18.04 Recitation 3
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$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$f' = u_x + i v_x$

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. We write $f(x, y) = u(x, y) + iv(x, y)$. Suppose that u and v are C^2 i.e. all partial derivatives of u and v of order up to (and including) 2 exist, and are continuous. Show that $f' = \frac{df}{dz} : \mathbb{C} \to \mathbb{C}$ is also analytic.

$u_{xx} = v_{yx} = v_{xy}$

$u_{xy} = u_{yx} = -v_{xx}$

2.1. Show that $\int \bar{z} dz$ is not path independent in $\mathbb{C}$. Why does this not contradict the fundamental theorem for complex line integrals?

$\int_\gamma \bar{z} dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = 2\pi i \neq 0$

2.2. For each $n \in \mathbb{Z}$, compute $\int_\gamma z^n dz$, where γ is the unit circle centered at the origin. Are your answers consistent with the fundamental theorem?

2.3. Do any of the answers in 2.2. change if γ is a circle such that the disk bounded by the circle does not contain the origin? $n = -1 \Rightarrow 0$

3. Recall from Recitation 2 that $\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.

3.1. Consider the region $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi\}$. What are the images of horizontal and vertical lines in $\mathcal{R}$? Is the mapping $z \mapsto \cos(z)$ restricted to $\mathcal{R}$ a one-to-one mapping?

3.2. To $\mathcal{R}$, add the half lines $x = 0, y \geq 0$ and $x = \pi, y > 0$ to produce a new region $\mathcal{R}_1$. What is the image of $\mathcal{R}_1$ under the map $z \mapsto \cos(z)$? Is the map still one-to-one on $\mathcal{R}_1$?

3.3. Note that $\mathcal{R}_1$ gives a branch of the multi-valued function $\cos^{-1}(z)$. What are the branch cuts in the domain of $\cos^{-1}(z)$ for this branch?