18.04 Problem Set 4, Spring 2018

Calendar
M Feb. 26: Finish topic 3 notes
W Feb. 28: Reading: topic 4 notes
R Mar. 1: Recitation
F Mar. 2: Reading: topic 4 notes

Coming next

Mar. 5-Mar. 9: Harmonic functions, fluid flow, Taylor series

Problem 1. (20: 5,5,5,5 points)
(a) Use Cauchy's integral formula to compute

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz, \quad 4\pi i$$

where C is the circle of radius 4: $|z| = 4$.

(b) Compute $\int_C \frac{z^2}{z^2+1}\, dz$, where C is the circle of radius 1 centered at $z = i$.

$= \int \frac{z^2}{(z+i)(z-i)}\, dz = 2\pi i \cdot \frac{(i)^2}{2i} = -\pi$

(c) Let C be the circle of radius 2: $|z| = 2$. Use Cauchy's integral formula to compute

$z = 0.5 + yi$

$$\int_C \frac{\overline{z}}{z^2 - 1}\, dz. = \int_C \frac{4}{z(z+1)(z-1)}\, dz = 2\pi i \cdot \frac{4}{-1} + 2\pi i \cdot \frac{4}{1 \times 2} + 2\pi i \cdot \frac{4}{1 \cdot 2} = 0$$

Be careful: $\overline{z}$ is not analytic, but there is a way around this.

(d) Let $\theta = \arg(z)$ Take C to be the wavy contour in the z-plane described by $0 \le \arg(z) \le \pi$; $|z| = 1 - 0.1\cos(100\theta)$. Compute the integral $\int_C z^2\, dz$.

$\frac{z^3}{3} \Big|_{0.9}^{-0.9}$

Problem 2. (15: 10,5 points)
(a) Let $f(z) = z^n$, where n is a positive integer. By directly computing the integral, show that Cauchy's integral formula holds for $f(z_0)$ and Cauchy's formula for derivatives holds for $f'(z_0)$.

You may need the binomial formula for expanding $(a+b)^n$. As a hint: you may want to make a short argument, based on Cauchy's theorem, reducing the integrals to circles centered on the point of interest.

(b) Let $P(z) = c_o + c_1 z + c_2 z^2 + c_3 z^3$. Let C be the circle $|z| = a$, for $a > 0$. Compute the integral

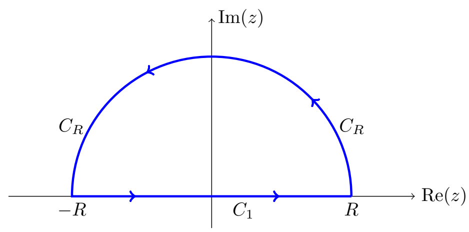
$$\int_C P(z) z^{-n} dz \text{ for n = 0, 1, 2, ...} = \int_C \left(c_0 z^{-n} + c_1 z^{1-n} + \ldots + c_3 z^{3-n}\right) dz$$

Problem 3. (15: 5,5,5 points)
(a) Compute $\int_C \frac{|z| e^z}{z^2}\, dz$ where C is the circle $|z| = 2$.

$= \int \frac{2e^z}{z^2}\, dz = 2\pi i \cdot 2e^0 = 4\pi i$

(b) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$.

Hint: integrate over the closed path shown below. Show that as R goes to infinity the contribution of the integral over C_R becomes 0.



(c) Show that $\displaystyle\int_{|z|=2} \frac{1}{z^2(z-1)^3}\,dz = 0$ in two different ways.

$= 2\pi i \cdot \frac{-3}{(0-1)^3} + 2\pi i \cdot \frac{(-2)\cdot(-3)}{2!\,1^4}$

(i) Use Cauchy's integral formula. You'll need to divide the contour to isolate each of the singularities of the integrand.

$= 0$

(ii) First, show that the integral doesn't change if you replace the contour by the curve $|z| = R$ for $R > 2$. Next, show that this integral must go to 0 as R goes to infinity.

$\leq \int_{C_R} \frac{R}{R^2(R-1)^3}\,d\theta$

Problem 4. (5 points)
Suppose f is analytic on and inside a simple closed curve C. Assume $f(z) = 0$ for z on C. Show $f(z) = 0$ for all z inside C.

$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz = 0$

Problem 5. (10 points)
Let γ be a simple closed curve that goes through the point $1 + i$. Let

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\cos(w)}{w-z}\,dw.$$

Find the following limits:
(i) $\displaystyle\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from outside γ. $= 0$
(ii) $\displaystyle\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from inside γ. $= \cos(1+i)$

Problem 6. (10: 5,5 points)
(a) Suppose that $f(z)$ is analytic on a region A that contains the disk $|z - z_0| \le r$. Use Cauchy's integral formula to prove the mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

(b) Prove the more general formula

$z = z_0 + re^{i\theta}$

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}\,d\theta.$$

Problem 7. (20: 4,4,4,4,4 points)
Let C be the curve $|z| = 2$. Explain why each of the following integrals is 0.

(a) $\displaystyle\int_C \frac{z}{z^2+35}\,dz.$ Cauchy's theorem

(b) $\displaystyle\int_C \frac{\cos(z)}{z^2-6z+10}\,dz.$

(c) $\displaystyle\int_C \mathrm{e}^{-z}(2z+1)\,dz.$

(d) $\displaystyle\int_C \log(z+3)\,dz$ (principal branch of log).

(e) $\displaystyle\int_C \sec(z/2)\,dz.$

Extra problems not to be scored. If you want someone to look at them, please turn them in separately to Jerry.

Problem 8. (0 points)

Show $\displaystyle\int_0^{\pi} \mathrm{e}^{\cos\theta}\cos(\sin(\theta))\,d\theta = \pi$. Hint, consider e^z/z over the unit circle.

$2\pi i = \displaystyle\int_{-\pi}^{\pi} \frac{e^{\cos\theta + i\sin\theta}}{e^{i\theta}} \cdot i \cdot e^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{\cos\theta}\cdot e^{i\sin\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{\cos\theta}\left(\cos(\sin\theta) + i\sin(\sin\theta)\right)d\theta$

Problem 9. (0 points)

(a) Suppose $f(z)$ is analytic on a simply connected region A and γ is a simple closed curve in A. Fix z_0 in A, but not on γ. Use the Cauchy integral formulas to show that

$$\int_\gamma \frac{f'(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz. = 2\pi i f'(z_0)$$

(b) Challenge: Redo part (a), but drop the assumption that A is simply connected.

$g(z) = \dfrac{f(z)}{z-z_0}$

Problem 10. (0 points)

Suppose $f(z)$ is entire and $\displaystyle\lim_{z\to\infty} \frac{f(z)}{z} = 0$. Show that $f(z)$ is constant.

You may use Morera's theorem: if $g(z)$ is analytic on $A - \{z_0\}$ and continuous on A, then f is analytic on A.

Problem 11. (0 points)

(a) Compute $\displaystyle\int_C \frac{\cos(z)}{z}\,dz$, where C is the unit circle. $2\pi i \cos(0)$

(b) Compute $\displaystyle\int_C \frac{\sin(z)}{z}\,dz$, where C is the unit circle. $2\pi i \sin(0)$

(c) Compute $\displaystyle\int_C \frac{z^2}{z-1}\,dz$, where C is the circle $|z|=2$. $2\pi i\ 1^2$

(d) Compute $\displaystyle\int_C \frac{\mathrm{e}^z}{z^2}\,dz$, where C is the circle $|z|=1$. $2\pi i\ e^0$

(e) Compute $\displaystyle\int_C \frac{z^2-1}{z^2+1}\,dz$, where C is the circle $|z|=2$. $2\pi i \cdot \frac{z^2-1}{z+i}\Big|_{z=i} + 2\pi i \cdot \frac{z^2-1}{z-i}\Big|_{z=-i}$

(f) Compute $\displaystyle\int_C \frac{1}{z^2+z+1}\,dz$ where C is the circle $|z|=2$. $= 0$

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