Distribution function

Denoted by $F(x)$. The distr func of a random variable X, is defined by $F(x) = P(X \le x)$

Function $F(x)$ gives prob of events that X takes a value less than or equal to a specific value x. Dist funct is abbreviated d.f and it is also called cumulative distrib function and it is cumulative prob function of the X, form (CDF). the smallest upp the specific

Since $F(x)$ is a prob. ...

→ $F(-\infty) = 0$, $F(+\infty) = P(S) = 1$

→ Let a & b be 2 real no. such that $a < b$ the prob of interval $[a, b]$ is

$$F(b) - F(a) = P(x \le b) - P(x \le a)$$
$$= P(a < X \le b)$$

The dist funct of discrete r.v is

$$F(x) = \sum_i f(x_i)$$

also

$$f(+\infty) = \sum_i f(x_i) = 1.$$

PDF $\xrightarrow{\text{integrate}}$ CDF

CDF $\xrightarrow{\text{Differentiate}}$ PDF

for Continuous

→ $\dfrac{d\,F(x)}{d(x)} = f(x)$

→ $\int_{-\infty} f(x)\,dx = F(x)$

No. of Heads x	$f(x_i)$	$F(x)$
0	$\frac{1}{8}$	$\frac{1}{8}$
1	3/8	$\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$
2	3/8	$\frac{4}{8} + \frac{3}{8} = \frac{7}{8}$
3	1/8	$\frac{7}{8} + \frac{1}{8} = 1$

Results ?

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1/8, & 0 \le x < 1 \\ 4/8, & 1 \le x \le 2 \\ 7/8, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

if $x < 0$, $P(X < x) = 0$

if $0 < x < 1$, $P(X < x) = P(X = 0) = \frac{1}{8}$

if $1 < x < 2$, $P(X < x) = P(x=0) + P(x=1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$

if $2 < x < 3$, $P(X < x) = P(x=0) + P(x=1) + P(x=2) = \frac{1}{8} + \frac{3}{8} = \frac{7}{8}$

if $x \ge 3$, $P(X < x) = \sum_{i=0}^{3} P(x=i) = 1$

CDF → PDF

$$F(x) = 0 \quad \text{for } x<0$$
$$= \frac{2x^2}{5} \quad 0<x\le 1$$
$$= \frac{-3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right), \quad 1<x\le 2$$
$$= 1 \quad x>2$$

find pdf $P(|x| < 1.5)$

Sol

$$f(x) = \frac{d}{dx}F(x)$$

$$f(x) = \frac{4x}{5} \quad , \; 0<x\le 1$$
$$= \frac{2}{5}(3-x), \quad 1<x\le 2$$
$$= 0 \quad \text{elsewhere}$$

$$P(-1.5 < x < 1.5) = \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{4x}{5}\,dx + \int_{1}^{1.5} \frac{2}{5}(3-x)\,dx$$

3.20 $f(x) = \frac{2}{27}(1+x) \quad x=2 \text{ and } x=5$

Find $F(x)$ and evaluate $3 \le x < 4$.

Sol

$$= \int_{2}^{x} \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^x$$

$$= \frac{2}{27}\left(x + \frac{x^2}{2} - 2 - 2\right)$$

$$= \frac{2}{27}\left(x + \frac{x^2}{2} - 4\right)$$

$$= \frac{2}{27}\left(\frac{2x + x^2 - 8}{2}\right)$$

$$= \frac{x^2 + 2x - 8}{27}$$

$$= \frac{(x+4)(x-2)}{27}$$

$$F(x) = \begin{cases} 0, & x<2 \\ \frac{(x-2)(x+4)}{27}, & 2 \le x < 5 \\ 1, & x \ge 5 \end{cases}$$

3.19 $f(x) = \frac{1}{2} \quad , x=1 \text{ and } x=3$

$F(x)$? Evaluate $P(2 < x < 2.5)$

Sol: $F(x) = P(X \le x)$

$$= \int_{1}^{x} \frac{1}{2}\,dx$$

x will be always upper limit and 1 will be lower limit

$$= \frac{1}{2}x\Big]_1^x = \frac{x-1}{2} = F(x)$$

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$$F(x) = \begin{cases} 0, & x<1 \\ \frac{x-1}{2}, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$\therefore F(b) - F(a)$

$$F(2.5) - F(2) = \frac{2.5-1}{2} - \frac{2-1}{2}$$
$$= \frac{1.5-1}{2} = \frac{0.5}{2}$$

$$\int (f(x) + g(x))\,dx$$

$$\frac{[f(x)]^{n+1}}{n+1}$$

$$F(4) - F(3)$$

$$= \frac{(4-2)(4+4)}{27} - \frac{(3-2)(3+4)}{27}$$
$$= \frac{16}{27} - \frac{7}{27}$$
$$= \frac{11}{27}$$

$E(x) = \infty$

$E(x) = \int x f(x)$ → Continuous case

Binomial Prob Distribution ~ $P(X=x) = \binom{n}{x} p^x q^{n-x}$ $\quad x = 0, 1, 2, \ldots, n$

$q = 1 - p$

It has 2 parameter n and p.

Q. $p = \frac{3}{8}$, find complete binomial dist for $n=5$. Trails

X	$P(X=x) = \binom{5}{x}\left(\frac{3}{8}\right)^x\left(\frac{5}{8}\right)^{5-x}$
0	$\binom{5}{0}\left(\frac{3}{8}\right)^0\left(\frac{5}{8}\right)^{5-0} = \left(\frac{5}{8}\right)^5$
1	$\binom{5}{1}\left(\frac{3}{8}\right)^1\left(\frac{5}{8}\right)^4$
2	
3	
4	

4.11

Q. Let X have a binomial dist with $n=4$ and $p = \frac{1}{3}$. Find $P(X=1)$, $P(X=\frac{3}{2})$, $P(X=6)$, $P(x \le 2)$.

$P(X=1) = \binom{n}{x} p^x q^{n-x}$

$= \binom{4}{1}\left(\frac{1}{3}\right)\left(1-\frac{1}{3}\right)^{4-1}$

$P(X = \frac{3}{2}) = P(X = 1.5) = 0$

We can't use fractional or decimal value because we can't do it in discrete prob

$P(x=6) = 0$, Because it can't go greater than 4, as only allow, 0, 1, 2, 3, 4.

$P(X \le 2) = P(X=0) + P(x=1) + P(X=2)$

Q. $P(A) = \frac{2}{3}$. In a set of 8 games $n=8$. What is prob that A will be winning (a) exactly 4 game (c) 6 or more games

(a) $P(X=4) = \binom{n}{x} p^x q^{n-x}$

$= \binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{8-4}$

(b) atleast 4 games

$P(X \ge 4) = P(X=4) + P(X=5) + P(X=6) + P(x=7) + P(x=8)$

$= 1 - P(x<4)$

$= 1 - [P(X)=0 + P(x=1) + P(x=2) + P(x=3)]$

$^2P_4$

$^2C_4$

$P(x; \lambda t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}$, $x = 0, 1, 2, ---$

$e = 2.71828$

Expectation of a random Variable.

Q: $\left\{ f(x) = \frac{3}{4}(3-x)(x-5) \quad , 3 \le x \le 5 \right.$
$\quad = 0 \quad$, elsewhere

Find mean, variance and S.D(x)?

Mean = $E(X) = \int_3^5 x f(x) dx$

$Var(X) = E(x^2) - (E(x))^2 \to$ ©

$E(x^2) = \int_3^5 x^2 f(x) dx = \frac{81}{5}$

$Var(x) = \frac{81}{5} - 4^2 = 0.2$

$S.D(x) = \sqrt{Var(x)} = 0.447$

$f(x) = \left\{ \frac{4}{\pi(1+x^2)} \quad , 0 < x < 1 \right.$

Find expected value of X . elsewhere

$E(x) = \int_0^1 x f(x) dx$

$= \int_0^1 x \frac{4}{\pi(1+x^2)} dx$

$= \frac{2}{\pi} \int_0^1 \frac{2x}{1+x^2} dx$

$= \frac{2}{\pi} \left| \ln|1+x^2| \right|_0^1 = \frac{2}{\pi}\left[\ln 2\right] - ~~\ln 1~~$

$= \frac{2 \ln 2}{\pi}$

$= \frac{\ln 2^2}{\pi} = \frac{\ln 4}{\pi}$

$P(n;u) = \frac{}{x!}$

Expectation of a random Variable:

$E(X) = \sum_{\infty} X f(x)$ → Discrete case

$E(X) = \int_{-\infty}^{\infty} X f(x)$ → Continuous case

Q- What is the mathematical expectation of no. of heads when 3 pairs of coins (3) are tossed?

X	f(X)	Xf(x)
0	1/8	0
1	3/8	3/8
2	3/8	6/8
3	1/8	3/8

$E(X) = \sum x f(x) = \frac{12}{8} = 1.5$

Q. If it rains and umbrella 30 $ per day. If it is fair it have 6$ per day. What is his expectation in the probability of rain is

	X	f(x)	Xf(x)
rain	$30	0.3	9.0
	$-6	0.7	-4.2

$E(X) = \sum x f(x)$
$= \$4.8$ per day

$var(X) = E(X^2) - [E(X)]^2$

Properties
- Var(X) cannot be negative
- Var(a) = 0
- $Var(aX) = a^2 Var(X)$
- $Var(aX+b) = a^2 Var(X)$

X_i	$f(x_i)$	$3x-1$	$(3x-1)f(x)$	$x^2 f(x)$
1	0.2	2	0.4	
2	0.3	5	1.5	
3	0.2	8	1.6	
4	0.2	11	2.2	
5	0.1	14	1.4	

Find $E(3x-1)$, $E(x^2)$ and $E(x^2+2)$

$E(3x-1) = \sum(3x-1)f(x) = 7.1$

$E(X^2) = \sum x^2 f(x) =$

Q. Let X be a r.v with pdf

$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$

find expected value of $2x-1$ and x^2.

Sol. $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(2x-1) = \int (2x-1) \cdot 2(x-1)\,dx$

$= 2\int (2x-1)(x-1)\,dx$

$= 2\int_1^2 (2x^2 - 2x - x + 1)\,dx$

$= 2\int_1^2 (2x^2 - 3x + 1)\,dx$

$= \int_1^2 (4x^2 - 6x + 2)\,dx$

$= \left.\frac{4x^3}{3} - \frac{6x^2}{2} + 2x\right]_1^2$

$E(x^2) = \int_1^2 x^2 f(x)\,dx = \frac{17}{6}$

$$P(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

$e = 2.71828$

3.21) $f(x) = \begin{cases} K\sqrt{x}, & 0 < x < 1 \\ 0, & \text{else where} \end{cases}$

(a) Evaluate K

$K \int_0^1 x^{1/2}\, dx = 1$

$K \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_0^1 = 1$

$K \left[\frac{x^{3/2}}{3/2} \right]_0^1 = 1$

$\frac{K(1-0)}{3} = 1$

$\boxed{K = 3/2}$

(b) find $F(x)$, and use it to find $P(0.3 < x < 0.6)$

$f(x) = \frac{3}{2}\sqrt{x}$

$F(x) = \frac{3}{2} \int_0^x x^{1/2}\, dx$

$= \frac{3}{2} \cdot \frac{x^{1/2+1}}{\frac{1}{2}+1}$

$= \frac{3}{2} \cdot \frac{x^{3/2}}{3/2}$

$= x^{3/2}$

$F(x) = \begin{cases} 0, & x < 0 \\ x^{3/2}, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$

3.29 $f(x) = \begin{cases} 3x^{-4}, & x > 1 \\ 0, & \text{else where} \end{cases}$

(a) verify this is valid density

(b) Evaluate $F(x)$

(c) what is prob that it exceed 4?

(a) Valid Density. $3\int_1^x x^{-4}\, dx$

$= 1 - x^{-3}$

(b) $F(x) = \begin{cases} 0, & x < 1 \\ 1 - x^{-3}, & x \geq 1 \end{cases}$

3.24 $n = 4$

5 jazz + 2 classical + 3 rocks = 10

$X \to$ jazz

Sol:

X (Jazz)	$P(X = x)$
0	$\frac{\binom{5}{0}\binom{5}{4}}{\binom{10}{4}} =$
1	$\frac{\binom{5}{1}\binom{5}{3}}{\binom{10}{4}} =$
2	$\frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} =$
3	
4	

$f(x) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}} \quad x = 0, \dots 4$

3.27 $f(x) = \begin{cases} \frac{1}{2000} \exp(-x/2000) & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) $F(x) = ?$

$F(x) = \frac{1}{2000} \int_0^\infty e^{(-x/2000)}\, dx$

$= \frac{1}{2000} \int_0^\infty e^{(-x/2000)} \left(\frac{-1}{2000}\right) \times (-2000)\, dx$

$= \frac{-2000}{2000} e^{-x/2000}$

$= -e^{-x/2000}$

(b) $P(x > 1000) = 1 - F(1000)$

$= 1 + e^{\left(-\frac{1000}{2000}\right)}$

(c) $P(x < 2000) = F(2000)$

$= -e^{-\frac{2000}{2000}} = -e^{-1}$

6-5-24 - Poisson dist. and Poisson Process

$$P(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad x=0,1,2,\ldots$$

$e = 2.718281$

Poisson Dist

$$P(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}$$

Q- 200 passengers have made ... . If prob that passenger who has reservation will not show up is 0.01. what is prob that exactly 3 will not show up?

$\mu = np = 200 \times 0.01$

$\boxed{\mu = 2}$

$$P(x=3) = \frac{e^{-\mu}\mu^x}{x!}$$

$$= \frac{e^{-2}2^x}{x!} = \frac{e^{-2}2^3}{3!} = 0.1804$$

Q Telephone calls are being placed through a certain at random time on average at 4 per min. Assuming a Poisson process, determine the prob that in a 15 sec interval there are 3 or more calls

$\lambda = 4$ per min

$t = \frac{15}{60} = \frac{1}{4}$

$\mu = \lambda t = 4 \times \frac{1}{4} = 1$

$$P(x \geq 3) = \sum_{x=3}^{\infty} \frac{e^{-1}(1)^x}{x!}$$

$$= 1 - P(x<3)$$

$$= 1 - \sum_{x=0}^{2} \frac{e^{-1}}{x!}$$

$$= 1 - \left[e^{-1} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!}\right]$$

$= 1-0.8925$

Q. Suppose an average 1 person in every 1000 is an alcoholic. Find prob that a r.s of 8000 people will yield fewer than 7 alcoholic?

$P(x < 7) = ?$ $n = 8000$ $p = \frac{1}{1000} = 0.001$ $\mu = np = 8000 \times 0.001$ $\mu = 8$

$P(x<7) = \sum_{x=0}^{6} \frac{e^{-8} \cdot 8^{x}}{x!}$

Q9 hypergeometric $N \to \infty$

$\frac{n}{N} \leq 0.05$

we use binomial

$\frac{5}{50} \leq 0.05$

$0.1 \nleq 0.05$

Normal Dist: A continuous r.v 'X' is said to be normally distributed if the prob density fn has following

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

S.D

Properties
- Total area under curve is unity 1.
- Its mean = med = mode that why it is called symmetrical data.
- $Q.D = 0.67456 \approx \frac{2}{3}\sigma$
- $M.D = 0.7979\ \sigma \approx \frac{4}{5}\sigma$
- $Mean = E(X) = \mu$

  variable $= Var(X) = \sigma^2$
- If $\mu = 0$, $\sigma = 1$, then it is known as standard normal distri.

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Negative Binomial Distribution

Q Given a normal dist $\mu = 300$, $\sigma = 50$ find prob that x assumes a value greater than 362?

$P(x > 362) = ?$

$z = \frac{x-\mu}{\sigma} = \frac{362-300}{50} = 1.24$

$P(z > 1.24) = 1 - P(z < 1.24)$
$= 1 - 0.8925$
$= 0.1075$

Q $\mu = 800$, $\sigma = 40$, $P(778 < x\ 834) = ?$

$z_1 = \frac{778-800}{40} = -0.55$

$z_2 = \frac{834-800}{40} = .85$

$P(778 < x < 834) = P(-0.55 < z < 0.85)$
$= P(z < 0.85) - P(z < -0.55)$
$= 0.8023 - 0.2912$
$= 0.5111$

Q Given a ND with $\mu = 40$, $\sigma = 6$ find x

(a) 38 % of area bellow it (b) 5 % of area above it

(a)

$P(z < ?) = 0.38$

$z = \frac{x-\mu}{\sigma}$

$-0.31 = \frac{x-40}{6}$

$x = (6 \times -0.31) + 40$

$x = 38.14$

(b)

$P(z > \ ) = 0.05$

$1 - 0.05 = 0.95$

$P(z < 1.645) = 0.95$

$z = \frac{x-\mu}{\sigma}$

$1.645 = \frac{x-40}{6}$

$x = ?$

when z is greater sub from 1 -

Negative Binomial Distribution

* Z: $\mu = 30$, $SD = 6$

1 2 values ? that contain middle 75% of normal curve Area?

2. $P(x_1 < z < x_2) = 0.75$

3

4 $1 - 0.75 = 0.25$

$\div$ by 2 $\frac{0.25}{2} = 0.125$

$x_1 = -$

$x_2 =$

Negative Binomial Distribution

1- Outcome of each trial is classified into one of 2 categories success or failure.

2- Prob of success remain constant.

3- The successive trial are independent.

4- The experiment is repeated a variable no. of time to obtain a fixed no. of success.

$$P(X=x) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$\text{Mean} = \frac{kq}{p}$$

$$\text{Var} = \frac{kq}{p^2}$$

~~KAMRAZ~~

~~AMRAZ~~

Q- Find the prob that a person tossing 3 coins will get either all heads or all tails for 2nd time on 5th toss.

$$P = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(X=5) = \; ?$$

Q- A person throws a pair of dice. What is prob that he will get a total of 7 for 2nd time on the 8th throw.

$$k=2, \quad P = \frac{6}{36} = \frac{1}{6}$$

$$P(X=8) =$$

Q. The prob that a person will believe in a rumours about the retirement of a politician is 0.25. what PROB.

(a) 6th person to hear the rumours will be the 1st to believe it.

(b) the 12th person to have rumours will be 4th to believe it.

Sol → (a) $P(x=6) = pq^{n-1}$
$= (0.25)(0.75)^{6-1}$

(b) $P(x=12) =$

(5.50)

I

Find the prob that a person flipping a coin gets

(a) 3rd head on the 9th flip.

(b) 1st head on 4th flip

$k=1$

$P = 1/2 \quad pq^{x-1}$

Q4 $N = 100,\ n = 10,\ k = 3,\ p = 1/2$. find

Distribution	mean	Variance
binomial	np	npq
hypergeometric	$np,\ p = \frac{k}{N}$	$npq\left(\frac{N-n}{N-1}\right)$
Neg binomial	$(q\,k)/p$	qk/p^2
Geometric	q/p	q/p^2

$q = \frac{N-k}{N}$