ASSIGNMENT #5

Please do not write your answers on a copy of this assignment. As with all assignments, there will conceptual and computational questions. For computational problems you may check your work using any tool you wish; however **you must clearly explain each step that you make in your computation.**

For this assignment I encourage you to work with others; however, you are expected to **submit your own work in your own words**. I will grade a subset of these problems and will take completion of the ungraded problems into account for the final grade of this assignment. Completion is worth 20% of the final grade of this assignment. To emphasize: **you must make an honest attempt on each problem for full points on the completion aspect of your grade.**

(1) Answer each of the following questions. Also, the following parts are not related to one another.

(a) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ under T.

(b) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$. Find a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$.

(c) Let $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 7 & 1 & 1 & 2 \end{bmatrix}$. Define $T : \mathbb{R}^4 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find all vectors $\mathbf{x} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = 0$.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Using what it means for T to be a linear transformation, find $T(3\mathbf{u})$, $T(2\mathbf{v})$, and $T(3\mathbf{u} + 2\mathbf{v})$.

(2) Assume $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. For each situation below find a matrix that represents T.

(a) T rotates points about the origin by $\frac{-\pi}{4}$ radians

(b) T is a vertical shear that maps $\mathbf{e}_1$ to $3\mathbf{e}_1 + 2\mathbf{e}_2$ but leaves $\mathbf{e}_2$ unchanged (i.e $T(\mathbf{e}_2) = \mathbf{e}_2$). Recall that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$

(c) T first reflects points through the vertical x_2 axis and then rotates points by $\frac{3}{2\pi}$ radians about the origin. *Hint:* You might consider each transformation one at a time, and then multiply the matrices you find.

(3) For each of the matrices below with $*$'s for entries, fill in the $*$'s with numbers that make each a true statement.

(a) $$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}.$$

(b) $$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ x_1 \\ 7x_2 \end{bmatrix}.$$

(c) Find a matrix that realizes the linear tranformation $T : \mathbb{R}^4 \to \mathbb{R}^2$, where

$$T(x_1, x_2, x_3, x_4) = (2x_1 + x_3, x_2 - x_4).$$

Hint: What you did for parts (a) and (b) may be helpful here.

(4) Let $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,, and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find all vectors in $\mathbb{R}^3$ that are fixed by A_1. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_1\mathbf{x} = \mathbf{x}$

(b) Find all vectors in $\mathbb{R}^3$ that are fixed by A_2. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_2\mathbf{x} = \mathbf{x}$

(c) Find all vectors in $\mathbb{R}^3$ that are fixed by A_3. In other words, find all $\mathbf{x} \in \mathbb{R}^3$ such that $A_3\mathbf{x} = \mathbf{x}$

(d) Use parts (a) through (c) to find all vectors in $\mathbb{R}^3$ that are fixed by A_1, A_2, A_3.

(e) Congratulate yourself! You have done a little bit of representation theory! These matrices generate the group of permutations of a three element set. This exercise is part of the steps you would take to find all irreducible representations of this permutation group (whatever that means).

(5) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Answer the following questions.

(a) Find a matrix A that represents T. In other words, find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

(b) Describe the set of vectors, $\mathbf{x}$, that satisfy $\mathbb{A}\mathbf{x} = 0$ in vector parametric notation.

(c) Does A have an inverse? If so, what is it?

(d) Is T an isomorphism? Why or why not?

(6) Answer the following as true or false. No justification is needed.

(a) Every matrix determines a linear transformation.

(b) Every linear transformation is determined by a matrix.

(c) There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is surjective.

(d) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that is injective.

(e) If $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations, then $T \circ S : \mathbb{R}^p \to \mathbb{R}^n$ is a linear transformation.

(f) If T is a linear transformation given by a $n \times m$ matrix, then its codomain is $\mathbb{R}^n$.