ASSIGNMENT #1

As with all assignments, there will conceptual and computational questions. For computational problems you may check your work using any tool you wish; however **you must clearly explain each step that you make in your computation.**

For this assignment I encourage you to work with others; however, you are expected to **submit your own work in your own words**. I will grade a subset of these problems and will take completion of the ungraded problems into account for the final grade of this assignment. Completion is worth 20% of the final grade of this assignment. To emphasize: **you must make an honest attempt on each problem for full points on the completion aspect of your grade.**

1. For each of the following, determine whether the equation is a linear equation in the variables x_1, x_2, x_3, and x_4. If it is not, explain why it is not.

(a) $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$
(b) $x_1^2 - x_4 = 2$
(c) $\sqrt{x_1 + 2x_2} - x_3 + 10x_4 = 17$
(d) $7x_1 - 2x_3 + 11x_4 = 22$

2. For each of the following collections of equations in the variables x_1, x_2, x_3, and x_4, determine whether the collection is a system of linear equations. If it is not, explain why it is not.

(a) $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 7x_1 + 2x_3 - 11x_4 = 22 \\ x_1 - x_3 = 2 \end{cases}$ (b) $\begin{cases} e^{x_1 - x_2} = 14 \\ x_2 - 5x_4 = 0 \\ \frac{1}{x_1} = 15 \end{cases}$ (c) $\begin{cases} x_1 = 2 - x_2 \\ x_3 = 3 - 2x_4 \\ x_4 = 0 \end{cases}$

3. For each of the following systems of equations in variables x_1, x_2, x_3, and x_4, write the augmented matrix corresponding to it.

(a) $\begin{cases} x_1 + x_2 - x_3 + x_4 = 0 \\ 2x_2 + x_4 - 7x_3 = 2 \end{cases}$ (b) $\begin{cases} 2x_1 = 2 - x_2 \\ x_2 - 4x_1 + x_3 = 4 \\ 2x_1 + 4x_4 - x_3 = 0 \end{cases}$

4. For each of the augmented matrices you found in problem 3, find an Echelon Form and the Reduced Echelon form for it. Use one of these two forms to determine whether or not the system is consistent. Be sure to indicate what row operations you are using at each step.

5. Consider the following system of linear equation in variables x_1, x_2, x_3, x_4, and x_5:

$$\begin{cases} x_1 - 2x_3 + 4x_5 = 0 \\ 2x_1 + 3x_2 - 9x_4 = 6 \\ x_2 - 5x_3 + x_4 - 4x_5 = 20 \end{cases}.$$

The following questions will guide you in finding the solution set to the above system.

(a) Write down the augmented matrix corresponding to the system of equations.
(b) Find a echelon form for the matrix you found in part (a). Be sure to indicate what row operations you are using at each step.
(c) Find the reduced echelon form for the matrix you found in part (a). Be sure to indicate what row operations you are using at each step.
(d) Identify the pivots and pivot columns of the matrix you found in part (a).
(e) Identify which variables of the system of equations are basic and which are free.
(f) Write down the general solution for the system of equations using a parametric description.

6. Answer the following true/false questions. No justification is required.

(a) The row echelon form of a matrix is unique.
(b) The reduced row echelon form of a matrix is uniqe.
(c) The following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(d) If two matrices are row equivalent, then their reduced row echelon forms are the same.
(e) Elementary row operations never change the solution set to a system of linear equations.
(f) An inconsistent equation has more than one solution.
(g) If a system of linear equations has a free variable, then it has a unique solution.
(h) The following matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

(i) A free variable is a variable that corresponds to a pivot column.
(j) Consider a system of linear equations in n variables and m equations with $n \geqslant m$. If the **coefficient matrix** of the system of linear equations has m pivot columns, then the system is consistent.
(k) Consider a system of linear equations in n variables consisting of m equations with $n \geqslant m$. If the **augmented matrix** of the system of linear equations has m pivot column, then the system is consistent.

7. We have seen that every system of *linear* equations has either: no solution, one solution, or infinitely many solutions. In particular, a system of *linear* equations cannot have only 2, 3, 4, ect... solutions. The purpose of this exercise is to show that this property fails for a system of equations consisting of some non-linear equations. **Fun fact:** The study of such systems of equations forms the foundation of *Classical Algebraic Geometry.* Now, the problem:

Find exactly how many solutions there are for following system of equations in the variables x and y

$$\begin{cases} xy = 1 \\ x - y = 0 \end{cases}.$$

Hint: Graph $xy = 1$ and $x - y = 0$ in the cartesian plane.

8. Briefly explain how you can efficiently determine if two matrices are row equivalent.

9. **Extra Credit:** The purpose of this question is to prove that elementary row operations do not change solutions sets to systems of equations.

   (a) Suppose $(s_1, \ldots, s_n)$ is a solution to your original system of linear equations. Show that after any elementary row operation, $(s_1, \ldots, s_n)$ is a solution to the new system of linear equations obtained by the elementary row operation. This will show that solutions remain solutions.

   (b) Show that if $(s_1, \ldots, s_n)$ is a solution to the new system of linear equations obtained by an elementary row operation, then $(s_1, \ldots, s_n)$ is a solution to the original system of linear equations. This will show that we do not get any new solutions when we perform elementary row operations.

   (c) Briefly explain why 10(a) and 10(b) show that elementary row operations do not change solutions sets to systems of equations.

   (d) Congratulate yourself! You have proven a key fact of Linear Algebra.