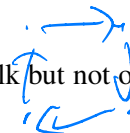


## In-Class Problems Week 7, Mon.



**Problem 1. (a)** Give an example of a digraph in which a vertex  $v$  is on a positive even-length closed walk, but *no* vertex is on an even-length cycle.

**(b)** Give an example of a digraph in which a vertex  $v$  is on an odd-length closed walk but not on an odd-length cycle.



**(c)** Prove that every odd-length closed walk contains a vertex that is on an odd-length cycle.

$$W = C_1 \oplus C_2 \oplus \dots \oplus C_n$$

$|C_2|$  odd

**Problem 2.**

In the course textbook Lemma 9.2.5 states that  $\text{dist}(u, v) \leq \text{dist}(u, x) + \text{dist}(x, v)$ . It also states that equality holds iff  $x$  is on a shortest path from  $u$  to  $v$ .

**(a)** Prove the “iff” statement from left to right.

**(b)** Prove the “iff” from right to left.

**Problem 3.**

A 3-bit string is a string made up of 3 characters, each a 0 or a 1. Suppose you’d like to write out, in one string, all eight of the 3-bit strings in any convenient order. For example, if you wrote out the 3-bit strings in the usual order starting with 000 001 010..., you could concatenate them together to get a length  $3 \cdot 8 = 24$  string that started 000001010....

But you can get a shorter string containing all eight 3-bit strings by starting with 00010.... Now 000 is present as bits 1 through 3, and 001 is present as bits 2 through 4, and 010 is present as bits 3 through 5, ....

**(a)** Say a string is *3-good* if it contains every 3-bit string as 3 consecutive bits somewhere in it. Find a 3-good string of length 10, and explain why this is the minimum length for any string that is 3-good.

000101100

**(b)** Explain how any walk that includes every edge in the graph shown in Figure 1 determines a string that is 3-good. Find the walk in this graph that determines your 3-good string from part (a).

$v = ab \rightarrow ba = v$   
 $a_1 \rightarrow b_1 = v$

**(c)** Explain why a walk in the graph of Figure 1 that includes every edge *exactly once* provides a minimum-length 3-good string.

**(d)** Generalize the 2-bit graph to a  $k$ -bit digraph,  $B_k$ , for  $k \geq 2$ , where  $V(B_k) ::= \{0, 1\}^k$ , and any walk through  $B_k$  that contains every edge exactly once determines a minimum length  $(k + 1)$ -good bit-string.

What is this minimum length?

$2^{k+1} + k$

Define the transitions of  $B_k$ . Verify that the in-degree and out-degree of every vertex is even, and that there is a positive path from any vertex to any other vertex (including itself) of length at most  $k$ .

$v = a_1 a_2 \dots a_k$

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<sup>1</sup>The 3-good strings explained here generalize to  $n$ -good strings for  $n \geq 3$ . They were studied by the great Dutch mathematician/logician Nicolaas de Bruijn, and are known as *de Bruijn sequences*. de Bruijn died in February, 2012 at the age of 94.

<sup>2</sup>Problem 9.23 explains why such “Eulerian” paths exist.

$v' = b_1 b_2 \dots b_k$   
 every step  $a_i \rightarrow b_i$   
 $i = 1, 2, \dots, k$