Chapter 1

What Happened to my Algebra and why is it Linear?

This class will likely be very different from the majority of math classes you have taken in your academic career. In your previous classes, there was likely an emphasis on computations and less so on theory/concepts; in this class we will put equal emphasis on computations and theory/concepts. That is to say: you will be expected to ably compute examples while, at the same time, demonstrate a working knowledge of the theory. I do not mention this to alarm or frighten you. Rather, it is important for you to know that this class may diverge from your usual notion of a math class, and that you persevere in the face of difficulty. With that being said you might be wondering:

Question: Why Take this class?

Answer: The cop-out answer is that this depends on what you want to use linear algebra for (I suggest you think about this as the class progresses), but as this may be your first introduction to the subject, it might be hard to answer this. In light of this, I will do my best to provide an answer that will encompass as many backgrounds as possible. You should study linear algebra because it is useful in many different areas of study, and, most importantly, you will strengthen your problem solving abilities and being comfortable with and overcoming mathematical/intellectual hurdles. I am here to help you when things get challenging; as such I encourage you to unitize office hours. In addition, I encourage you to collaborate with and befriend with your classmates. **I want this to be an enjoyable and worthwhile endeavor for each of you!**

With all of that said, lets answer the question posed by the title of the chapter: What is Linear Algebra?

There are many correct answers to what is linear algebra, each of which depends strongly on what you will use linear algebra for; for now, we will keep in mind a down to earth answer. However, I strongly encourage you to come back periodically and think about what linear algebra means to you as you learn more about this beautiful subject.

Answer: Linear algebra is the study of n-dimensional spaces, the functions between them, and how they fit inside each other. By an n-dimensional space, I mean a 0-dimensional space is a point, a 1-dimensional space is a line, a 2-dimensional is the plane, and so on.

Now, you might be wondering:

Question: How do I succeed in this class.

Answer: To succeed in this class, you should, at minimum, do the following:

1. Study the online lecture notes and/or textbook before and after each class.

2. Come to class with questions.
3. Start homework early! It will be challenging.
4. Come to office hours. **You do not need to have questions to come to office hours; it can be a space for you to do homework with others.**
5. Talk and study with your peers.
6. Don't give up when things get hard (I can't stress this enough)!
7. Most importantly, have fun!

With all of that said, lets embark on our journey into the fantastic world of linear algebra!

Chapter 2

Systems of Linear Equations and Matrices

2.1 Systems of Linear Equations

In the last chapter, we said that linear algebra is the study of n-dimensional spaces, the functions between them, and how they fit inside of each other. Our first step toward making sense of this is through systems of linear equations. First, a couple of definitions, and then we will look at why systems of linear equations are the right thing to look at if we want to study n-dimensional spaces.

Linear Equation

Definition 2.1. A linear equation is in the variables $x_1, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where all a_i and b are complex numbers.

Example 2.1. The following are examples of linear equations:

1. $x_1 + 2x_2 = \sqrt{2}$
2. $x_2 - 8x_3 - 2x_5 = 3x_1 + 1$
3. $21x_2 + 5x_4 = 0$

Example 2.2. The following **are not** examples of linear equations:

1. $x_1^2 - x_2 = 5$
2. $e^{x_1 - x_2} + x_3 = 2$
3. $x_1x_4 - x_2x_3 = 0$
4. $\frac{1}{x_1} = 2$
5. $\sin(x_1^2 - x_2^2) = \sqrt{2}$
6. $\sqrt{x_1} + 2x_2 = \sqrt{7}$

Linear equations are very good models of n-dimensional spaces. Indeed, the graph of a linear equation in 2 variables is a line (or 1-dimensional space), the graph of a linear equation in 3 variables is a plane (or 2-dimensional space), and so on. We will want to see how these equations (spaces) interact when they are in the same ambient space; that is: we want to figure out when they intersect. This leads us to the following definition.

System of Linear Equations

Definition 2.2. A system of linear equations is a collection of one or more linear equations involving the same variables.

Example 2.3. The following are examples of systems of linear equations:

1. $\begin{cases} 4x_1 + x_2 = 7 \\ 2x_1 - 2x_2 = \sqrt{3} \\ x_2 + \sqrt{14}x_2 = 0 \end{cases}$

2. $\begin{cases} 2x_1 + x_3 = 0 \\ 2x_1 - 2x_2 = \sqrt{3} \\ x_1 - x_4 = 0 \end{cases}$

Solution

Definition 2.3. A solution to a system of linear equations is a tuple $(s_1, \ldots, s_n)$ of complex numbers that make each linear equation in the system a true statement.

Example 2.4. $(3, 2)$ is a solution for the following system of linear equations and $(0, 1)$ is not a solution.

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

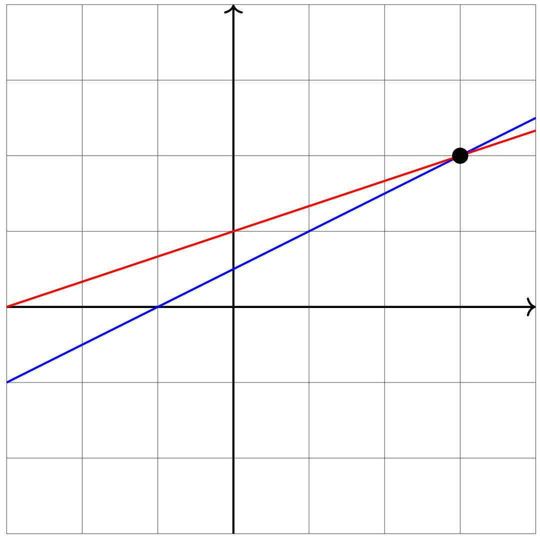
Solution Set

Definition 2.4. The set of all possible solutions to a system of linear equations is called the solution set. Two linear systems are said to be equivalent if they have the same solution sets.

Example 2.5. 1. The solution set to the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

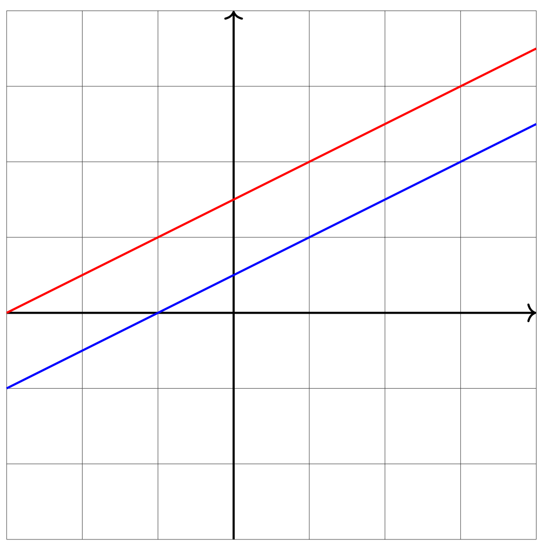
is the set {(3,2)}. Here's a geometric picture of what is going on:



2. The solution set to the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$$

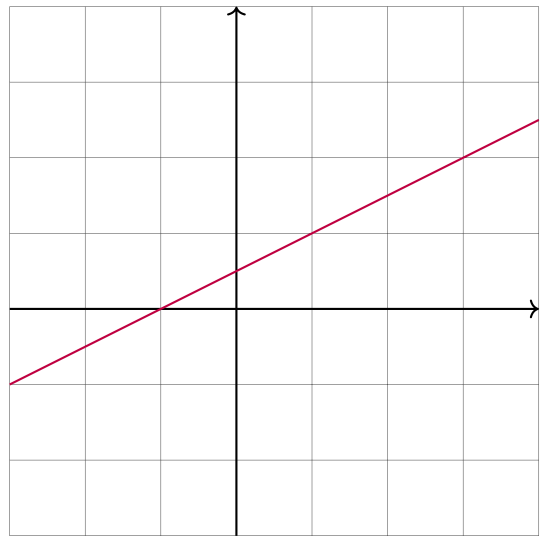
is the empty set, meaning there is no solution. Here is a geometric picture of what is going on:



3. The solution set to the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ 2x_1 - 4x_2 = -2 \end{cases}$$

is the line carved out by $x_1 - 2x_2 = -1$. Here is a geometric picture of what is going on:



These three examples illustrate the following fact:

Theorem 2.6. *The solution set to any system of linear equations in any number of variables is either*

1. *The empty set (i.e no solution).*
2. *One and only one point.*
3. *Infinitely many points.*

Proof. Postponed for now. □

Consistent and Inconsistent Systems

Definition 2.5. If a system of linear equations has at least one solution, it is called consistant. If it has no solutions, it is called inconsistant.

Using an online graphing convince yourself that Theorem 2.6 is true for systems of linear equations of three variables.

2.1.1 Matrix Notation

It is very convinient to encode information about a system of linear equations into a rectangular array called a **matrix**. There are two ways to do this, which we will demonstrate through an example.

Example 2.7. Consider the following system of linear equations

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 0 \\ x_1 + x_3 = 2 \\ -2x_1 + 2x_2 = 7 \end{cases},$$

The **coefficient matrix** of the system of linear equations is made by arranging the coefficients of the system into the following matrix

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & 2 & 0 \end{bmatrix}.$$

We will find it useful to define the **augmented matrix** of a system of linear equations by also including the values on the right handside of the equal signs:

$$\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ -2 & 2 & 0 & 7 \end{bmatrix}$$

Definition 2.6. A matrix is an $m \times n$ matrix if it has m rows (horizontal) and n columns (vertical).

So, why should we look at matrices? It turns out that the augmented matrix of a system of linear equations is sufficient information to find the solution set. We will dive more into this soon. For now, we will talk about *elementary row operations* as these will very useful tools in our venture to find solution sets for any size system of linear equations!

Definition 2.7. The *elementary row operations* are the following:

1. Replacement: replace one row by the sum of itself and a multiple of another row.
2. Interchange: Switch the position of two rows.
3. : Multiply all entries in a row by a **nonzero** constant.

Here is why we consider these operations:

Theorem 2.1. Suppose a matrix, which we will call A, is the augmented matrix for a system of linear equations. If we create a new matrix, which we will call B, by any elementary row operation (see Definition 2.1.1), then the system of linear equations represented by B has the same solution set as the original system

Proof. The proof is outlined as a bonus question on Homework 1. □

This is incredibly powerful! We can simplify an augmented matrix in such a way that we can determine whether or not the solution is consistent, as we shall now see.

Example 2.8. Consider the system of linear equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}.$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}.$$

Since elementary row operations do not change the solution set to a system of linear equations, lets simplyfy the above matrix to something more tame.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{-5R_1+R_3 \mapsto R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \xrightarrow{\frac{1}{10}R_3 \mapsto R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2 \mapsto R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -6 & 6 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{6}R_2 \mapsto R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

We may stop here (although we can go further, which we will do next time). From this new matrix, we can pick out a solution to the original system of linear equations. Indeed, row 2 tells us that $x_3 = -1$. Using this, row 3 tells us that $x_2 = 0$. Finally, row 1 tells us that $x_1 = 1$. Therefore, the original system is consistent with $(1, 0, -1)$ being a solutions.

Helpful Tip: It is helpful to think of matrices and systems of linear equations as the same thing under different guises. What we mean by this is that if you see a matrix you should think about it as being the augmented matrix of some system of equations, and, on the other hand, when you see a system of linear equations, you should think of its augmented matrix. **As with many aspects of mathematics, different points of view of the same thing is a power that cannot be overestimated.**

2.2 More Row Reduction and the Echelon Forms

In the last section we introduced elementary row operations and saw how they can aid us in solving a system of linear equations. It seems reasonable, at this point, to ask

Question: Can we find the solution set to any system of linear equations by writing down its augmented matrix and performing row operations?

Answer: Yes!

Not only can we find the solution set to any system of linear equations using row operations, but there are *always* two forms we can row reduce the augmented matrix into that yields valuable information about our system of linear equations called **Row Echelon Form (REF)** and **Reduced Row Echelon Form (RREF)**. These forms and the information they posses will be the content of this section, and

Definition 2.8.

1. A matrix is said to be in row Echelon form (REF), or simply Echelon form, if it satisfies the following properties:
   (a) All nonzero rows are above any rows consisting of all zeros.
   (b) Each leading entry **(i.e the left most nonzero entry)** of a row is in a column tot he right of the leading entry of the row above it.
   (c) All entries below a leading term are 0.
2. A matrix is said to be in row reduced Echelon form (RREF) if it satisfies the following conditions:
   (a) It satisfies all properties of being in REF.
   (b) The leading entry of each row is 1.
   (c) Each leading 1 is the only nonzero entry in its column.

If a matrix is in RREF, then it is in REF. However, if a matrix is in REF, then it may not be true that it is in RREF, as we shall see in examples 2.9 and 2.10.

Example 2.9. The following matrix is in REF but not in RREF:

$$\begin{bmatrix} 2 & 0 & 7 & 9 & 1 & 10 \\ 0 & 0 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The following matrix is not in REF:

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 2.10. The following matrix is in RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The following matrix is in REF but not RREF:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 2.2. Given any matrix A, there exists a series of row operations that put a matrix into a REF. Similarly, there is a series of row operations that put a matrix into a RREF.

Proof. We will postpone this for now. Later, we will describe an algorithm that shows this. □

Theorem 2.3. Given any matrix A, the RREF of A is **unique**. That is to say there is only one RREF we can row reduce A to.

Proof. We won't prove this; however, I encourage you to try! □

The following is a homework exercise, but as it is important, we will state it here.

A matrix A has a unique RREF by Theorem 2.2. However, a matrix A can have many different REF's. Construct an example of a matrix with multiple REF's.

Before we discuss an algorithm to row reduce a matrix to a REF or RREF, we will find it helpful to define a couple of terms.

Definition 2.9. A **pivot position** in a matrix A (not necessarily in RREF) is a location in A that corresponds to a leading 1 in the RREF of A. A pivot column of A is a column of A that contains a **pivot column**.

This is, perhaps, a strange definition; so, lets do an example and identify the pivot positions and pivot columns of a matrix.

Example 2.11. We will find all of the pivot positions and pivot columns of the following matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

Lets perform some row operations to put A into RREF:

$$\begin{bmatrix} 3 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 0 & 2 & 1 \end{bmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-2R_1+R_2 \mapsto R_2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_2 \mapsto R_2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[R_2+R_1 \mapsto R_1]{-3R_2+R_3 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

This is the RREF of A. Below are circled the leading ones, which correspond to the pivot positions of A. Also, highlighted, are the pivot columns of A.

As the above example shows **not all columns of a matrix will be pivot columns!**.

2.2.1 The Row Reduction Algorithm (Gaussian Elimination)

We will describe the algorithm of Gaussian Elimination, which yields a way to put a matrix into REF or RREF. Given a matrix A, we proceed in the following steps:

Gaussian Elimination

Algorithm 2.1.

1. Start with the left most nonzero column. Select a nonzero entry in the column to be a pivot. If necessary, interchange two rows so the pivot is at the top of the matrix.
2. Use row operations to get zeros in all positions below the pivot you found in step 1.
3. Cover the first row of the new matrix and apply 1-2 to the new matrix with the first row deleted. Keep doing this until you cannot.
4. Starting form the right most column, create zeros above each pivot.

Let's return to systems of linear equations! Given an augmented matrix, we may put it into REF or RREF, which yields information about the solution set of the system of linear equations. Before we see an example of this, we will take a brief detour through the world of solution sets. As we said in the previous section, there are systems of equations with infinitely many solutions; for these systems it would be very inefficient (and impossible) to enumerate all solutions by hand. To remedy this, we can describe all solutions using parameters (or free variables). To this end, we will find the next definition to be useful.

Definition 2.10. Suppose a matrix is A is the augmented matrix of a system of linear equations. The variables corresponding to pivot columns are called basic variables and the other variables are called free variables.

Example 2.12. We will find the solution set of the following system of linear equations by putting its augmented matrix into RREF and identifying pivot positions/columns:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 4x_3 + x_4 = 8 \\ 2x_1 + 6x_3 + 8x_4 = 4 \end{cases}.$$

Let's apply Gaussian elimination to put the augmented matrix into RREF:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 4 & 0 & 4 & 1 & 8 \\ 2 & 0 & 6 & 8 & 4 \end{bmatrix} \xrightarrow[-2R_1+R_3\mapsto R_3]{-4R_1+R_2\mapsto R_2} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & -8 & -8 & 1 & 8 \\ 0 & -4 & 0 & 8 & 4 \end{bmatrix} \xrightarrow{\frac{-1}{4}R_3\mapsto R_3} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & -8 & -8 & 1 & 8 \\ 0 & 1 & 0 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -8 & -8 & 1 & 8 \end{bmatrix} \xrightarrow{8R_2+R_3\mapsto R_3} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & -8 & -15 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{-1}{8}R_3\mapsto R_3} \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & \frac{15}{8} & 0 \end{bmatrix} \xrightarrow{-2R_3+R_1\mapsto R_1} \begin{bmatrix} 1 & 2 & 0 & \frac{-15}{4} & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & \frac{15}{8} & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1\mapsto R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & \frac{15}{8} & 0 \end{bmatrix}$$

We then see that x_1, x_2, and x_3 are basic variables for our system, and x_4 is a free variable. From the RREF we found, we find that $x_3 = -\frac{15}{8}x_4$, $x_2 = 2x_2 - 1$, and $x_1 = -\frac{1}{4}x_4 + 2$. A sufficient way to write this is to say the following is a **parametric description** of the general solution:

$$\begin{cases} x_1 - \frac{1}{4}x_4 + 2 \\ x_2 = 2x_2 - 1 \\ x_3 = -\frac{15}{8}x_4 \\ x_4 \text{ is free} \end{cases}$$

Note: sometimes people replace free variables with letters like t or s; we won't do that here, but we might in the future. Here, we see that x_4 can be any number, and it will determine what x_1, x_2, and x_3 need to be to give a solution to the system (this is why we say that x_4 is a free variable). For example, set $x_4 = 0$. Then, $x_1 = 2$, $x_2 = -1$ and $x_3 = 0$. Thus, $(2, -1, 0, 0)$ is a solution to our original solution..

The number of free variables of a system is the dimension of the solution set. To make sense of this, we need to agree on a notion of dimension, which will come up later.

We will end this section with a beautiful theorem that determines whether or not a system is consistent just by analyzing the REF of a matrix!

Theorem 2.4. A system of linear equations is consistent if and only if the right most olumn of the augmented matrix is not a pivot column. In other words, the REF of the augmented matrix has *no row* of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$$

where b is any *nonzero* number.