Chebyshev polynomials

Chebyshev polynomials of the first kind are solutions to differential equation:

$$(1-x^2)y'' - xy' + n^2 y = 0. \tag{1}$$

They are defined by recursion formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{with} \quad T_0(x) = 0 \quad \text{and} \quad T_1(x) = x. \tag{2}$$

On an interval $[-1, 1]$ they form a complete basis set, similar to $\cos(x)$ and $\sin(x)$ for $x \in [-\pi, \pi]$. Of course one can expand these intervals to arbitrary intervals $[-a, a]$ by a simple transformation $x \to x/a$. Basis polynomials $T_n(x)$ are orthogonal with respect to the following scalar product:

$$\langle T_m(x)|T_n(x)\rangle = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\mathrm{d}x = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

or, written for arbitrary interval $[-a, a]$:

$$\langle T_m(x/a)|T_n(x/a)\rangle = \int_{-a}^{a} \frac{T_m(x/a)T_n(x/a)}{\sqrt{1-(x/a)^2}}\frac{\mathrm{d}x}{a} = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

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One can now expand sin and cos functions into series on an interval $[-\pi, \pi]$ using transformation $x \to x/\pi$:

$$cos(x) = \sum_n a_n T_n\left(\frac{x}{\pi}\right). \tag{3}$$

Multiplying both sides by $\frac{1}{\pi}T_m(\frac{x}{\pi})/\sqrt{1-(x/\pi)^2}$ and integrating over the interval $[-\pi, \pi]$ yields for $m \neq 0$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(x)T_m(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}}\mathrm{d}x = \frac{\pi}{2}a_m \tag{4}$$

and for $m = 0$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(x)T_0(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}}\mathrm{d}x = \pi a_0. \tag{5}$$

Using $x/\pi = cos(z)$ in equations 6 and 7 leads to:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(\pi\cos(z))T_m(\cos(z))}{\sqrt{1-\cos^2(z)}}\mathrm{d}(\pi\cos(z)) \tag{6}$$

and for $m = 0$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(\pi\cos(z))T_0(\cos(z))}{\sqrt{1-\cos^2(z)}}\mathrm{d}(\pi\cos(z)) \tag{7}$$

To simplify the integral calculation we use relation between $T_n(x)$ and $\cos(x)$:

$$T_n(\cos\theta) = \cos n\theta, \tag{8}$$

which yields:

$$a_m = \frac{2\pi}{\pi^2} \int_{-1}^{1} \frac{\cos(\pi \cos z) \cos(mz)}{\sqrt{1 - \cos^2 z}} \mathrm{d}(cosz) = -\frac{2}{\pi} \int_{\pi}^{0} \cos(\pi \cos z) \cos(mz) \mathrm{d}z, \tag{9}$$

where we took into account that $\cos z = -1$ when $x = -\pi$ and 1 when $x = \pi$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi^2} \int_{-1}^{1} \frac{\cos(\pi \cos z)}{\sqrt{1 - \cos^2 z}} \mathrm{d}(\pi cos\theta) = -\frac{1}{\pi} \int_{\pi}^{0} \cos(\pi \cos z) \mathrm{d}z. \tag{10}$$

Chebyshev polynomial can be divided into even and odd polynomials. For even n polynomial is even and for odd n polynomial is odd. Thus the integral $\int \cos(x) T_n(x\pi)$ is zero for all odd n. Only even n polynomials will participate in expansion of $\cos(x)$ into a series using Chebyshev polynomials. The opposite happens when we expand $\sin(x)$ into such series. Therefore only odd n polynomials will participate in series expansion. For $m \neq 0$ we get:

$$a_m = \frac{2}{\pi} \int_{0}^{\pi} \sin(\pi \cos z) \cos(mz) \mathrm{d}z, \tag{11}$$

where we took into account that $\cos z = -1$ when $z = \pi$ and 1 when $z = -\pi$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} \sin(\pi \cos z) \mathrm{d}z. \tag{12}$$

1 Legendre polynomials

Legendre polynomials are solutions to a differential eqaution:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[(1 - x^2) \frac{\mathrm{d}P_n(x)}{\mathrm{d}x} + n(n+1) P_n(x) \right] = 0. \tag{13}$$

They can also be obtained from a taylor series:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \tag{14}$$

Legendre polynomials are defined by equation:

$$P_0 = 1 \quad P_1 = 1 \quad P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x). \tag{15}$$

They form an orthonormal basis set on the interval $[-1, 1]$ with respect to the following dot product:

$$\langle P_i, P_j \rangle = \int_{-1}^{1} P_i(x) P_j(x) \mathrm{d}x = \begin{cases} 0 & i \neq j \\ \frac{2}{2i+1} & \text{otherwise} \end{cases}$$

2 Laguerre polynomials

Laguerre poylnomials are a solution to the following differential equation:

$$xy'' + (1-x)y' + ny = 0, \tag{16}$$

where n is non-negative integer. The generating function for these polynomials is:

$$\frac{1}{1-t}e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x). \tag{17}$$

The polynomials can be calculated using derivation:

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}x^n\right) \tag{18}$$

or recursive relation:

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}. \tag{19}$$

They are orthonormal with respect to the following scalar product:

$$\langle L_k(x), L_j(x)\rangle = \int_0^{\infty} L_k(x)L_j(x)e^{-x}\mathrm{d}x = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

3 Tan approximation

For numerical tangens calculation is best to use integrals involving Legendre polynomials:

$$I(n,a) = \int_{t=0}^{t=1} P_{2n}(t)\cos(at)\mathrm{d}t = N(n,a)\sin(a) + M(n,a)\cos(a) \tag{20}$$

and

$$J(n,a) = \int_{t=0}^{t=1} P_{2n+1}(t)\sin(at)\mathrm{d}t = K(n,a)\sin(a) + L(n,a)\cos(a), \tag{21}$$

where M,N,K,L are polynomials. Their power depends on parameter n. P are Legendre polynomials. As $n \to \infty$ and a stays small, integrals go to zero and we get the following approximations:

$$\tan(a) \approx \frac{M(n,a)}{N(n,a)} = C(n,a) \quad \text{and} \quad \tan(a) \approx \frac{L(n,a)}{K(n,a)} = S(n,a). \tag{22}$$

Using the above expression we can obtain for increasing n the following expressions:

$$C(1,a) = \frac{3a}{3-a^2} \tag{23}$$

$$S(1,a) = \frac{15a - a^3}{15 - 6a^2} \tag{24}$$

$$C(2,a) = \frac{105a - 10a^3}{105 - 45a^2 + a^4} \tag{25}$$

$$S(2,a) = \frac{945a - 105a^3 + a^5}{945 - 420a^2 + 15a^4} \tag{26}$$

$$C(3,a) = \frac{10395a - 1260a^3 + 21a^5}{10395 - 4725a^2 + 210a^4 - a^6} \tag{27}$$

$$S(3,a) = \frac{135135a - 17325a^3 + 378a^5 - a^7}{135135 - 62370a^2 + 3150a^4 - 28a^6} \tag{28}$$

$$C(4,a) = \frac{2027025a - 270270a^3 + 6930a^5 - 36a^7}{2027025 - 945945a^2 + 51975a^4 - 630a^6 + a^8} \tag{29}$$

$$S(4,a) = \frac{34459425a - 4729725a^3 + 1351355a^5 - 990a^7 + a^9}{34459425 - 16216200a^2 + 945945a^4 - 138600a^6 + 45a^8} \tag{30}$$

$$S(5,a) = \frac{3604742431983t - 600790405215t^3 + 30039519510t^5 - 715224510t^7 + 9930635t^9 - 88179t^{11}}{33\,(29393t^{10} - 2708355t^8 + 151714290t^6 - 4551442350t^4 + 54617309565t^2 - 109234619151)} \tag{31}$$

This library implements S(4,a) routine to calculate tangens.

4 ArcTan approximation

To calculate arctan we start with the basic definition:

$$F(a) = \frac{1}{a}\arctan(1/a) = \int_{t=0}^{t=1} \frac{\mathrm{d}t}{t^2 + a^2} \tag{32}$$

and then look at the formula:

$$I(n,a) = \int_{t=0}^{t=1} \frac{P_{2n}(t)\mathrm{d}t}{t^2 + a^2}, \tag{33}$$

where P_{2n} represents the even Legendre polynomial which can be defined by Rodrigues formula:

$$P_{2n}(t) = \frac{1}{2^{2n}(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}(t^2 - 1)^{2n}. \tag{34}$$

If one then expands equation 33 one finds:

$$I(n,a) = \int_{t=0}^{t=1} Q(n,a,t)\mathrm{d}t + M(n,a)F(a), \tag{35}$$

where Q(n,a,t) is a polynomial obtained by dividing $P_{2n}(t)$ by t^2+a^2. The integration then produces the following identity:

$$I(n,a) = N(n,a) + M(n,a)F(a). \tag{36}$$

As in the case of tangent, when n gets large and $a \ll n$, $I(n,a)$ approaches zero. Thus, for large n one can write $I(n,a) = 0$ and obtain:

$$F(a) = \frac{1}{a}\arctan(1/a) = -\frac{N(n,a)}{M(n,a)} \tag{37}$$

and further:

$$\arctan(1/a) \approx -a\frac{N(n,a)}{M(n,a)} \rightarrow \arctan(a) \approx -\frac{N(n,1/a)}{aM(n,1/a)} \tag{38}$$