MAT120: Monthly Assignment #Set-D

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Answer to the Question Number One

Part (a)

$$\textit{Given,}$$
$$P(a \leq x \leq b) = \int_a^b f(x)dx$$
$$f(x) = \begin{cases} 0, & x \leq 0 \\ kc^{-kx}, & x \geq 0 \end{cases}$$

Since accidents are occurring at a rate one of every 3 months,

$$k = 3$$
$$\therefore f(x) = ke^{-kx}$$
$$= 3e^{-3x}$$

Probability of no accident during 8 month interval is,

$$\begin{aligned} P(x \geq 8) &= 1 - (8 \geq x \geq 0) \\ &= 1 - \int_0^8 3e^{-3x}dx \\ &= 1 - 3\left[\frac{e^{-3x}}{-3}\right]_0^8 \\ &= 1 + \left[e^{-3x}\right]_0^8 \\ &= 1 + \left[e^{24} - e^0\right] \\ &= 1 + e^{24} - 1 \\ &= 3.8 \times 10^{-11} \end{aligned}$$

The Probability or chance of no accident in 8 month is 3.8 $\times 10^{-11}$**.**
Since, the value is very small, it can be said that the changes were effective.

Answer to the Question Number Two

Part (a)

$$d[HBr] = K[H_2][Br_2]^{\frac{1}{2}}$$

for this reaction, Differential Equation comes,

$$\frac{dx}{dt} = k(a-x)(b-x)^{\frac{1}{2}}$$

where,

$x = [HBr]$ & a, b *are the initial concentrations of hydrogen and bromine*

(a)

$$\frac{dx}{dt} = k(a-x)(b-x)^{\frac{1}{2}}$$
$$= k(a-x)^{\frac{3}{2}}$$
$$\Rightarrow \frac{dx}{(a-x)^{\frac{3}{2}}}dx = kdt$$

Let,

$$u = a - x$$
$$\Rightarrow du = -dx$$
$$\Rightarrow -dx = du$$
$$\therefore dx = -du$$

Integrating both sides and plugging the value

$$\Rightarrow -\int \frac{1}{u^{\frac{3}{2}}}du = \int kdt$$
$$\Rightarrow \frac{-u^{-\frac{1}{2}}}{-\frac{1}{2}} = \int kdt$$
$$\Rightarrow 2u^{-\frac{1}{2}} = kt + c$$
$$\Rightarrow 2(a-x)^{-\frac{1}{2}} = kt + c$$
$$\Rightarrow (a-x)^{\frac{1}{2}} = \frac{kt+c}{2}$$
$$\Rightarrow \sqrt{a-x} = \frac{2}{kt+c}$$
$$\Rightarrow x = a - \frac{4}{(kt+c)^2}$$

$$Now,$$
$$x(0) = 0$$
$$So,$$
$$x = 0, \qquad t = 0$$
$$x = a - \frac{4}{(kt+c)^2}$$
$$\Rightarrow 0 = a - \frac{4}{c^2}$$
$$\Rightarrow a = \frac{4}{c^2}$$
$$\Rightarrow c = \sqrt{\frac{4}{a}}$$
$$c = \frac{2}{\sqrt{a}}$$

$$\therefore x = a - \frac{4}{\left(kt + \frac{2}{\sqrt{a}}\right)^2}$$

[Answer]

Part=b

The differential equation is

$$\frac{dx}{dt} = k(a-x)(b-x)^{\frac{1}{2}}$$
$$\Rightarrow \frac{dx}{(a-x)(b-x)^{\frac{1}{2}}} = kdt$$
$$\Rightarrow \int \frac{dx}{(a-x)\sqrt{b-x}} = \int kdt \qquad \textit{[Integrating both sides]}$$

$$Let,$$
$$u = \sqrt{b-x}$$
$$\Rightarrow u^2 = b - x$$
$$\Rightarrow x = b - u^2$$
$$\Rightarrow dx = -2udu$$
$$\Rightarrow -dx = 2udu$$

$$\Rightarrow -2\int \frac{1}{(a-b+u^2)}du = kt + c$$

$$\Rightarrow \frac{1}{(\sqrt{a-b})^2+u^2}du = kt + c$$

$$\Rightarrow -\frac{2}{\sqrt{a-b}}tan^{-1}\frac{u}{\sqrt{a-b}} = kt + c \qquad \left[\because \int \frac{1}{a^2-u^2}du = \frac{1}{a}tan^{-1}(\frac{u}{a}) + c\right]$$

$$\Rightarrow -\frac{2}{\sqrt{a-b}}tan^{-1}\sqrt{\frac{b-x}{a-b}} = kt + c$$

$x(0) = 0$

So,

$x = 0 \qquad t = 0$

$$\Rightarrow \frac{-2}{\sqrt{a-b}}tan^{-1}\sqrt{\frac{b-0}{a-b}} = k \cdot (0) + c$$

$$\Rightarrow \frac{-2}{\sqrt{a-b}}tan^{-1}\sqrt{\frac{b}{a-b}} = c$$

Therefore,

$$\frac{-2}{\sqrt{a-b}}tan^{-1}\sqrt{\frac{b-x}{a-b}} = kt - \frac{2}{\sqrt{a-b}}tan^{-1}\sqrt{\frac{b}{a-b}}$$

$$\therefore t = \frac{2}{k\sqrt{a-b}}$$

[Answer]

Answer to the Question Number Three

(a)

$$
\begin{aligned}
&\textit{Given that,}\\
&\qquad m\frac{dv}{dt} = mg - cv\\
&\qquad \Rightarrow m\frac{dv}{mg-cv} = dt\\
&\qquad \Rightarrow m\int\frac{dv}{mg-cv} = \int dt\\
&\qquad \Rightarrow -\frac{m}{c}ln|mg-cv| = t + C\\
&\qquad \Rightarrow -\frac{m}{c}ln(mg-cv) = t + C \ \ .....(1)
\end{aligned}
$$

As the object dropped from rest, here-

v=0 and t=0

$$
\begin{aligned}
&So,\\
&\quad \Rightarrow -\frac{m}{c}\, ln\, mg = C\\
&\quad Now,\ putting\ value\ of\ C\ in\ Eq.(1)\\
&\quad \Rightarrow -\frac{m}{c}\, ln\,(mg-cv) = t - \frac{m}{c}ln\, mg\\
&\quad \Rightarrow \frac{m}{c}\, ln\, \frac{mg}{(mg-cv)} = t\\
&\quad \Rightarrow ln\, \frac{mg}{(mg-cv)} = \frac{c}{m}t\\
&\quad \Rightarrow \frac{mg}{(mg-cv)} = e^{ct/m}\\
&\quad \Rightarrow mg - cv = \frac{mg}{e^{ct/m}}\\
&\quad \Rightarrow mg - cv = mge^{-ct/m}\\
&\quad \Rightarrow cv = mg - mge^{-ct/m}\\
&\quad \Rightarrow v = \frac{mg}{c}(1-e^{-ct/m})\\
&So,\\
&\quad v = \frac{mg}{c}(1-e^{-\frac{c}{m}t}) \quad\ [Solved]
\end{aligned}
$$

(b)

Given that,

v(t)=s'(t)

$$So,$$

$$s(t) = \int_0^t v(\tau) d\tau$$

$$We\ have\ found\ from\,(a),$$

$$v = \frac{mg}{c}(1 - e^{-\frac{c}{m}t})$$

$$putting\ the\ value\ of\ v,$$

$$s(t) = \int_0^t \frac{mg}{c}(1 - e^{-\frac{c}{m}\tau})\,dt$$

$$= \frac{mg}{c}\left[\tau + \frac{m}{c}e^{-\frac{c}{m}\tau}\right]_0^t$$

$$= \frac{mg}{c}t + \frac{m^2 g}{c^2}e^{-\frac{c}{m}t} - \frac{m^2 g}{c^2}$$

Ans.

Answer to the Question Number Four

Part (a)

$$
\begin{aligned}
&Given, \\
&\int_{-\infty}^{+\infty} \rho(x) = 1 \\
&\therefore \rho(x) = Ae^{-\lambda(x-a)^2}
\end{aligned}
$$

As, A, a and λ are constants

$$
\begin{aligned}
&let, \\
&u = \lambda(x-a) \\
&u^2 = (\sqrt{\lambda})^2(x-a)^2 \\
\therefore \frac{du}{dx} &= \sqrt{\lambda} \\
\Rightarrow dx &= \frac{du}{\sqrt{\lambda}}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \rho(x) = 1 \\
&\Rightarrow \int_{-\infty}^{+\infty} Ae^{-\lambda(x-a)^2} = 1 \\
&\Rightarrow A \int_{-\infty}^{+\infty} e^{-u^2} \frac{du}{\sqrt{\lambda}} = 1 \qquad [Plugging\ the\ value] \\
&\Rightarrow \frac{A}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-u^2} du = 1 \\
&\Rightarrow \frac{A}{\sqrt{\lambda}} \times \sqrt{\pi} = 1 \\
&\Rightarrow A\sqrt{\pi} = \sqrt{\lambda} \\
&\therefore A = \sqrt{\frac{\lambda}{\pi}}
\end{aligned}
$$

Part (b)

$$\begin{aligned}
< x > &= \int_{-\infty}^{\infty} x\rho(x)dx \\
&= \int_{-\infty}^{\infty} xAe^{-\lambda(x-a)^2}dx \\
&= A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx \\
&= \sqrt{\frac{\lambda}{\pi}} xe^{-\lambda(x-a)^2}dx \\
Let, \\
u &= x-a; \qquad ; x = u+a \\
&\Rightarrow \frac{du}{dx} = 1 \\
\Rightarrow dx = du \\
< x > &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{+\infty} (u+a)e^{-\lambda u^2}du \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{+\infty} \left[-\frac{1}{2\lambda}e^{-\lambda u^2}du + a\int_{-\infty}^{+\infty} e^{-\lambda u^2}du\right] \\
Here, \\
\int_{-\infty}^{\infty} ue^{-\lambda u^2}du &= \int_{-\infty}^{0} ue^{-\lambda u^2}du + \int_{0}^{+\infty} ue^{-\lambda u^2}du \\
Let, \\
v = -\lambda u^2 \\
\Rightarrow \frac{dv}{du} = -2u\lambda \\
\therefore \int -ue^2\frac{dv}{2u\lambda} &= \int -ue^2\frac{dv}{2\lambda} \\
&= -\frac{1}{2\lambda}\int e^2 dv \\
&= -\frac{1}{2\lambda}e^2 \\
&= -\frac{1}{2\lambda}e^{-\lambda u^2}
\end{aligned}$$

$$
\begin{aligned}
\therefore \int_{-\infty}^{+\infty} ue^{-\lambda u^2} du &= \left[-\frac{1}{2\lambda}e^{-\lambda u^2}\right]_{-\infty}^{0} + \left[-\frac{1}{2\lambda}e^{-\lambda u^2}\right]_{0}^{+\infty} \\
&= -\frac{1}{2\lambda}e^0 - 0 + 0 + \frac{1}{2\lambda}e^0 \\
&= -\frac{1}{2\lambda}e^0 + \frac{1}{2\lambda}e^0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\therefore < x > &= \sqrt{\frac{\lambda}{\rho}}[0 + a \cdot \frac{\pi}{\lambda}] \\
&= \frac{\sqrt{\lambda}}{\pi} \cdot a \cdot \frac{\sqrt{\pi}}{\lambda} \\
&= a
\end{aligned}
$$

$$
\begin{aligned}
< x^2 > &= A \int_{-\infty}^{+\infty} (u+a)^2 e^{-\lambda u^2} du \\
&= A \int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} + 2aue^{-\lambda u^2} + a^2 e^{-\lambda u^2} du \\
&= A \left[\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} du + 2a \int_{-\infty}^{+\infty} ue^{-\lambda u^2} du + \int_{-\infty}^{+\infty} e^{-\lambda u^2} du\right] \\
&= A \left[\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} du + 0 + a^2 \sqrt{\frac{\pi}{\lambda}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&We\ know, \\
\frac{d}{dx}\int_a^b f(x,t)dt &= \int_a^b \frac{\partial}{\partial x} f(x,t)dt \\
&Let, \\
I(\lambda) &= \int_{-\infty}^{\infty} e^{-\lambda u^2} du \dots\dots\dots\dots\dots\dots (i) \\
\Rightarrow \frac{dI}{d\lambda} &= \int_{-\infty}^{\infty} \frac{\partial}{\partial \lambda} e^{-\lambda u^2} du \\
&= \int_{-\infty}^{\infty} -u^2 e^{-\lambda u^2} du
\end{aligned}
$$

$$
\begin{aligned}
I(\lambda) &= \int_{-\infty}^{+\infty} e^{-\lambda u^2} du \\
&= \sqrt{\frac{\pi}{\lambda}} \\
\frac{dI}{d\lambda} &= \sqrt{\pi}\frac{d}{d\lambda}(\frac{1}{\lambda}) \\
&= \frac{\sqrt{\pi}}{2}\lambda^{-\frac{3}{2}}
\end{aligned}
$$

$So,$

$$
-\frac{\sqrt{\pi}}{2}\lambda^{-\frac{3}{2}} = -\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} du
$$

$$
\begin{aligned}
\therefore < x^2 > &= A\left[\frac{\pi}{2}\lambda^{-\frac{3}{2}} + a^2\sqrt{\frac{\pi}{\lambda}}\right] \\
&= \frac{\sqrt{\lambda}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot \lambda^{-\frac{3}{2}} + \sqrt{\frac{\lambda}{\pi}} \cdot a^2\sqrt{\frac{\pi}{\lambda}} \\
&= \frac{1}{2\lambda} + a^2
\end{aligned}
$$

[Answer]

Answer to the Question Number Five

Part(a)

$$Here,\ given\ a\ wave-function\ of\ a\ particle\ of\ constant\ mass\ m,$$

$$\psi = Aexp^{(\frac{-amx^2}{\hbar})}$$

$$\therefore\ = Ae^{(\frac{-amx^2}{\hbar})} \quad [notation\ e^x\ =\ exp(x)]$$

$$Given\ that$$

$$\int_{-\infty}^{\infty} \psi.\psi dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} Ae^{(\frac{-amx^2}{\hbar})}.Ae^{(\frac{-amx^2}{\hbar})}dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} A^2e^{(\frac{-2amx^2}{\hbar})}dx = 1$$

$$\Rightarrow A^2 \int_{-\infty}^{\infty} e^{-(\frac{\sqrt{2am}}{\sqrt{\hbar}}\ .x)^2}dx = 1$$

$$We\ know, \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$Now, \int_{-\infty}^{\infty} e^{-cx^2}dx = \frac{1}{c}\int_{-\infty}^{\infty} e^{-x^2}dx$$

$$\Rightarrow \frac{1}{c}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{1}{c}\sqrt{\pi}$$

$$So,$$

$$A^2 \int_{-\infty}^{\infty} e^{-(\frac{\sqrt{2am}}{\sqrt{\hbar}}\ .x)^2}dx = 1$$

$$\Rightarrow A^2(\frac{1}{\frac{\sqrt{2am}}{\sqrt{\hbar}}}\ .\sqrt{\pi}) = 1$$

$$\Rightarrow A^2\frac{\sqrt{\pi}}{\sqrt{\frac{2am}{\hbar}}} = 1$$

$$\Rightarrow A^2\sqrt{\pi}.\sqrt{\frac{\hbar}{2am}} = 1$$

$$\Rightarrow A^2\sqrt{\frac{\pi\hbar}{2am}} = 1$$

$$\Rightarrow A^2 = \sqrt{\frac{2am}{\pi\hbar}}$$

$$\Rightarrow A = \sqrt[4]{\frac{2am}{\pi\hbar}}$$

$So,$

$$A = \sqrt[4]{\frac{2am}{\pi\hbar}} \quad [m, \hbar \; and \; a \; are \; constants]$$

$$Answer : A = \sqrt[4]{\frac{2am}{\pi\hbar}}$$

Part(b)

$$Here,$$
$$< x >= \int_{-\infty}^{\infty} \psi \hat{x} \psi dx$$
$$< p >= \int_{-\infty}^{\infty} \psi \hat{p} \psi dx$$

$$Now\ computing,$$
$$< x >= \int_{-\infty}^{\infty} \psi x \psi dx \quad [since, \hat{x} = x]$$
$$\Rightarrow < x >= \int_{-\infty}^{\infty} x|\psi|^2 dx$$
$$= \int_{-\infty}^{\infty} x\ Ae^{(\frac{-amx^2}{\hbar})}.Ae^{(\frac{-amx^2}{\hbar})} dx \quad [since, \psi = Ae^{(\frac{-amx^2}{\hbar})}]$$
$$= \int_{-\infty}^{\infty} x\ A^2 e^{(\frac{-2amx^2}{\hbar})} dx$$

$$let\ substitute, u = \frac{-2amx^2}{\hbar}$$
$$\Rightarrow du = -\frac{2am}{\hbar}. \frac{d}{dx}(x^2)$$
$$\Rightarrow \frac{du}{dx} = -\frac{2(2)amx}{\hbar}$$
$$\Rightarrow \frac{du}{dx} = -\frac{4amx}{\hbar}$$
$$\Rightarrow dx = -\frac{\hbar}{4amx} du$$
$$so,$$
$$\int_{-\infty}^{\infty} x\ A^2 e^{(\frac{-2amx^2}{\hbar})} dx$$

$$= \int_{-\infty}^{\infty} x\ A^2 e^u (-\frac{\hbar}{4amx}) du$$

$$= \int_{-\infty}^{\infty} A^2 e^u (-\frac{\hbar}{4am}) du$$

$$= -\frac{A^2\hbar}{4am} \int_{-\infty}^{\infty} e^u du$$

$$= -\frac{A^2\hbar}{4am} \int_{-\infty}^{0} e^u du - \frac{A^2\hbar}{4am} \int_{0}^{\infty} e^u du$$

$$= -\frac{A^2\hbar}{4am} \lim_{t\to-\infty} \int_{t}^{0} e^u du - \frac{A^2\hbar}{4am} \lim_{t\to\infty} \int_{0}^{t} e^u du$$

$$= -\frac{A^2\hbar}{4am} \lim_{t\to-\infty} [e^u]_t^0 - \frac{A^2\hbar}{4am} \lim_{t\to\infty} [e^u]_0^t$$

$$= -\frac{A^2\hbar}{4am}(e^0 - e^{-\infty}) - \frac{A^2\hbar}{4am}(e^{\infty} - e^0)$$

$$= -\frac{A^2\hbar}{4am}(1 - e^{-\infty}) - \frac{A^2\hbar}{4am}(e^{\infty} - 1)$$

$$= -\frac{A^2\hbar}{4am} + \frac{A^2\hbar}{4am}e^{-\infty} - \frac{A^2\hbar}{4am}e^{\infty} + \frac{A^2\hbar}{4am})$$

$$= 0$$

$So,$

$$< x >= 0$$

$$Now\ computing,$$

$$<p>=\int_{-\infty}^{\infty}\psi\ \hat{p}\ \psi dx$$

$$<p>=\int_{-\infty}^{\infty}\psi\ \ -i\hbar\frac{d}{dx}\ (\psi)dx \quad [since, \hat{p}=-i\hbar\frac{d}{dx}]$$

$$here,$$

$$i\hbar\frac{d}{dx}\ (\psi)=i\hbar\frac{d}{dx}(Ae^{(\frac{-amx^2}{\hbar})})$$

$$=i\hbar Ae^{(\frac{-amx^2}{\hbar})}\frac{d}{dx}(\frac{-amx^2}{\hbar})$$

$$=i\hbar Ae^{(\frac{-amx^2}{\hbar})}(-\frac{am}{\hbar}\ .\frac{d}{dx}x^2)$$

$$=i\hbar Ae^{(\frac{-amx^2}{\hbar})}(-\frac{am}{\hbar}\ )2x$$

$$=-\frac{i\hbar Ae^{(\frac{-amx^2}{\hbar})}am2x}{\hbar}$$

$$=-2iAamxe^{-\frac{amx^2}{\hbar}}$$

$$So,\ now,$$

$$\int_{-\infty}^{\infty}\psi\ \hat{p}\ \psi dx$$

$$=\int_{-\infty}^{\infty}\psi\ \ -i\hbar\frac{d}{dx}\ (\psi)dx$$

$$=\int_{-\infty}^{\infty}Ae^{(\frac{-amx^2}{\hbar})}\ 2iAamxe^{-\frac{amx^2}{\hbar}}\ dx$$

$$=2A^2iam\int_{-\infty}^{\infty}xe^{(\frac{-amx^2}{\hbar})}.\ e^{(\frac{-amx^2}{\hbar})}dx$$

$$= 2iamA^2 \int_{-\infty}^{\infty} x e^{(\frac{-2amx^2}{\hbar})} dx$$

$$since < x >= A^2 \int_{-\infty}^{\infty} x \ e^{(\frac{-2amx^2}{\hbar})} dx$$

$$\Rightarrow < x >= 0 \quad [we \ got \ by \ computing \ \ < x >]$$

$therefore,$

$$< p >= 2iamA^2 \int_{-\infty}^{\infty} x e^{(\frac{-2amx^2}{\hbar})} dx$$

$$= 2iamA^2 \ .0$$

$$= 0$$

$$So, < p >= 0$$

$$Answer :< x >= 0, < p >= 0$$

Part(c)

$$< x^2 >= \int_{-\infty}^{\infty} \psi \hat{x^2} \psi dx$$
$$< p^2 >= \int_{-\infty}^{\infty} \psi \hat{p^2} \psi dx$$

$$Now\ computing,$$
$$< x^2 >= \int_{-\infty}^{\infty} \psi x^2 \psi dx \quad [since, \hat{x^2} = x^2]$$

$$= \int_{-\infty}^{\infty} x^2 \ Ae^{(\frac{-amx^2}{\hbar})}.Ae^{(\frac{-amx^2}{\hbar})} dx \quad [since, \psi = Ae^{(\frac{-amx^2}{\hbar})}]$$

$$= \int_{-\infty}^{\infty} x^2 \ A^2 e^{(\frac{-2amx^2}{\hbar})} dx$$

$$= \int_{-\infty}^{\infty} x^2 \ A^2 e^{(-\sqrt{\frac{2am}{\hbar}}x)2} dx$$

$$let\ substitute,$$
$$u = \sqrt{\frac{2am}{\hbar}} x$$

$$\Rightarrow \frac{du}{dx} = \sqrt{\frac{2am}{\hbar}} \ .1$$

$$\Rightarrow du = \sqrt{\frac{2am}{\hbar}} dx$$

$$\Rightarrow dx = \sqrt{\frac{\hbar}{2am}} du$$

$$here,$$

$$u = \sqrt{\frac{2am}{\hbar}}x$$

$$\Rightarrow x\sqrt{\frac{2am}{\hbar}} = u$$

$$\Rightarrow x = u\ \sqrt{\frac{\hbar}{2am}}$$

$$so\ now,\ we\ get,$$

$$< x^2 >= \int_{-\infty}^{\infty} x^2\ A^2 e^{(-\sqrt{\frac{2am}{\hbar}}x)2} dx$$

$$= \int_{-\infty}^{\infty} x^2\ A^2 e^{-u^2} \sqrt{\frac{\hbar}{2am}}\ du$$

$$= A^2 \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} u^2 \frac{(\sqrt{\hbar})^2}{(\sqrt{2am})^2}\ e^{-u^2}\ du$$

$$= A^2 \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} u^2 \frac{\hbar}{2am}\ e^{-u^2}\ du$$

$$= A^2 \sqrt{\frac{\hbar}{2am}}\ \frac{\hbar}{2am} \int_{-\infty}^{\infty} u^2\ e^{-u^2}\ du \quad [equation : 01]$$

$$here, let\ us\ take, \int_{-\infty}^{\infty} u^2\ e^{-u^2}\ du$$

$$We\ know$$

$$Integral\ by\ parts,$$

$$\int f dg = fg - \int g df$$

$$let, f = u$$

$$\Rightarrow df = du$$

$And,$

$dg = ue^{-u^2}$

$\Rightarrow g = \int ue^{-u^2} du$

$let\ substitute, v = -u^2$

$\Rightarrow dv = -2udu$

$\Rightarrow -2udu = dv$

$\Rightarrow udu = -\frac{dv}{2}$

$Here,$

$g = \int ue^{-u^2} du\ [equation : 2]$

$= \int -e^v \frac{dv}{2}$

$= -\frac{1}{2} \int e^v dv$

$= -\frac{1}{2} e^v$

$g = -\frac{1}{2} e^{-u^2} + C$

$when\ putting\ limits\ on\ equation\ 2\ after\ integration$

$= -\frac{1}{2} \lim_{t \to -\infty} [e^v]_t^0 - \frac{1}{2} \lim_{t \to \infty} [e^v]_0^t \quad [substituting\ back, -\frac{1}{2} e^v = -\frac{1}{2} e^{-u^2}]$

$= -\frac{1}{2} \lim_{t \to -\infty} [e^{-u^2}]_t^0 - \frac{1}{2} \lim_{t \to \infty} [e^{-u^2}]_0^t$

$= -\frac{1}{2} (e^0 - e^{-\infty^2}) - -\frac{1}{2} (e^{\infty^2} - e^0)$

$= 0$

$Thus, -\frac{1}{2} e^{-u^2} = 0$

$So,\ now,\ from\ equation\ 1\ our\ Integral\ Part$

$\int_{-\infty}^{\infty} u^2\ e^{-u^2}\ du \quad [Integral\ by\ parts, \int fdg = fg - \int gdf]$

$= -\frac{ue^{-u^2}}{2} - \int_{-\infty}^{\infty} -\frac{e^{-u^2}}{2} du$

$= 0 + \frac{1}{2} \sqrt{\pi}\ \ [we\ know, \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}\ and\ putting\ limits\ on\ equation 02\ ]$

$= \frac{\sqrt{\pi}}{2}$

$After\ using\ integration\ by\ parts\ and\ now\ putting\ back\ the\ values\ on\ equation 1,$

$$< x^2 >= A^2\sqrt{\frac{\hbar}{2am}}\ \frac{\hbar}{2am}\int_{-\infty}^{\infty} u^2\ e^{-u^2}\ du$$

$$= A^2\sqrt{\frac{\hbar}{2am}}\ \frac{\hbar}{2am}\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\frac{2am}{\pi\hbar}}\ \sqrt{\frac{\hbar}{2am}}\ \frac{\hbar}{2am}\ \frac{\sqrt{\pi}}{2}\quad [putting\ the\ value\ of\ A^2\ from\ Part\ (a)]$$

$$< x^2 >= \frac{\hbar}{4am}$$

$$So, < x^2 >= \frac{\hbar}{4am}$$

$Now\ computing,$

$$< p^2 >= \int_{-\infty}^{\infty}\psi\ \hat{p^2}\ \psi dx$$

$$< p >= \int_{-\infty}^{\infty}\psi\ \ -\hbar^2\frac{d^2}{dx^2}\ (\psi)dx \quad [since, \hat{p} = -\hbar^2\frac{d^2}{dx^2}]$$

$here,$

$$\frac{d}{dx}\ (\psi) = \frac{d}{dx}(Ae^{(\frac{-amx^2}{\hbar})})$$

$$= Ae^{(\frac{-amx^2}{\hbar})}\frac{d}{dx}(\frac{-amx^2}{\hbar})$$

$$= Ae^{(\frac{-amx^2}{\hbar})}(-\frac{am}{\hbar}\ .\frac{d}{dx}x^2)$$

$$= Ae^{(\frac{-amx^2}{\hbar})}(-\frac{am}{\hbar}\ )2x$$

$$= -\frac{2Aamxe^{(\frac{-amx^2}{\hbar})}}{\hbar}$$

$$\frac{d^2}{dx^2}\ (\psi) = \frac{d^2}{dx^2}\ (-\frac{2Aamxe^{(\frac{-amx^2}{\hbar})}}{\hbar})$$

$$= -\frac{2Aam}{\hbar}\ .\frac{d}{dx}(\frac{xe^{(\frac{-amx^2}{\hbar})}}{\hbar})$$

$$= -\frac{2Aam}{\hbar}\ .\frac{d}{dx}(x)\ .e^{(\frac{-amx^2}{\hbar})} + x\frac{d}{dx}e^{(\frac{-amx^2}{\hbar})}\quad [Product\ rule]$$

$$= -\frac{2Aam}{\hbar}\ .(1.e^{(\frac{-amx^2}{\hbar})} + (xe^{(\frac{-amx^2}{\hbar})}\frac{d}{dx}(\frac{-amx^2}{\hbar})))$$

$$= -\frac{2Aam}{\hbar}\ .(e^{(\frac{-amx^2}{\hbar})} + (xe^{(\frac{-amx^2}{\hbar})}(\frac{-am}{\hbar}\ .\frac{d}{dx}x^2)))$$

$$= -\frac{2Aam}{\hbar}\ .(e^{(\frac{-amx^2}{\hbar})} + (xe^{(\frac{-amx^2}{\hbar})}\frac{-am}{\hbar}\ .2x))$$

$$= -\frac{2Aam}{\hbar}(e^{(\frac{-amx^2}{\hbar})} - \frac{2amx^2e^{(\frac{-amx^2}{\hbar})}}{\hbar})$$

$$= -\frac{2Aam}{\hbar}(\frac{\hbar e^{(\frac{-amx^2}{\hbar})} - 2amx^2e^{(\frac{-amx^2}{\hbar})}}{\hbar})$$

$$= \frac{2Aam}{\hbar}\ .\frac{(2amx^2-\hbar)e^{\frac{-amx^2}{\hbar}}}{\hbar}$$

$So,\ now,$

$$<p^2>= \int_{-\infty}^{\infty}\psi\ \ -\hbar^2\frac{d^2}{dx^2}\ (\psi)dx$$

$$= \int_{-\infty}^{\infty}\psi\ (-\hbar^2)\frac{2Aam}{\hbar}\ .\frac{(2amx^2-\hbar)e^{\frac{-amx^2}{\hbar}}}{\hbar}$$

$$= \int_{-\infty}^{\infty}\psi\ (-\hbar^2)\frac{2Aame^{\frac{-amx^2}{\hbar}}}{\hbar}\ .(\frac{2amx^2}{\hbar}-1)$$

$$= \int_{-\infty}^{\infty} \psi \ (-\hbar^2) - \frac{2Aame^{\frac{-amx^2}{\hbar}}}{\hbar} \ .(1 - \frac{2amx^2}{\hbar})dx$$

$$= \int_{-\infty}^{\infty} \psi \ (\hbar^2)\frac{2Aame^{\frac{-amx^2}{\hbar}}}{\hbar} \ .(1 - \frac{2amx^2}{\hbar})dx$$

$$= \hbar^2 \int_{-\infty}^{\infty} \psi \ \frac{2Aame^{\frac{-amx^2}{\hbar}}}{\hbar} \ .(1 - \frac{2amx^2}{\hbar})$$

$$= \hbar^2 \int_{-\infty}^{\infty} \psi \ \psi \ \frac{2am}{\hbar} \ .(1 - \frac{2amx^2}{\hbar}) \quad [since, \psi = Ae^{(\frac{-amx^2}{\hbar})}]$$

$$= \hbar^2 \int_{-\infty}^{\infty} \psi^2 \ \frac{2am}{\hbar} \ .(1 - \frac{2amx^2}{\hbar})$$

$$= \hbar^2 \int_{-\infty}^{\infty} \psi^2 \ (\frac{2am}{\hbar} - \frac{4a^2m^2x^2}{\hbar^2})dx$$

$$= \hbar^2 \int_{-\infty}^{\infty} \psi^2 \ \frac{2am}{\hbar}dx + \hbar^2 \int_{-\infty}^{\infty} -\psi^2 \ \frac{4a^2m^2x^2}{\hbar^2}dx$$

$$= \hbar^2 \frac{2am}{\hbar} \int_{-\infty}^{\infty} \psi^2 dx - \hbar^2 \frac{4a^2m^2}{\hbar^2} \int_{-\infty}^{\infty} \psi^2 x^2 dx$$

$$= 2am\hbar \int_{-\infty}^{\infty} \psi^2 dx - 4a^2m^2 \int_{-\infty}^{\infty} \psi^2 x^2 dx$$

$$= 2am\hbar \int_{-\infty}^{\infty} \psi^2 dx - 4a^2m^2 \int_{-\infty}^{\infty} x^2 A^2 e^{(\frac{-2amx^2}{\hbar})} dx \quad [since, \psi = Ae^{(\frac{-amx^2}{\hbar})}]$$

$we \ got \ from \ part(a), \int_{-\infty}^{\infty} \psi^2 dx = 1$

$$and,\ from\ part(c), \int_{-\infty}^{\infty} x^2\ A^2 e^{(\frac{-2amx^2}{\hbar})} dx = \frac{\hbar}{4am}$$

$so,$

$$= 2am\hbar \int_{-\infty}^{\infty} \psi^2 dx - 4a^2m^2 \int_{-\infty}^{\infty} x^2 A^2 e^{(\frac{-2amx^2}{\hbar})} dx$$

$$= 2am\hbar.1 - 4a^2m^2(\frac{\hbar}{4am})$$

$$= 2am\hbar - am\hbar$$

$$= am\hbar$$

$$So, < p^2 >= am\hbar$$

$$Therefore, < x^2 >= \frac{\hbar}{4am}\ and,\ < p^2 >= am\hbar$$

$$Answer :< x^2 >= \frac{\hbar}{4am},$$
$$< p^2 >= am\hbar$$

Part(d)

$Here,$

$${\sigma_x}^2 =< x^2 > - < x >^2$$

$$\sigma_x = \sqrt{< x^2 > - < x >^2}$$

$$= \sqrt{\frac{\hbar}{4am} - 0} \quad [since, we\ got\ < x^2 >\ from\ part\ (c)\ and\ < x >\ from\ part(b)]$$

$$= \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_x = \sqrt{\frac{\hbar}{4am}}$$

$And,$

$${\sigma_p}^2 =< p^2 > - < p >^2$$

$$\sigma_p = \sqrt{< p^2 > - < p >^2}$$

$$= \sqrt{am\hbar - 0} \quad [since, we\ got\ < p^2 >\ from\ part\ (c) and\ < p >\ from\ part(b)]$$

$$= \sqrt{am\hbar}$$

$$\sigma_p = \sqrt{am\hbar}$$

$Now, we\ have\ to\ prove,$

$$\sigma_x\sigma_p \geqslant \frac{\hbar}{2}$$

$So,\ here,$

$\sigma_x\sigma_p$

$$= \sqrt{\frac{\hbar}{4am}}\,\sqrt{am\hbar}$$

$$= \sqrt{\frac{\hbar am\hbar}{4am}}$$

$$= \sqrt{\frac{\hbar^2}{4}}$$

$$= \frac{\hbar}{2}$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

$Thus,$

$\sigma_x\sigma_p \geqslant \frac{\hbar}{2} \qquad [Proved]$

$Therefore,\ as,\ \ \sigma_x\sigma_p = \frac{\hbar}{2}, so\ it\ states\ that,$
$the\ minimum\ uncertainty\ is\ only\ achieved\ by\ Gaussian\ wave\ function.$
$Our\ Gaussian\ wave-function\ holds\ Heisenberg's\ Uncertainty\ principle$
$and\ so\ our\ particle's\ wave-function\ is\ quantum\ mechanical.$

$||Thank\ you||$