Homogeneous System

Definition 0.1. A system of equation is said to be homogeneous, if it can be written as $A\mathbf{x} = \mathbf{0}$. A homogeneous system always has a solution, namely $\mathbf{0}$, which we call the trivial solution. Any other solution, if it exists is called a nontrivial solution.

1. Consider the system

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 - 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 - x_4 = 0 \end{cases}.$$

   (a) Does the system have a nontrivial solution?

   (b) Find a parametric description of its solution set.

   (c) Think of a way to rewrite your answer in (b) as a vector equation. Hint: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ be a solution, and use your answer in (b) to find something this vector is equal to.

Problem 1(c) inspires us to make the following definition.

Parametric Vector Equations

Definition 0.2. Suppose that $x_1, \dots, x_n$ be the basic variables and $t_1, \dots, t_k$ be the free variables of a system of linear equations. As we have done before, we have a parametric description of the systems solution set:

$$\begin{cases} x_1 & = a_{1,1}t_1 + \dots + a_{1,k}t_k \\ x_2 & = a_{2,1}t_1 + \dots + a_{2,k}t_k \\ \vdots & \vdots \\ x_n & = a_{n,1}t_1 + \dots + a_{n,k}t_k \\ t_1 & = \text{ free} \\ \vdots & \vdots \\ t_k & = \text{free} \end{cases}$$

where all $a_{i,j}$ are real numbers. As we did in problem 1(c), we may rewrite the parametric description above as a vector equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} a_{1,1}t_1 + \dots + a_{1,k}t_k \\ a_{2,1}t_1 + \dots + a_{2,k}t_k \\ \vdots \\ a_{n,1}t_1 + \dots + a_{n,k}t_k \\ t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix} = t_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + t_k \begin{bmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

we call the vector equation above a *parametric vector equation*

Remark 1. *I would make sure that this definition makes sense and lines ups with the work we did in problem 1(c). A definition is only as good as the examples that accompany it (don't quote me on that when I forget to include examples)!*

2. Fill in the blank: the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one ________ variable. Hint: see problem 1.

Non-Homogeneous System

Definition 0.3. A system of equation is said to be non-homogeneous, if it can be written as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$. In other words, a system of equations is said to be non-homogeneous if it is not homogeneous. **We have seen a few of these already!**

3. Come up with an example of a non-homogeneous system of equations. You don't need to solve it.

4. Come up with a non-homogeneous system of equations that does not have a solution. How is this different than homogeneous systems?

5. Consider the homogeneous system of equations

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 1 \\ -3x_1 - 2x_2 - 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 - x_4 = 2 \end{cases} .$$

   (a) Is the system consistent?

   (b) Find a parametric description of its solution set.

(c) Think of a way to rewrite your answer in (b) as a vector equation (i.e a parametric vector equation).

Hint: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ be a solution, and use your answer in (b) to find something this vector is equal to.

6. **Notice that the system in Problem 5 is very similar to the system we saw in problem 1.** What is the difference between the two systems?

It turns out that homogeneous systems and non-homogeneous systems with the same coefficient matrix have a nice relation between their solution sets. In short, once we know **all** solution of the homogeneous system and **one** solution of the non-homogeneous system, then we can find all solutions of the non-homogeneous system. This is nice since solving homogeneous equation is typically easier as we don't have to worry about how the row operations affect the last column of the augmented matrix (since this column is all zeros)! The method for finding these non-homogeneous solutions is described in much more detail in the next theorem:

Theorem 0.1. Suppose $A\mathbf{x} = \mathbf{b}$ is *consistent* with a solution $\mathbf{p}$ (it can be any solution you want). Then any solution, $\mathbf{w}$, to $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{w} = \mathbf{v}_h + \mathbf{p},$$

where $\mathbf{v}_h$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. **Warning:** the choice of $\mathbf{v}_h$ depends on $\mathbf{w}$.

7. This exercise will outline the proof of the above theorem. There are two main parts to the proof.

   (a) Suppose that $\mathbf{w} = \mathbf{v}_h + \mathbf{p}$, where $\mathbf{v}_h$ is a solution to the homogeneous system and $\mathbf{p}$ is a solution to the non-homogeneous system. Show that $\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

   (b) We aren't done yet! We still need to show that every solution to $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{v}_h + \mathbf{p}$ for some solution, $\mathbf{v}_h$, to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

      i. Show that $\mathbf{w} - \mathbf{p}$ is a homogeneous solution to $A\mathbf{x} = \mathbf{0}$.

      ii. Set $\mathbf{v}_h := \mathbf{w} - \mathbf{p}$, and conclude $\mathbf{w} = \mathbf{v}_h + \mathbf{p}$ (this should be very short).

   (c) Briefly explain why parts (a) and (b) complete the proof of the theorem.

8. Boron Sulfide reacts with water to create boric acid and hydrogen sulfide gas. We will use linear algebra to balance the following chemical equation that illustrates this reaction:

$$x_1 B_2 S_3 + x_2 H_2 O \rightarrow x_3 H_3 B O_3 + x_4 H_2 S.$$

   To do so, find whole numbers whole numbers x_1, x_2, x_3, and x_4 such that the total number of Boron (B), Sulfur (S), Hydrogen (H), and Oxygen (0) on the left matches the number on the right. Hint: Try and set up a system of linear equations.