18.04 Problem Set 3, Spring 2018

Calendar
T Feb. 20: Finish topic 2 notes
W Feb. 21: Reading: Review of 18.02
R Feb. 22: Recitation
F Feb. 23: Reading: Topic 3 notes

Coming next

Feb. 26-Mar. 2: Cauchy's theorem, Cauchy's integral formula

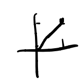
Problem 1. (30: 10,10,10 points)
(a) Compute $\int_C \frac{1}{z}\, dz$, where C is the unit circle around the point $z = 2$ traversed in the counterclockwise direction. 0

(b) Show that $\int_C z^2\, dz = 0$ for any simple closed curve C in 2 ways.
(i) Apply the fundamental theorem of complex line integrals $= \frac{z^3}{3}\Big|_{start}^{end} = 0$
(ii) Write out both the real and imaginary parts of the integral as 18.02 integrals of the form $\int_C M\, dx + N\, dy$ and apply Green's theorem to each part.

(c) Consider the integral $\int_C \frac{1}{z}\, dz$, where C is the unit circle. Write out both the real and imaginary parts as 18.02 integrals, i.e. of the form $\int_C M(x, y)\, dx + N(x, y)\, dy$. $(0,0) \in C \Rightarrow 2\pi i$ otherwise 0

Problem 2. (20: 10,10 points)
(a) Let C be the unit circle traversed counterclockwise. Directly from the definition of complex line integrals compute $\int_C \overline{z} dz$. $= \int_0^{2\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta = 2\pi i$

Is this the same as $\int_C z\, dz$? : 0

(b) Compute $\int_C \overline{z}^2\, dz$ for each of the following paths from 0 to $1 + i$. 

(i) The straight line connecting the two points. $\int_0^1 t^2 (1-i)^2 (1+i)\, dt =$

(ii) The path consisting of the line from 0 to 1 followed by the line from 1 to $1 + i$.
$\int_0^1 t^2\, dt + \int_0^1 (1-it)^2\, i\, dt$

Problem 3. (20: 10,10 points)
Let C be the circle of radius 1 centered at $z = -4$. Let $f(z) = 1/(z + 4)$. and consider the line integral

$$I = \int_C f(z)\, dz.$$

(a) Does Cauchy's Theorem imply that $I = 0$? Why or why not? $= 2\pi i$

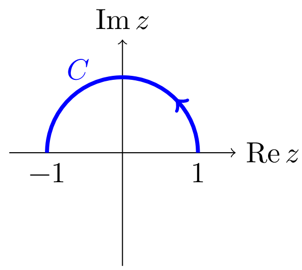
(b) Parametrize the curve C and carry out the calculation to find the value of I. Check that the answer confirms your excellent reasoning in part (a).

Problem 4. (10 points)
Let C be a path from the point $z_1 = 0$ to the point $z_2 = 1 + i$. Find

$$I = \int_C z^9 + \cos(z) - e^z \, dz$$

in the form $I = a + ib$. Justify your steps. $= \frac{z^{10}}{10}\Big|_0^{1+i} + \sin z\Big|_0^{1+i} - e^z\Big|_0^{1+i}$

Problem 5. (15: 10,5 points)
(a) Compute $\int_C z^{1/3} \, dz$, where C the unit semicircle shown. Use the principal branch of $\arg(z)$ to compute the cube root.



$\int_C z^{\frac{1}{3}} dz$

$= \frac{3}{4} z^{\frac{4}{3}}$

$= \frac{3}{4} e^{\frac{4}{3}\ln z}\Big|_1^{-1}$

(b) Repeat using the branch with $\pi \le \arg(z) < 3\pi$.

Problem 6. (10 points)
Use the fundamental theorem for complex line integrals to show that $f(z) = 1/z$ cannot possibly have an antiderivative defined on $\mathbf{C} - \{0\}$.

$= 2\pi i$

Problem 7. (10 points)
Does $\mathrm{Re}\left(\int_C f(z) \, dz\right) = \int_C \mathrm{Re}(f(z)) \, dz$? If so prove it, if not give a counterexample.

NO

Problem 8. (10 points)
Are the following simply connected?
(i) The punctured plane. N
(ii) The cut plane: $\mathbf{C} - \{\text{nonnegative real axis}\}$. Y
(iii) The part of the plane inside a circle. Y
(iv) The part of the plane outside a circle. N