

(*0*)

$$\text{OP} \frac{4 \Rightarrow 4 \quad 5 \Rightarrow 5 \quad 4 + 5 \Rightarrow 9}{4 + 5 \Rightarrow 9}$$

Axiome $v \Rightarrow v$ werde ich in ab hier meistens weglassen

$$\text{APP} \frac{\text{GD} \frac{f = \text{fun } x \rightarrow 2 * x}{f \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{OP} \frac{2 * 4 \Rightarrow 8}{2 * 4 \Rightarrow 8}}{f \ 4 \Rightarrow 8}$$

(*1*)

$$\text{APP} \frac{\text{GD} \frac{f = \text{fun } x \rightarrow 2 * x}{f \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{APP} \frac{\text{GD} \frac{g = \text{fun } x \rightarrow x - 1}{g \Rightarrow \text{fun } x \rightarrow x - 1} \quad \text{OP} \frac{2 - 1 \Rightarrow 1}{2 - 1 \Rightarrow 1}}{g \ 2 \Rightarrow 1} \quad \text{OP} \frac{2 * 1 \Rightarrow 2}{2 * 1 \Rightarrow 2}}{f \ (g \ 2) \Rightarrow 2}$$

$$\text{APP} \frac{\text{GD} \frac{\text{diff} = \text{fun } x \rightarrow \text{fun } y \rightarrow x - y}{\text{diff} \Rightarrow \text{fun } x \rightarrow \text{fun } y \rightarrow x - y} \quad \text{OP} \frac{3 - 2 \Rightarrow 1}{3 - 2 \Rightarrow 1}}{\text{diff} \ 3 \ 2 \Rightarrow 1}$$

(*2*)

$$\text{APP} \frac{\text{APP} \frac{\text{GD} \frac{\text{diff} = \text{fun } x \rightarrow \text{fun } y \rightarrow x - y}{\text{diff} \Rightarrow \text{fun } x \rightarrow \text{fun } y \rightarrow x - y} \quad \text{APP} \frac{\text{GD} \frac{f = \text{fun } x \rightarrow 2 * x}{f \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{OP} \frac{2 * 5 \Rightarrow 10}{2 * 5 \Rightarrow 10}}{f \ 5 \Rightarrow 10}}{\text{diff} \ (f \ 5) \Rightarrow \text{fun } y \rightarrow 10 - y} \quad \text{OP} \frac{10 - 5 \Rightarrow 5}{10 - 5 \Rightarrow 5}}{\text{diff} \ (f \ 5) \ 5 \Rightarrow 5}$$

$$\text{LD} \frac{5 \Rightarrow 5 \quad \text{APP} \frac{\text{GD} \frac{g = \text{fun } x \rightarrow x - 1}{g \Rightarrow \text{fun } x \rightarrow x - 1} \quad \text{APP} \frac{\text{GD} \frac{f = \text{fun } x \rightarrow 2 * x}{f \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{OP} \frac{2 * 5 \Rightarrow 10}{2 * 5 \Rightarrow 10}}{f \ 5 \Rightarrow 10} \quad \text{OP} \frac{10 - 1 \Rightarrow 9}{10 - 1 \Rightarrow 9}}{g \ (f \ 5) \Rightarrow 9} \quad \text{let } x = 5 \text{ in } g \ (f \ x) \Rightarrow 9$$

$$\text{LD} \frac{\text{APP} \frac{\text{fun } a \rightarrow a + 1 \Rightarrow \text{fun } a \rightarrow a + 1 \quad \text{OP} \frac{2 + 1 \Rightarrow 3}{2 + 1 \Rightarrow 3}}{(\text{fun } a \rightarrow a + 1) \ 2 \Rightarrow 3}}{\text{let } fa = \text{fun } a \rightarrow a + 1 \text{ in } fa \ 2 \Rightarrow 3}$$

(* 3 *)

$$\begin{array}{c}
\text{GD} \frac{\text{diff} = \text{fun } x \rightarrow \text{fun } y \rightarrow x - y}{\text{diff} \Rightarrow \text{fun } x \rightarrow \text{fun } y \rightarrow x - y} \quad \text{GD} \frac{f = \text{fun } x \rightarrow 2 * x}{f \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{Op} \frac{2 * 5 \Rightarrow 10}{2 * 5 \Rightarrow 10} \\
\text{App} \frac{\text{diff} \Rightarrow \text{fun } x \rightarrow \text{fun } y \rightarrow x - y}{\text{diff} (f \ 5) \Rightarrow \text{fun } y \rightarrow 10 - y} \quad \text{App} \frac{\text{double} = f}{\text{double} \Rightarrow \text{fun } x \rightarrow 2 * x} \quad \text{GD} \frac{g = \text{fun } x \rightarrow x - 1}{g \Rightarrow \text{fun } x \rightarrow x - 1} \quad \text{Op} \frac{2 - 1 \Rightarrow 1}{2 - 1 \Rightarrow 1} \quad \text{Op} \frac{2 * 1 \Rightarrow 2}{2 * 1 \Rightarrow 2} \\
\text{App} \frac{\text{diff} (f \ 5) \Rightarrow \text{fun } y \rightarrow 10 - y}{\text{diff} (f \ 5) (\text{double}(g \ 2)) \Rightarrow 8} \quad \text{App} \frac{\text{double} (g \ 2) \Rightarrow 2}{\text{double} (g \ 2) \Rightarrow 2} \quad \text{Op} \frac{10 - 2 \Rightarrow 8}{10 - 2 \Rightarrow 8}
\end{array}$$

$$\begin{array}{c}
\text{App} \frac{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \Rightarrow (\text{fun } x \rightarrow \text{fun } y \rightarrow x + y)}{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \ 5 \Rightarrow \text{fun } y \rightarrow 5 + y} \quad \text{Op} \frac{5 + 2 \Rightarrow 7}{5 + 2 \Rightarrow 7} \\
\text{App} \frac{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \Rightarrow (\text{fun } x \rightarrow \text{fun } y \rightarrow x + y)}{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) ((\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \ 5 \ 2) \Rightarrow \text{fun } y \rightarrow 7 + y} \quad \text{Op} \frac{7 + 8 \Rightarrow 15}{7 + 8 \Rightarrow 15} \\
\text{App} \frac{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) ((\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \ 5 \ 2) \Rightarrow \text{fun } y \rightarrow 7 + y}{(\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) ((\text{fun } x \rightarrow \text{fun } y \rightarrow x + y) \ 5 \ 2) \ 8 \Rightarrow 3} \\
\text{LD} \frac{\text{let } fb = \text{fun } x \rightarrow \text{fun } y \rightarrow x + y \text{ in } fb \ (fb \ 5 \ 2) \ 8 \Rightarrow 15}{}
\end{array}$$

$$\begin{array}{c}
\text{LD} \frac{3 \Rightarrow 3}{\text{let } x = 3 \text{ in let } x = \text{fun } x \rightarrow x \text{ in } x \ 2 \Rightarrow 2} \quad \text{LD} \frac{(\text{fun } x \rightarrow x) \Rightarrow (\text{fun } x \rightarrow x)}{(\text{fun } x \rightarrow x) \ 2 \Rightarrow 2} \quad \text{App} \frac{(\text{fun } x \rightarrow x) \Rightarrow (\text{fun } x \rightarrow x)}{(\text{fun } x \rightarrow x) \ 2 \Rightarrow 2}
\end{array}$$

Sei $e_{match} = \text{match } x \text{ with } | \ 0 \rightarrow \square \mid x \rightarrow x :: \text{init } (x-1)$

Sowie $\pi = \frac{\text{init} = \text{fun } x \rightarrow e_{match}}{\text{init} \Rightarrow \text{fun } x \rightarrow e_{match}}$ GD

$$\begin{array}{c}
\pi \text{ Op} \frac{1 - 1 \Rightarrow 0 \quad 0 \Rightarrow 0 \equiv 0 \quad \square \Rightarrow \square}{1 - 1 \Rightarrow 0 \quad e_{match}[0/x] \Rightarrow \square} \text{ PM} \\
\pi \text{ Op} \frac{1 \Rightarrow 1 \quad \text{init } (1 - 1) \Rightarrow \square}{2 :: \text{init } (1 - 1) \Rightarrow [1] \equiv 1 :: \square} \text{ App} \\
\pi \text{ Op} \frac{2 - 1 \Rightarrow 1 \quad 1 \Rightarrow 1 \equiv x[1/x]}{2 - 1 \Rightarrow 1 \quad e_{match}[1/x] \Rightarrow [1]} \text{ PM} \\
\pi \text{ Op} \frac{2 \Rightarrow 2 \quad \text{init } (2 - 1) \Rightarrow [1]}{3 - 1 \Rightarrow 2 \quad 2 \Rightarrow 2 \equiv x[2/x]} \text{ App} \\
\pi \text{ Op} \frac{3 - 1 \Rightarrow 2 \quad 2 \Rightarrow 2 \equiv x[2/x]}{3 - 1 \Rightarrow 2 \quad e_{match}[2/x] \Rightarrow [2; 1]} \text{ L} \\
\pi \text{ Op} \frac{3 \Rightarrow 3 \quad \text{init } (3 - 1) \Rightarrow [2; 1]}{3 \Rightarrow 3 \quad e_{match}[3/x] \Rightarrow [3; 2; 1]} \text{ PM} \\
\pi \text{ Op} \frac{3 \Rightarrow 3 \quad \text{init } (3 - 1) \Rightarrow [2; 1]}{3 \Rightarrow 3 \quad e_{match}[3/x] \Rightarrow [3; 2; 1]} \text{ App} \\
\pi \text{ Op} \frac{3 \Rightarrow 3 \quad \text{init } (3 - 1) \Rightarrow [2; 1]}{3 \Rightarrow 3 \quad e_{match}[3/x] \Rightarrow [3; 2; 1]} \text{ L} \\
\pi \text{ Op} \frac{3 \Rightarrow 3 \quad \text{init } (3 - 1) \Rightarrow [2; 1]}{3 \Rightarrow 3 \quad e_{match}[3/x] \Rightarrow [3; 2; 1]} \text{ PM} \\
\pi \text{ Op} \frac{3 \Rightarrow 3 \quad \text{init } (3 - 1) \Rightarrow [2; 1]}{3 \Rightarrow 3 \quad e_{match}[3/x] \Rightarrow [3; 2; 1]} \text{ App}
\end{array}$$