In-Class Problems Week 7, Mon.

Problem 1. **(a)** Give an example of a digraph in which a vertex v is on a positive even-length closed walk, but *no* vertex is on an even-length cycle.

(b) Give an example of a digraph in which a vertex v is on an odd-length closed walk but not on an odd-length cycle.

(c) Prove that every odd-length closed walk contains a vertex that is on an odd-length cycle.

$W = C_1 \oplus C_2 \oplus \dots \oplus C_n$ $|C_i|$ odd

Problem 2.
In the course textbook Lemma 9.2.5 states that $\text{dist}(u, v) \le \text{dist}(u, x) + \text{dist}(x, v)$. It also states that equality holds iff x is on a shortest path from u to v.

(a) Prove the "iff" statement from left to right.

(b) Prove the "iff" from right to left.

Problem 3.
A 3-bit string is a string made up of 3 characters, each a 0 or a 1. Suppose you'd like to write out, in one string, all eight of the 3-bit strings in any convenient order. For example, if you wrote out the 3-bit strings in the usual order starting with 000 001 010..., you could concatenate them together to get a length $3 \cdot 8 = 24$ string that started 000001010....

But you can get a shorter string containing all eight 3-bit strings by starting with 00010.... Now 000 is present as bits 1 through 3, and 001 is present as bits 2 through 4, and 010 is present as bits 3 through 5, ...

(a) Say a string is *3-good* if it contains every 3-bit string as 3 consecutive bits somewhere in it. Find a 3-good string of length 10, and explain why this is the minimum length for any string that is 3-good.

0001011100

$v = ab \to b0 = v'$

$\to b1 = v'$

(b) Explain how any walk that includes every edge in the graph shown in Figure 1 determines a string that is 3-good. Find the walk in this graph that determines your 3-good string from part (a).

(c) Explain why a walk in the graph of Figure 1 that includes every every edge *exactly once* provides a minimum-length 3-good string.[1]

(d) Generalize the 2-bit graph to a k-bit digraph, B_k, for $k \ge 2$, where $V(B_k) ::= \{0, 1\}^k$, and any walk through B_k that contains every edge exactly once determines a minimum length $(k + 1)$-good bit-string.[2]

What is this minimum length? $2^{k+1} + k$

Define the transitions of B_k. Verify that the in-degree and out-degree of every vertex is even, and that there is a positive path from any vertex to any other vertex (including itself) of length at most k.

$v = a_1 a_2 \dots a_k$

$v' = b_1 b_2 \dots b_k$

every step $a_i \to b_i$

$i = 1, 2, \dots, k$

[1] The 3-good strings explained here generalize to n-good strings for $n \ge 3$. They were studied by the great Dutch mathematician/logician Nicolaas de Bruijn, and are known as *de Bruijn sequences*. de Bruijn died in February, 2012 at the age of 94.

[2] Problem 9.23 explains why such "Eulerian" paths exist.