

## Topic 2 Notes

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## 2 Analytic functions

### 2.1 Introduction

The main goal of this topic is to define and give some of the important properties of complex analytic functions. A function  $f(z)$  is analytic if it has a complex derivative  $f'(z)$ . In general, the rules for computing derivatives will be familiar to you from single variable calculus. However, a much richer set of conclusions can be drawn about a complex analytic function than is generally true about real differentiable functions.

### 2.2 The derivative: preliminaries

In calculus we defined the derivative as a limit. In complex analysis we will do the same.

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

Before giving the derivative our full attention we are going to have to spend some time exploring and understanding limits. To motivate this we'll first look at two simple examples – one positive and one negative.

**Example 2.1.** Find the derivative of  $f(z) = z^2$ .

*Solution:* We compute using the definition of the derivative as a limit.

$$\lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} 2z + \Delta z = 2z.$$

That was a positive example. Here's a negative one which shows that we need a careful understanding of limits.

**Example 2.2.** Let  $f(z) = \bar{z}$ . Show that the limit for  $f'(0)$  does not converge.

*Solution:* Let's try to compute  $f'(0)$  using a limit:

$$f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Here we used  $\Delta z = \Delta x + i\Delta y$ .

Now,  $\Delta z \rightarrow 0$  means both  $\Delta x$  and  $\Delta y$  have to go to 0. There are lots of ways to do this. For example, if we let  $\Delta z$  go to 0 along the  $x$ -axis then,  $\Delta y = 0$  while  $\Delta x$  goes to 0. In this case, we would have

$$f'(0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand, if we let  $\Delta z$  go to 0 along the positive  $y$ -axis then

$$f'(0) = \lim_{\Delta y \rightarrow 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

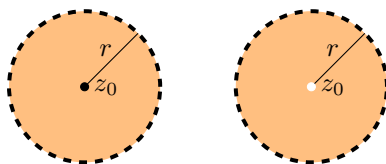
The limits **don't** agree! The problem is that the limit depends on how  $\Delta z$  approaches 0. If we came from other directions we'd get other values. There's nothing to do, but agree that the limit does not exist.

Well, there is something we can do: explore and understand limits. Let's do that now.

### 2.3 Open disks, open deleted disks, open regions

**Definition.** The **open disk of radius  $r$  around  $z_0$**  is the set of points  $z$  with  $|z - z_0| < r$ , i.e. all points within distance  $r$  of  $z_0$ .

The **open deleted disk of radius  $r$  around  $z_0$**  is the set of points  $z$  with  $0 < |z - z_0| < r$ . That is, we remove the center  $z_0$  from the open disk. A deleted disk is also called a **punctured disk**.



Left: an open disk around  $z_0$ ; right: a deleted open disk around  $z_0$

**Definition.** An **open region** in the complex plane is a set  $A$  with the property that every point in  $A$  can be surrounded by an open disk that lies entirely in  $A$ . We will often drop the word open and simply call  $A$  a region.

In the figure below, the set  $A$  on the left is an open region because for every point in  $A$  we can draw a little circle around the point that is completely in  $A$ . (The dashed boundary line indicates that the boundary of  $A$  is not part of  $A$ .) In contrast, the set  $B$  is not an open region. Notice the point  $z$  shown is on the boundary, so every disk around  $z$  contains points outside  $B$ .



Left: an open region  $A$ ; right:  $B$  is not an open region

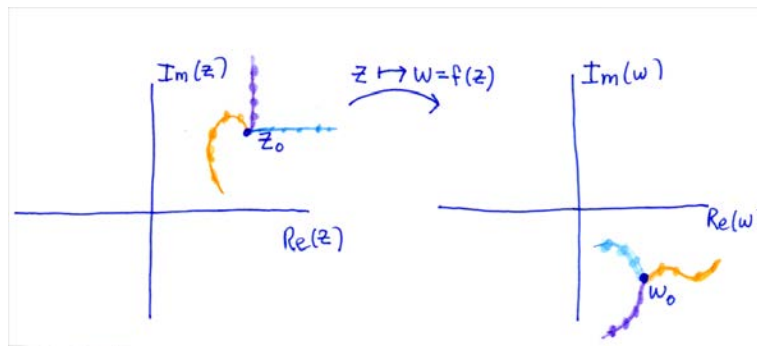
### 2.4 Limits and continuous functions

**Definition.** If  $f(z)$  is defined on a punctured disk around  $z_0$  then we say

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

if  $f(z)$  goes to  $w_0$  no matter what direction  $z$  approaches  $z_0$ .

The figure below shows several sequences of points that approach  $z_0$ . If  $\lim_{z \rightarrow z_0} f(z) = w_0$  then  $f(z)$  must go to  $w_0$  along each of these sequences.



Sequences going to  $z_0$  are mapped to sequences going to  $w_0$ .

**Example 2.3.** Many functions have obvious limits. For example:

$$\lim_{z \rightarrow 2} z^2 = 4$$

and

$$\lim_{z \rightarrow 2} (z^2 + 2)/(z^3 + 1) = 6/9.$$

Here is an example where the limit doesn't exist because different sequences give different limits.

**Example 2.4. (No limit)** Show that

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{z \rightarrow 0} \frac{x + iy}{x - iy}$$

does not exist.

*Solution:* On the real axis we have

$$\frac{z}{\bar{z}} = \frac{x}{x} = 1,$$

so the limit as  $z \rightarrow 0$  along the real axis is 1.

By contrast, on the imaginary axis we have

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1,$$

so the limit as  $z \rightarrow 0$  along the imaginary axis is -1. Since the two limits do not agree the limit as  $z \rightarrow 0$  does not exist!

### 2.4.1 Properties of limits

We have the usual properties of limits. Suppose

$$\lim_{z \rightarrow z_0} f(z) = w_1 \text{ and } \lim_{z \rightarrow z_0} g(z) = w_2$$

then

- $\lim_{z \rightarrow z_0} f(z) + g(z) = w_1 + w_2.$
- $\lim_{z \rightarrow z_0} f(z)g(z) = w_1 \cdot w_2.$

- If  $w_2 \neq 0$  then  $\lim_{z \rightarrow z_0} f(z)/g(z) = w_1/w_2$
- If  $h(z)$  is continuous and defined on a neighborhood of  $w_1$  then  $\lim_{z \rightarrow z_0} h(f(z)) = h(w_1)$   
(Note: we will give the official definition of continuity in the next section.)

We won't give a proof of these properties. As a challenge, you can try to supply it using the formal definition of limits given in the appendix.

We can restate the definition of limit in terms of functions of  $(x, y)$ . To this end, let's write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

and abbreviate

$$P = (x, y), \quad P_0 = (x_0, y_0), \quad w_0 = u_0 + iv_0.$$

Then

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{iff} \quad \begin{cases} \lim_{P \rightarrow P_0} u(x, y) = u_0 \\ \lim_{P \rightarrow P_0} v(x, y) = v_0. \end{cases}$$

**Note.** The term 'iff' stands for 'if and only if' which is another way of saying 'is equivalent to'.

### 2.4.2 Continuous functions

A function is continuous if it doesn't have any sudden jumps. This is the gist of the following definition.

**Definition.** If the function  $f(z)$  is defined on an open disk around  $z_0$  and  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$  then we say  $f$  is continuous at  $z_0$ . If  $f$  is defined on an open region  $A$  then the phrase ' $f$  is continuous on  $A$ ' means that  $f$  is continuous at every point in  $A$ .

As usual, we can rephrase this in terms of functions of  $(x, y)$ :

**Fact.**  $f(z) = u(x, y) + iv(x, y)$  is continuous iff  $u(x, y)$  and  $v(x, y)$  are continuous as functions of two variables.

#### Example 2.5. (Some continuous functions)

(i) A polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

is continuous on the entire plane. Reason: it is clear that each power  $(x + iy)^k$  is continuous as a function of  $(x, y)$ .

(ii) The exponential function is continuous on the entire plane. Reason:

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y).$$

So the both the real and imaginary parts are clearly continuous as a function of  $(x, y)$ .

(iii) The principal branch  $\text{Arg}(z)$  is continuous on the plane minus the non-positive real axis. Reason: this is clear and is the reason we defined branch cuts for arg. We have to remove the negative real axis because  $\text{Arg}(z)$  jumps by  $2\pi$  when you cross it. We also have to remove  $z = 0$  because  $\text{Arg}(z)$  is not even defined at 0.

(iv) The principal branch of the function  $\log(z)$  is continuous on the plane minus the non-positive real axis. Reason: the principal branch of  $\log$  has

$$\log(z) = \log(r) + i \operatorname{Arg}(z).$$

So the continuity of  $\log(z)$  follows from the continuity of  $\operatorname{Arg}(z)$ .

### 2.4.3 Properties of continuous functions

Since continuity is defined in terms of limits, we have the following properties of continuous functions.

Suppose  $f(z)$  and  $g(z)$  are continuous on a region  $A$ . Then

- $f(z) + g(z)$  is continuous on  $A$ .
- $f(z)g(z)$  is continuous on  $A$ .
- $f(z)/g(z)$  is continuous on  $A$  except (possibly) at points where  $g(z) = 0$ .
- If  $h$  is continuous on  $f(A)$  then  $h(f(z))$  is continuous on  $A$ .

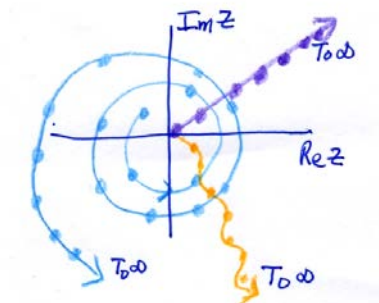
Using these properties we can claim continuity for each of the following functions:

- $e^{z^2}$
- $\cos(z) = (e^{iz} + e^{-iz})/2$
- If  $P(z)$  and  $Q(z)$  are polynomials then  $P(z)/Q(z)$  is continuous except at roots of  $Q(z)$ .

## 2.5 The point at infinity

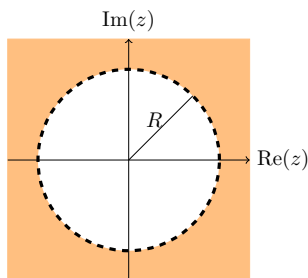
By definition the **extended complex plane**  $= \mathbf{C} \cup \{\infty\}$ . That is, we have **one point at infinity** to be thought of in a limiting sense described as follows.

A sequence of points  $\{z_n\}$  goes to infinity if  $|z_n|$  goes to infinity. This “point at infinity” is approached in any direction we go. All of the sequences shown in the figure below are growing, so they all go to the (same) “point at infinity”.



Various sequences all going to infinity.

If we draw a large circle around 0 in the plane, then we call the region **outside** this circle a **neighborhood of infinity**.



The shaded region outside the circle of radius  $R$  is a neighborhood of infinity.

### 2.5.1 Limits involving infinity

The key idea is  $1/\infty = 0$ . By this we mean

$$\lim_{z \rightarrow \infty} \frac{1}{z} = 0$$

We then have the following facts:

- $\lim_{z \rightarrow z_0} f(z) = \infty \Leftrightarrow \lim_{z \rightarrow z_0} 1/f(z) = 0$
- $\lim_{z \rightarrow \infty} f(z) = w_0 \Leftrightarrow \lim_{z \rightarrow 0} f(1/z) = w_0$
- $\lim_{z \rightarrow \infty} f(z) = \infty \Leftrightarrow \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0$

**Example 2.6.**  $\lim_{z \rightarrow \infty} e^z$  is not defined because it has different values if we go to infinity in different directions, e.g. we have  $e^z = e^x e^{iy}$  and

$$\lim_{x \rightarrow -\infty} e^x e^{iy} = 0$$

$$\lim_{x \rightarrow +\infty} e^x e^{iy} = \infty$$

$$\lim_{y \rightarrow +\infty} e^x e^{iy} \text{ is not defined, since } x \text{ is constant, so } e^x e^{iy} \text{ loops in a circle indefinitely.}$$

**Example 2.7.** Show  $\lim_{z \rightarrow \infty} z^n = \infty$  (for  $n$  a positive integer).

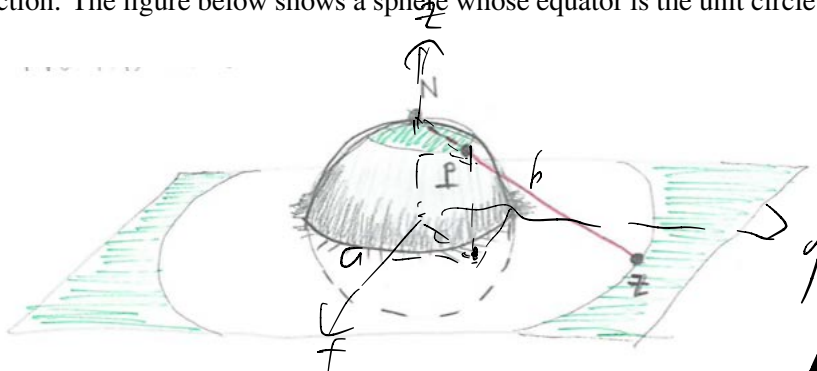
*Solution:* We need to show that  $|z^n|$  gets large as  $|z|$  gets large. Write  $z = R e^{i\theta}$ , then

$$|z^n| = |R^n e^{in\theta}| = R^n = |z|^n$$

### 2.5.2 Stereographic projection from the Riemann sphere

This is a lovely section and we suggest you read it. However it will be a while before we use it in 18.04.

One way to visualize the point at  $\infty$  is by using a (unit) Riemann sphere and the associated stereographic projection. The figure below shows a sphere whose equator is the unit circle in the complex plane.



Stereographic projection from the sphere to the plane.

$$\frac{01}{1-c} = \frac{x}{01}$$

Stereographic projection from the sphere to the plane is accomplished by drawing the secant line from the north pole  $N$  through a point on the sphere and seeing where it intersects the plane. This gives a 1-1 correspondence between a point on the sphere  $P$  and a point in the complex plane  $z$ . It is easy to see show that the formula for stereographic projection is

$$P = (a, b, c) \mapsto z = \frac{a}{1-c} + i \frac{b}{1-c}.$$

The point  $N = (0, 0, 1)$  is special, the secant lines from  $N$  through  $P$  become tangent lines to the sphere at  $N$  which never intersect the plane. We consider  $N$  the point at infinity.

In the figure above, the region outside the large circle through the point  $z$  is a neighborhood of infinity. It corresponds to the small circular cap around  $N$  on the sphere. That is, the small cap around  $N$  is a neighborhood of the point at infinity on the sphere!

The figure below shows another common version of stereographic projection. In this figure the sphere sits with its south pole at the origin. We still project using secant lines from the north pole.



## 2.6 Derivatives

The definition of the complex derivative of a complex function is similar to that of a real derivative of a real function: For a function  $f(z)$  the derivative  $f$  at  $z_0$  is defined as

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Provided, of course, that the limit exists. If the limit exists we say  $f$  is analytic at  $z_0$  or  $f$  is differentiable at  $z_0$ .

**Remember:** The limit has to exist and be the same no matter how you approach  $z_0$ !

If  $f$  is analytic at all the points in an open region  $A$  then we say  $f$  is analytic on  $A$ .

As usual with derivatives there are several alternative notations. For example, if  $w = f(z)$  we can write

$$f'(z_0) = \left. \frac{dw}{dz} \right|_{z_0} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

**Example 2.8.** Find the derivative of  $f(z) = z^2$ .

*Solution:* We did this above in Example 2.1. Take a look at that now. Of course,  $f'(z) = 2z$ .

**Example 2.9.** Show  $f(z) = \bar{z}$  is not differentiable at any point  $z$ .

*Solution:* We did this above in Example 2.2. Take a look at that now.

**Challenge.** Use polar coordinates to show the limit in the previous example can be any value with modulus 1 depending on the angle at which  $z$  approaches  $z_0$ .

### 2.6.1 Derivative rules

It wouldn't be much fun to compute every derivative using limits. Fortunately, we have the same differentiation formulas as for real-valued functions. That is, assuming  $f$  and  $g$  are differentiable we have:

- Sum rule:  $\frac{d}{dz}(f(z) + g(z)) = f' + g'$
- Product rule:  $\frac{d}{dz}(f(z)g(z)) = f'g + fg'$
- Quotient rule:  $\frac{d}{dz}(f(z)/g(z)) = \frac{f'g - fg'}{g^2}$
- Chain rule:  $\frac{d}{dz}g(f(z)) = g'(f(z))f'(z)$
- Inverse rule:  $\frac{df^{-1}(z)}{dz} = \frac{1}{f'(f^{-1}(z))}$

To give you the flavor of these arguments we'll prove the product rule.

$$\begin{aligned} \frac{d}{dz}(f(z)g(z)) &= \lim_{z \rightarrow z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{(f(z) - f(z_0))g(z) + f(z_0)(g(z) - g(z_0))}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} g(z) + f(z_0) \frac{g(z) - g(z_0)}{z - z_0} \\ &= f'(z_0)g(z_0) + f(z_0)g'(z_0) \end{aligned}$$

Here is an important fact that you would have guessed. We will prove it in the next section.

**Theorem.** If  $f(z)$  is defined and differentiable on an open disk and  $f'(z) = 0$  on the disk then  $f(z)$  is constant.



## 2.7 Cauchy-Riemann equations

The Cauchy-Riemann equations are our first consequence of the fact that the limit defining  $f(z)$  must be the same no matter which direction you approach  $z$  from. [The Cauchy-Riemann equations will be one of the most important tools in our toolbox.](#)

### 2.7.1 Partial derivatives as limits

Before getting to the Cauchy-Riemann equations we remind you about partial derivatives. If  $u(x, y)$  is a function of two variables then the partial derivatives of  $u$  are defined as

$$\frac{\partial u}{\partial x}(x, y) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x},$$

i.e. the derivative of  $u$  holding  $y$  constant.

$$\frac{\partial u}{\partial y}(x, y) = \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y},$$

i.e. the derivative of  $u$  holding  $x$  constant.

### 2.7.2 The Cauchy-Riemann equations

The Cauchy-Riemann equations use the partial derivatives of  $u$  and  $v$  to allow us to do two things: first, to check if  $f$  has a complex derivative and second, to compute that derivative. We start by stating the equations as a theorem.

**Theorem 2.10. (Cauchy-Riemann equations)** If  $f(z) = u(x, y) + iv(x, y)$  is analytic (complex differentiable) then

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

In particular,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This last set of partial differential equations is what is usually meant by the Cauchy-Riemann equations.

Here is the short form of the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

*Proof.* Let's suppose that  $f(z)$  is differentiable in some region  $A$  and

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We'll compute  $f'(z)$  by approaching  $z$  first from the horizontal direction and then from the vertical direction. We'll use the formula

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

where  $\Delta z = \Delta x + i\Delta y$ .

Horizontal direction:  $\Delta y = 0$ ,  $\Delta z = \Delta x$

$$\begin{aligned}
 f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(u(x + \Delta x, y) + iv(x + \Delta x, y)) - (u(x, y) + iv(x, y))}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\
 &= \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y)
 \end{aligned}$$

Vertical direction:  $\Delta x = 0$ ,  $\Delta z = i\Delta y$  (We'll do this one a little faster.)

$$\begin{aligned}
 f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
 &= \lim_{\Delta y \rightarrow 0} \frac{(u(x, y + \Delta y) + iv(x, y + \Delta y)) - (u(x, y) + iv(x, y))}{i\Delta y} \\
 &= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \\
 &= \frac{1}{i} \frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y) \\
 &= \frac{\partial v}{\partial y}(x, y) - i \frac{\partial u}{\partial y}(x, y)
 \end{aligned}$$

We have found two different representations of  $f'(z)$  in terms of the partials of  $u$  and  $v$ . If put them together we have the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

It turns out that the converse is true and will be very useful to us.

**Theorem.** Consider the function  $f(z) = u(x, y) + iv(x, y)$  defined on a region  $A$ . If  $u$  and  $v$  satisfy the Cauchy-Riemann equations and have continuous partials then  $f(z)$  is differentiable on  $A$ .

The proof of this is a tricky exercise in analysis. It is somewhat beyond the scope of this class, so we will skip it. If you're interested, with a little effort you should be able to grasp it.

### 2.7.3 Using the Cauchy-Riemann equations

The Cauchy-Riemann equations provide us with a direct way of checking that a function is differentiable and computing its derivative.

**Example 2.11.** Use the Cauchy-Riemann equations to show that  $e^z$  is differentiable and its derivative is  $e^z$ .

*Solution:* We write  $e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$ . So

$$u(x, y) = e^x \cos(y) \quad \text{and} \quad v(x, y) = e^x \sin(y).$$

Computing partial derivatives we have

$$\begin{aligned} u_x &= e^x \cos(y), & u_y &= -e^x \sin(y) \\ v_x &= e^x \sin(y), & v_y &= e^x \cos(y) \end{aligned}$$

We see that  $u_x = v_y$  and  $u_y = -v_x$ , so the Cauchy-Riemann equations are satisfied. Thus,  $e^z$  is differentiable and

$$\frac{d}{dz} e^z = u_x + i v_x = e^x \cos(y) + i e^x \sin(y) = e^z.$$

**Example 2.12.** Use the Cauchy-Riemann equations to show that  $f(z) = \bar{z}$  is not differentiable.

*Solution:*  $f(x + iy) = x - iy$ , so  $u(x, y) = x$ ,  $v(x, y) = -y$ . Taking partial derivatives

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1$$

Since  $u_x \neq v_y$  the Cauchy-Riemann equations are not satisfied and therefore  $f$  is not differentiable.

**Theorem.** If  $f(z)$  is differentiable on a disk and  $f'(z) = 0$  on the disk then  $f(z)$  is constant.

*Proof.* Since  $f$  is differentiable and  $f'(z) \equiv 0$ , the Cauchy-Riemann equations show that

$$u_x(x, y) = u_y(x, y) = v_x(x, y) = v_y(x, y) = 0$$

We know from multivariable calculus that a function of  $(x, y)$  with both partials identically zero is constant. Thus  $u$  and  $v$  are constant, and therefore so is  $f$ .

#### 2.7.4 $f'(z)$ as a $2 \times 2$ matrix

Recall that we could represent a complex number  $a + ib$  as a  $2 \times 2$  matrix

$$a + ib \quad \leftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \quad (1)$$

Now if we write  $f(z)$  in terms of  $(x, y)$  we have

$$f(z) = f(x + iy) = u(x, y) + i v(x, y) \quad \leftrightarrow \quad f(x, y) = (u(x, y), v(x, y)).$$

We have

$$f'(z) = u_x + i v_x,$$

so we can represent  $f'(z)$  as

$$\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}.$$

Using the Cauchy-Riemann equations we can replace  $-v_x$  by  $u_y$  and  $u_x$  by  $v_y$  which gives us the representation

$$f'(z) \quad \leftrightarrow \quad \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

i.e.,  $f'(z)$  is just the Jacobian of  $f(x, y)$ .

For me, it is easier to remember the Jacobian than the Cauchy-Riemann equations. Since  $f'(z)$  is a complex number I can use the matrix representation in Equation 1 to remember the Cauchy-Riemann equations!

## 2.8 Cauchy-Riemann all the way down

We've defined an analytic function as one having a complex derivative. The following theorem shows that if  $f$  is analytic then so is  $f'$ . Thus, there are derivatives all the way down!

**Theorem 2.13.** Assume the second order partials of  $u$  and  $v$  exist and are continuous. If  $f(z) = u + iv$  is analytic, then so is  $f'(z)$ .

*Proof.* To show this we have to prove that  $f'(z)$  satisfies the Cauchy-Riemann equations. If  $f = u + iv$  we know

$$u_x = v_y, \quad u_y = -v_x, \quad f' = u_x + iv_x.$$

Let's write

$$f' = U + iV, \quad \begin{matrix} -iu_y & v_y - u_x \\ \text{ } & \end{matrix}$$

so, by Cauchy-Riemann,

$$U = u_x = v_y, \quad V = v_x = -u_y. \quad (2)$$

We want to show that  $U_x = V_y$  and  $U_y = -V_x$ . We do them one at a time.

To prove  $U_x = V_y$ , we use Equation 2 to see that

$$U_x = v_{yx} \quad \text{and} \quad V_y = v_{xy}.$$

Since  $v_{xy} = v_{yx}$ , we have  $U_x = V_y$ .

Similarly, to show  $U_y = -V_x$ , we compute

$$U_y = u_{xy} \quad \text{and} \quad V_x = -u_{yx}.$$

So,  $U_y = -V_x$ . QED.

**Technical point.** We've assumed as many partials as we need. So far we can't guarantee that all the partials exist. Soon we will have a theorem which says that an analytic function has derivatives of all order. We'll just assume that for now. In any case, in most examples this will be obvious.

## 2.9 Gallery of functions

In this section we'll look at many of the functions you know and love as functions of  $z$ . For each one we'll have to do three things.

1. Define how to compute it.
2. Specify a branch (if necessary) giving its range.
3. Specify a domain (with branch cut if necessary) where it is analytic.
4. Compute its derivative.

Most often, we can compute the derivatives of a function using the algebraic rules like the quotient rule. If necessary we can use the Cauchy-Riemann equations or, as a last resort, even the definition of the derivative as a limit.

Before we start on the gallery we define the term "entire function".

**Definition.** A function that is analytic at every point in the complex plane is called an **entire function**. We will see that  $e^z$ ,  $z^n$ ,  $\sin(z)$  are all entire functions.

### 2.9.1 Gallery of functions, derivatives and properties

The following is a concise list of a number of functions and their complex derivatives. None of the derivatives will surprise you. We also give important properties for some of the functions. The proofs for each follow below.

1.  $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$ .

Domain = all of  $\mathbf{C}$  ( $f$  is entire).

$$f'(z) = e^z.$$

2.  $f(z) \equiv c$  (constant)

Domain = all of  $\mathbf{C}$  ( $f$  is entire).

$$f'(z) = 0.$$

3.  $f(z) = z^n$  ( $n$  an integer  $\geq 0$ )

Domain = all of  $\mathbf{C}$  ( $f$  is entire).

$$f'(z) = nz^{n-1}.$$

4.  $P(z)$  (polynomial)

A polynomial has the form  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ .

Domain = all of  $\mathbf{C}$  ( $P(z)$  is entire).

$$P'(z) = na_n z^{n-1} + (n-1)a_{n-1} z^{n-2} + \dots + 2a_2 z + a_1.$$

5.  $f(z) = 1/z$

Domain =  $\mathbf{C} - \{0\}$  (the punctured plane).

$$f'(z) = -1/z^2.$$

6.  $f(z) = P(z)/Q(z)$  (rational function).

When  $P$  and  $Q$  are polynomials  $P(z)/Q(z)$  is called a rational function.

If we assume that  $P$  and  $Q$  have no common roots, then:

Domain =  $\mathbf{C} - \{\text{roots of } Q\}$

$$f'(z) = \frac{P'Q - PQ'}{Q^2}.$$

7.  $\sin(z), \cos(z)$

**Definition.**  $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

(By Euler's formula we know this is consistent with  $\cos(x)$  and  $\sin(x)$  when  $z = x$  is real.)

Domain: these functions are entire.

$$\frac{d \cos(z)}{dz} = -\sin(z), \quad \frac{d \sin(z)}{dz} = \cos(z).$$

Other key properties of  $\sin$  and  $\cos$ :

- $\cos^2(z) + \sin^2(z) = 1$
- $e^z = \cos(z) + i \sin(z)$   $\frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} i = e^{iz}$
- Periodic in  $x$  with period  $2\pi$ , e.g.  $\sin(x + 2\pi + iy) = \sin(x + iy)$ .
- They are not bounded!
- In the form  $f(z) = u(x, y) + iv(x, y)$  we have

$$\cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$$

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

(cosh and sinh are defined below.)

- The zeros of  $\sin(z)$  are  $z = n\pi$  for  $n$  any integer.
- The zeros of  $\cos(z)$  are  $z = \pi/2 + n\pi$  for  $n$  any integer.
- (That is, they have only real zeros that you learned about in your trig. class.)

## 8. Other trig functions $\cot(z)$ , $\sec(z)$ etc.

**Definition.** The same as for the real versions of these function, e.g.  $\cot(z) = \cos(z)/\sin(z)$ ,  $\sec(z) = 1/\cos(z)$ .

Domain: The entire plane minus the zeros of the denominator.

Derivative: Compute using the quotient rule, e.g.

$$\frac{d \tan(z)}{dz} = \frac{d}{dz} \left( \frac{\sin(z)}{\cos(z)} \right) = \frac{\cos(z) \cos(z) - \sin(z)(-\sin(z))}{\cos^2(z)} = \frac{1}{\cos^2(z)} = \sec^2 z$$

(No surprises there!)

## 9. $\sinh(z)$ , $\cosh(z)$ (hyperbolic sine and cosine)

**Definition.**

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

Domain: these functions are entire.

$$\frac{d \cosh(z)}{dz} = \sinh(z), \quad \frac{d \sinh(z)}{dz} = \cosh(z)$$

Other key properties of cosh and sinh:

- $\cosh^2(z) - \sinh^2(z) = 1$
- For real  $x$ ,  $\cosh(x)$  is real and positive,  $\sinh(x)$  is real.
- $\cosh(iz) = \cos(z)$ ,  $\sinh(z) = -i \sin(iz)$ .

## 10. $\log(z)$ (See Topic 1.)

**Definition.**  $\log(z) = \log(|z|) + i \arg(z)$ .

Branch: Any branch of  $\arg(z)$ .

Domain:  $\mathbb{C}$  minus a branch cut where the chosen branch of  $\arg(z)$  is discontinuous.

$$\frac{d}{dz} \log(z) = \frac{1}{z}$$

- 11.
- $z^a$
- (any complex
- $a$
- ) (See Topic 1.)

**Definition.**  $z^a = e^{a \log(z)}$ .Branch: Any branch of  $\log(z)$ .Domain: Generally the domain is  $\mathbf{C}$  minus a branch cut of  $\log$ . If  $a$  is an integer  $\geq 0$  then  $z^a$  is entire. If  $a$  is a negative integer then  $z^a$  is defined and analytic on  $\mathbf{C} - \{0\}$ .

$$\frac{dz^a}{dz} = az^{a-1}.$$

- 12.
- $\sin^{-1}(z)$

**Definition.**  $\sin^{-1}(z) = -i \log(iz + \sqrt{1 - z^2})$ .The definition is chosen so that  $\sin(\sin^{-1}(z)) = z$ . The derivation of the formula is as follows. Let  $w = \sin^{-1}(z)$ , so  $z = \sin(w)$ . Then,

$$z = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0$$

Solving the quadratic in  $e^{iw}$  gives

$$e^{iw} = \frac{2iz + \sqrt{-4z^2 + 4}}{2} = iz + \sqrt{1 - z^2}.$$

Taking the log gives

$$iw = \log(iz + \sqrt{1 - z^2}) \Leftrightarrow w = -i \log(iz + \sqrt{1 - z^2}). \quad \blacksquare$$

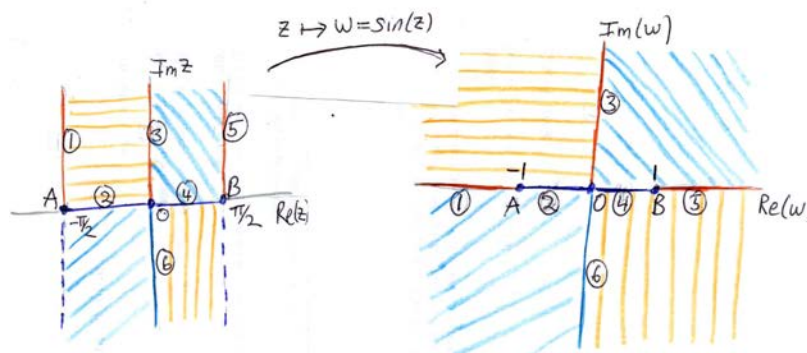
From the definition we can compute the derivative:

$$\frac{d}{dz} \sin^{-1}(z) = \frac{1}{\sqrt{1 - z^2}}.$$

Choosing a branch is tricky because both the square root and the log require choices. We will look at this more carefully in the future.

For now, the following discussion and figure are for your amusement.

Sine (likewise cosine) is not a 1-1 function, so if we want  $\sin^{-1}(z)$  to be single-valued then we have to choose a region where  $\sin(z)$  is 1-1. (This will be a branch of  $\sin^{-1}(z)$ , i.e. a range for the image.) The figure below shows a domain where  $\sin(z)$  is 1-1. The domain consists of the vertical strip  $z = x + iy$  with  $-\pi/2 < x < \pi/2$  together with the two rays on boundary where  $y \geq 0$  (shown as red lines). The figure indicates how the regions making up the domain in the  $z$ -plane are mapped to the quadrants in the  $w$ -plane.

A domain where  $z \mapsto w = \sin(z)$  is one-to-one

### 2.9.2 A few proofs

Here we prove at least some of the facts stated in the list just above.

1.  $f(z) = e^z$ . This was done in Example 2.11 using the Cauchy-Riemann equations.
2.  $f(z) \equiv c$  (constant). This case is trivial.
3.  $f(z) = z^n$  ( $n$  an integer  $\geq 0$ ): show  $f'(z) = nz^{n-1}$

It's probably easiest to use the definition of derivative directly. Before doing that we note the factorization

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \dots + z^2z_0^{n-3} + zz_0^{n-2} + z_0^{n-1})$$

Now

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{z^n - z_0^n}{z - z_0} \\ &= \lim_{z \rightarrow z_0} (z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \dots + z^2z_0^{n-3} + zz_0^{n-2} + z_0^{n-1}) \\ &= nz_0^{n-1}. \end{aligned}$$

Since we showed directly that the derivative exists for all  $z$ , the function must be entire.

4.  $P(z)$  (polynomial). Since a polynomial is a sum of monomials, the formula for the derivative follows from the derivative rule for sums and the case  $f(z) = z^n$ . Likewise the fact the  $P(z)$  is entire.
5.  $f(z) = 1/z$ . This follows from the quotient rule.
6.  $f(z) = P(z)/Q(z)$ . This also follows from the quotient rule.
7.  $\sin(z)$ ,  $\cos(z)$ . All the facts about  $\sin(z)$  and  $\cos(z)$  follow from their definition in terms of exponentials.
8. Other trig functions  $\cot(z)$ ,  $\sec(z)$  etc. Since these are all defined in terms of  $\cos$  and  $\sin$ , all the facts about these functions follow from the derivative rules.
9.  $\sinh(z)$ ,  $\cosh(z)$ . All the facts about  $\sinh(z)$  and  $\cosh(z)$  follow from their definition in terms of exponentials.
10.  $\log(z)$ . The derivative of  $\log(z)$  can be found by differentiating the relation  $e^{\log(z)} = z$  using the chain rule. Let  $w = \log(z)$ , so  $e^w = z$  and

$$\frac{d}{dz}e^w = \frac{dz}{dz} = 1 \quad \Rightarrow \quad \frac{de^w}{dw} \frac{dw}{dz} = 1 \quad \Rightarrow \quad e^w \frac{dw}{dz} = 1 \quad \Rightarrow \quad \frac{dw}{dz} = \frac{1}{e^w}$$

Using  $w = \log(z)$  we get

$$\frac{d \log(z)}{dz} = \frac{1}{z}.$$

11.  $z^a$  (any complex  $a$ ). The derivative for this follows from the formula

$$z^a = e^{a \log(z)} \quad \Rightarrow \quad \frac{dz^a}{dz} = e^{a \log(z)} \cdot \frac{a}{z} = \frac{az^a}{z} = az^{a-1}$$



## 2.10 Branch cuts and function composition

We often compose functions, i.e.  $f(g(z))$ . In general in this case we have the chain rule to compute the derivative. However we need to specify the domain for  $z$  where the function is analytic. And when branches and branch cuts are involved we need to take care.

**Example 2.14.** Let  $f(z) = e^{z^2}$ . Since  $e^z$  and  $z^2$  are both entire functions, so is  $f(z) = e^{z^2}$ . The chain rule gives us

$$f'(z) = e^{z^2}(2z).$$

**Example 2.15.** Let  $f(z) = e^z$  and  $g(z) = 1/z$ .  $f(z)$  is entire and  $g(z)$  is analytic everywhere but 0. So  $f(g(z))$  is analytic except at 0 and

$$\frac{df(g(z))}{dz} = f'(g(z))g'(z) = e^{1/z} \cdot \frac{-1}{z^2}.$$

**Example 2.16.** Let  $h(z) = 1/(e^z - 1)$ . Clearly  $h$  is entire except where the denominator is 0. The denominator is 0 when  $e^z - 1 = 0$ . That is, when  $z = 2\pi ni$  for any integer  $n$ . Thus,  $h(z)$  is analytic on the set

$$\mathbf{C} - \{2\pi ni, \text{ where } n \text{ is any integer}\}$$

The quotient rule gives  $h'(z) = -e^z/(e^z - 1)^2$ . A little more formally:  $h(z) = f(g(z))$ , where  $f(w) = 1/w$  and  $w = g(z) = e^z - 1$ . We know that  $g(z)$  is entire and  $f(w)$  is analytic everywhere except  $w = 0$ . Therefore,  $f(g(z))$  is analytic everywhere except where  $g(z) = 0$ .

**Example 2.17.** It can happen that the derivative has a larger domain where it is analytic than the original function. The main example is  $f(z) = \log(z)$ . This is analytic on  $\mathbf{C}$  minus a branch cut. However

$$\frac{d}{dz} \log(z) = \frac{1}{z}$$

is analytic on  $\mathbf{C} - \{0\}$ . [The converse can't happen.](#)

**Example 2.18.** Define a region where  $\sqrt{1-z}$  is analytic.

*Solution:* Choosing the principal branch of argument, we have  $\sqrt{w}$  is analytic on

$$\mathbf{C} - \{x \leq 0, y = 0\}, \text{ (see figure below.)}$$

So  $\sqrt{1-z}$  is analytic except where  $w = 1-z$  is on the branch cut, i.e. where  $w = 1-z$  is real and  $\leq 0$ . It's easy to see that

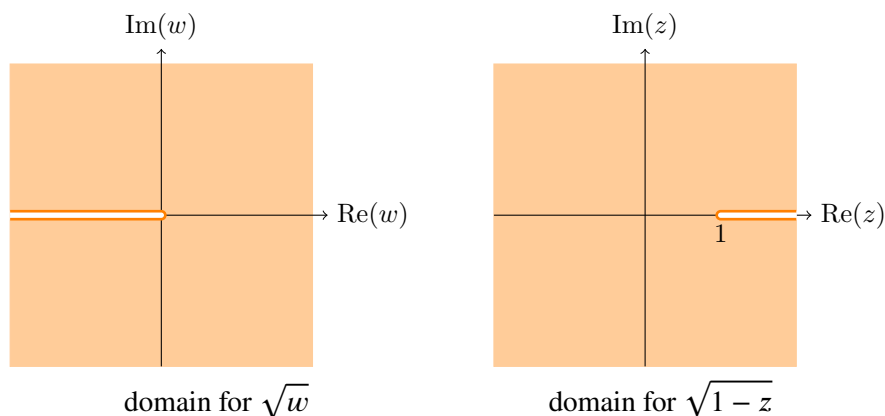
$$w = 1-z \text{ is real and } \leq 0 \Leftrightarrow z \text{ is real and } \geq 1.$$

So  $\sqrt{1-z}$  is analytic on the region (see figure below)

$$\mathbf{C} - \{x \geq 1, y = 0\}$$

**Note.** A different branch choice for  $\sqrt{w}$  would lead to a different region where  $\sqrt{1-z}$  is analytic.

The figure below shows the domains with branch cuts for this example.



**Example 2.19.** Define a region where  $f(z) = \sqrt{1 + e^z}$  is analytic.

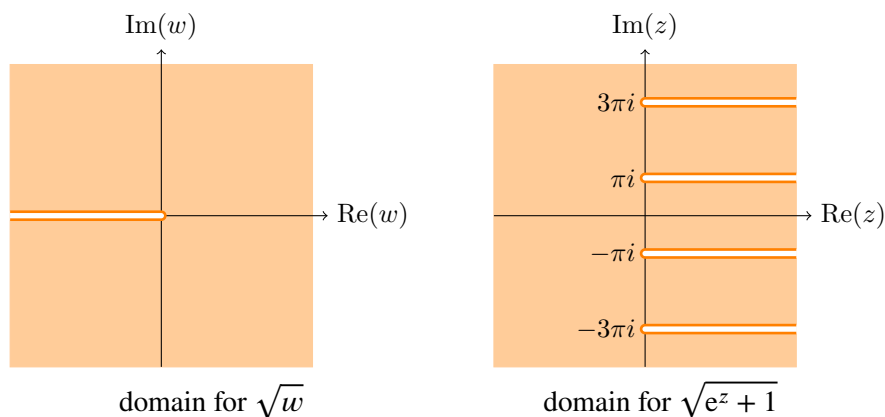
*Solution:* Again, let's take  $\sqrt{w}$  to be analytic on the region

$$\mathbf{C} - \{x \leq 0, y = 0\}$$

So,  $f(z)$  is analytic except where  $1 + e^z$  is real and  $\leq 0$ . That is, except where  $e^z$  is real and  $\leq -1$ . Now,  $e^z = e^x e^{iy}$  is real only when  $y$  is a multiple of  $\pi$ . It is negative only when  $y$  is an odd multiple of  $\pi$ . It has magnitude greater than 1 only when  $x > 0$ . Therefore  $f(z)$  is analytic on the region

$$\mathbf{C} - \{x \geq 0, y = \text{odd multiple of } \pi\}$$

The figure below shows the domains with branch cuts for this example.



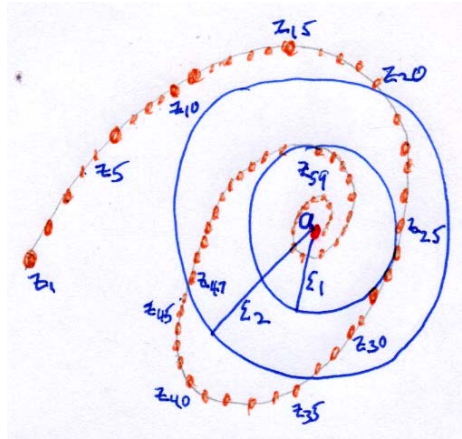
## 2.11 Appendix: Limits

The intuitive idea behind limits is relatively simple. Still, in the 19th century mathematicians were troubled by the lack of rigor, so they set about putting limits and analysis on a firm footing with careful definitions and proofs. In this appendix we give you the formal definition and connect it to the intuitive idea. In 18.04 we will not need this level of formality. Still, it's nice to know the foundations are solid, and some students may find this interesting.

### 2.11.1 Limits of sequences

Intuitively, we say a sequence of complex numbers  $z_1, z_2, \dots$  converges to  $a$  if for large  $n$ ,  $z_n$  is really close to  $a$ . To be a little more precise, if we put a small circle of radius  $\epsilon$  around  $a$  then eventually the sequence should stay inside the circle. Let's refer to this as [the sequence being captured by the circle](#). This has to be true for any circle no matter how small, though it may take longer for the sequence to be 'captured' by a smaller circle.

This is illustrated in the figure below. The sequence is strung along the curve shown heading towards  $a$ . The bigger circle of radius  $\epsilon_2$  captures the sequence by the time  $n = 47$ , the smaller circle doesn't capture it till  $n = 59$ . Note that  $z_{25}$  is inside the larger circle, but since later points are outside the circle we don't say the sequence is captured at  $n = 25$



A sequence of points converging to  $a$

**Definition.** The sequence  $z_1, z_2, z_3, \dots$  converges to the value  $a$  if for every  $\epsilon > 0$  there is a number  $N_\epsilon$  such that  $|z_n - a| < \epsilon$  for all  $n > N_\epsilon$ . We write this as

$$\lim_{n \rightarrow \infty} z_n = a.$$

Again, the definition just says that eventually the sequence is within  $\epsilon$  of  $a$ , no matter how small you choose  $\epsilon$ .

**Example 2.20.** Show that the sequence  $z_n = (1/n + i)^2$  has limit  $-1$ .

*Solution:* This is clear because  $1/n \rightarrow 0$ . For practice, let's phrase it in terms of epsilons: given  $\epsilon > 0$  we have to choose  $N_\epsilon$  such that

$$|z_n - (-1)| < \epsilon \text{ for all } n > N_\epsilon$$

One strategy is to look at  $|z_n + 1|$  and see what  $N_\epsilon$  should be. We have

$$|z_n - (-1)| = \left| \left( \frac{1}{n} + i \right)^2 + 1 \right| = \left| \frac{1}{n^2} + \frac{2i}{n} \right| < \frac{1}{n^2} + \frac{2}{n}$$

So all we have to do is pick  $N_\epsilon$  large enough that

$$\frac{1}{N_\epsilon^2} + \frac{2}{N_\epsilon} < \epsilon$$

Since this can clearly be done we have proved that  $z_n \rightarrow -1$ .

This was clearly more work than we want to do for every limit. Fortunately, most of the time we can apply general rules to determine a limit without resorting to epsilons!

**Remarks.**

1. In 18.04 we will be able to spot the limit of most concrete examples of sequences. The formal definition is needed when dealing abstractly with sequences.
2. To mathematicians  $\epsilon$  is one of the go-to symbols for a small number. The prominent and rather eccentric mathematician Paul Erdos used to refer to children as epsilons, as in ‘How are the epsilons doing?’
3. The term ‘captured by the circle’ is not in common usage, but it does capture what is happening.

### 2.11.2 $\lim_{z \rightarrow z_0} f(z)$

Sometimes we need limits of the form  $\lim_{z \rightarrow z_0} f(z) = a$ . Again, the intuitive meaning is clear: as  $z$  gets close to  $z_0$  we should see  $f(z)$  get close to  $a$ . Here is the technical definition

**Definition.** Suppose  $f(z)$  is defined on a punctured disk  $0 < |z - z_0| < r$  around  $z_0$ . We say  $\lim_{z \rightarrow z_0} f(z) = a$  if for every  $\epsilon > 0$  there is a  $\delta$  such that

$$|f(z) - a| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

This says exactly that as  $z$  gets closer (within  $\delta$ ) to  $z_0$  we have  $f(z)$  is close (within  $\epsilon$ ) to  $a$ . Since  $\epsilon$  can be made as small as we want,  $f(z)$  must go to  $a$ .

**Remarks.**

1. Using the punctured disk (also called a **deleted neighborhood**) means that  $f(z)$  does not have to be defined at  $z_0$  and, if it is then  $f(z_0)$  does not necessarily equal  $a$ . If  $f(z_0) = a$  then we say the  $f$  is continuous at  $z_0$ .
2. Ask any mathematician to complete the phrase “For every  $\epsilon$ ” and the odds are that they will respond “there is a  $\delta \dots$ ”

### 2.11.3 Connection between limits of sequences and limits of functions

Here’s an equivalent way to define limits of functions: the limit  $\lim_{z \rightarrow z_0} f(z) = a$  if, for every sequence of points  $\{z_n\}$  with limit  $z_0$  the sequence  $\{f(z_n)\}$  has limit  $a$ .

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## 18.04 Complex Variables with Applications

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