Mathematic modelling of mechanical systems dynamic

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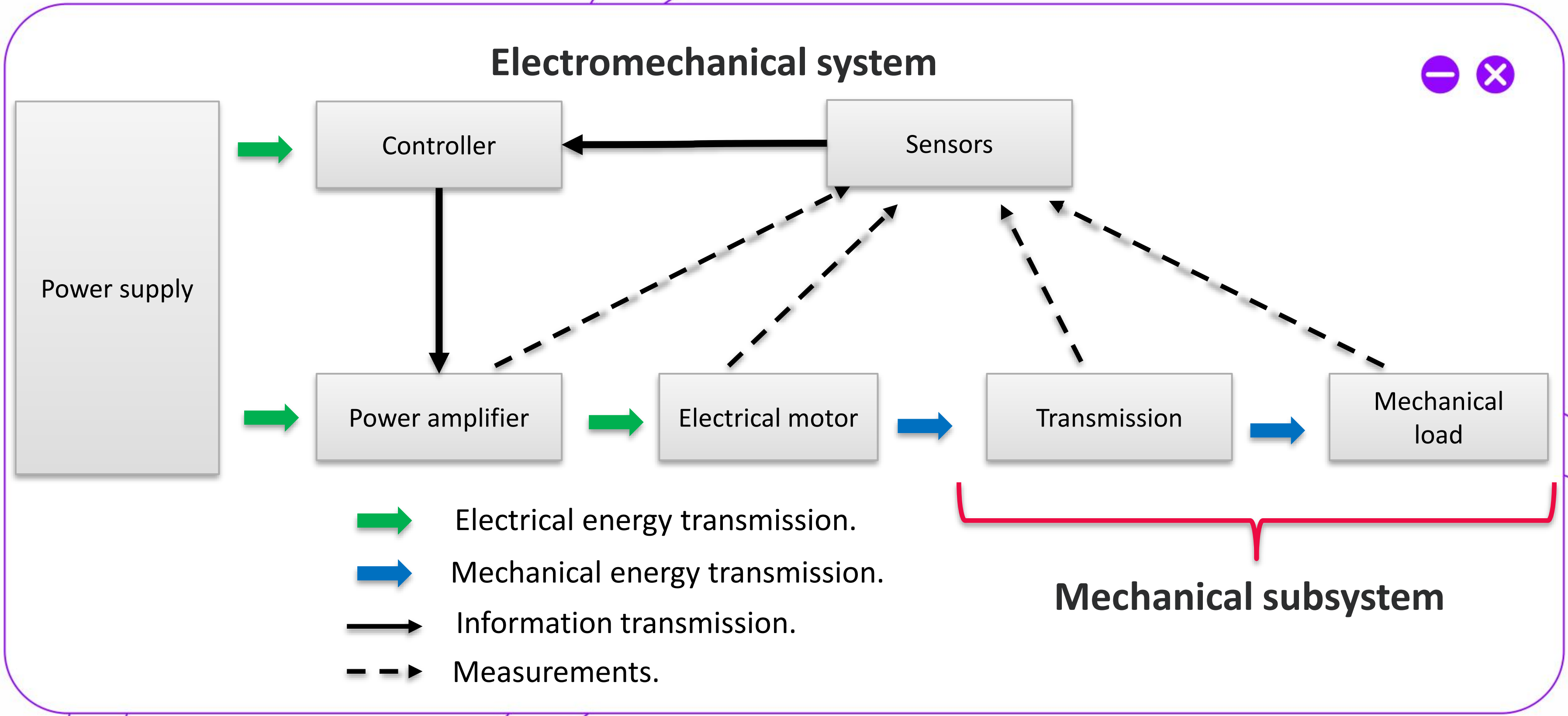
Outline

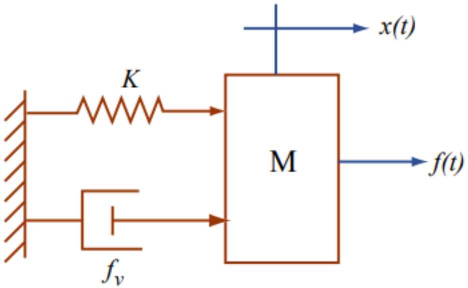
1. Introduction
2. Lumped mass method
3. Basic dynamic elements of mechanical systems
4. Dynamical models of mechanical systems

Introduction

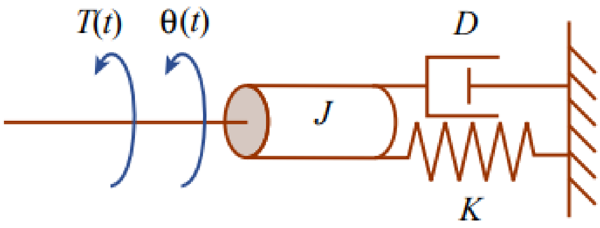


Introduction

- Mechanical systems are modelled, basically, as systems with **lumped-parameter elements**;
- From the energy point of view mechanical systems can be described with dissipative elements, potential energy storage elements, kinetic energy storage elements;
- Forces and moments which drive mechanical system are typically applied by actuators but might represent other loads applied by the environment.



Translational movement



Rotational movement

Lumped mass method

- Display of physical properties in a dynamic model as a collection of discrete elements.
- All elements are simple. Each relates to only one physical property.
- The state of a simple element is characterized by one phase variable of the flow type and one variable of the potential type.
- A component equation is an equation that describes the physical property of an element in the form of a relationship between phase variables.

Lumped mass method

- ❑ The main physical properties of technical objects of any physical nature:
- ✓ inertial ;
- ✓ elastic ;
- ✓ dissipative.

- ❑ Friction and transformer elements display specific properties that are not characteristic of all technical objects. Those elements should be considered separately

Lumped mass method – Component equations

- Approximation of models.

An example of replacing partial derivatives of phase variables with respect to spatial coordinates by ratios of finite differences:

$$\frac{\partial \varphi}{\partial x} = \frac{\varphi_1 - \varphi_2}{l_x}$$

- Obtaining equations based on physical laws:

✓ for inertial element: $$\varphi_J = J \frac{d\Phi_J}{dt}$$

✓ for a dissipative element: $$\varphi_D = D\Phi_D$$

✓ for elastic element: $$\varphi_E = E\int \Phi_E dt$$

Lumped mass method – Topological equations

- **Equilibrium condition:**

$$\sum_i \varphi_i = 0$$

- **Continuity condition:**

$$\sum_k \Phi_k = 0$$

- The form of component and topological equations is the same for systems of different physical nature.

Lumped mass method – Topological equations

- **Translational movement:**
  - ✓ linear speed v;
  - ✓ force F.

- **Rotational movement:**
  - ✓ angular velocity ω;
  - ✓ torque T.

- **Flow type phase variables (Φ):**
  - ✓ force F;
  - ✓ torque T.

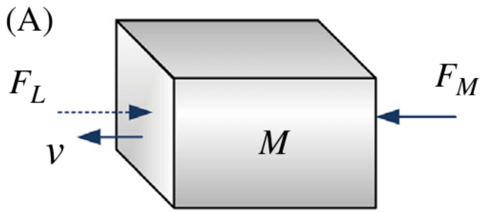
- **Phase variables of potential type (ϕ):**
  - ✓ linear speed v ;
  - ✓ angular velocity ω.

Dynamic equation of motion

Consider that a driving force F_M acts on an object of mass m, so that the object moves at a speed v.

From Newton's second law: $$F_M - F_L = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$$

If the mass m of the load is constant: $$F_M - F_L = m\frac{dv}{dt} = ma$$

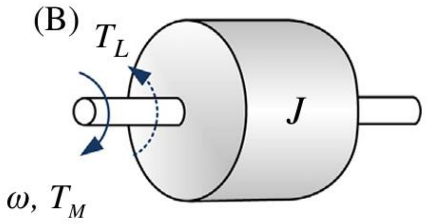


Dynamic equation of motion

For a rotational motion:

From Newton's second law: $$T_M - T_L = \frac{d}{dt}\left(J\omega\right) = J\frac{d\omega}{dt} + \omega\frac{dJ}{dt}$$

If the inertia *J* assumed to be constant: $$T_M - T_L = J\frac{d\omega}{dt} = J\varepsilon$$



We can see that if $T_M > T_L$, then the object will accelerate, and if $T_M > T_L$, then it will decelerate.

In the case of $T_M = T_L$, the speed will not be changed.

Dynamic equation of motion

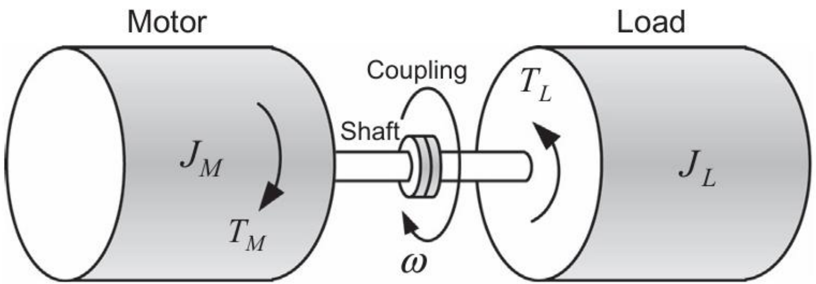
From the motor's viewpoint, a mechanical load can be seen as a load torque T_L connected to its shaft. The equation of motion of a motor drive system:

$$T_M = (J_M + J_L)\frac{d\omega}{dt} + T_L$$

When the motor is driving a mechanical load, the torque required to run the load is the load torque T_L, which varies with the type of load. When a load begins to move and is in motion, a friction force occurs, resisting the motion. The equation of motion needs to include the friction force T_F as

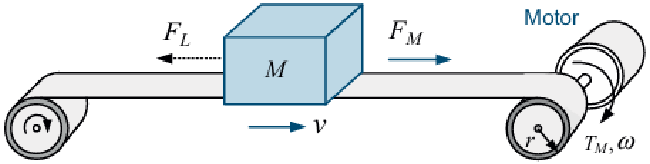
$$T_M = (J_M + J_L)\frac{d\omega}{dt} + T_L + T_F$$

This equation is also known one-mass model of mechanical subsystem



Combination system of translational motion and rotational motion

Many motions in the industry such as an elevator, a conveyor, or electric vehicles are a combination of translational motion and rotational motion. In this case, we need to convert the load parameters such as load inertia to the motor shaft.



The expression for the translational motion:

$$F_M - F_L = m\frac{dv}{dt}$$

The relationships between force and torque, and velocity and angular velocity are:

$$T_M = rF_M \quad T_L = rF_L \quad \omega = \frac{v}{r}$$

Resulting equation from the motor's point of view:

$$T_M - T_L = mr^2\frac{d\omega}{dt} = J_e\frac{d\omega}{dt}$$

where J_e represents the equivalent moment of inertia, which is reflected to the motor shaft side of the mass m in translational motion.

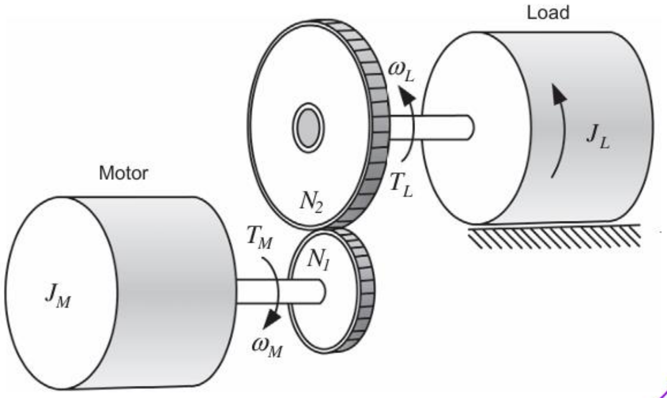
System with gears or pulleys

Often, the speed required by the load is too low compared to the nominal speed of the motor. Gears or pulleys between the motor and the load being driven are most often used to change the speed. In this case, we need to know how the load will be seen through the gears or pulleys at the motor side.

The equation of motion at the load side: $$T_L = J_L \frac{d\omega_L}{dt}$$

Because the two gears will travel at an equal distance, we have: $$\omega_L = \omega_M \frac{N_1}{N_2}$$

where N_1 and N_2 are the number of teeth on the gears of the motor and load sides, respectively.



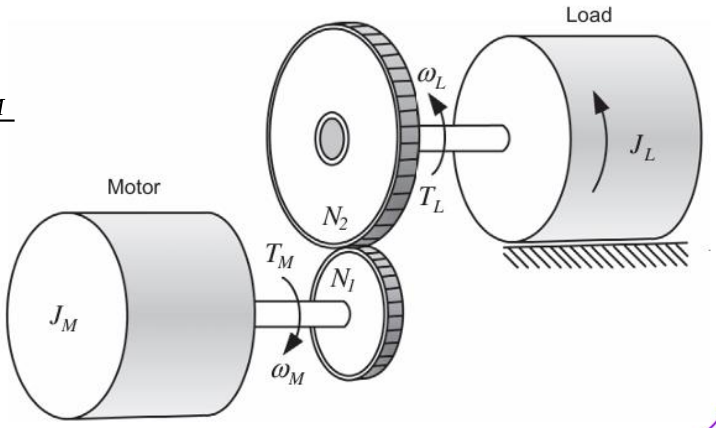
System with gears or pulleys

When neglecting friction and losses of the gears, the power is the same at the input and at the output of the gear, i.e.:

$$P = T_L \omega_L = T_M \omega_M \quad \rightarrow \quad T_L = \frac{N_2}{N_1} T_M$$

The equation of motion at the motor side is given by:

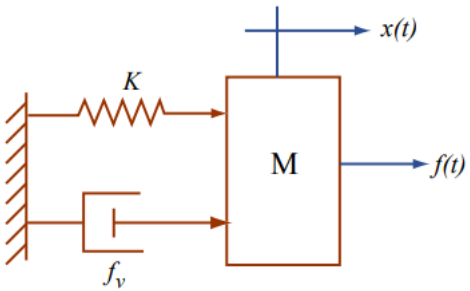
$$T_M = J_L \left(\frac{N_1}{N_2} \right)^2 \frac{d\omega_M}{dt} + J_M \frac{d\omega_M}{dt} = \left(J_L \left(\frac{N_1}{N_2} \right)^2 + J_M \right) \frac{d\omega_M}{dt}$$



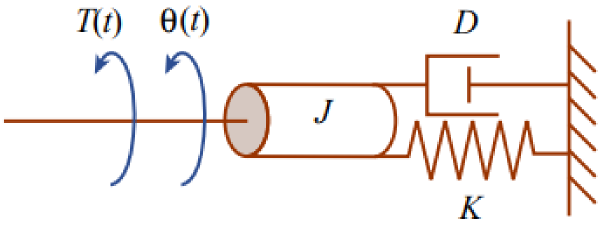
Basic dynamic elements of mechanical systems

Mechanical systems consist **three** basic elements:

- ❑ Inertia elements
- ❑ Spring elements (elastic)
- ❑ Damper elements (dissipative)



Translational movement

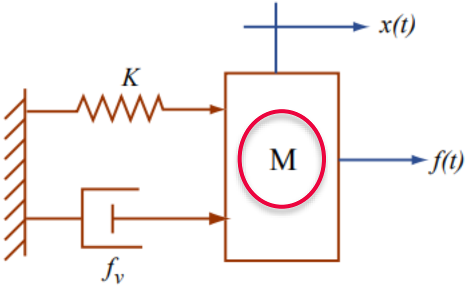


Rotational movement

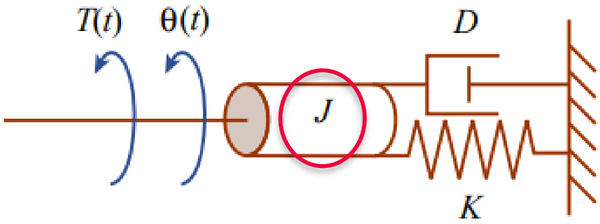
Inertia elements

Inertia elements:

- **Mass** for translational movement
- **Rotational inertia** for rotational movement
- Each inertia element with independent motion needs its own differential equation
- Inertia elements store kinetic energy

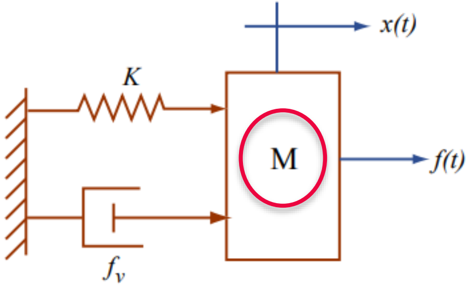


Translational movement

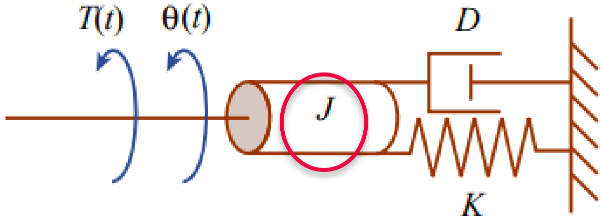


Rotational movement

Inertia elements



Translational movement



Rotational movement

Mass, m, [kg] ⇐ Element ⇒ *Rotational inertia, J, [kg·m^2]*

Newton 2nd law

$$\sum F = ma$$

⇐ Force (torque) equation ⇒

Euler 2nd law

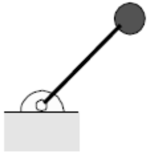
$$\sum T = J\varepsilon$$

$$E = \frac{1}{2}mv^2$$

⇐ Kinetic energy ⇒

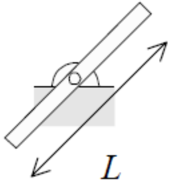
$$E = \frac{1}{2}J\omega^2$$

Inertia elements



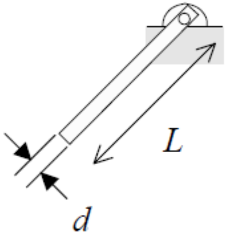
Point mass at radius r

$$J = mr^2$$



Rod or bar about centroid

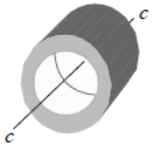
$$J = \frac{mL^2}{12}$$



Short bar about pivot

$$J = \frac{m}{12}\left(d^2 + 4l^2\right)$$

Slender bar case, *d*=0

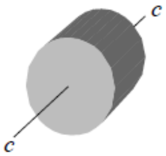


Cylindrical shell about axis c-c (inner radius *r*)

$$J = mr^2$$

If outer radius is R, and not thin shell,

$$J = \frac{1}{2}m\left(R^2 + r^2\right)$$



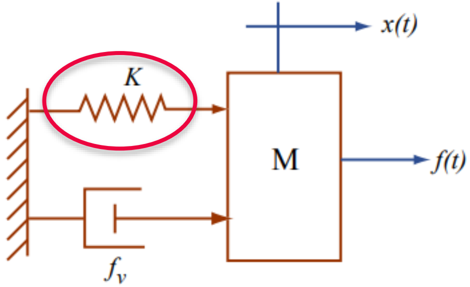
Cylinder about axis c-c (radius *r*)

$$J = \frac{1}{2}mr^2$$

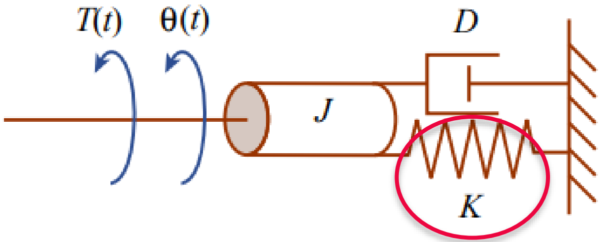
Spring elements

Spring elements:

- ❑ Force (torque) is generated to resist deflection
- ❑ Examples: translational and rotational springs. Even steel rod has stiffness
- ❑ Spring elements store potential energy

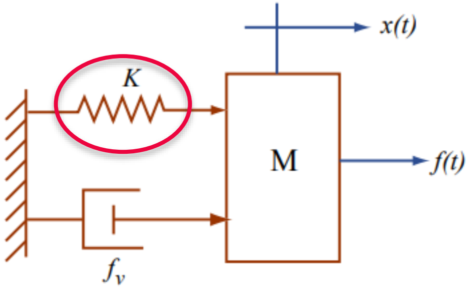


Translational movement

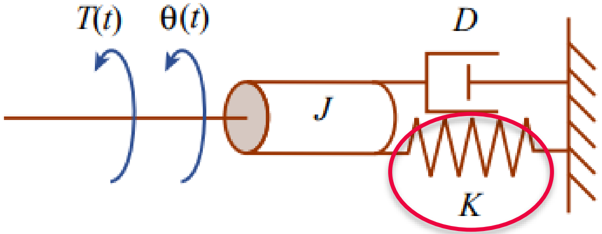


Rotational movement

Spring elements



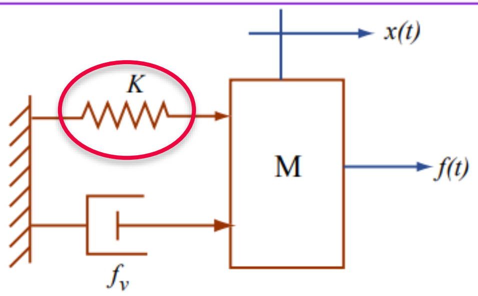
Translational movement



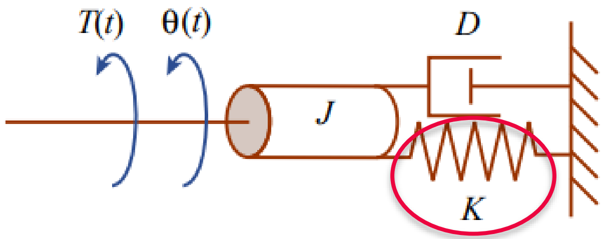
Rotational movement

Stiffness, k, [N/m]	⇐	Element	⇒	*Stiffness, K, [N·m/rad]*
$F = k(x_1 - x_2)$	⇐	Force (torque) equation	⇒	$T = K(\theta_1 - \theta_2)$
$E = \frac{1}{2}kx^2$	⇐	Potential energy	⇒	$E = \frac{1}{2}K\theta^2$

Spring elements



Translational movement

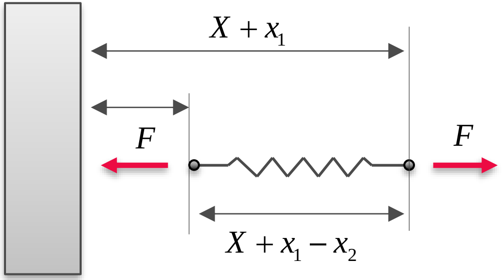


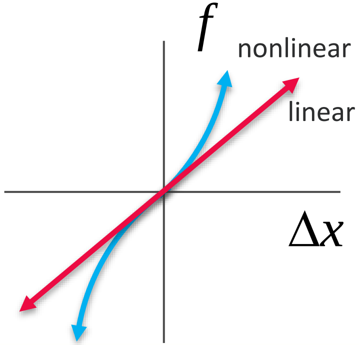
Rotational movement

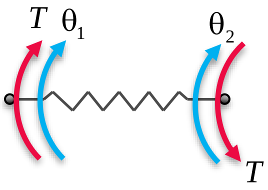
$$F = k(x_1 - x_2)$$

Force (torque) equation

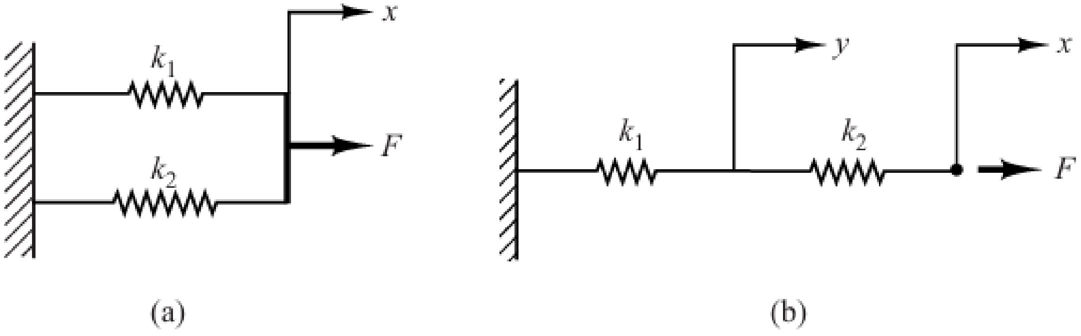
$$T = K(\theta_1 - \theta_2)$$







Connection of spring elements



$$k_1 x + k_2 x = F = k_{eq} x$$

$$k_{eq} = k_1 + k_2$$

$$k_1 y = F \quad k_2 \left(x - y \right) = F$$

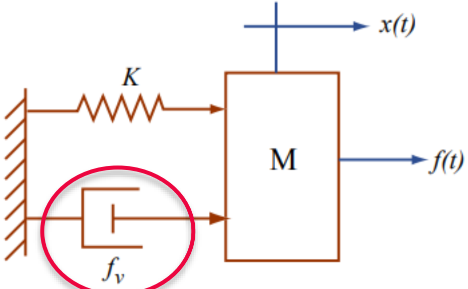
$$k_2 \left(x - \frac{F}{k_1} \right) = F$$

$$k_{eq} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

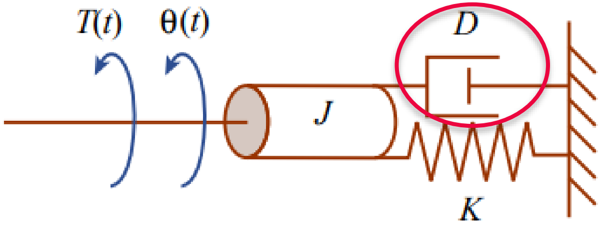
Damper elements

Damper elements:

- ❑ Force (torque) is generated to resist motion
- ❑ Examples: dashpots, friction, wind drag
- ❑ Damper elements dissipate energy

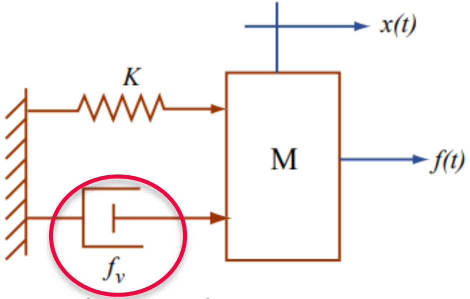


Translational movement

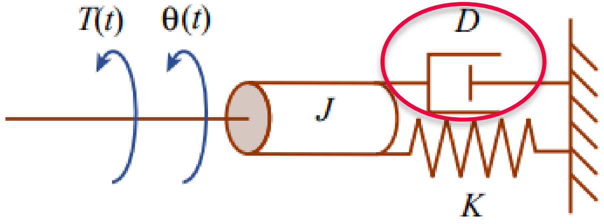


Rotational movement

Damper elements



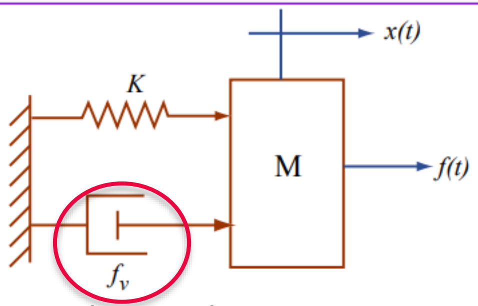
Translational movement



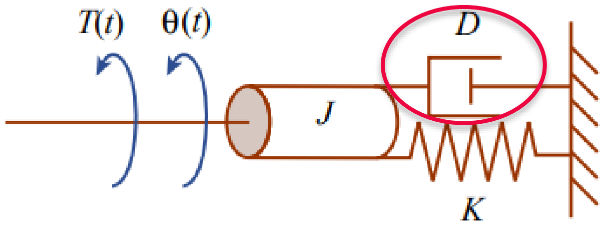
Rotational movement

Damping, b, [N·s/m]	Element	*Damping, b, [N·m·s/rad]*
$F = b(\dot{x}_1 - \dot{x}_2)$	Force (torque) equation	$T = b(\dot{\theta}_1 - \dot{\theta}_2)$
$E = bV^2$	Dissipated energy	$E = b\omega^2$

Damper elements



Translational movement

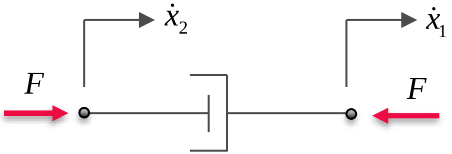


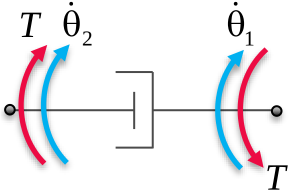
Rotational movement

$$F = b(\dot{x}_1 - \dot{x}_2)$$

Force (torque) equation

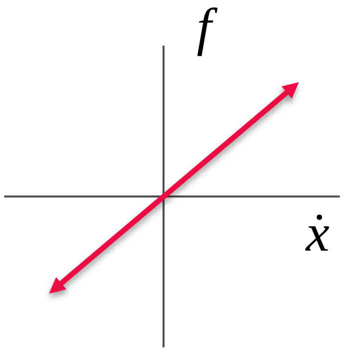
$$T = b(\dot{\theta}_1 - \dot{\theta}_2)$$





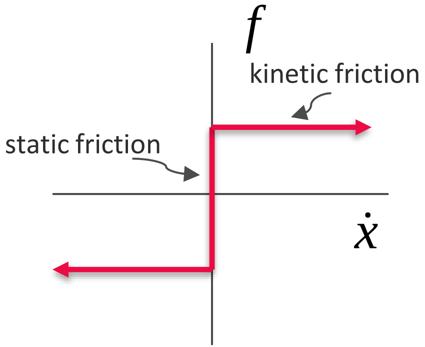
Damper elements

Linear damping:



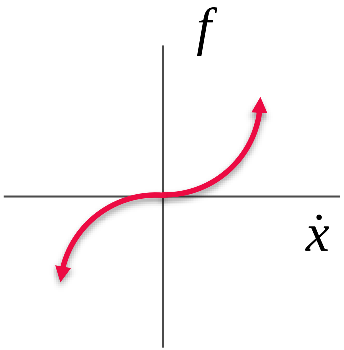
$$F = b\dot{x}$$

Coulomb friction:



$$F = b \cdot \mathrm{sgn}(\dot{x})$$

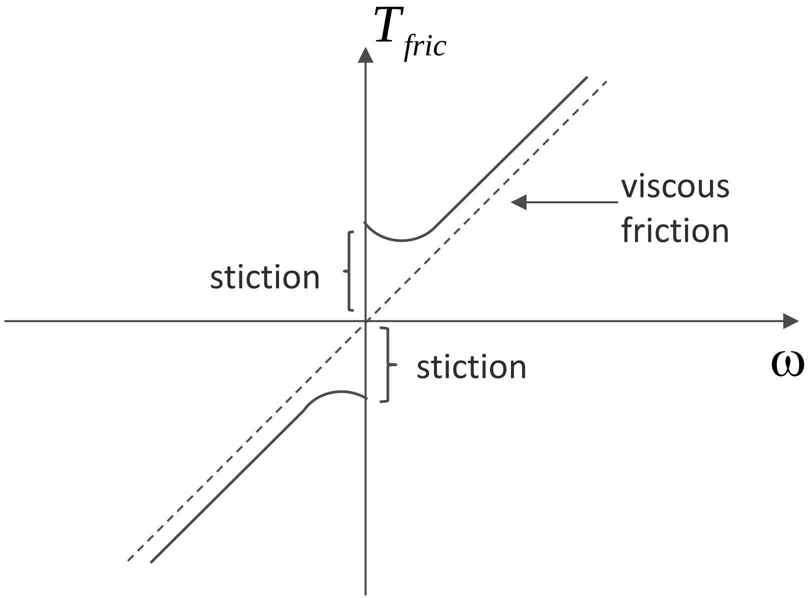
Drag:



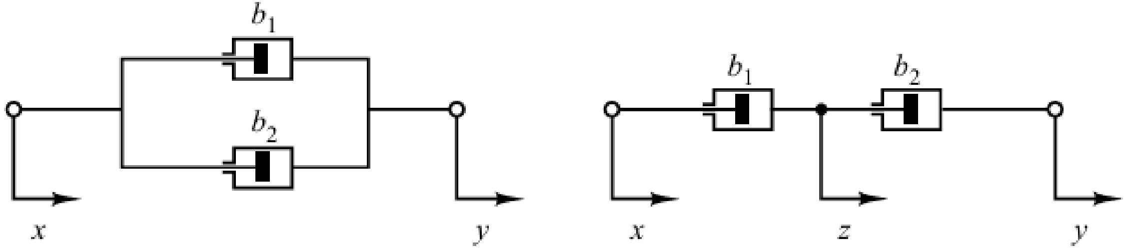
$$F = b\dot{x}^2$$

Damper elements

Real friction function combines aspects of multiple model types



Connection of damper elements



(a) (b)

$$F = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) =$$

$$= (b_1 + b_2)(\dot{y} - \dot{x})$$

$$F = b_{eq}(\dot{y} - \dot{x})$$

$$b_{eq} = b_1 + b_2$$

$$F = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z})$$

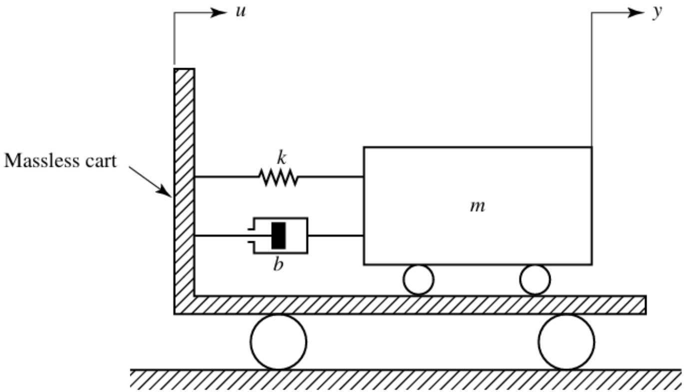
$$\dot{z} = \frac{1}{b_1 + b_2}\left(b_2\dot{y} + b_1\dot{y}\right)$$

$$F = b_{eq}\left(\dot{y} - \dot{x}\right) = b_2\left[\dot{y} - \frac{1}{b_1 + b_2}\left(b_2\dot{y} + b_1\dot{y}\right)\right] = \frac{b_1 b_2}{b_1 + b_2}\left(\dot{y} - \dot{x}\right)$$

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

Examples

Consider the spring-mass-dashpot system mounted on a massless cart. Let us obtain mathematical models of this system by assuming that the cart is standing still for t < 0 and the spring-mass-dashpot system on the cart is also standing still for t < 0. In this system, u(t) is the displacement of the cart and is the input to the system. At t=0, the cart is moved at a constant speed, or constant. The displacement y(t) of the mass is the output. In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k de notes the spring constant.



For translational systems, Newton's second law states that:

$$\sum F = ma$$

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

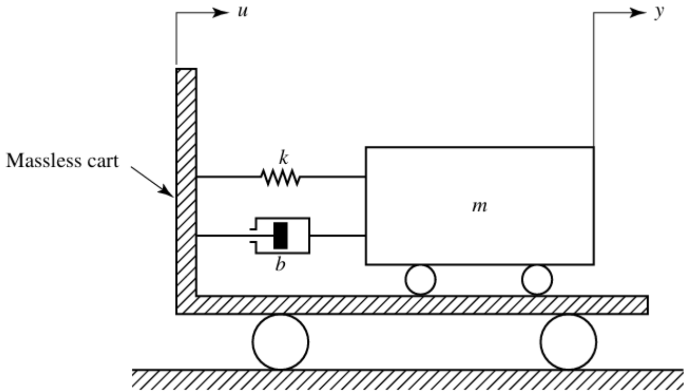
Examples

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives:

$$\left(ms^2+bs+k\right)Y(s)=(bs+k)U(s)$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{bs+k}{ms^2+bs+k}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.



Examples

Next, we shall obtain a state-space model of this system. We shall first compare the differential equation for this system:

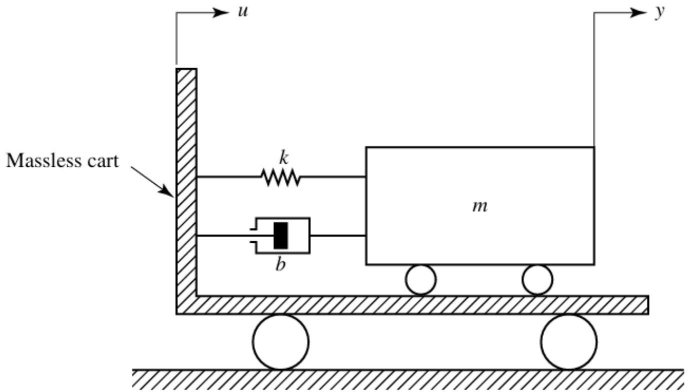
$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m} \quad a_2 = \frac{k}{m} \quad b_0 = 0 \quad b_1 = \frac{b}{m} \quad b_2 = \frac{k}{m}$$



Examples

Then, we can obtain state space model using the follows technique

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

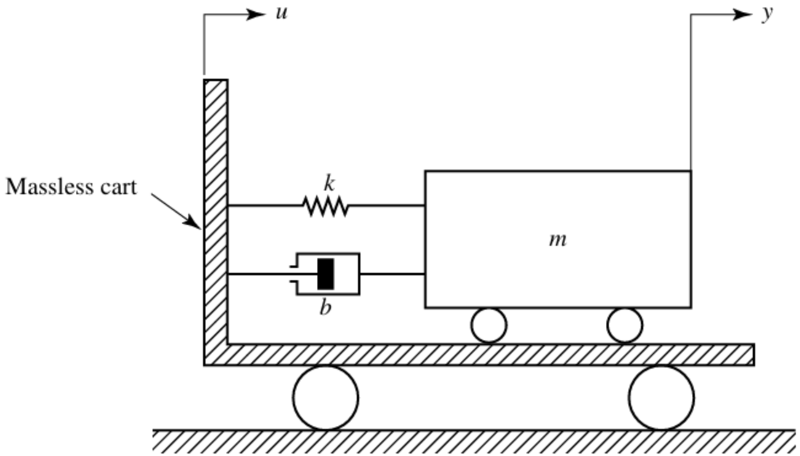
$$y = x_1$$

Examples

Vector-matrix representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

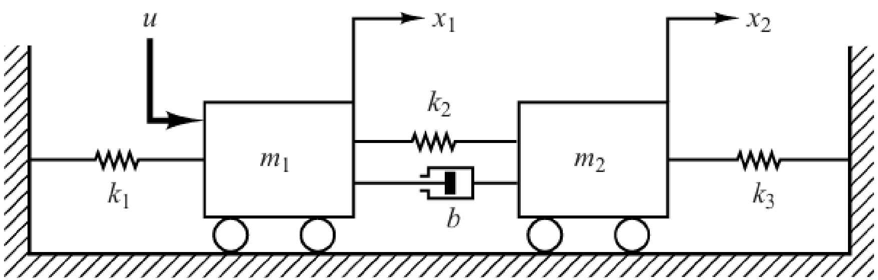


Examples

Consider the mechanical system shown in Figure and find transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$.

The equations of motion for the system:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2\left(x_1 - x_2\right) - b\left(\dot{x}_1 - \dot{x}_2\right) + u$$
$$m_2\ddot{x}_2 = -k_3x_2 - k_2\left(x_2 - x_1\right) - b\left(\dot{x}_2 - \dot{x}_1\right)$$



Simplifying, we obtain:

$$m_1\ddot{x}_1 + b\dot{x}_1 + \left(k_1 + k_2\right)x_1 = b\dot{x}_2 + k_2x_2 + u$$
$$m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_2 + k_3\right)x_2 = b\dot{x}_1 + k_2x_1$$

Examples

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain:

$$\left(m_1 s^2 + bs + \left(k_1 + k_2\right)\right) X_1(s) = \left(bs + k_2\right) X_2(s) + U(s)$$

$$\left(m_2 s^2 + bs + \left(k_2 + k_3\right)\right) X_2(s) = \left(bs + k_2\right) X_1(s)$$

Transfer functions:

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + bs + k_2 + k_3}{\left(m_1 s^2 + bs + k_1 + k_2\right)\left(m_2 s^2 + bs + k_2 + k_3\right) - \left(bs + k_2\right)^2}$$

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{\left(m_1 s^2 + bs + k_1 + k_2\right)\left(m_2 s^2 + bs + k_2 + k_3\right) - \left(bs + k_2\right)^2}$$

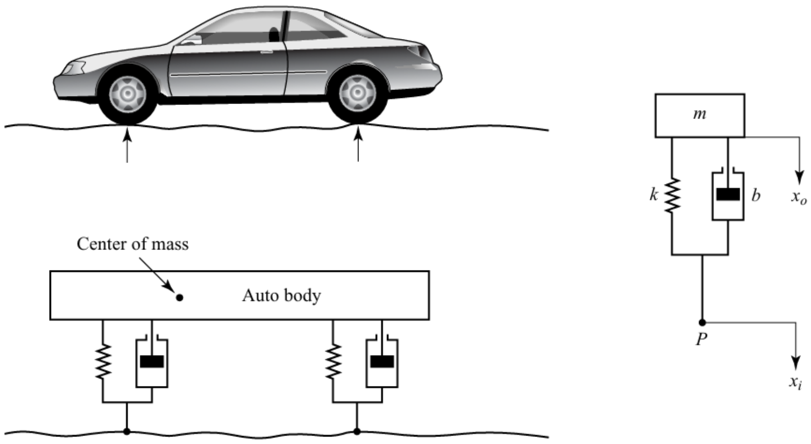
Examples

Figure (a) shows a schematic diagram of an automobile suspension system.

As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the auto mobile suspension system.

The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass.

Mathematical modeling of the complete system is quite complicated.



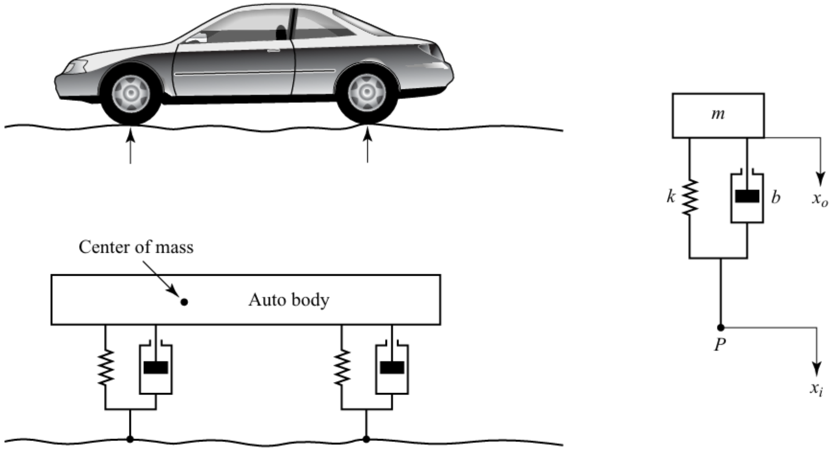
(a) (b)

Examples

A very simplified version of the suspension system is shown in Figure (b).

Assuming that the motion x_i at point P is the input to the system and the vertical motion x_o of the body is the output, obtain the transfer function $X_o(s)/X_i(s)$. (Consider the motion of the body only in the vertical direction.)

Displacement x_o is measured from the equilibrium position in the absence of input x_i.



(a) (b)

Examples

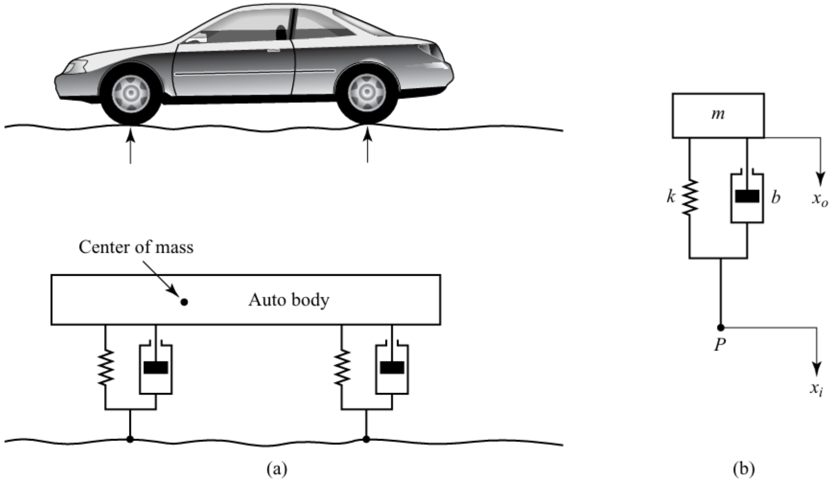
The equation of motion for the system shown in (b) is

$$m\ddot{x}_o + b\left(\dot{x}_o - \dot{x}_i\right) + k\left(\dot{x}_o - \dot{x}_i\right) = 0$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain:

$$\left(ms^2 + bs + k\right)X_o(s) = \left(bs + k\right)X_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

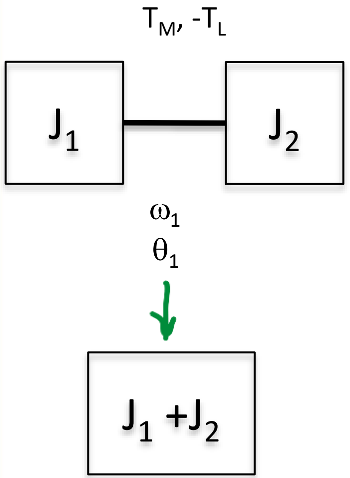


Dynamical models of rotating mechanical systems

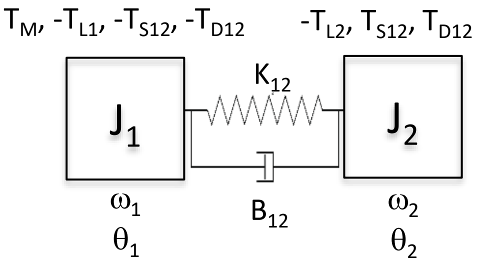
ITMO

Basic kinematic diagrams for electromechanical systems:

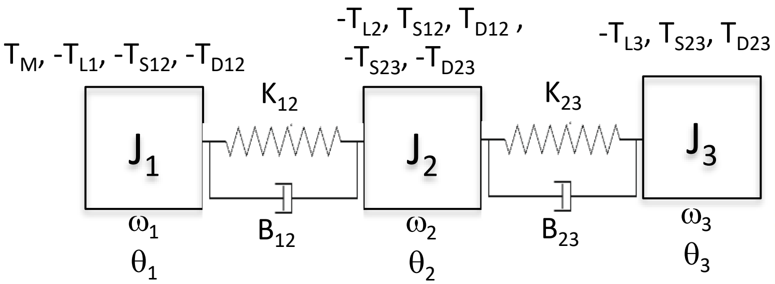
One-mass system



Two-mass system



Three-mass system



T_M – motor torque

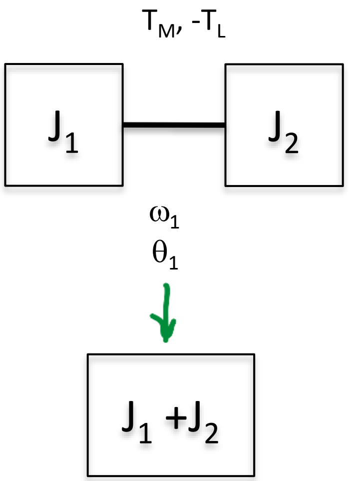
T_L – load torque

T_S – spring torque

T_D – damper torque

Dynamical models of mechanical systems

One-mass system

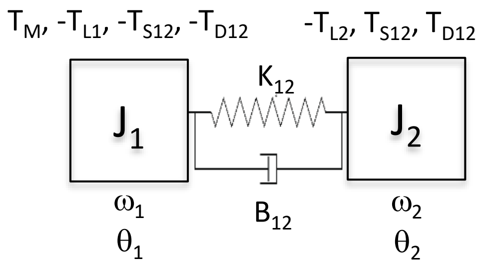


$$\begin{cases} J_\Sigma \dfrac{d\omega_1(t)}{dt} = T(t) - T_L(t), \\ J_\Sigma = J_1 + J_2 \end{cases}$$

$$W_{mech2}(s) = \frac{\omega_1(s)}{T(s)} = \frac{1}{J_1 + J_2} \cdot \frac{1}{s}.$$

Dynamical models of mechanical systems

Two-mass system



$$\begin{cases} J_1 \dfrac{d\omega_1(t)}{dt} = T(t) - T_{S12}(t) - b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{L1}(t), \\ \dfrac{dT_{s12}(t)}{dt} = K_{12}(\omega_1(t) - \omega_2(t)) \\ J_2 \dfrac{d\omega_2(t)}{dt} = T_{s12}(t) + b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{L2}(t). \end{cases}$$

Sensor is installed on the first inertia mass

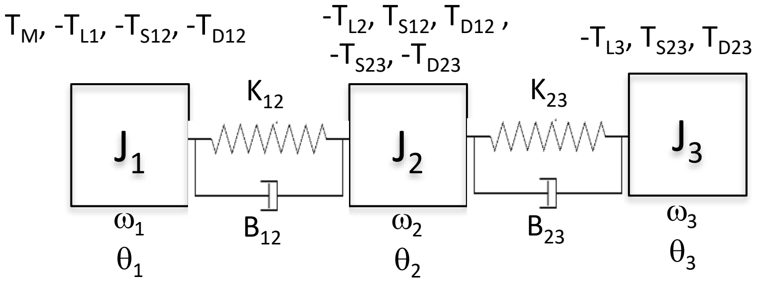
$$W_{mech2}(s) = \frac{\omega_1(s)}{T(s)} = \frac{1}{J_1 + J_2} \cdot \frac{\dfrac{J_2}{K_{12}} s^2 + \dfrac{b_{12}}{K_{12}} s + 1}{s\left(\dfrac{J_1 J_2}{K_{12}(J_1 + J_2)} s^2 + \dfrac{b_{12}}{K_{12}} s + 1\right)}.$$

Sensor is installed on the second inertia mass

$$W_{mech3}(s) = \frac{\omega_2(s)}{T(s)} = \frac{1}{J_1 + J_2} \cdot \frac{\dfrac{b_{12}}{K_{12}} s + 1}{s\left(\dfrac{J_1 J_2}{K_{12}(J_1 + J_2)} s^2 + \dfrac{b_{12}}{K_{12}} s + 1\right)}.$$

Dynamical models of mechanical systems

Three-mass system



$$\begin{cases} J_1 \dfrac{d\omega_1(t)}{dt} = T(t) - T_{s12}(t) - b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{L1}(t), \\ \dfrac{dT_{s12}(t)}{dt} = K_{12}(\omega_1(t) - \omega_2(t)), \\ J_2 \dfrac{d\omega_2(t)}{dt} = T_{s12}(t) + b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{s23}(t) - b_{23}\left(\omega_2(t) - \omega_3(t)\right) - T_{L2}(t), \\ \dfrac{dT_{s23}(t)}{dt} = K_{23}(\omega_2(t) - \omega_3(t)), \\ J_3 \dfrac{d\omega_3(t)}{dt} = T_{s23}(t) + b_{23}\left(\omega_2(t) - \omega_3(t)\right) - T_{L3}(t). \end{cases}$$

Dynamical models of mechanical systems

Multi-mass system

$$
\begin{cases}
J_1 \dfrac{d\omega_1(t)}{dt} = T(t) - T_{s12}(t) - b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{L1}(t), \\
\dfrac{dT_{s12}(t)}{dt} = K_{12}(\omega_1(t) - \omega_2(t)), \\
J_2 \dfrac{d\omega_2(t)}{dt} = T_{s12}(t) + b_{12}\left(\omega_1(t) - \omega_2(t)\right) - T_{s23}(t) - b_{23}\left(\omega_2(t) - \omega_3(t)\right) - T_{L2}(t), \\
\dfrac{dT_{s23}(t)}{dt} = K_{23}(\omega_2(t) - \omega_3(t)), \\
\dots \\
J_{n-1} \dfrac{d\omega_{n-1}(t)}{dt} = T_{sn-2n-1}\left(t\right) + b_{n-2n-1}\left(\omega_{n-2}(t) - \omega_{n-1}(t)\right) - T_{sn-1n}\left(t\right) - \\
-b_{n-1n}\left(\omega_{n-1}(t) - \omega_n(t)\right) - T_{Ln-1}(t), \\
\dfrac{dT_{sn-1n}(t)}{dt} = K_{n-1n}(\omega_{n-1}(t) - \omega_n(t)), \\
J_n \dfrac{d\omega_n(t)}{dt} = T_{sn-1n}\left(t\right) + b_{n-1n}\left(\omega_{n-1}(t) - \omega_n(t)\right) - T_{Ln}(t),
\end{cases}
$$

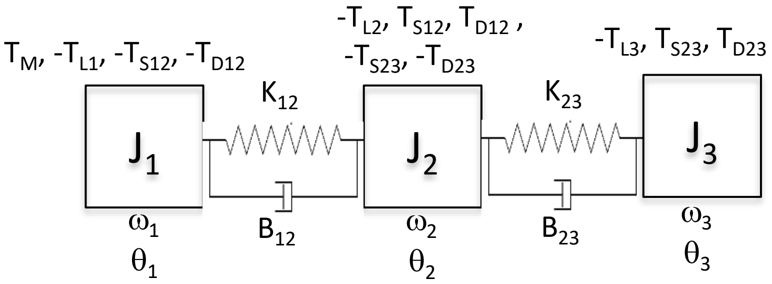
If the sensor is installed on the first inertia mass:

$$
W_{mech}(s) = \frac{\omega_1(s)}{T(s)} = \frac{K_\omega \prod\limits_{i=2,4,6,..}^{n} T_i^2 s^2 + 2\nu T_i s + 1}{s \prod\limits_{j=1,3,5,..}^{n-1} T_j^{\,2} s^2 + 2\varsigma T_j s + 1},
$$

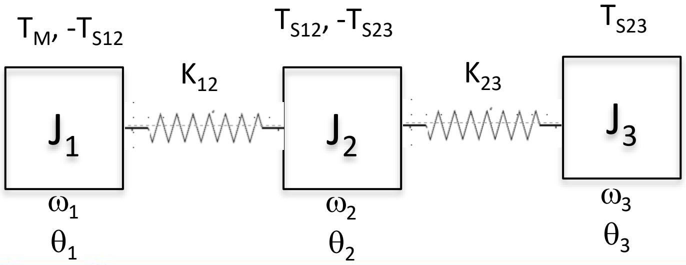
$$
K_\omega = \frac{1}{J_1 + J_2 + \ldots + J_n} = \frac{1}{J_\Sigma}.
$$

Three-mass system analysis

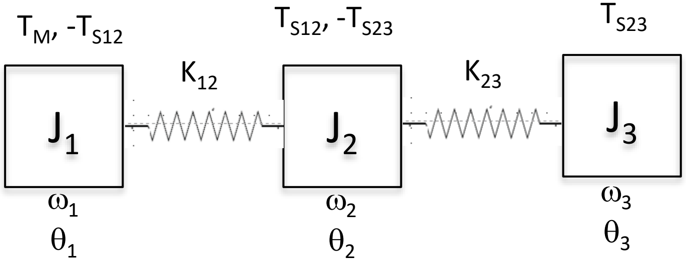
Three-mass mechanical system is considered



Assume $T_{L1} = T_{L2} = T_{L3} = 0$, $B_{12} = B_{23} = 0$



Three-mass system analysis



$$
\begin{cases}
J_1 \dfrac{d\omega_1(t)}{dt} = T(t) - T_{s12}(t), \\
\dfrac{dT_{s12}(t)}{dt} = K_{12}(\omega_1(t) - \omega_2(t)), \\
J_2 \dfrac{d\omega_2(t)}{dt} = T_{s12}(t) - T_{s23}(t), \\
\dfrac{dT_{s23}(t)}{dt} = K_{23}(\omega_2(t) - \omega_3(t)), \\
J_3 \dfrac{d\omega_3(t)}{dt} = T_{s23}(t).
\end{cases}
$$

$$
W_{mech2}(s) = \frac{\omega_1(s)}{T(s)} = \frac{J_2 J_3 s^4 + \left[K_{23}\left(J_2 + J_3\right) + K_{12} J_3\right] s^2 + K_{12} K_{23}}{s(J_1 J_2 J_3 s^4 + \left[J_1 K_{23}\left(J_2 + J_3\right) + K_{12} J_3\left(J_1 + J_2\right)\right] s^2 + K_{12} K_{23}\left(J_1 + J_2 + J_3\right))}.
$$

Characteristic polynomial:

$$
D\left(s\right) = s(J_1 J_2 J_3 s^4 + \left[J_1 K_{23}\left(J_2 + J_3\right) + K_{12} J_3\left(J_1 + J_2\right)\right] s^2 + K_{12} K_{23}\left(J_1 + J_2 + J_3\right)).
$$

Three-mass system analysis

Characteristic equation:

$$D(s)=s(J_1J_2J_3s^4+\left[J_1K_{23}(J_2+J_3)+K_{12}J_3(J_1+J_2)\right]s^2+K_{12}K_{23}(J_1+J_2+J_3))=0.$$

Poles of the transfer function:

$$s_1=0;$$

$$s_{2,3}=\pm j\sqrt{\frac{a}{2}\left(1-\sqrt{1-\frac{4b}{a^2}}\right)}=\pm j\omega_{R1};$$

$$s_{4,5}=\pm j\sqrt{\frac{a}{2}\left(1+\sqrt{1-\frac{4b}{a^2}}\right)}=\pm j\omega_{R2}.$$

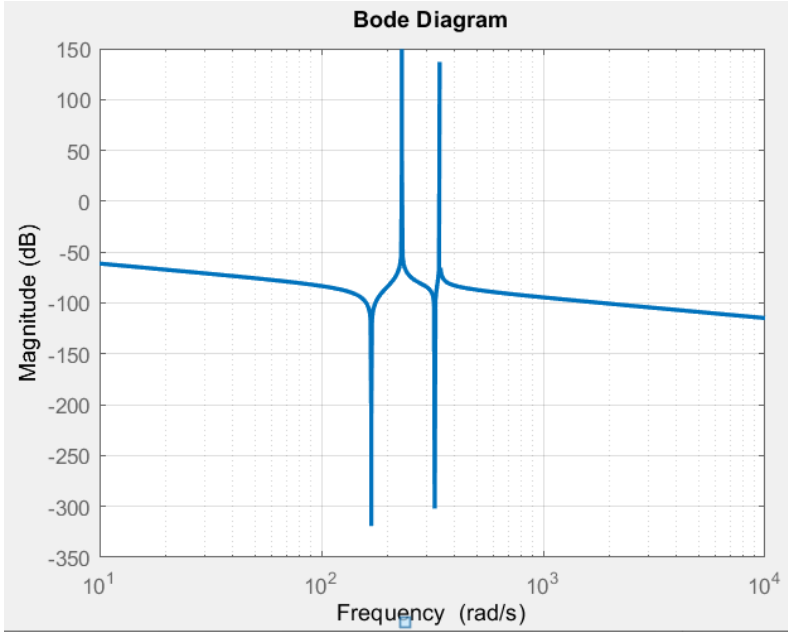
$$a=\frac{J_3K_{12}(J_1+J_2)+J_1K_{23}(J_2+J_3)}{J_1J_2J_3};$$

$$b=\frac{K_{12}K_{23}(J_1+J_2+J_3)}{J_1J_2J_3}.$$

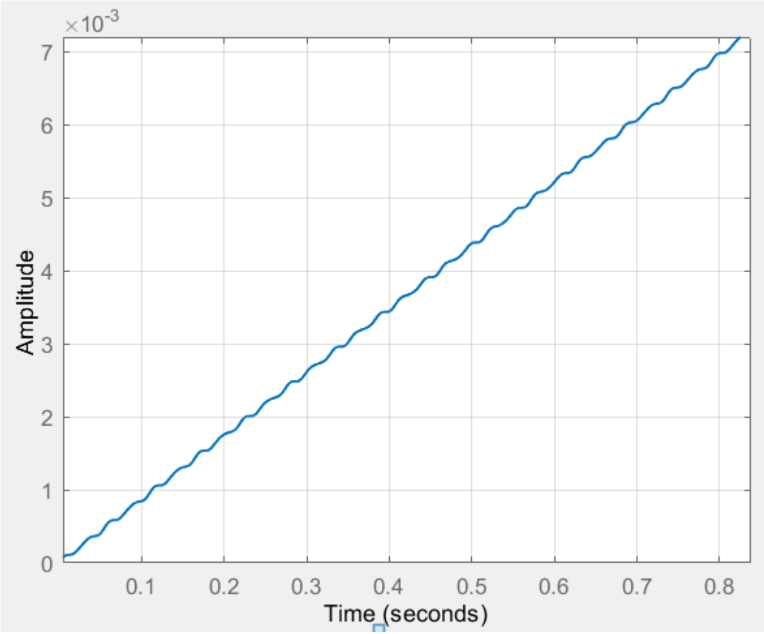
ω_{R1}, ω_{R2}. - resonant frequencies of the three-mass system

Three-mass system analysis

Frequency response function (Bode plot)

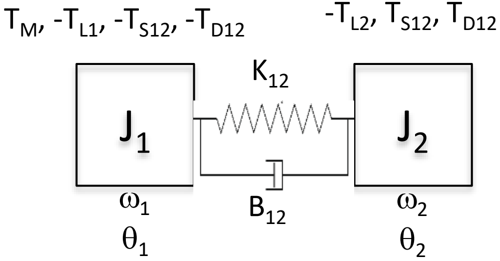


Step response

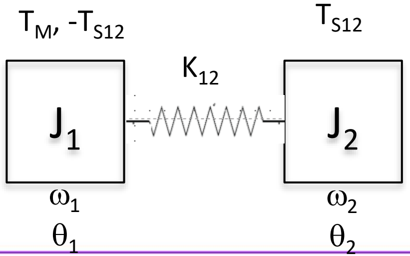


Two-mass system analysis

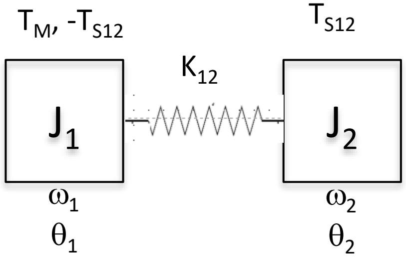
Two-mass mechanical system is considered



Assume $T_{L1} = T_{L2} = 0$, $B_{12} = 0$



Two-mass system analysis



$$\begin{cases} J_1 \dfrac{d\omega_1(t)}{dt} = T(t) - T_{s12}(t), \\ \dfrac{dT_{s12}(t)}{dt} = K_{12}(\omega_1(t) - \omega_2(t)), \\ J_2^{'} \dfrac{d\omega_2(t)}{dt} = T_{s12}(t) \end{cases}$$

$$W_{mech2}(s) = \frac{\omega_1(s)}{T(s)} = \frac{\dfrac{J_2}{K_{12}} s^2 + 1}{\left(J_1 + J_2\right) s \left(\dfrac{J_1 J_2}{K_{12}\left(J_1 + J_2\right)} s^2 + 1 \right)}.$$

Characteristic polynomial:

$$D(s) = \left(J_1 + J_2\right) s \left(\frac{J_1 J_2}{K_{12}\left(J_1 + J_2^{'}\right)} s^2 + 1 \right).$$

Two-mass system analysis

Characteristic equation:

$$D(s)=(J_1+J_2)s\left(\frac{J_1J_2}{K_{12}(J_1+J_2^{'})}s^2+1\right)=0.$$

Poles of the transfer function:

$$s_1=0;$$

$$s_{2,3}=\pm j\sqrt{\frac{K_{12}(J_1+J_2)}{J_1J_2^{'}}}=\pm j\omega_{R1};$$

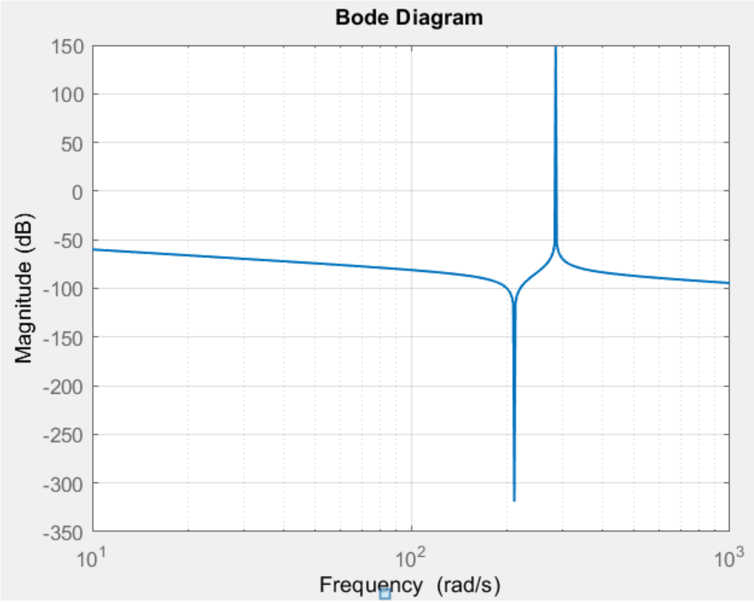
ω_{R1}. - resonant frequency of the two-mass system

$$\omega_{R1}=\sqrt{\frac{K_{12}(J_1+J_2)}{J_1J_2}}=\sqrt{\frac{K_{12}\gamma}{J_2}}$$

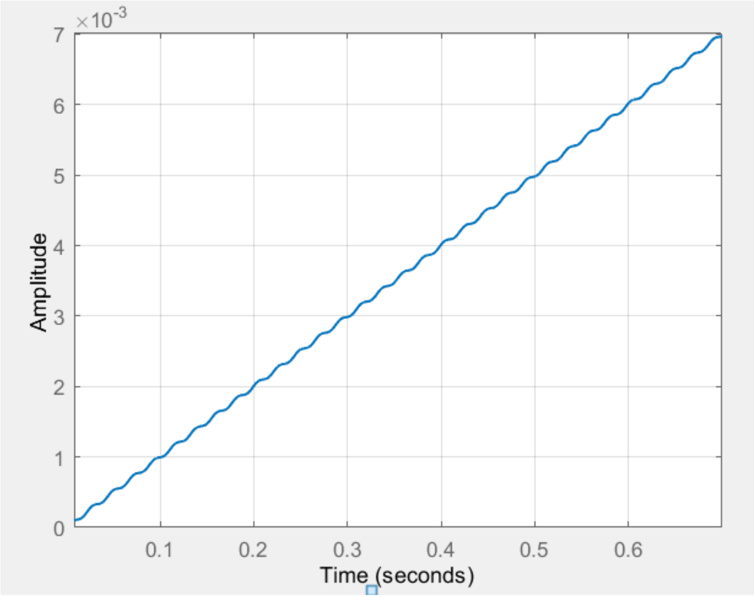
$$\gamma=\frac{J_1+J_2}{J_1}$$ - mass ratio

Two-mass system analysis

Frequency response function
(Bode plot)

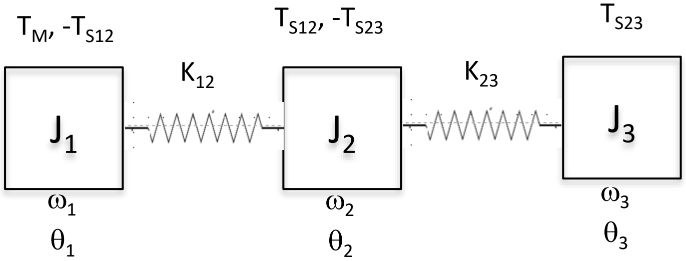


Step response

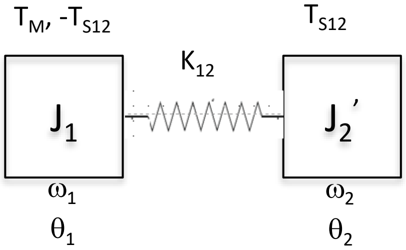


Two-mass system analysis

Transformation of the three-mass system into two-mass system

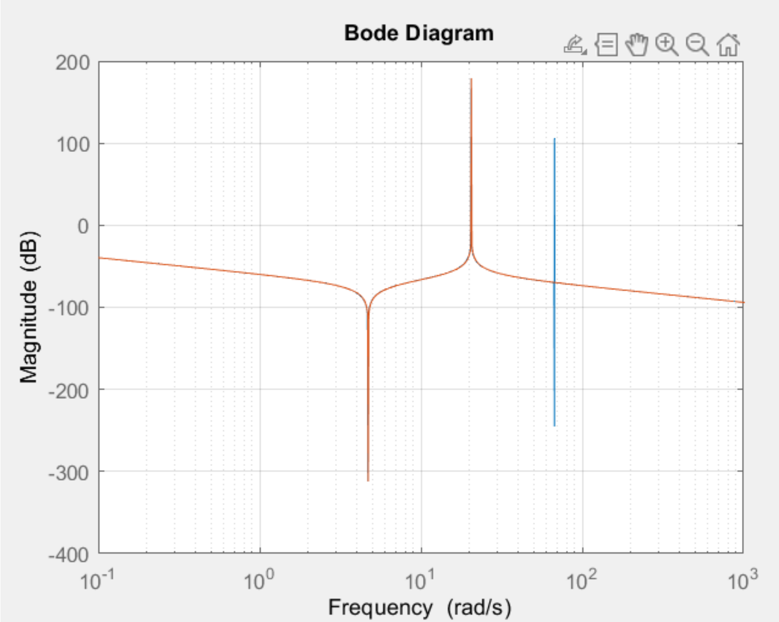


Assume $K_{23} = \infty$, $J_2' = J_2 + J_3$

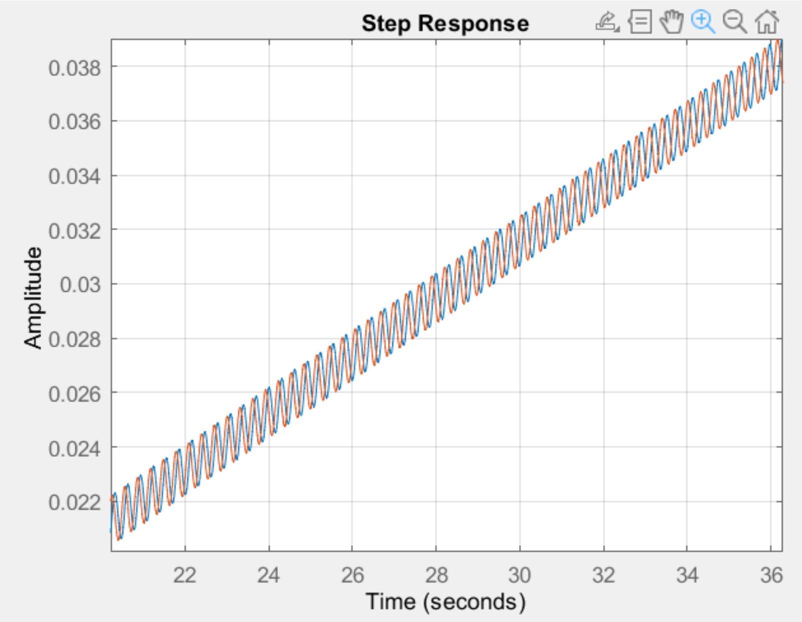


Case 1

Frequency response function
(Bode plot)



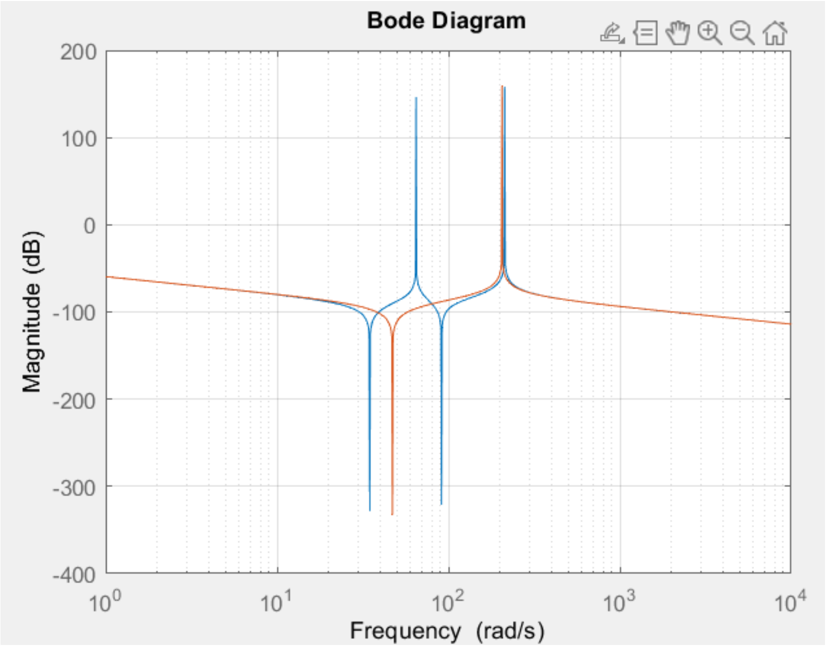
Step response



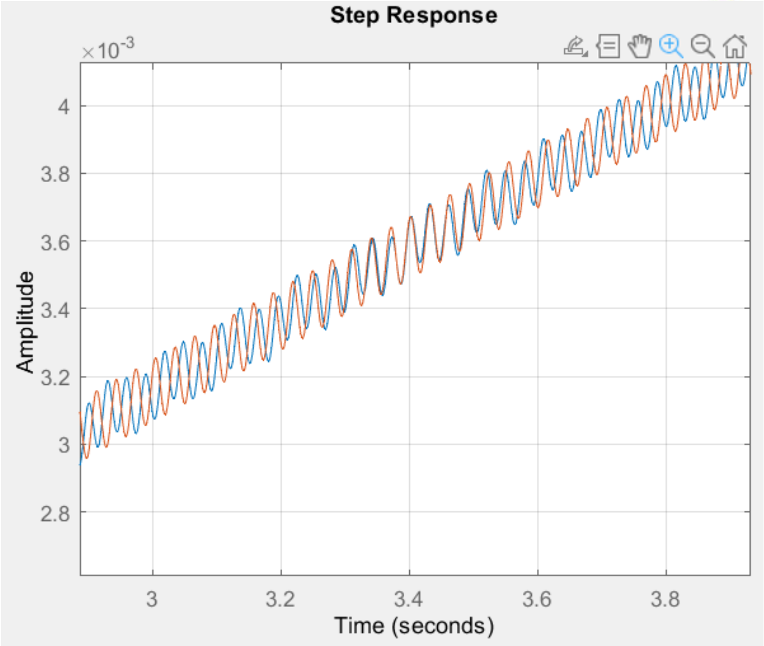
Transformation is relevant

Case 2

Frequency response function
(Bode plot)



Step response



Transformation is irrelevant

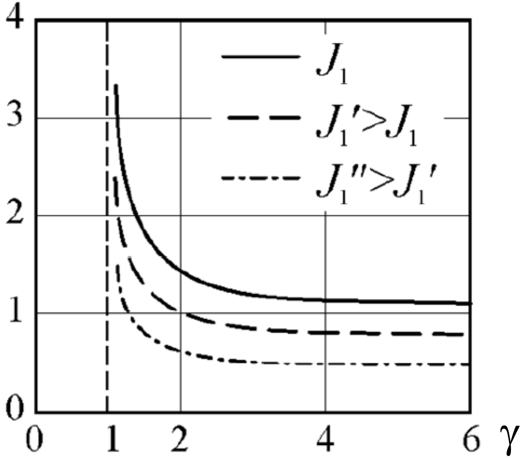
Two-mass system analysis

$$\omega_{R1} = \sqrt{\frac{K_{12}\left(J_1 + J_2\right)}{J_1 J_2}} = \sqrt{\frac{K_{12}\gamma}{J_2}}$$

$$\gamma = \frac{J_1 + J_2}{J_1}$$ - mass ratio

- At $\gamma > 3$ mass ratio has no effect on resonant frequency;
- At $\gamma < 1.5$ the value of the resonant frequency increases extremely.

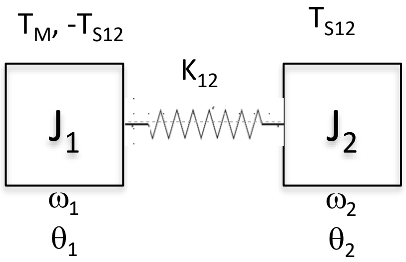
$$\bar{\omega}_{R1} = \frac{\omega_{R1}}{\sqrt{K_{12}}}$$



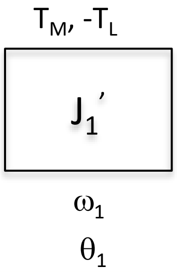
If $J_1 >> J_2$ elastic joint has little effect on dynamical properties of the mechanical system

Two-mass system analysis

Transformation of the two-mass system into one-mass system



Assume $K_{12} = \infty$, $J_1' = J_1 + J_2$

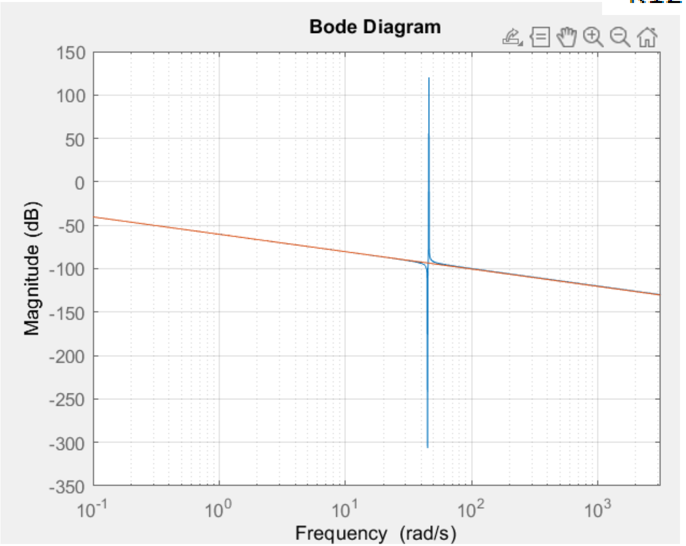


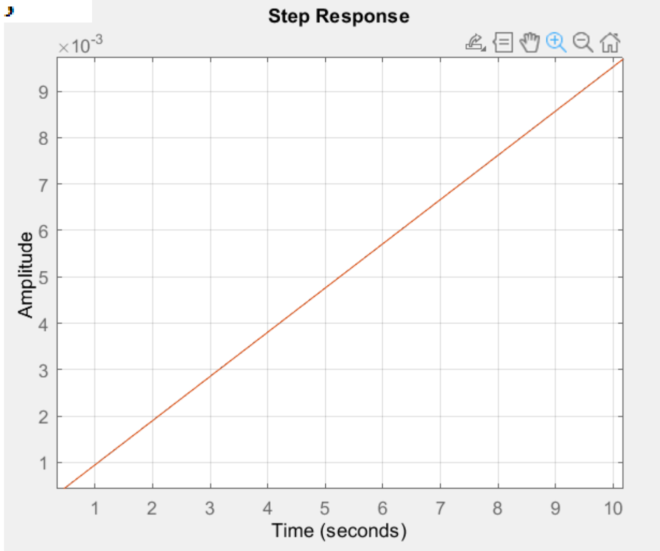
Case 1

Frequency response function (Bode plot)

```
J1 = 1000;
J2 = 50;
K12 = 1e5;
```

Step response

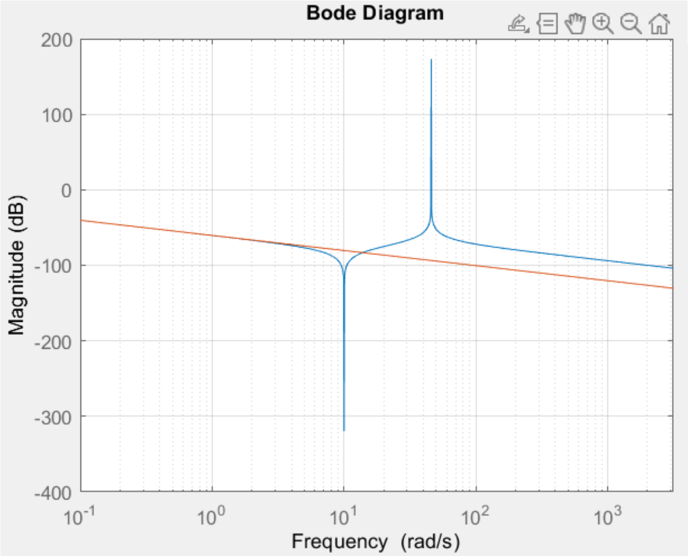




Transformation is relevant

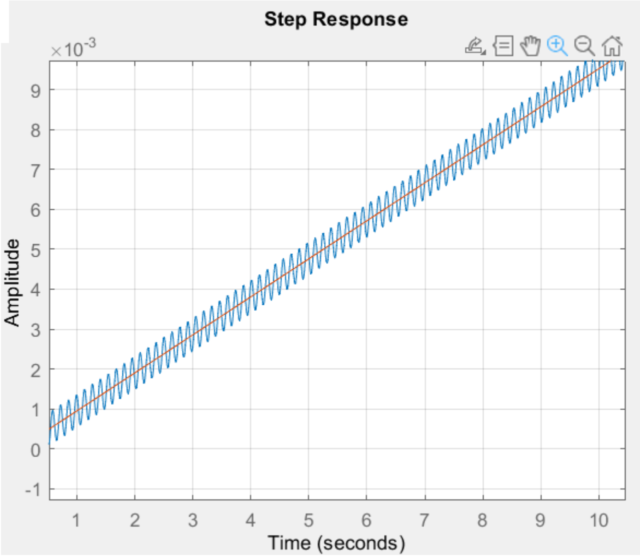
Case 2

Frequency response function (Bode plot)




```
J1 = 50;
J2 = 1000;
K12 = 1e5;
```

Step response



Transformation is irrelevant

THANK YOU
FOR YOUR TIME!

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