ASSIGNMENT #9

Please do not write your answers on a copy of this assignment, use blank paper. As with all assignments, there will conceptual and computational questions. For computational problems you may check your work using any tool you wish; however **you must clearly explain each step that you make in your computation.**

For this assignment I encourage you to work with others; however, you are expected to **submit your own work in your own words**. I will grade a subset of these problems and will take completion of the ungraded problems into account for the final grade of this assignment. Completion is worth 20% of the final grade of this assignment. To emphasize: **you must make an honest attempt on each problem for full points on the completion aspect of your grade.**

(1) For each of the linear transformations $T : V \to W$ of vector spaces below, i) give a description of $\ker(T)$, ii) find $\text{nullity}(T)$, and iii) use the rank-nullity theorem to find $\text{rank}(T)$.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$, where $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 7 \end{bmatrix}$.

(b) $T : \mathbb{R}[x]_{\leqslant 3} \to \mathbb{R}[x]_{\leqslant 2}$, where $T(f) = \frac{df}{dx}$.

(2) Below, for each basis $\mathcal{B}$ for a vector V, use $[\mathbf{x}]_{\mathcal{B}}$ to write $\mathbf{x} \in V$ as a linear combinations of elements of $\mathcal{B}$

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(b) $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}$ and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$.

(3) The set $\mathcal{B} = \{1 - x^2, x - x^2, 2 - 2x + x^2\}$ is a basis for $V = \mathbb{R}[x]_{\leqslant 2}$.

(a) Find the coordinate vector of $p(x) = 1$ with respect to $\mathcal{B}$.

(b) Find the coordinate vector of $p(x) = x$ with respect to $\mathcal{B}$.

(c) Find the coordinate vector of $p(x) = x^2$ with respect to $\mathcal{B}$.

(d) Find the coordinate vector of $p(x) = 3 + x - 6x^2$ with respect to $\mathcal{B}$.

(4) Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V. Suppose that

$$\begin{aligned}\mathbf{f}_1 &= 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3\\ \mathbf{f}_2 &= 3\mathbf{d}_2 + \mathbf{d}_3\\ \mathbf{f}_3 &= -3\mathbf{d}_1 + 2\mathbf{d}_3.\end{aligned}$$

(a) Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$

(b) Find the base-change matrix $P_{\mathcal{D}\to\mathcal{F}}$.

(c) Find the base-change matrix $P_{\mathcal{F}\to\mathcal{D}}$.

(5) Let $V = \mathbb{R}[x]_{\leqslant 2}$. The sets $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1 - 3x^2, 2 + x - 5x^2, 1 + 2x\}$ are two different bases for V.

(a) Find $P_{\mathcal{B}\to\mathcal{C}}$.

(b) Find $P_{\mathcal{C}\to\mathcal{B}}$.

(6) This is a new type of question and is open ended. Take a moment to think of an instance where using something like a "change of basis" would be a good idea. Write down what you have thought of; it does not need to be linear algebra related, but you should convince me why what you have written has the characteristics of a "change of basis".

(7) Answer the following true and false questions. You do not need to provide justification.

(a) Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be bases for a vector space V. Then $P_{\mathcal{C}\to\mathcal{D}} \cdot P_{\mathcal{B}\to\mathcal{C}} = P_{\mathcal{B}\to\mathcal{D}}$. Hint: you might consider looking at a few examples.

(b) There is a finite dimensional vector space V, of dimension n, that is not isomorphic to $\mathbb{R}^n$.

(c) The kernel of a linear transformation $T : V \to W$ is a subspace of V.

(d) The image of a linear transformation $T : V \to W$ is not a subspace of W.