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A NEW METHOD FOR OBTAINING HIGHLY ACCURATE APPROXIMATIONS TO THE ARCTAN FUNCTION

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INTRODUCTION:

Computing the arctangent is needed in many applied fields, for example, measuring distance on the ellipsoid for navigation, calculating phase in digital signal processing and determining angles in computer graphics. Each of these applications requires that the simple definition of the tangent on a right triangle be inverted. However, this is difficult as the arctangent exhibits a highly non-linear relationship with angle. Methods to calculate the arctangent are usually based upon summation of a Taylor series. This summation is inconvenient because the series converges slowly outside of a narrow range ($x \ll 1$). The conventional approach performs range reduction and attempts to shorten the series using minimax methods. Nevertheless, for high accuracy, many terms with lengthy coefficients must be summed (Hart et al, 1968 and Muller, 2005). Even in the last 10 years, there has been only marginal progress on computing arctan. Medina found a somewhat shorter series using Hermite polynomials (Medina, 2006). Other workers have sidestepped the convergence issue by deriving the arctangent from the arcsine (Markstein, 2005) or using brute force computer power to iterate to the solution using either the Cordic (Muller, 2005) or KDF9 methods (Findlay, 1964). Recent work for situations with thousands of measurements that need to be efficiently inverted has focused on low degree but inaccurate formulas covering the entire range [0,1] with just a few simple coefficients (Rajan et al, 2006). Some of these formulas try to avoid division which is a costly instruction in a CPU. In summary, previous methods have not been entirely satisfactory and the atan2 function as implemented in a computer library is slow and expensive.

It is the purpose of this note to introduce a new approach for finding highly accurate approximations for arctan(1/a) valid for the entire range $-\infty<a<+\infty$ suited for computer sub-routines. We also demonstrate a multipoint form of range reduction that enables us to calculate any arctangent starting from a nearby argument for which the arctangent is known.

NEW ARCTAN APPROXIMATION

Although very accurate values for arctan(x) expressed in fractions of π are known for x=0, ±1/sqrt(3), ±1, and ±sqrt(3) , other points in the range -∞<x<∞ rely on tables or computer sub-routines based on the basic series expansion

$$\arctan(x)=\sum_{k=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots \quad \textbf{(1)}$$

and the identity

$$\arctan(x)+\arctan(1/x)=\pi/2 \quad (2)$$

Since arctan(x) is an odd function one has arctan(-x)= -arctan(x) and so one needs to only find values for x>0 to know the value of arctan(x) over the entire range [-∞,∞]. The problem with the series expansion shown in (1) is that it converges very slowly because it lacks a factorial term in the denominator of the series. This makes it difficult to obtain high precision for arbitrary values of x with standard computer sub-routines.

We show here a new approach to obtaining very accurate approximations to arctan (x) based on an integral evaluation method involving Legendre polynomials. The procedure, already outlined earlier (Kurzweg, 2009 and Kurzweg, 2011), is to start with the basic definition

$$F(a)=(1/a)\arctan(1/a)=\int_{t=0}^{1}\frac{dt}{t^2+a^2} \quad (3)$$

and then look at the related integral

$$I(n,a)=\int_{t=0}^{1}\frac{P_{2n}(t)}{(t^2+a^2)}dt \quad (4)$$

Here $P_{2n}(t)$ represents the even Legendre Polynomials which can be defined by the Rodrigues Formula as

$$P_{2n}(t)=\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dt^{2n}}(t^2-1)^{2n} \quad (5)$$