the Question Number One

[Part a]

$$\int_0^1 \int_0^x 4y\sqrt{x^2-y^2}\text{dy dx}$$

Let ,
$x^2 - y^2 = u$
$\Rightarrow (0-2y)$ dy = du
$\Rightarrow -2y$ dy = du
$\Rightarrow 4y$ dy = -2 du

If , $y = 0$; $u = x^2$

If , $y = x$; $u = 0$

$\int_0^1 \int_{x^2}^0 (-2\sqrt{u})$ du dx

$= -2\int_0^1 \left[\frac{(u)^{(\frac{1}{2}+1)}}{\frac{3}{2}}\right]_{x^2}^0$ dx

$= -2\int_0^1 \left[\frac{(u)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{x^2}^0$ dx

$= -2\int_0^1 \left[\frac{(x^2-y^2)^{\left(\frac{3}{2}\right)}}{\frac{3}{2}}\right]_0^x$ dx

$= -\frac{-2*2}{3}\int_0^1 \left[[x^2-x^2]^{\left(\frac{3}{2}\right)} - (x^2-0)^{\frac{3}{2}}\right]$ dx

$= -\frac{4}{3}\int_0^1 -(x^2)^{\frac{3}{2}}$

$= \frac{4}{3}\int_0^1 x^3\text{dx}$

$= \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1$

$= \frac{4}{3} * \frac{1}{4}\left[(1^4) - 0\right]$

$= \frac{1}{3}$

[ANSWER]

[Part b]

$$\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{2x^2+2y^2} \text{ dx dy dz}$$

Let ,
$u = \frac{x}{y}$; $x = uy$
$\Rightarrow dx = du$

If , x= $\sqrt{3}y$; u = $\sqrt{3}$

If , x= 0 ; u = 0

$\int_1^2 \int_z^2 \int_0^{\sqrt{3}} \frac{y*ydu}{2u^2y^2+2y^2}$ dy dz

$= \int_1^2 \int_z^2 \int_0^{\sqrt{3}} \frac{y^2du}{2y^2(u^2+1)}$ dy dz

$= \frac{1}{2} \int_1^2 \int_z^2 \int_0^{\sqrt{3}} \frac{du}{(u^2+1)}$ dy dz

$= \frac{1}{2} \int_1^2 \int_z^2 \left[tan^{-1}(u)\right]_0^{\sqrt{3}}$ dy dz

$= \frac{1}{2} \int_1^2 \int_z^2 \left[tan^{-1}(\frac{x}{y})\right]_0^{\sqrt{3}y}$ dy dz

$= \frac{1}{2} \int_1^2 \int_z^2 \left[tan^{-1}(\sqrt{3}) - tan^{-1}(0)\right]$ dy dz

$= \frac{1}{2} \int_1^2 \int_z^2 \left(\frac{\pi}{3}\right)$ dy dz

$= \left(\frac{1}{2}\right) \left(\frac{\pi}{3}\right) \int_1^2 \int_z^2$ dy dz

$= \left(\frac{\pi}{6}\right) \int_1^2 [y]_z^2$ dz

$= \left(\frac{\pi}{6}\right) \int_1^2 [2 - z]$ dz

$= \left(\frac{\pi}{6}\right) \left[[2z]_1^2 - \left[\frac{z^2}{2}\right]_1^2\right]$

$= \left(\frac{\pi}{6}\right) \left[(4-2) - \left(\frac{4}{2} - \frac{1}{2}\right)\right]$

$= \frac{\pi}{6}(2 - \frac{3}{2})$

$= \frac{\pi}{6} \left(\frac{4-3}{2}\right)$

$= \frac{\pi}{6} * \frac{1}{2}$

$= \frac{\pi}{12}$

[ANSWER]

Answer to the Question Number Two

Solve the following 1st order DE:

[Part a]

$\frac{2}{3}x\frac{dy}{dx} - \frac{2}{3}y = \frac{2}{3}x;\ y(1) = 2$

Given equation,

$$\frac{2}{3}x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}y = \frac{2}{3}x$$
$$\frac{2}{3}[x\frac{\mathrm{d}y}{\mathrm{d}x} - y] = \frac{2}{3}x$$
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = 1$$

Comparing with general equation of 1st order DE :

$y' + P(x)y = Q(x)$

$Here,\ P(x) = \frac{1}{x}\ ;\ Q(x) = 1$

general solution :

$$yI(x) = \int I(x)Q(x)dx + c$$

$Here,\ I(x) = e^{\int P(x)dx}$

$= e^{\int -\frac{1}{x}dx}$

$= e^{-\ln x}$

$= e^{\ln \frac{1}{x}}$

$= \frac{1}{x}$

Solution :

$$y\frac{1}{x} = \int 1 * \frac{1}{x}dx + c$$
$$\frac{y}{x} = \ln x + c$$
$$y = x(\ln x + c).....[1]$$

$Again\ ,\ y(1) = 2\ then,$

$$2 = 1 * (\ln 1 + c)$$
$$2 = \ln 1 + c$$
$$\therefore c = 2$$

Putting the value of c into [1]

$$y = x(\ln x + 2)$$

`[ANSWER]`

[Part b]

$(32y - 8cosy)\frac{dy}{dx} - 24x^2 = 0$; $y(0) = 0$

given equation,

$$(32y - 8\cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 24x^2$$
$$8(4y - \cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 24x^2$$
$$(4y - \cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$
$$(4y - \cos y)dy = 3x^2dx$$
$$\int(4y - \cos y)dy = \int 3x^2dx$$
$$\int 4ydy - \int \cos ydy = 3\int x^2dx$$
$$4\frac{y^2}{2} - \sin y = 3\frac{x^3}{3} + c$$
$$2y^2 - \sin y = x^3 + c..........[1]$$

given condition, $y(0) = 0$

$$2 * 0 - \sin 0 = 0 + c$$
$$\therefore c = 0;$$

putting value of c in to [1],

$$2y^2 - \sin y = x^3 + 0$$
$$2y^2 - \sin y = x^3$$

[ANSWER]

<u>**Answer to the Question Number Three**</u>

[Part a]

$Given,$

$(2e^{2y} - 2ycos(xy))dx + (4xe^{2y} - 2xcos(xy) + 4y)dy = 0$

$here,$

$M(x, y) = (2e^{2y} - 2ycos(xy))$

$N(x, y) = (4xe^{2y} - 2xcos(xy) + 4y)$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2e^{2y} - 2ycos(xy))$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2e^{2y})\frac{\partial}{\partial y}(2ycos(xy))$$
$$\Rightarrow \frac{\partial M}{\partial y} = 4e^{2y} - 2(cosxy - xy(sinxy)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4xe^{2y} - 2xcos(xy) + 4y)$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4xe^{2y}) - \frac{\partial}{\partial x}(2xcos(xy) + 4y)$$
$$\frac{\partial N}{\partial x} = 4e^{2y} - 2(cos(xy) - xysin(xy))$$

As, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,we have an exact equation

$We\,know,$

$\psi(x, y) = C$

$Again,$

$$\psi(x, y) = \int N(x, y)dy$$
$$= \int (4xe6(2y) - 2xcos(xy) + 4y)dy$$
$$= \int 4ydy - \int 2cos(xy)dy + \int 4xe^(2y)dy$$
$$= 2y^2 - 2sin(xy) + 2xe6(2y) + c_1$$

$\psi(x, y) = c_2$

$2y^2 - 2sin(xy) + 2xe^(2y) + c_1 = c_2$

$\therefore 2y^2 - 2sin(xy) + 2xe^(2y) = C$

`[ANSWER]`

[Part b]

$$
\begin{aligned}
&(8x^2+12y^2-80)dy = -4xydx\\
&\Rightarrow (8x^2+12y^2-80)dy + 4xydx = 0\\
&\Rightarrow (8x^2+12y^2-80)\frac{dy}{dx} + 4xy = 0; \quad [divided\, by\, dx]\\
&\Rightarrow (8x^2+12y^2-80)y' + 4xy = 0\\
&\Rightarrow 4xy^4 + y^3(8x^2+12y^2-80)y' = 0; \quad [multiplied\, by\, y^3]\\
&\Rightarrow 4xy^4dx + y^3(8x^2+12y^2-80)dy = 0; \quad [multiplied\, by\, dx]
\end{aligned}
$$

$$
\begin{aligned}
&Here,\\
&M(x,y) = 4xy^4\\
&N(x,y) = y^3(8x^2+12y^2-80)\\
&\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial y}4xy^4 = 16xy^3\\
&\frac{\partial}{\partial x}(N) = \frac{\partial}{\partial x}y^3(8x^2+12y^2-80)\\
&\Rightarrow \frac{\partial}{\partial x}(N) = y^3\frac{\partial}{\partial x}(8x^2+12y^2-80)\\
&\Rightarrow \frac{\partial}{\partial x}(N) = y^3 16x = 16xy^3
\end{aligned}
$$

$$
As,\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},\ we\ get\ an\ exact\ equation.
$$

$$
\begin{aligned}
&we\ know,\ \psi(x,y) = c\\
&so,\\
&\psi(x,y) = \int N(x,y)dy\\
&\qquad = \int y^3(8x^2+12y^2-80)dy\\
&\qquad = \int 4(-20+3y^2+2x^2)y^3dy\\
&\qquad = 4\int(-20+3y^2+2x^2)y^3dy\\
&Here,\ Let,\ u = (-20+3y^2+2x^2) and\, v' = y^3
\end{aligned}
$$

Applying integration by parts

$$\begin{aligned}
&4\int(-20+3y^2+2x^2)y^3dy \\
&=4(\frac{1}{4}y^4(-20+3y^2+2x^2)-\int\frac{3y^5}{2}dy) \\
&=4(\frac{1}{4}y^4(-20+3y^2+2x^2-\frac{y^6}{4}) \\
&=2y^6+2x^2y^4-20y^4
\end{aligned}$$

$$\therefore \psi(x,y)=\int N(x,y)dy=2y^6+2x^2y^4-20y^4+c_1$$

$\psi(x,y)=c_2$

$2y^6+2x^2y^4-20y^4+c_1=c_2$

$\therefore 2y^6+2x^2y^4-20y^4=C$

[ANSWER]

<u>**Answer to the Question Number Four**</u>

[Part a]

$y'' + y' + \frac{17}{4}y = 0; y(0) = -1, y'(0) = 2$

Let,

$$
\begin{aligned}
&y = e^{rt} \quad where \quad r = root \quad of \quad the \quad auxiliary \quad equation\\
&\therefore y'' = r^2e^{rt}, y' = re^{rt}\\
&\therefore r^2e^{rt} + re^{rt} + \frac{17}{4}e^{rt} = 0\\
&\Rightarrow r^2 + r + \frac{17}{4} = 0 && [divided \quad by \quad e^{rt}]\\
&\therefore r = \frac{-1 \pm \sqrt{1-17}}{2} = -\frac{1}{2} \pm 2i
\end{aligned}
$$

The two solutions of the differential equation,

$$
\begin{aligned}
&y_1(t) = e^{(-\frac{1}{2}+2i)t} = e^{-\frac{t}{2}}[\cos(2t) + i\ \sin(2t)]\\
&y_2(t) = e^{(-\frac{1}{2}-2i)t} = e^{-\frac{t}{2}}[\cos(2t) - i\ \sin(2t)]\\
&Let,\\
&u(t) = e^{-\frac{t}{2}}\cos(2t)\\
&v(t) = e^{-\frac{t}{2}}\sin(2t)
\end{aligned}
$$

$$
\begin{aligned}
\therefore y_c(t) &= c_1u(t) + c_2v(t)\\
&= e^{-\frac{t}{2}}(c_1\cos(2t) + c_2\sin(2t))
\end{aligned}
$$

For $y(0) = -1$,

$$
\begin{aligned}
-1 &= (c_1\cos 0 + c_2\sin 0)\\
\therefore c_1 &= -1
\end{aligned}
$$

For $y'(0) = 2$,

$$
\begin{aligned}
y'(t) &= \frac{d}{dt}(e^{-\frac{t}{2}})(c_1\cos(2t) + c_2\sin(2t)) + \frac{d}{dt}(c_1\cos(2t) + c_2\sin(2t))e^{-\frac{t}{2}}\\
&= -\frac{1}{2}e^{-\frac{t}{2}}(c_1\cos(2t) + c_2\sin(2t)) + (-2c_1\sin(2t) + 2c_2\cos(2t))e^{-\frac{t}{2}}\\
\Rightarrow 2 &= -\frac{1}{2}c_1 + 2c_2\\
\Rightarrow 2c_2 &= \frac{1}{2}(-1) + 2\\
\therefore c_2 &= \frac{3}{4}
\end{aligned}
$$

$$
\therefore y_c = e^{-\frac{t}{2}}(-\cos(2t) + \frac{3}{4}\sin(2t))
$$

`[ANSWER]`

[Part b]

$\frac{1}{2}\frac{d^2y}{dy^2} - 5\frac{dy}{dx} + \frac{25}{2}y = 0; y(0) = 1, y'(0) = 1$

Let,

$$\begin{aligned}
&y = e^{rt} \quad where \quad r = root \quad of \quad the \quad auxiliary \quad equation \\
&\therefore y'' = r^2 e^{rt}, y' = re^{rt} \\
&\frac{1}{2}\frac{d^2y}{dy^2} - 5\frac{dy}{dx} + \frac{25}{2}y = \frac{1}{2}y'' - 5y' + \frac{25}{2}y = 0 \\
&\Rightarrow \frac{1}{2}r^2e^{rt} - 5re^{rt} + \frac{25}{2}e^{rt} = 0 \\
&\Rightarrow \frac{1}{2}r^2 - 5r\frac{25}{2} = 0 \qquad [divided \quad by \quad e^{rt}] \\
&\Rightarrow r^2 - 10r + 25 = 0 \\
&\Rightarrow (r-5)^2 = 0
\end{aligned}$$

The roots $r_1 = r_2 = 5$ are real and equal.
We know, if a differential equation is linear and have constant coefficient, then for root $\lambda_1 = \lambda_2$, the general solution, $y_c = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x}$

$$y_c = c_1e^{5t} + c_2te^{5t}$$

For $y(0) = 1$,

$$1 = c_1$$

For $y'(0) = 1$,

$$\begin{aligned}
y' &= 5c_1e^{5t} + c_2e^{5t} + 5c_2te^{5t} \\
\Rightarrow 1 &= 5c_1 + c_2 \\
\Rightarrow c_2 &= 1 - 5 \\
&= -4
\end{aligned}$$

$$\therefore y_c = e^{5t} - 4te^{5t}$$

`[ANSWER]`

Answer to the Question Number Five

Solve the following Second Order Differential Equation

Part (a)

$$Given,$$

$$2y'' - 4y' + 2y = 2e^x$$

$$\Rightarrow y'' - 2y' + y = e^x \qquad \text{[dividing both side by 2]}$$

$$let,$$

$$y = e^{mx}$$

$$y' = me^{mx}$$

$$y'' = m^2e^{mx}$$

For complimentary function:

$$\therefore m^2e^{mx} - 2me^{mx} + e^{mx} = 0$$

$$\Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow m^2 - m - m + 1 = 0$$

$$\Rightarrow m(m-1) - 1(m-1) = 0$$

$$\Rightarrow (m-1)(m-1) = 0$$

$$\therefore m_1 = m_2 = 1$$

$$\therefore y_c = c_1e^x + c_2xe^x$$

For particular solution:

$$y_p = x^2(Ae^x) \qquad [to\ make\ it\ distinct\ from\ y_c]$$

$$\begin{aligned}
{y_p}' &= (Ae^x x^2)\frac{d}{dx} \\
&= Ae^x x^2 + 2Ae^x x
\end{aligned}$$

$$\begin{aligned}
{y_p}'' &= (Ae^x x^2)\frac{d^2}{dx^2} \\
&= (Ae^x x^2)\frac{d}{dx} + (2Ae^x x)\frac{d}{dx} \\
&= A2e^x + Ae^x x^2 + A2e^x x + A2e^x x \\
&= A(2e^x + e^x x^2 + 4e^x x)
\end{aligned}$$

$$\begin{aligned}
&\therefore y'' - 2y' + y = e^x \\
&\Rightarrow A2e^x + Ae^x x^2 + A4e^x x - 2Ae^x x + Ae^x x^2 = e^x \\
&\therefore 2Ae^x = e^x \\
&\Rightarrow 2A = 1 \\
&\therefore A = \frac{1}{2} \\
&\therefore y_P = \frac{1}{2} \times e^x x^2
\end{aligned}$$

$$\begin{aligned}
\therefore y_G(x) &= y_C(x) + y_P(x) \\
&= c_1 e^x + c_2 x e^x + \frac{e^x x^2}{2}
\end{aligned}$$

[ANSWER]

Part (b)

$$Given,$$
$$5\frac{d^2y}{dx^2} + 20y = 15sin2x$$
$$\Rightarrow 5y'' + 20y = 15sin2x$$
$$\Rightarrow y'' + 4y = 3sin2x \qquad [dividing\ both\ side\ by\ 5]$$

$$let,$$
$$y = e^{mx}$$
$$\Rightarrow y' = me^{mx}$$
$$\Rightarrow y'' = m^2e^{mx}$$

For complimentary function:

$$m^2e^{mx} + 4e^{mx} = 0$$
$$\Rightarrow m^2 + 4 = 0$$
$$\Rightarrow m^2 = -4$$
$$\therefore m = \pm\sqrt{-4}$$
$$= \pm i\sqrt{4}$$
$$= 0 \pm i\sqrt{4}$$

$$y_c = e^{0 \cdot x}(c_1 cos\sqrt{4}x + c_2 sin\sqrt{4}x)$$
$$y_c = c_1 cos\sqrt{4}x + c_2 sin\sqrt{4}x$$
$$= c_1 cos2x + c_2 sin2x$$

For particular solution:

$$y_p = e^{0 \cdot x}(Axcos2x + Bxsin2x)$$
$$= Axcos2x + Bxsin2x$$
$$\Rightarrow y_p{}' = Acos2x - 2Asin2x + Bsin2x + 2Bxcos2x$$
$$\Rightarrow y_p{}'' = -2Asin2x - 2Asin2x - 4Axcos2x + 2Bcos2x + 2Bcos2x - 4Bsin2x$$
$$= -4Asin2x - 4Axcos2x + 4Bcos2x - 4Bxsin2x$$

$$\therefore y'' + 4y = -4Asin2x - 4Axcos2x + 4Bcos2x - 4Bxsin2x + 4cos2x + 4Bxsin2x$$

$$\Rightarrow 3sin2x = -4Asin2x + 4Bcos2x. \dots\dots\dots\dots (i)$$

equate coefficient of sin(2x) on both sides of equation (i):

$$-4A = 3$$
$$A = -\frac{3}{4}$$

equate coefficient of cos(2x) on both sides of equation (i):

$$4B = 0$$
$$\therefore B = 0$$

$$\begin{aligned}\therefore y_p &= Acos2x + 0 \times sin2x \\ &= -\frac{3}{4}xcos2x\end{aligned}$$

$$\begin{aligned}\therefore y_G(x) &= y_c(x) + y_p(x) \\ &= c_1cos2x + c_2sin2x - \frac{3}{4}xcos2x\end{aligned}$$

[ANSWER]

[THANK YOU]