ASSIGNMENT #4

Please do not write your answers on a copy of this assignment. As with all assignments, there will conceptual and computational questions. For computational problems you may check your work using any tool you wish; however **you must clearly explain each step that you make in your computation.**

For this assignment I encourage you to work with others; however, you are expected to **submit your own work in your own words**. I will grade a subset of these problems and will take completion of the ungraded problems into account for the final grade of this assignment. Completion is worth 20% of the final grade of this assignment. To emphasize: **you must make an honest attempt on each problem for full points on the completion aspect of your grade.**

(1) Answer each of the following questions. Also, the following parts are not related to one another.

(a) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ under T.

(b) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$.

(c) Let $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 7 & 1 & 1 & 2 \end{bmatrix}$. Define $T : \mathbb{R}^4 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find all vectors $\mathbf{x} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = 0$.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Using what it means for T to be a linear transformation, find $T(3\mathbf{u})$, $T(2\mathbf{v})$, and $T(3\mathbf{u} + 2\mathbf{v})$.

(2) Assume $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. For each situation below find a matrix that represents T.

(a) T rotates points about the origin by $\frac{-\pi}{4}$ radians

(b) T is a vertical shear that maps $\mathbf{e}_1$ to $3\mathbf{e}_1 + 2\mathbf{e}_2$ but leaves $\mathbf{e}_2$ unchanged (i.e $T(\mathbf{e}_2) = \mathbf{e}_2$). Recall that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$

(c) T first reflects points through the vertical x_2 axis and then rotates points by $\frac{3}{2\pi}$ radians about the origin. *Hint:* You might consider each transformation one at a time, and then multiply the matrices you find.

(3) For each of the matrices below with $*$'s for entries, fill in the $*$'s with numbers that make each a true statement.

(a) $$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}.$$

(b) $$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ x_1 \\ 7x_2 \end{bmatrix}.$$

(c) Find a matrix that realizes the linear tranformation $T : \mathbb{R}^4 \to \mathbb{R}^2$, where

$$T(x_1, x_2, x_3, x_4) = (2x_1 + x_3, x_2 - x_4).$$

Hint: What you did for parts (a) and (b) may be helpful here.

(4) Let $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,, and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find all vectors in $\mathbb{R}^3$ that are fixed by A_1. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_1\mathbf{x} = \mathbf{x}$

(b) Find all vectors in $\mathbb{R}^3$ that are fixed by A_2. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_2\mathbf{x} = \mathbf{x}$

(c) Find all vectors in $\mathbb{R}^3$ that are fixed by A_3. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_3\mathbf{x} = \mathbf{x}$

(d) Use parts (a) through (c) to find all vectors in $\mathbb{R}^3$ that are fixed by A_1, A_2, A_3.

(e) Congratulate yourself! You have done a little bit of representation theory! These matrices generate the group of permutations of a three element set. This exercise is part of the steps you would take to find all irreducible representations of this permutation group (whatever that means).

(5) Find the determinants of the following matrices using cofactor expansion along the row or column of your choice.

(a) $$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

(b) $$\begin{bmatrix} 1 & 0 & 2 & 8000 \\ 11 & 0 & 1 & 121 \\ 0 & 0 & 0 & 21 \\ 12 & 0 & 21 & 90 \end{bmatrix}$$

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 7 & 1 & 0 \\ 23 & 9 & 3 \end{bmatrix}$

(d) For each of the matrices above, determine which ones represent linear transformations (via multiplication on the left) that are isomorphisms (i.e bijective funtions).

(6) Answer the following as true or false. No justification is needed.
   (a) Every matrix determines a linear transformation.
   (b) Every linear transformation is determined by a matrix.
   (c) There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is surjective.
   (d) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that is injective.
   (e) If $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations, then $T \circ S : \mathbb{R}^p \to \mathbb{R}^n$ is a linear transformation.
   (f) If T is a linear transformation given by a $n \times m$ matrix, then its codomain is $\mathbb{R}^n$.
   (g) For $n \times n$ matrices A and B, we have $\det(A + B) = \det(A) + \det(B)$.
   (h) Let A be an $n \times n$ matrix. IF $\det(A) = 0$, then two rows or two columns are the same, or a row or a column is zero. *Hint*: Consider some cocrete examples of 2×2 matrices.
   (i) Let A and P be $n \times n$ matrices. Assume that P is invertible. Then, $\det(PAP^{-1}) = \det(A)$.