Lecture 1.

References

1. Introduction to mathematical logic by Jerome Malitz
2. Logic for computer Scientist, by Uwe Schöning.

Def. 1. A grammatical sentence is called a sentence in logical point of view if it is true or false

We use notation to denote the sentences:

$p, q, r, s, t, \dots$

The symbol $w(p)=1$ it means that the sentence p is true and $w(p)=0$ it means that the sentence p is false

To get the new sentences we use the logical operators:

- $\wedge$ – conjunction
- $\vee$ – alternative
- $\Rightarrow$ – implication
- $\Leftrightarrow$ equivalence
- $\neg$ ($\sim$) negation

If p, q are sentences we can create the new ones

$p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$, $\sim p$

$p \Rightarrow q$

predecessor

successor

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

p	$\sim p$
0	1
1	0

Def 2. (Logical law / tautology) An expression consisting of sentences, brackets and logical operators is logical law (tautology) if it stands always true for every sentences.

Some logical laws

1° $p \vee q \Leftrightarrow q \vee p$ } commutativity law
$p \wedge q \Leftrightarrow q \wedge p$ }

2° $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ associativity "
$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

3° $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$ distributivity
$(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$

4° $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ } de'Morgan law
$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ }

5 $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$
$p \Rightarrow q \Leftrightarrow \sim p \vee q$

6° $(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow r)$
$\sim(p \Rightarrow q) \Leftrightarrow p \wedge \sim q$

Proof: 5 $p \Rightarrow q \Leftrightarrow \sim p \vee q$ (*)

p	q	$p \Rightarrow q$	$\sim p \vee q$	*
1	1	1	1	1
0	1	1	1	1
1	0	0	0	1
0	0	1	1	1

The methods of prooving

1° $p \Rightarrow q$ direct proof

Ex If n and m are even numbers then $n+m$ is even number, either

2° a) $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$ } A proof by
b) $p \Rightarrow q \Leftrightarrow p \wedge \sim q \Rightarrow s,\ w(s)=0$ } contradiction

Ex. a) if $\frac{x+y}{2} > 1 \Rightarrow x > 1 \vee y > 1$

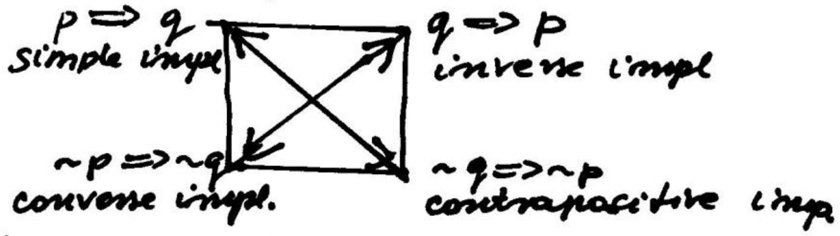
b) $\sqrt{2}$ is irrational number

3° $p \vee q \Rightarrow s \Leftrightarrow p \Rightarrow s \vee q \Rightarrow s$ – exausting the cases

Ex. $|x| \geq x$

$x \geq 0 \Rightarrow |x| = x \geq x$, $x < 0 \Rightarrow \cancel{|x| = -x}\ x < 0 \leq |x|$

The logical square



Theorem $A \Leftrightarrow B$

Necessity $A \Rightarrow B$ Sufficiency: $B \Rightarrow A$

B is neccessary for B B is sufficied for B

Ex. If a positive integer n is divided over 4 then is divided over 2

Sets

The sets are denoted by the big letters $A, B, C, \dots,$ and elements of set by small letters $a, b, c, x, \dots$ the fact that x is an element of A we denote by $x \in A$ and the fact that x is not element of A by $\sim(x \in A)$ or $x \notin A$.

The equality rule.

The sets A and B are equall if and only if they consists of the same elements:

$A = B \Leftrightarrow$ for every x $(x \in A \Leftrightarrow x \in B)$

Def 2 We shall say that A is contained in B if and only if for every x

$x \in A \Rightarrow x \in B$

Theorem 1. $A = B \Leftrightarrow A \subset B \wedge B \subset A$

If $A \subset B$ and $A \neq B$ we denote $A \subsetneq B$

Def 3. A set which has no elements is called the empty set. (denoted by ϕ)

Property 1: $\phi \subset A$ for every set A.

If $A \subset X$ then ~~the set~~ we say that A is a subset of X. The family of all subsets of ~~X~~ is denoted by 2^X or $P(A)$

Remark: The axioms of set theory say that there exist an empty set and for every X the set 2^X called the power set.

Operations on sets

$\cup$ union, $\cap$ intersection, $-$ difference, Δ — symmetrical difference

$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$

$A \cup B$

$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$

$A \cap B$

$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

$A - B$

$A \Delta B = (A - B) \cup (B - A)$

$A \cap B = \phi$ — then A is disjoint from B

If $A \subset X$ then the set $X - A$ is called the complement A with respect to X and is denoted by A', then for every $x \subset X$ we have that $x \in A' \Leftrightarrow x \notin A$

Theorem 2. For every set $A, B, C \subset X$

1° $A \cup B = B \cup A$, $A \cap B = B \cap A$ } commutativity

2° $(A\cup B)\cup C = A\cup(B\cup C)$, $(A\cap B)\cap C = A\cap(B\cap C)$ } associativity

3° $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$, $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ } distributivity

4° $A\cup A' = X$, $A\cap A' = \emptyset$

5 $(A')' = A$

6 $(A\cup B)' = A'\cap B'$, $(A\cap B)' = A'\cup B'$ } de Morgan laws

7. $A\cup\emptyset = A$

8. $A\cap X = A$

proof 6. $x\in(A\cup B)' \Leftrightarrow \sim x\in(A\cup B) \Leftrightarrow \sim(x\in A \vee x\in B) \Leftrightarrow x\notin A \wedge x\notin B \Leftrightarrow x\in A' \wedge x\in B' \Leftrightarrow x\in A'\cap B'$.

Remark: If A has finite number of elements we use notation $A=\{a,b,x,y\}$

Ex. Using only the empty set find the set which consists of 1 element, 2 elements, 3-elements, ...

Theorem 3. For every set $A, B, C, D\subset X$ we have

1° $A\subset A\cup B$

2° $A\cap B\subset A$

3° $A\subset B \Rightarrow A'\subset A'$

4° $A\subset B \wedge C\subset D \Rightarrow A\cup C\subset B\cup D$

5° $A\subset B \wedge C\subset D \Rightarrow A\cap C\subset B\cap D$

6° $A\subset B \wedge C\subset D \Rightarrow A\setminus D\subset B\setminus C$

Theorem 4. the following conditions are equivalent for every sets A and B

a) $A\subset B$

b) $A\cap B = A$

c) $A\cup B = B$

Theorem 5 (The properties of the symmetrical difference)

1° $A \triangle A = \emptyset \iff A = B$

2° $A \triangle B = B \triangle A$

3° $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

Theorem 6 For an arbitrary a and b there exists an unique set consisting only from a and b. It is denoted by $\{a, b\}$

If $a = b$ then $\{a, b\} = \{a\}$

Definition. The set consisting from $\{a, b\}$, $\{b\}$, it means $\{\{a, b\}, \{b\}\}$ is called an ordered pair and is denoted by $\langle a; b \rangle$. a is called predeccessor an b successor

Def 4. If $A, B \neq \emptyset$, then the cartesian product of the sets A and B is the set $\{\langle a, b \rangle : a \in A, b \in B\}$ and is denoted by $A \times B$. If $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$

Theorem. $\langle a, b \rangle = \langle c, d \rangle \iff a = c \wedge b = d$.

theorem 7 For evey $A, B, \subset X$ and $C \subset Y$.

a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

b) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

c) $(A - B) \times C = (A \times C) \cap ((X - B) \times C)$

Remark: It is not true that

$$A \times B = B \times A$$

$A = \{1, 2\}$ $B = \{3\}$ $A \times B = \{\langle 1, 3 \rangle, \langle 2, 3 \rangle$

$B \times A = \{\langle 3, 1 \rangle, \langle 3, 2 \rangle\}$. $(A \times B \cap (B \times A)) = \emptyset$.

Mathematical induction

Let $\varphi(n)$ be an expression which is becoming a sentence if n denot the natural number.

Let us assume, that

1° $\varphi(1)$ is true

2° if $\varphi(n)$ is true then $\varphi(n+1)$ is true, as well for every $n \in N$.

Then $\varphi(n)$ is true for every $n \in N$

Exercise: Prove by mathematical induction that

1) $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1) \quad n \in N$

2) $2^n > n \quad n \in N.$