18.04 Recitation 4
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1. We will compute $I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$ using Cauchy's integral formula. It will be helpful to recall the triangle inequality for integrals: $\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz|$.

1.1. Consider the semicircle C in the upper half plane which is centered at 0 and has radius R. Use Cauchy's integral formula to compute $\int_C \frac{1}{(1+z^2)^2} dz$. $= \int_C \frac{1}{(z+i)^2 (z-i)^2} dz = 2\pi i \left(\frac{1}{(z+i)^2}\right)'$ $z=i$

1.2. Decompose $C = C_1 \cup C_2$, where C_1 denotes the segment between $-R$ and R on the x-axis, and C_2 denotes the remaining part of C. Use the triangle inequality for integrals to give an upper bound on $\left|\int_{C_2} \frac{1}{(1+z^2)^2} dz\right|$. $\ge \left|\int_0^{\pi} \frac{iRe^{i\theta}}{(1+R^2e^{2i\theta})^2} d\theta\right| \le \frac{\pi}{R^3}$

1.3. Use the results of the previous two parts to obtain an estimate $\int_{C_1} \frac{1}{(1+z^2)^2} dz$. What happens as you take $R \to \infty$?

0

2.1. *(Cauchy's inequality)* Let C_R be the circle of radius R centered at the point z_0, and suppose that f is analytic on C_R and its interior. Further, let $M_R = \max_{z \in C_R} |f(z)|$. Use Cauchy's integral formula for derivatives, and the triangle inequality for integrals to show that

$$\left|f^{(n)}(z_0)\right| \le \frac{n! M_R}{R^n}.$$

$|f^{(n)}(z_0)| \le \frac{n!}{2\pi i} \int_{C_R} \left|\frac{f(z)}{(z-z_0)^{n+1}}\right| dz$

2.2. *(Liouville's Theorem)* Now, suppose f is an entire function and $|f(z)| \le M$ for all $z \in \mathbb{C}$. By analyzing the $n = 1$ case in the previous part, what can you say about f?

$f \equiv C$

3. *(Fundamental Theorem of Algebra)* Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a degree n polynomial with $a_n \ne 0$. We will show that $P(z)$ has exactly n roots (counting multiplicities) over $\mathbb{C}$.

3.1. Assume for contradiction that $P(z) \ne 0$ for all $z \in \mathbb{C}$. Show that under this assumption, $f(z) := 1/P(z)$ is entire and bounded. Use Liouville's theorem to get a contradiction.

$a_0 \ne 0 \Rightarrow f(z) = C \Rightarrow P(z) = \frac{1}{C}$ ×

3.2. The previous part shows that P must have at least one root. Iterate it to show that P has exactly n roots (counting multiplicites).

$P(z) = (Q(z))(z - z_0)$

4. *(Mean value property)* Let C_R be the circle of radius R centered at the point z_0, and suppose that f is analytic on C_R and its interior. Use Cauchy's integral formula to show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta.$$

$z = z_0 + Re^{i\theta}$

$f(z) = f(z_0) + f'(z_0)(z - z_0)$

$$+ \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n +$$

$$\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz =$$

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$$\frac{n!}{2\pi i}$$