ASSIGNMENT #7

Please do not write your answers on a copy of this assignment, use blank paper. As with all assignments, there will conceptual and computational questions. For computational problems you may check your work using any tool you wish; however **you must clearly explain each step that you make in your computation.**

For this assignment I encourage you to work with others; however, you are expected to **submit your own work in your own words**. All questions except the true and false will be graded for completion. **If you would like feedback on a problem, please indicate it somehow.** You must make an honest attempt on each problem for full points on the completion aspect of your grade.

(1) Show carefully that each of the following spaces are vector spaces. That is check that each of the 10 conditions hold.
   (a) Let $\mathrm{Diff}(\mathbb{R})$ be the space of all differentiable functions on $\mathbb{R}$ with addition as function addition and scalar multiplication as function scalar multiplication.
   (b) Let $\mathrm{Int}(\mathbb{R})$ be the space of all integrable functions on $\mathbb{R}$ with addition as function addition and scalar multiplication as function scalar multiplication.
   (c) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with entries in $\mathbb{R}$ where addition is matrix addition and scalar multiplication is matrix scalar multiplication.

(2) In class we discussed the notion of set-builder notation (page 75 in the notes). This problem will have you practice writing sets of things in set-builder notation; we will be using this often in this class, so it's important to become comfortable with it. **Write a set-builder description for each of the sets of elements below.**
   (a) The set of all integers divisible by 3.
   (b) The set of all polynomials in $\mathbb{R}[x]$ that have a zero at 1.
   (c) The set of all continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.
   (d) The set of points on the y-axis of the Cartesian xy-plane.

(3) Answer the following questions. **Justify your answers using the subspace criteria.** I know there's a lot, but this will help you develop an intuition for determining if you have a subspace or not.
   (a) Is $\mathrm{SL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid |\det(M)| = 1\}$ a subspace of $M_n(\mathbb{R})$?
   (b) Is the set $\{f \in \mathbb{R}[x] \mid \deg(f) = 5\}$ a subspace of $\mathbb{R}[x]$?
   (c) Is the set $\{f \in C(\mathbb{R}) \mid f \text{ is differentiable on } (1,2)\}$ a subspace of $C(\mathbb{R})$?
   (d) Is the set $\{(v_1, v_2, 0) \mid v_1, v_2 \in R\}$ a subspace of $\mathbb{R}^3$?
   (e) Is the set $\{(v_1, v_2, 0, 1) \mid v_1, v_2 \in R\}$ a subspace of $\mathbb{R}^4$?
   (f) Let A be any $m \times n$ matrix. Is the solution set of $A\mathbf{x} = \mathbf{0}$ a subspace of $\mathbb{R}^n$?

(g) Let A be any $m \times n$ matrix. Is the image of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ (recall T_A is multiplication by A on the left) a subspace of $\mathbb{R}^m$?

(h) Is the set of points inside and on the unit circle in $\mathbb{R}^2$ a subspace of $\mathbb{R}^2$? *Hint:* the set of points inside on the unit circle can be described as $H = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$?

(i) Is the set of all polynomials in $\mathbb{R}[x]$ that have 1 as a root a subspace fo $\mathbb{R}[x]$?

(j) Is the set $W = \{ax^2+1 \in \mathbb{R}[x] \mid A \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$? What about $V = \{ax^2 \in \mathbb{R}[x] \mid a \in \mathbb{R}\}$?

(4) For each of sets of vectors below, determine if they form a basis for the vector space they live in. If the set does not form a basis, determine if the set is at linearly independent, or a spanning set, or neither.

(a) Determine if the set of vectors in $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$.

(b) Determine if the set of vectors in $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$

(c) Determine if the set of vectors $\{x^2 + 1, x_1, 2\}$ of $\mathbb{R}_{\leqslant 2}[x]$ form a basis for $\mathbb{R}_{\leqslant 2}[x]$.

(5) Let $W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \mathbf{w}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -4 \end{bmatrix}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -6 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 6 \\ -2 \end{bmatrix}.$$

Find bases for W and V. *Hint*: you might consider determining linear dependence relations among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and for $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and getting rid of vectors that are dependent on the others.

(6) Answer the following true and false questions. No justification is needed.

(a) The set consisting of one nonzero vector of a vector space is linearly dependent.

(b) If a finite set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ span a vector space V, then some subset of S forms a basis for V.

(c) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent vectors in some vector space V, and $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for W

(d) A subset W of a vector space V is a subspace if and only if the zero vector of V is in W.

(e) $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$.