Mathematic modelling of dynamic systems

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Outline

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2. Stages of Building Model
3. Static vs Dynamic systems
4. Electromechanical systems
5. Ways of modelling of dynamic systems
6. Laplace transform
7. Fourier transform & Bode diagram
8. Transfer functions
9. Block diagrams
10. State space representation
11. Analysis of transient processes in dynamical systems

Goals of modelling

- **Prediction:** Forecasting the future behavior of the system under different conditions.
- **Analysis:** Understanding how the system works and identifying critical parameters.
- **Optimization:** Finding the best parameters or inputs to achieve desired outcomes.

Stages of Building Model

- **Define the Problem:** Clearly state what you want to achieve with the model.
- **Choose Variables and Parameters:** Identify the key state variables and system parameters.
- **Formulate Equations:** Write down the mathematical relationships governing the system.
- **Validate the Model:** Compare the model's predictions with real-world data to ensure accuracy.

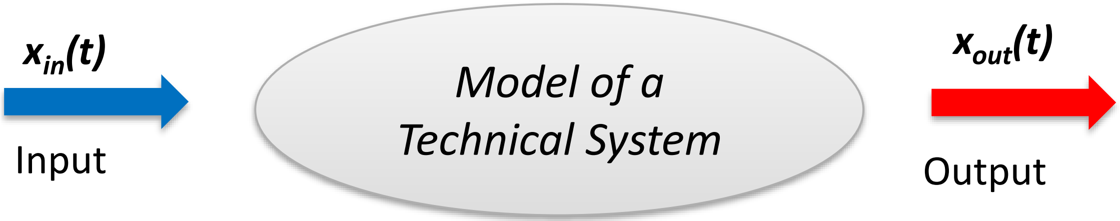
Static vs Dynamic models

Static model

- ❑ Output determined only by current input, reacts instantaneously;
- ❑ Relationship does not change;
- ❑ Relationship is represented by an algebraic equation

Dynamic model

- ❑ Output takes time to react;
- ❑ Relationship changes with time, depends on past inputs and initial conditions;
- ❑ Relationship is represented by a differential equation



Static vs Dynamic systems

DC motor from static viewpoint.

$$\omega = \frac{U}{\Psi} - \frac{r}{\Psi^2} T = \omega_0 - T / h$$

It is just an algebraic equation of torque-speed curve

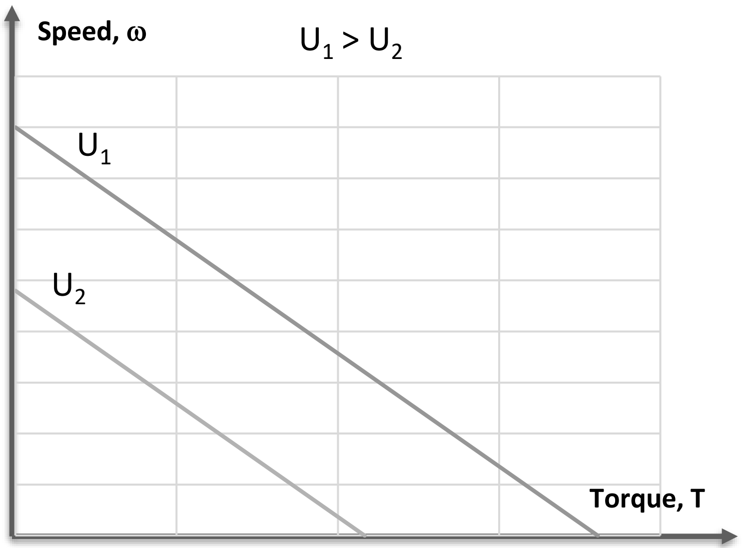
DC motor from dynamic viewpoint.

$$\begin{cases} L_a \cdot \dfrac{di_a(t)}{dt} = U - r \cdot i_a(t) - \Psi \cdot \omega(t) \\ J \cdot \dfrac{d\omega(t)}{dt} = \Psi \cdot i_a(t) - T_L \end{cases}$$

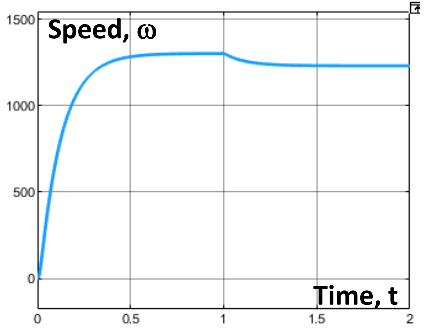
It is two differential equations (state space form)

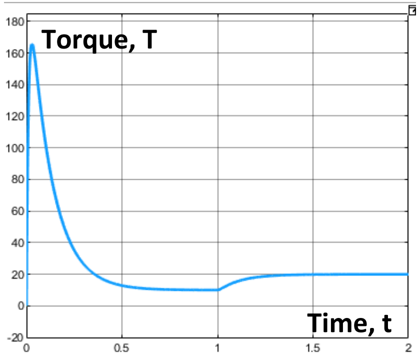
Static vs Dynamic systems

DC motor from static viewpoint.

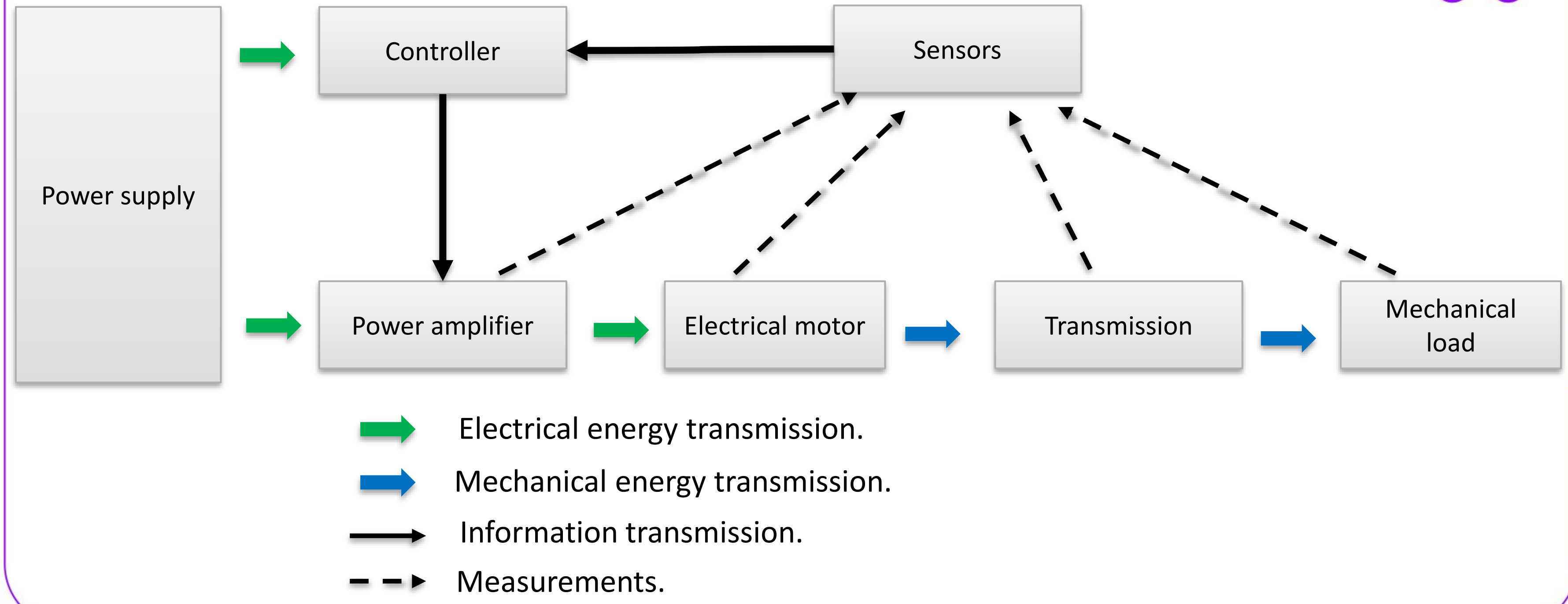


DC motor from dynamic viewpoint.

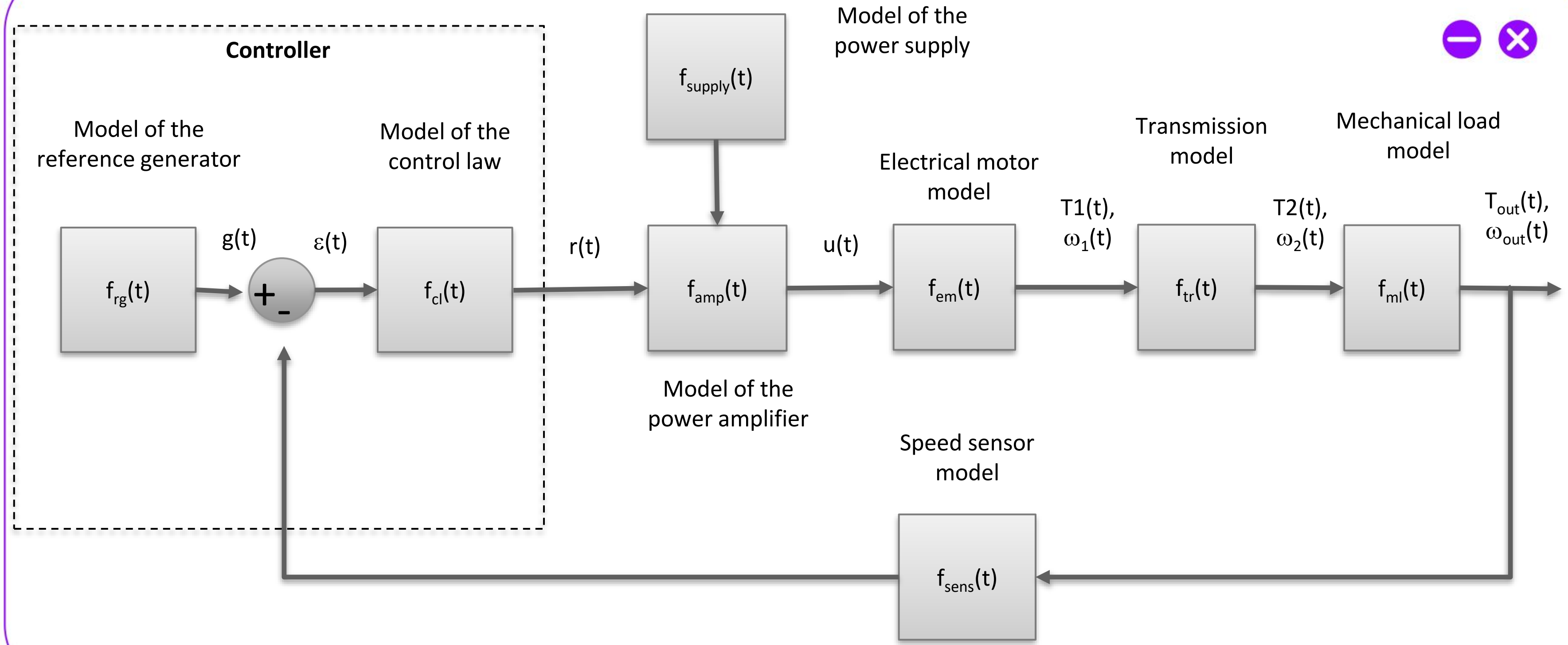




Electromechanical systems



Electromechanical systems



Ways of modelling of dynamic systems

Dynamical systems

Linear

Nonlinear

Linearization in some equilibrium points

Ways of modelling

- **Transfer function;**
- **State-space models;**
- Block diagram;
- Ready dynamic blocks in specialized programs like Simscape;

Ways of modelling

- Block diagram;
- Ready dynamic blocks in specialized programs;
- Special techniques (like finite element method) using specialized CAD programs;

All real-life dynamical systems are nonlinear!

Laplace transform

Laplace transform – is an integral transform that converts a function f(t) of a real variable, for example time t, to a function F(s) of a complex variable s = σ + jω (complex frequency).

The (unilateral) Laplace transform L is defined by the following equation:

$$F(s)=\int_{0}^{\infty} f(t)\cdot e^{-s\cdot t}dt \quad \Rightarrow \quad F(s)=L\{f(t)\}$$

Inverse Laplace transform – is an operation to recover a real function f(t), using the known function F(s) which is the Laplace transform of f(t).

The inverse Laplace transform L^{-1} is defined by the following equation*:

$$f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\cdot e^{s\cdot t}ds \quad \Rightarrow \quad f(t)=L^{-1}\{F(s)\}$$

** - this equation isn`t really useful in practice*

Laplace transform

Inverse Laplace transform can be found using **partial fractions decomposition** as the ***sum of residues in poles of the function F(s)est:***

$$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\cdot e^{s\cdot t}ds = \sum_{s_k} \operatorname*{Res}_{s_k}\left(F(s)e^{st}\right)$$

Residue of the function ***f(z)*** in pole $\boldsymbol{z_0}$ of order ***n*** can be found as follows:

$$\operatorname*{Res}_{z_0}\left(f(z)\right) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{(n-1)}}{dz^{(n-1)}}\left[(z-z_0)^n f(z)\right]$$

Residue of the function ***f(z)*** in pole $\boldsymbol{z_0}$ of order ***n=1*** can be found as follows:

$$\operatorname*{Res}_{z_0}\left(f(z)\right) = \lim_{z\to z_0}\left(f(z)(z-z_0)\right)$$

Properties of Laplace transform

Laplace transforms are only concerned with functions where t ≥ 0.

If *t < 0 f(t)* **must** be a zero

Linearity: $$L\{a\cdot f_1(t)+b\cdot f_2(t)\}=a\cdot L\{f_1(t)\}+b\cdot L\{f_2(t)\}$$

Time delay: for $\tau > 0$ $$L\{f(t-\tau)\}=e^{-s\cdot\tau}\cdot L\{f(t)\}=e^{-s\cdot\tau}F(s)$$

The first shifting theorem: $$L(e^{-\alpha t}f(t))=F(s+\alpha)$$

Time scaling: for *g(t) = f(at)* $$G(s)=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Properties of Laplace transform

Integration:

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}\cdot F(s)$$

Convolution:

$$L^{-1}\left\{F_1(s)\cdot F_2(s)\right\} = \int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau$$

The final value theorem:

$$\lim_{s\to\infty}\left(s\cdot F(s)\right) = \lim_{t\to 0}\left(f(t)\right) = f(0)$$

$$\lim_{s\to 0}\left(s\cdot F(s)\right) = \lim_{t\to \infty}\left(f(t)\right) = f(\infty)$$

Properties of Laplace transform

Derivation:

$$L\left\{\frac{dy}{dt}\right\} = s \cdot Y(s) - y(0)$$

$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2 \cdot Y(s) - s \cdot y(0) - \dot{x}(0)$$

$$L\left\{\frac{d^ny}{dt^n}\right\} = s^n \cdot Y(s) - \sum_{k=1}^{n} s^{n-k} \cdot y^{(k-1)}(0)$$

For zero initial conditions:

$$L\left\{\frac{d^ny}{dt^n}\right\} = s^n \cdot Y(s)$$

Fourier transform

The Fourier transform of a signal x(t) is the following function:

$$X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

The function value X(ω) is (in general) a complex number:

$$X(\omega)=\int_{-\infty}^{+\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)dt$$

|X(ω)| is called the amplitude spectrum of x(t); ∠ X(ω) is the phase spectrum of x(t)

The spectrum characterizes the ratio of amplitudes and phases of an infinite set of sinusoidal components that that are included in the signal x(t).

$$X(j\omega)=F\{x(t)\}$$

Fourier transform

A time function has a Fourier transform if:

- It must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
- It must have only a finite number of maxima and minima within one periodic;
- It must be integrable function: $$\int_{-\infty}^{+\infty} \left| x(t) \right| dt < \infty$$

Bode diagram

The frequency response function (FRF) of a LTI system **W(jω)** is a function of the complex argument **jω** obtained by formally replacing **s** with **jω** in the transfer function expression:

$$W(j\omega) = \left[W(s)\right]_{s=j\omega}$$

$$F\{x_{in}(t)\} = \left[X_{in}(s)\right]_{s=j\omega} = X_{in}(j\omega)$$

$$F\{x_{out}(t)\} = \left[X_{out}(s)\right]_{s=j\omega} = \left[X_{in}(s)\cdot W(s)\right]_{s=j\omega} = X_{in}(j\omega)\cdot W(j\omega)$$

$$\frac{F\{x_{out}(t)\}}{F\{x_{in}(t)\}} = \frac{X_{in}(j\omega)\cdot W(j\omega)}{X_{in}(j\omega)} = W(j\omega)$$

Bode diagram

Frequency response function (FRF) can be found as ratio of Fourier transform (spectrum) of the output signal to Fourier transform of the input signal:

$$\frac{F\{x_{out}(t)\}}{F\{x_{in}(t)\}} = \frac{X_{in}(j\omega)\cdot W(j\omega)}{X_{in}(j\omega)} = W(j\omega)$$

Also, frequency response function (FRF) can be found as ratio of the cross power spectral density of input and output signals to the power spectral density of the input signal:

$$W(\omega) = \frac{P_{X_{in}X_{out}}(\omega)}{P_{X_{in}X_{in}}(\omega)}$$

Spectrum of the output signal can be found using spectrum of the input signal and frequency response function (FRF):

$$F\{x_{out}(t)\} = F\{x_{in}(t)\}\cdot W(j\omega)$$

Bode diagram

Frequency response:

$$W(j\omega) = P(\omega) + jQ(\omega) = \left|W(j\omega)\right| e^{j\varphi(\omega)}$$

Magnitude response:

$$f_1(\omega) = \left|W(j\omega)\right| = \sqrt{P^2(\omega) + Q^2(\omega)}$$

This coefficient indicates how many times the output signal changes in relation to the input signal.

Logarithmic magnitude response:

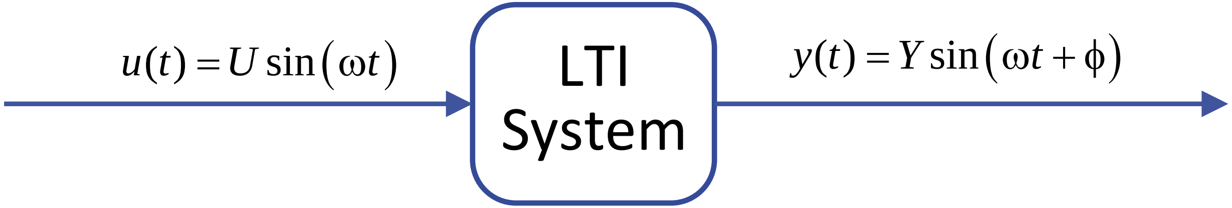
$$f_1^*(\omega) = 20\lg\left(f_1(\omega)\right) = 20\lg\left|W(j\omega)\right|$$

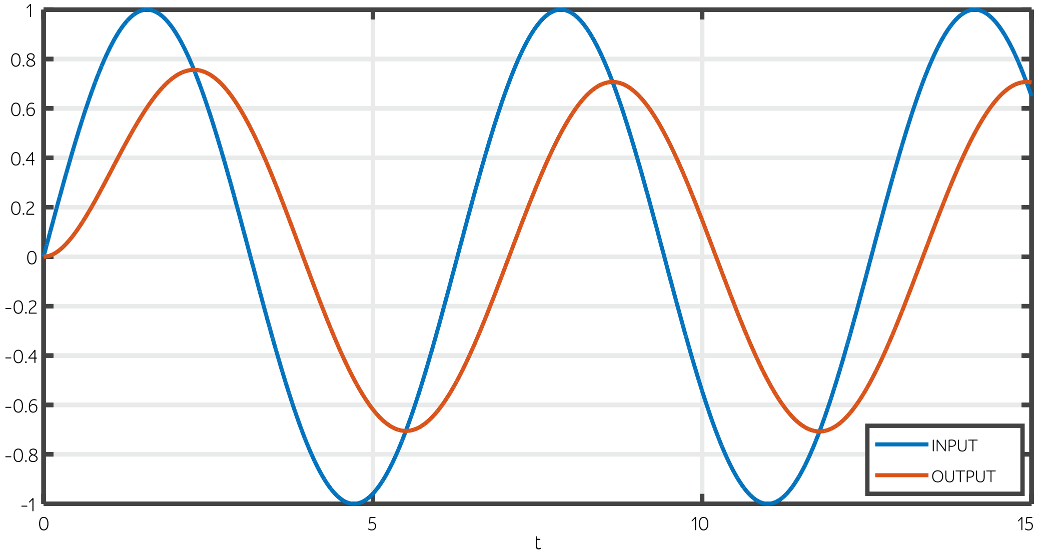
Phase response:

$$f_2(\omega) = \varphi(\omega) = arctg\,\frac{Q(\omega)}{P(\omega)}$$

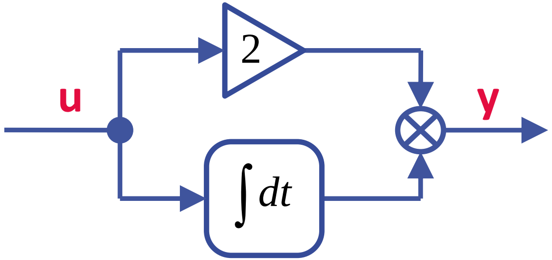
Polar (Nyquist) plot – Re vs. Im of W(jω) in complex plane

Bode diagram





Example

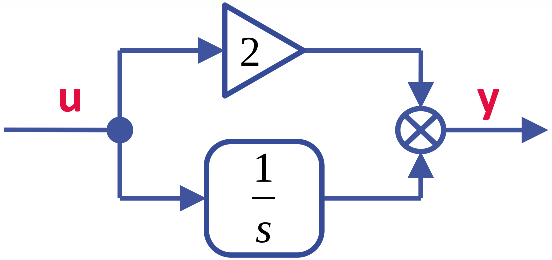


$$u = \sin\left(\frac{1}{2}t\right),\ \omega = \frac{1}{2}[rad/s]$$

$$y = 2\sin\left(\frac{1}{2}t\right) + \int \sin\left(\frac{1}{2}t\right)dt = 2\sin\left(\frac{1}{2}t\right) - 2\cos\left(\frac{1}{2}t\right) =$$

$$= \sqrt{8}\sin\left(\frac{1}{2}t + arctg(-1)\right)$$

Example

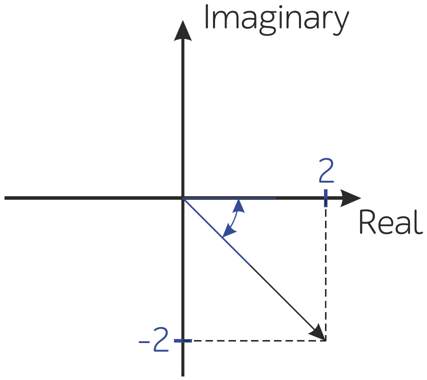


$$W(s) = \frac{2s+1}{s}$$

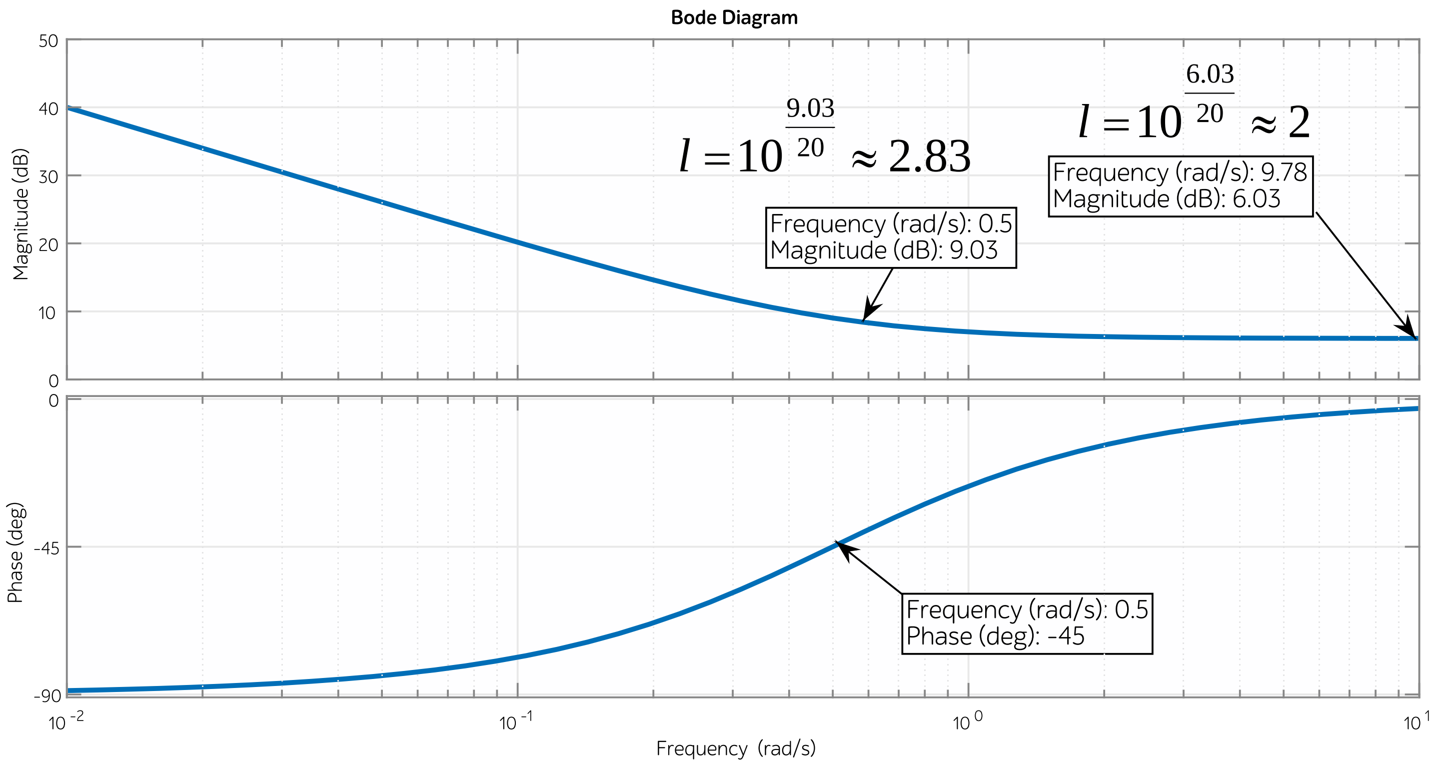
$$s = j\omega$$

$$W(j\omega) = \frac{2j\omega+1}{j\omega} = 2 - \frac{1}{\omega}j = 2-2j$$

$$l = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} \qquad \phi = arctg\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

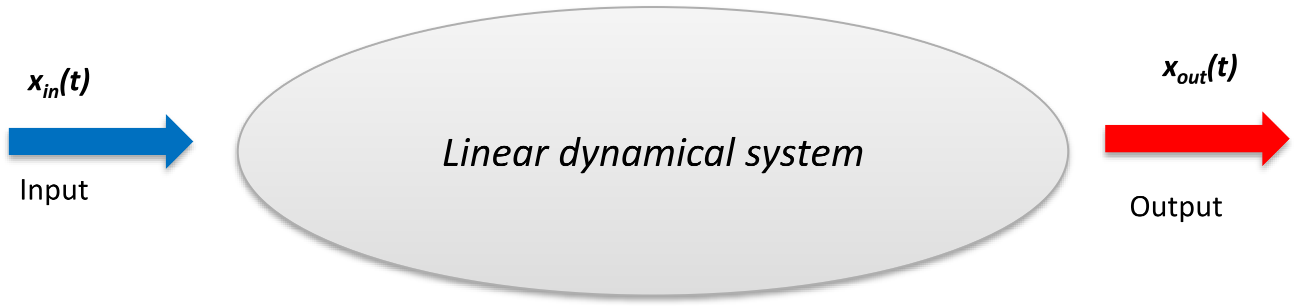


Example



Bode Diagram
$l = 10^{\frac{9.03}{20}} \approx 2.83$
$l = 10^{\frac{6.03}{20}} \approx 2$
Frequency (rad/s): 0.5
Magnitude (dB): 9.03
Frequency (rad/s): 9.78
Magnitude (dB): 6.03
Frequency (rad/s): 0.5
Phase (deg): -45
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
50
40
30
20
10
0
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}

Transfer functions



Linear dynamic system has at least one input and at least one output which connected by linear differential equations.
It is also called **linear time invariant (LTI)** system

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Transfer functions

LTI system:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$X(s) = L\{X(t)\}$$

$$Y(s) = L\{Y(t)\}$$

Laplace transform of input and output signals

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) =$$

$$= b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \ldots + b_1 s X(s) + b_0 X(s)$$

Transfer functions

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) =$$
$$= b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \ldots + b_1 s X(s) + b_0 X(s)$$

$$\Downarrow$$

$$\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) Y(s) =$$
$$= \left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right) X(s)$$

$$\Downarrow$$

$$\frac{Y(s)}{X(s)} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right)} = \frac{b(s)}{a(s)}$$

Transfer functions

$$\frac{Y(s)}{X(s)} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right)} = \frac{b(s)}{a(s)}$$

$$\Downarrow$$

$$W(s) = \frac{Y(s)}{X(s)}$$

Transfer function of LTI system is the ratio of Laplace transform of output signal to Laplace transform of input signal

a(s) – characteristic polynomial of LTI system

a(s) = 0 – characteristic equation of LTI system

Transfer functions

$$W(s)=\frac{Y(s)}{X(s)}=\frac{b(s)}{a(s)}$$

a(s) – characteristic polynomial of LTI system

a(s) = 0 – characteristic equation of LTI system

Roots of equation **a(s) = 0** are called **poles** of LTI system

Roots of equation **b(s) = 0** are called **zeros** of LTI system

Transfer functions

$$W(s) = \frac{Y(s)}{X(s)} = \frac{b(s)}{a(s)}$$

The applicability of the concept of the transfer function is limited to linear, time-invariant, differential equation systems. The transfer function approach, however, is extensively used in the analysis and design of such systems

1. The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.

2. The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.

Transfer functions

$$W(s) = \frac{Y(s)}{X(s)} = \frac{b(s)}{a(s)}$$

3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)

4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

Transfer functions

Convolutional integral

Transfer function:

$$W(s)=\frac{Y(s)}{X(s)}=\frac{b(s)}{a(s)}$$

Output signal:

$$Y(s)=W(s)X(s)$$

the multiplication in the complex domain is equivalent to convolution in the time domain

$$y(t)=\int_0^t x(\tau)w(t-\tau)d\tau=$$

$$=\int_0^t w(\tau)x(t-\tau)d\tau$$

Transfer functions

Impulse response

Consider the output (response) of a LTI system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is

$$Y(s) = W(s)$$

The inverse Laplace transform of the output gives the impulse response of the system:

$$L^{-1}\left[W(s)\right] = w(t)$$

The impulse-response function w(t) is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function.

Common forms of transfer functions

Static gain:

$$W(s) = k$$

The 1st order system transfer function:

$$W(s) = \frac{k}{T \cdot s + 1}$$

Integration:

$$W(s) = \frac{1}{T \cdot s}$$

The 2nd order system transfer function (damped oscillator):

$$W(s) = \frac{k}{T^2 \cdot s^2 + 2 \cdot T \cdot \xi \cdot s + 1}$$

Derivative:

$$W(s) = T \cdot s$$

Time delay:

$$W(s) = e^{-\tau s}$$

Common forms of transfer functions

The 1st order derivative transfer function: $$W(s) = Qs + 1$$

The 2nd order derivative transfer function: $$W(s) = Q^2 s^2 + 2\tau Qs + 1$$

A «real» derivative transfer function: $$W(s) = \frac{k \cdot T_1 \cdot s}{T_2 \cdot s + 1}$$

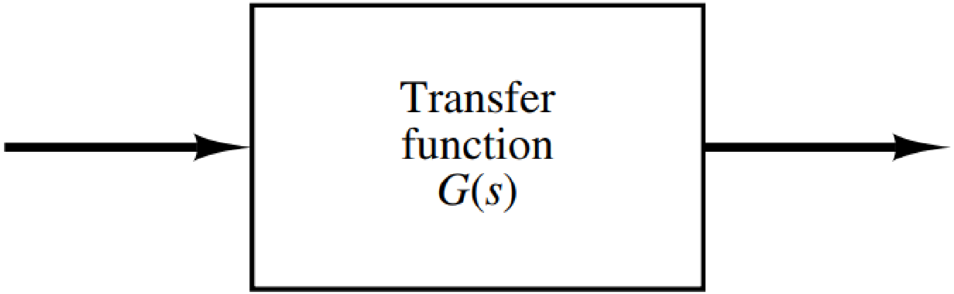
A proportional – integral controller: $$W(s) = \frac{1}{T_I \cdot s} + k_P$$

A proportional – integral – derivative controller: $$W(s) = k_P + \frac{1}{T_I \cdot s} + T_D \cdot s$$

Block diagrams

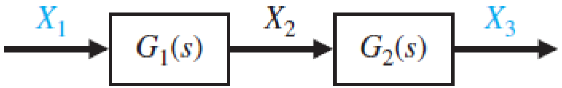
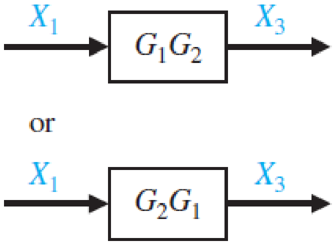
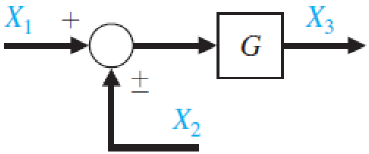
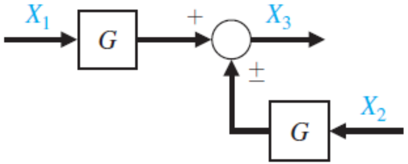
A **block diagram** of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.



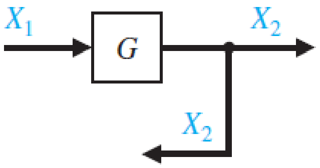
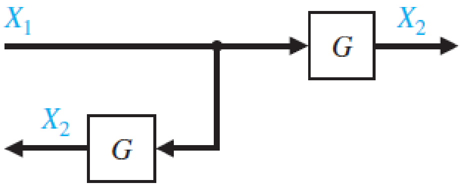
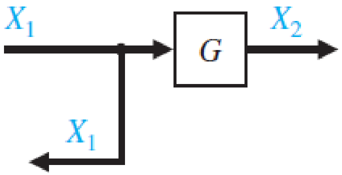
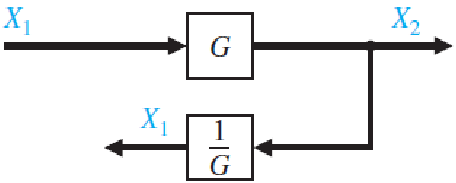
Block diagrams

How do we operate with block diagrams?

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		

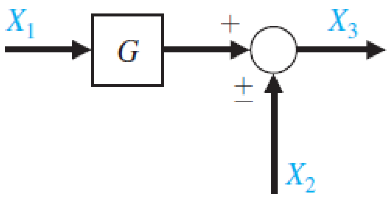
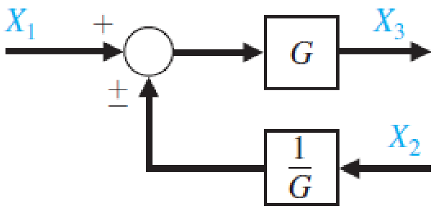
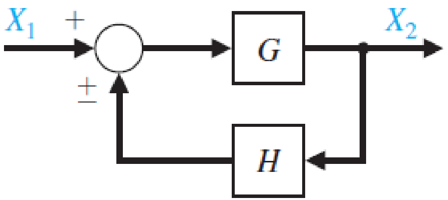
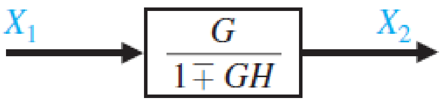
Block diagrams

How do we operate with block diagrams?

Transformation	Original Diagram	Equivalent Diagram
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		

Block diagrams

How do we operate with block diagrams?

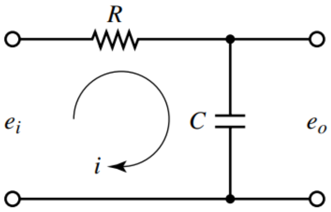
Transformation	Original Diagram	Equivalent Diagram
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Block diagrams

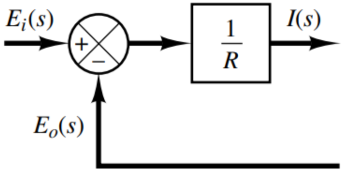
Procedure for Drawing a Block Diagram.

$$i = \frac{e_i - e_0}{R} \Rightarrow I(s) = \frac{E_i(s) - E_0(s)}{R}$$

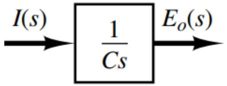
$$e_0 = \frac{\int i dt}{C} \Rightarrow E_0(s) = \frac{I(s)}{sC}$$



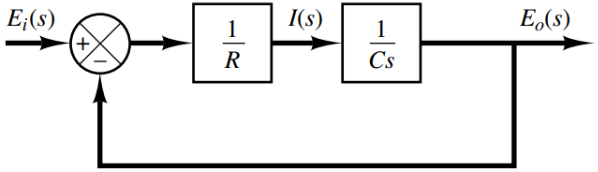
(a)



(b)



(c)

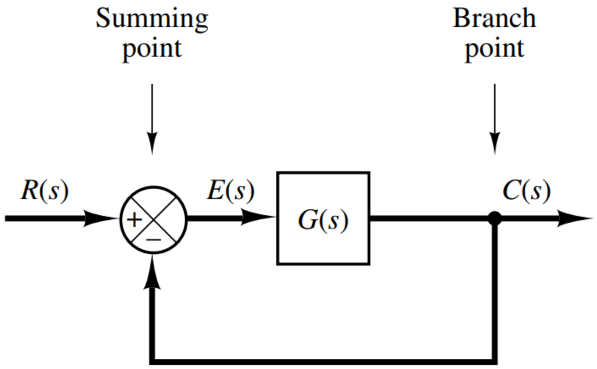


(d)

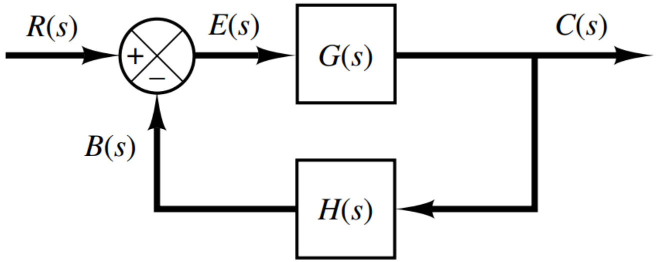
Block diagrams

Block Diagram of a Closed-Loop System

The output C(s) is fed back to the summing point, where it is compared with the reference input R(s).



Often the output signal needs to be converted before it can be compared with the input signal. This conversion is accomplished by the feedback element whose transfer function is H(s).



Block diagrams

Open-Loop Transfer Function and Feedforward Transfer Function

The ratio of the feedback signal B(s) to the actuating error signal E(s) is called the **open-loop transfer function**.

The ratio of the output C(s) to the actuating error signal E(s) is called the **feedforward transfer function**

$$Open-loop\ transfer\ function = \\ = \frac{B(s)}{E(s)} = G(s)H(s)$$

$$Feedforward\ transfer\ function = \\ = \frac{C(s)}{E(s)} = G(s)$$

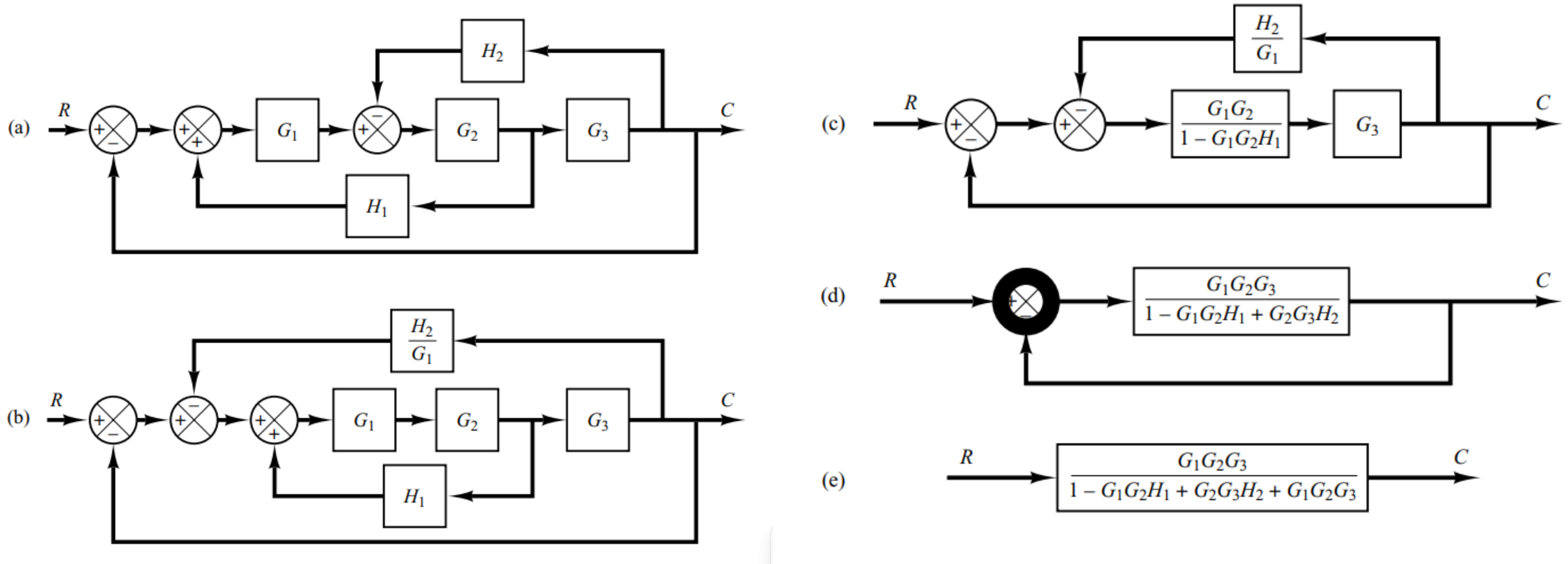
Block diagrams

Closed-Loop Transfer Function.

The transfer function relating C(s) to R(s) is called the closed-loop transfer function. It relates the closed-loop system dynamics to the dynamics of the feedforward elements and feedback elements.

$$Closed-loop\ transfer\ function = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Example



(a)
R
C
G_1
G_2
G_3
H_1
H_2
(b)
R
C
G_1
G_2
G_3
H_1
$\frac{H_2}{G_1}$
(c)
R
C
$\frac{G_1G_2}{1 - G_1G_2H_1}$
G_3
$\frac{H_2}{G_1}$
(d)
R
C
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2}$
(e)
R
C
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$

State space representation

State. The state of a dynamic system is the smallest set of variables (called state variables) such that knowledge of these variables at $t=t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

State Variables. The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables x_1 , x_2 , … , x_n are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t=t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

State space representation

State Vector. If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector x. Such a vector is called a state vector. A state vector is thus a vector that determines uniquely the system state x(t) for any time $t \geq t0$, once the state at t=t0 is given and the input u(t) for $t \geq t0$ is specified.

State Space. The n-dimensional space whose coordinate axes consist of the x1 axis, x2 axis, … , xn axis, where x1, x2, … , xn are state variables, is called a state space. Any state can be represented by a point in the state space.

State space representation

Assume that a multiple-input, multiple-output system involves n integrators. Assume also that there are r inputs $u_1(t)$, $u_2(t)$, … , $u_r(t)$ and m outputs $y_1(t)$, $y_2(t)$, … , $y_m(t)$.

Define n outputs of the integrators as state variables: $x_1(t)$, $x_2(t)$, … , $x_n(t)$ Then the system may be described by

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\;\;\vdots \\
\dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
$$

$$
\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\;\;\vdots \\
y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
$$

State space representation

State vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

Output vector

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}$$

Control (input) vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

Linear dynamic system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

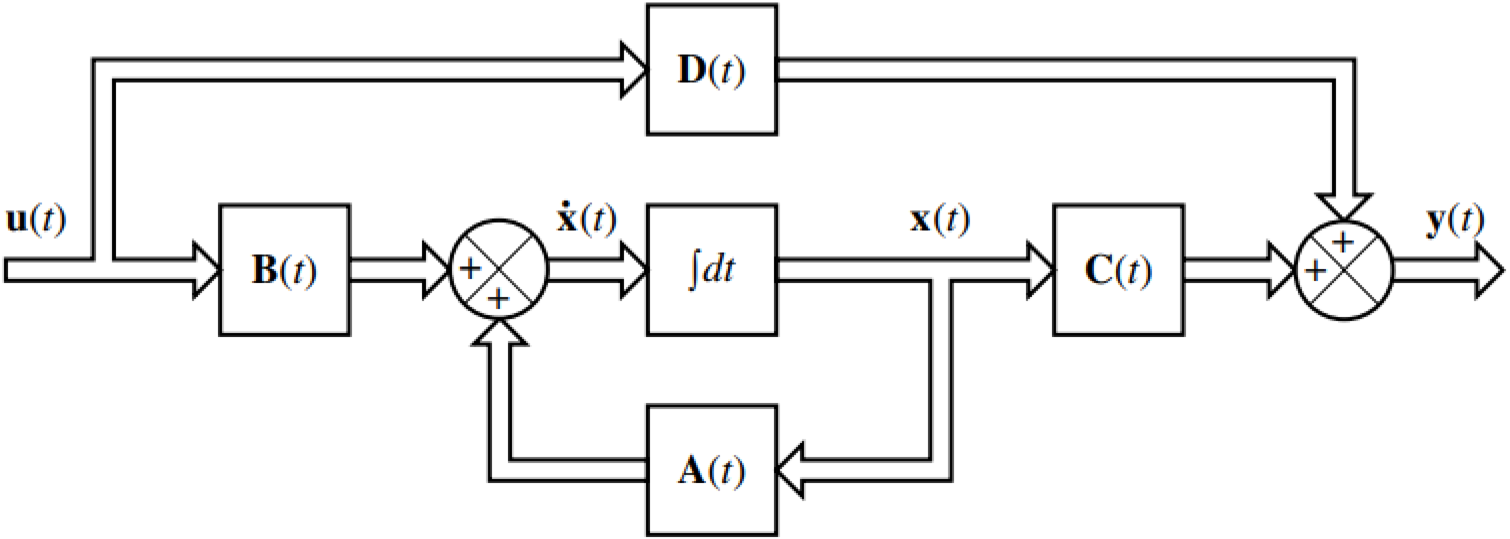
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

Linear time-invariant dynamic system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

State space representation



State space representation

Correlation Between Transfer Functions and State-Space Equations.

Transfer function:

$$\frac{Y(s)}{U(s)} = G(s)$$

State-space representation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Analysis of transient processes in dynamical systems

Unit-Step Response of First-Order Systems

Input signal $r(t)=1,\ L\left[r(t)\right]=R(s)=\dfrac{1}{s}$

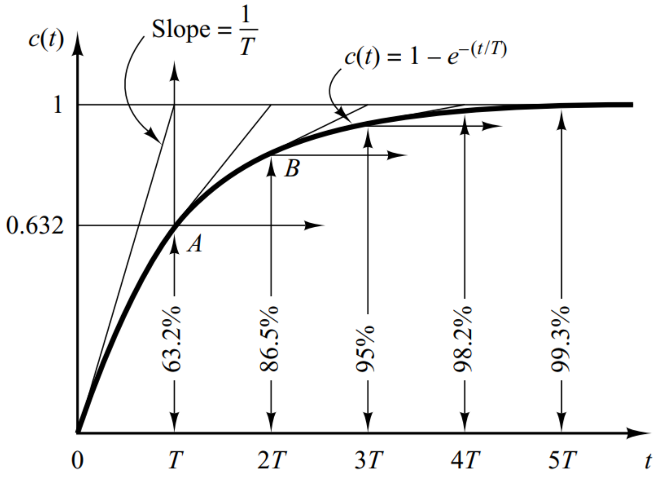
Output signal $C(s) = \dfrac{1}{Ts+1}\dfrac{1}{s}$

Expanding C(s) into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+(1/T)}$$

Taking the inverse Laplace transform we obtain

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$



Analysis of transient processes in dynamical systems

Unit-Ramp Response of First-Order Systems.

Input signal $r(t) = t,\ L[r(t)] = R(s) = \dfrac{1}{s^2}$

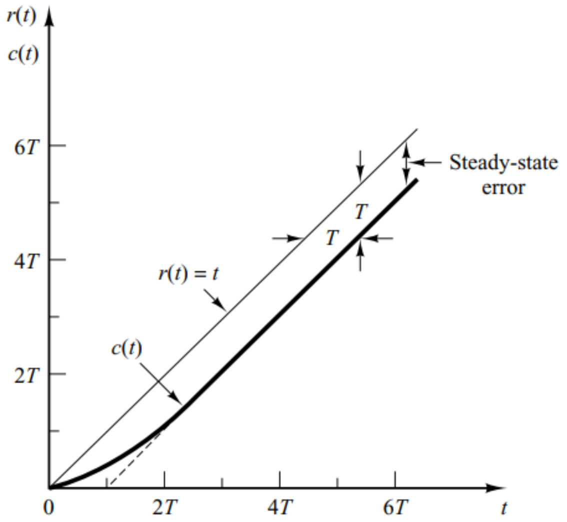
Output signal $C(s) = \dfrac{1}{Ts+1}\dfrac{1}{s^2}$

Expanding C(s) into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking the inverse Laplace transform we obtain

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$



Analysis of transient processes in dynamical systems

Consider the transfer function of the follows form $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

This form is called the **standard form** of the second-order system.
The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n.

If 0 < ζ < 1 the closed-loop poles are complex conjugates and lie in the left-half s plane. The system is then called **underdamped**, and the transient response is oscillatory.

If **ζ = 0**, the transient response does not die out.

If **ζ = 1**, the system is called **critically damped**.

Overdamped systems correspond to **ζ > 1**

Analysis of transient processes in dynamical systems

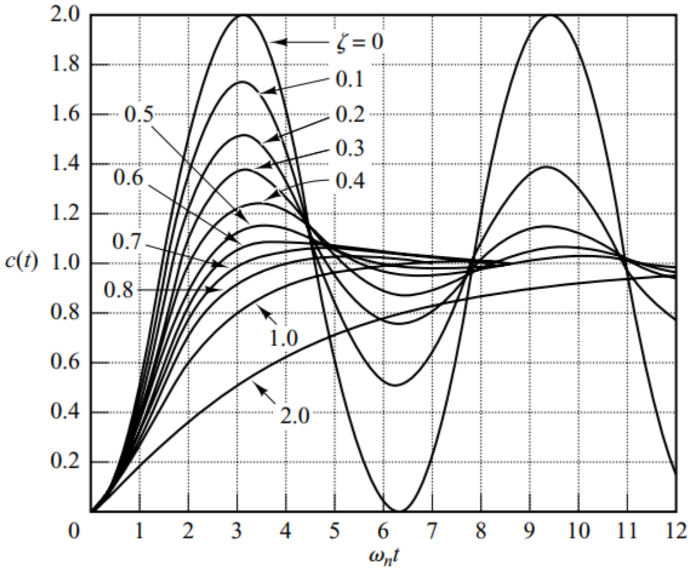
Step Response of Second-Order System.
(Underdamped case) $(0 < \zeta < 1)$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$ damped natural frequency

The output under unit step response:

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$



Analysis of transient processes in dynamical systems

The inverse Laplace transform can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Referring to the Laplace transform table

$$\mathscr{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathscr{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

The inverse Laplace transform:

$$
\begin{aligned}
\mathscr{L}^{-1}[C(s)] &= c(t) \\
&= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \\
&= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0
\end{aligned}
$$

Analysis of transient processes in dynamical systems

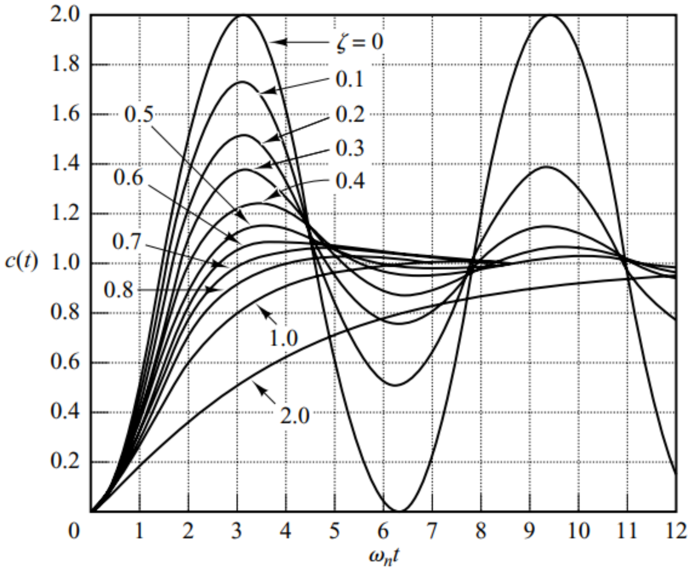
Step Response of Second-Order System.
(Critically damped case) $(\zeta = 1)$:

The output under unit step response:

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

The inverse Laplace transform:

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0$$



Step Response of Second-Order System.
(Overdamped case)

The output under unit step response:

$$C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s}$$

The inverse Laplace transform:

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta + \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta - \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0 \end{aligned}$$

$$s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

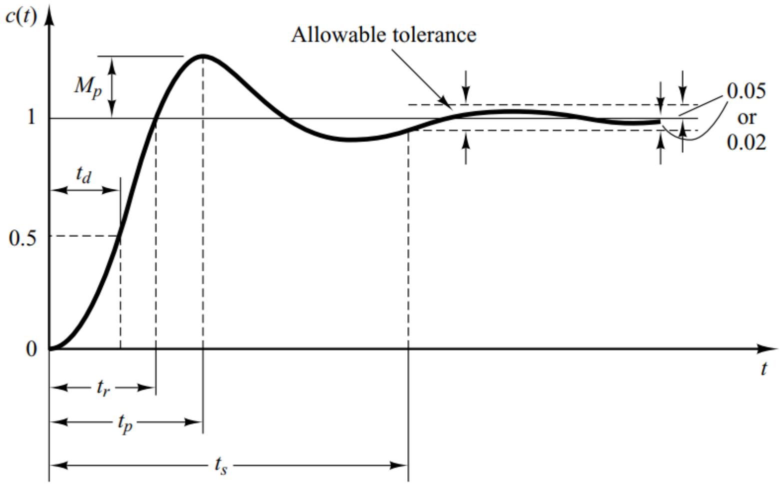
$$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

Analysis of transient processes in dynamical systems

1. **Delay time, td:** The delay time is the time required for the response to reach half the final value the very first time.

2. **Rise time, tr:** The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

3. **Peak time, tp**: The peak time is the time required for the response to reach the first peak of the overshoot.

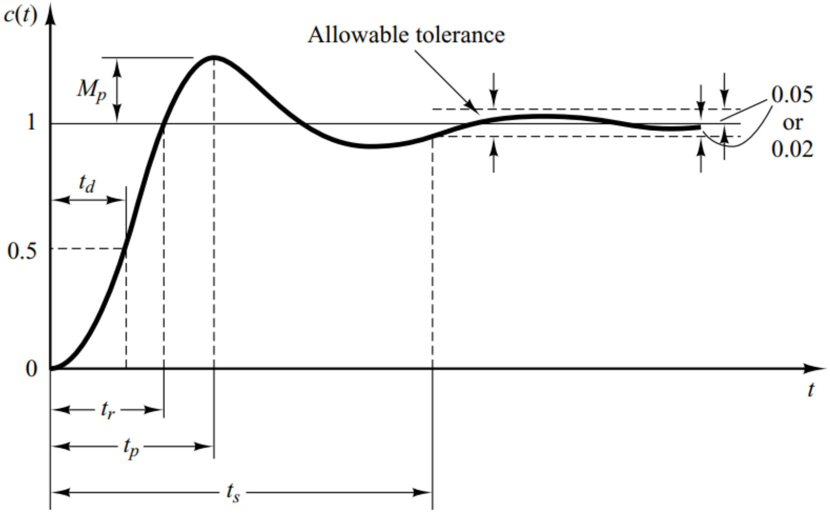


Analysis of transient processes in dynamical systems

4. Maximum (percent) overshoot, Mp: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

5. Settling time, ts : The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.



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