CS 325 - Homework 7 - Solutions

1. **(7 points 1 pt each)** Let X and Y be two decision problems. Suppose we know that X reduces to Y in polynomial time. Which of the following can we infer? Explain

a. If Y is NP-complete then so is X. False cannot be inferred
b. If X is NP-complete then so is Y. False cannot be inferred
c. If Y is NP-complete and X is in NP then X is NP-complete. False cannot be inferred
d. If X is NP-complete and Y is in NP then Y is NP-complete. TRUE
e. X and Y can't both be NP-complete. False cannot be inferred
f. If X is in P, then Y is in P. False cannot be inferred
g. If Y is in P, then X is in P. TRUE

2. **(4 points 1 pt each)** Clearly explain whether or not each of the following statements follows from that fact that COMPOSITE is in NP and SUBSET-SUM is NP-complete:

a. SUBSET-SUM $\leq_p$ COMPOSITE.

No. SUBSET-SUM is NP-complete and so may be reduced to any other NP-complete problem. However, we don't know that COMPOSITE is NP-complete, only that it is in NP.

b. If there is an $O(n^3)$ algorithm for SUBSET-SUM, then there is a polynomial time algorithm for COMPOSITE.

Yes. The given running time is polynomial. Since SUBSET-SUM is NP-complete, this implies P = NP. Hence, every algorithm in NP, including COMPOSITE, would have a polynomial-time algorithm.

c. If there is a polynomial algorithm for COMPOSITE, then P = NP.

No. COMPOSITE is in NP, but it is not known if it is in NP-complete.

d. If P $\neq$ NP, then ***no*** problem in NP can be solved in polynomial time.

No. All problems in P are also in NP and can be solved in polynomial time. Proving P 6= NP would show only that NP-complete problems cannot be solved in polynomial time.

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3. Two well-known NP-complete problems are 3-SAT and TSP, the traveling salesman problem. The 2-SAT problem is a SAT variant in which each clause contains at most two literals. 2-SAT is known to have a polynomial-time algorithm. Is each of the following statements true or false? Justify your answer.

a. 3-SAT $\leq_p$ TSP.

True. There exists a reduction from any NP-complete problem to any other such problem.

b. If P ≠ NP, then 3-SAT $\leq_p$ 2-SAT.

False. If P ≠NP, there is no polynomial-time algorithm for 3-SAT. However, 2-SAT is known to be in P; if the reduction existed, it would imply a polynomial-time algorithm for 3-SAT.

c. If P ≠ NP, then no NP-complete problem can be solved in polynomial time.

True. A polynomial-time algorithm for one NP-complete problem yields polynomialtime algorithms for all others. Hence, either all these problems are in P, or none are. P 6= NP implies the latter.

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4. A Hamiltonian path in a graph is a simple path that visits every vertex exactly once. Show that HAM-PATH = { (G, u, v): there is a Hamiltonian path from u to v in G} is NP-complete. You may use the fact that HAM-CYCLE is NP-complete

2 points

1) show HAM-PATH ∈NP

Given a graph G with n vertices, and a path from u to v, we can verify in polynomial time that path is a simple path with n vertices, by checking the adjacency list to verify the vertices are adjacent, and that there are n vertices.

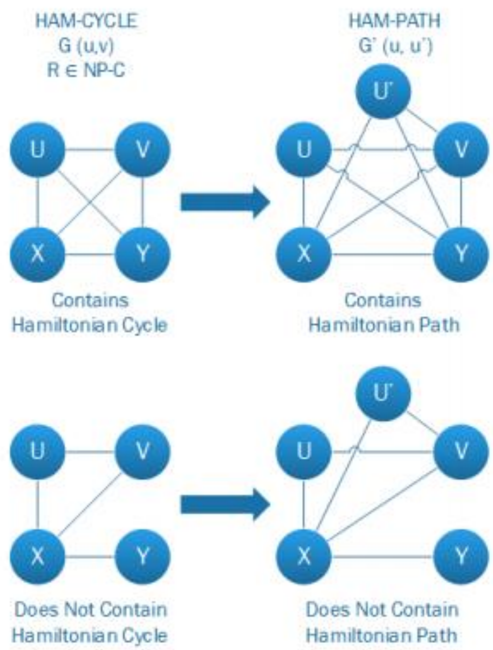
2) Show that R ≤p HAM-PATH for some R ∈NP-Complete **3 points**

a) Select R = HAM-CYCLE because it has a similar structure to HAM-PATH and we know HAM-CYCLE is NP-Complete, and therefore in NP.

b) Show that HAM-CYCLE reduces to HAM-PATH

Let HC = HAM-CYCLE and HP = HAM-PATH

HC (u-v) reduces to HP (u-u'). Given a graph G(u-v) having a Hamiltonian Cycle, where (u-v) is a set of vertices, we produce a new graph G'(u-u') by duplicating arbitrary vertex u along with all of it's connecting edges and naming it u'. This new graph, G'(u, u') now has a Hamiltonian Path from u to u'. This reduction occurs in polynomial time simply by adding the list of edges for u' to the edge list of G. See image below:



c) If G' has a Hamiltonian Path from u to u', then G has a Hamiltonian Cycle and conversely if G has a Hamiltonian Cycle, then G' has a Hamiltonian Path. Also IF G does not have a Hamiltonian Cycle, then G' does not have a Hamiltonian Path.

d) Since HC is NP-Complete, HP must be in NP-Hard.

Since 1 and 2 are true, HAM-PATH is NP-Complete.

1 point

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Alternative proof.

$\text{HAM-PATH} \in NP$ **2 points**

Let $p = \{u, ..., v\}$ be a certificate path.

Traverse p, and mark the number of times a vertex is visited (initially zero).

Confirm that every vertex $i \in V$ is visited exactly once, and each traversed edge $(i, j) \in E$.

$\text{HAM-PATH} \in NP\text{-hard}$ **3 points**

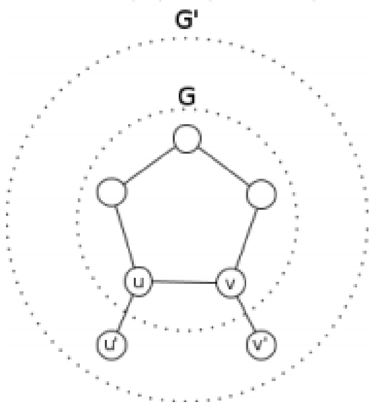
Let $G = (V, E)$ be an instance of the HAM-CYCLE problem $(\text{HAM-CYCLE} \in NP\text{-complete})$.

Let $G' = (V', E')$ be an instance of the HAM-PATH problem.

$V' = V \cup \{u', v'\}$

$E' = E \cup \{(u', u'), (v', v)\}$ for an edge $(u, v) \in E$.

Adding two vertices and two edges transforms $\langle G \rangle$ to $\langle G', u', v' \rangle$ in polynomial time.



Suppose that G has a Hamiltonian cycle.

A simple path $p = \{u, ..., v\}$visits each vertex in V exactly once, and $(u, v) \in E$.

A simple path $p' = \{u', u, ..., v, v'\}$ visits each vertex in V' exactly once.

Therefore, G' has a Hamiltonian path.

Suppose that G' has a Hamiltonian path from u' to v'.

A simple path $p' = \{u', u, ..., v, v'\}$ visits each vertex in V' exactly once.

A simple path $p = \{u, ..., v\}$is a subpath of p'.

p visits each vertex in V exactly once, and $(u, v) \in E$.

Therefore, G has a Hamiltonian cycle.

Having shown that $\text{HAM-PATH} \in NP$ and $\text{HAM-PATH} \in NP\text{-hard}$, this completes the proof that $\text{HAM-PATH} \in NP\text{-complete}$.

1 points

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5. LONG-PATH is the problem of, given (G, u, v, k) where G is a graph, u and v vertices and k an integer, determining if there is a simple path in G from u to v of length at least k. Prove that LONG-PATH is NP-complete.

To show that LONG-PATH is NP-complete.

Step 1: **(2 points)** LONG-PATH is NP. A certificate for LONG-PATH (G,u,v, k) would be a listing of the sequence of vertices in a path of length k.

- If $k \leq n-1$, a poly-time verifier can check that the sequence is a valid path in G, is simple, and has length k.
  - To check that the sequence of vertices is a path takes $O(n^2)$.
  - To check if a simple path takes $O(n^2)$.
  - To check length takes O(n).
- If $k > n-1$, there can be no simple path with k edges (as it would have $k+1 > n$ vertices on it), and we can reject the input whatever certificate is offered.

Step 2: **(2 points)** Show LONG- PATH is NP-hard. We can reduce HAM-PATH, proven in problem 4 to be NP-complete, to LONG-PATH. An input to HAM-PATH is (G, u, v), and is in HAM-PATH if and only if there exists a simple path in G starting at u, ending at v, containing all n vertices and thus having exactly n-1 edges. So (G,u,v) is in HAM-PATH if and only if (G,u,v,n-1) is in LONG-PATH, and clearly this reduction can be computed in polynomial time. We have that HAM-PATH $\leq_p$ LONG-PATH, and thus LONG-PATH is NP-hard.

(1 point) Since 1) and 2) are true LONG-PATH is in NP-Complete.

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1) show LP $\in$NP

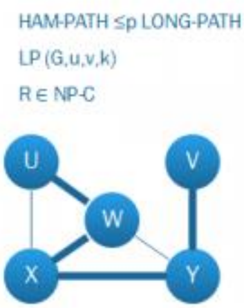
LP is a decision problem which can be easily verified in polynomial time by examining the adjacency list of the path, and verifying that there are more than k vertices in the adjacency list of the path.

2) R $\in$NPC R $\leq$p LP

a) Select R = HP (Hamiltonian Path), because a Hamiltonian path is a longest path. We know HP $\in$NP

b) show HP $\leq$p LP in polynomial time.

To reduce HP to LP, simply modify G increasing or decreasing HP by the number of vertices necessary such that k $\leq$ the length of HP which is n-1.



c) If HP exists, and k $\leq$ n-1, then LP exists. If LP exists, then HP exists for G' n-1 $\geq$ k.

d) Since HP is in NP-Complete, the LP must be in NP-Hard.

Since 1 and 2 are true, LP is NP-Complete.