Question 1. Assign 2

Primal form of hard-margin SVM:

$$\min_{w,b} \frac{1}{2}\|w\|_2^2 \quad s.t. \quad y_i(w^T x_i + b) \geq 1, \forall i$$

$w \in R^n$ is the weight vector, $b \in R$ is the bias term.

minimize $\frac{1}{2}\|w\|^2$ to maximize the margin: $\rho = \frac{1}{\|w\|_2}$

Derivation of the dual form:

1. Formulate the Lagrangian: Introduce Lagrage multipliers $\alpha_i \geq 0$ for each constraint $y_i(w^T x_i + b) \geq 1$

$$L(w, b, \alpha) = \frac{1}{2}\|w\|^2 - \sum_{i=1}^{m} \alpha_i [y_i(w^T x_i + b) - 1]$$

2. Find the derivatives with respect to w and b:

$$\frac{\partial L}{\partial w} = w - \sum_{i=1}^{m} \alpha_i y_i x_i = 0 \implies w = \sum_{i=1}^{m} \alpha_i y_i x_i$$

$$\frac{\partial L}{\partial b} = -\sum_{i=1}^{m} \alpha_i y_i = 0 \implies \sum_{i=1}^{m} \alpha_i y_i = 0$$

3. Substitude w and b back into the Lagrangian:

$$L = \frac{1}{2}\left(\sum_{i=1}^{m} \alpha_i y_i x_i\right)^T \left(\sum_{i=1}^{m} \alpha_i y_i x_i\right) - \sum_{i=1}^{m} \alpha_i \left[y_i\left(\sum_{i=1}^{m} \alpha_i y_i x_i^T x_i + b\right) - 1\right]$$

$$= \boxed{\frac{1}{2}\sum\sum \alpha_i \alpha_j y_i y_j x_i x_j - \sum\sum \alpha_i \alpha_j y_i y_j x_i x_j} - \cancel{\sum \alpha_i y_i b} + \sum \alpha_i$$

$$= \sum \alpha_i - \frac{1}{2}\sum\sum \alpha_i \alpha_j y_i y_j x_i x_j$$

Dual form of hard-margin SVM:

$$\max_{\alpha_i}\left[\sum_i \alpha_i - \frac{1}{2}\sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i x_j\right]$$

$$s.t. \quad \alpha_i \geq 0, \forall i$$

$$\sum_i \alpha_i y_i = 0$$