Chebyshev polynomials

Chebyshev polynomials of the first kind are solutions to differential equation:

$$(1-x^2)y'' - xy' + n^2 y = 0. \tag{1}$$

They are defined by recursion formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{with} \quad T_0(x) = 0 \quad \text{and} \quad T_1(x) = x. \tag{2}$$

On an interval $[-1, 1]$ they form a complete basis set, similar to $\cos(x)$ and $\sin(x)$ for $x \in [-\pi, \pi]$. Of course one can expand these intervals to arbitrary intervals $[-a, a]$ by a simple transformation $x \to x/a$. Basis polynomials $T_n(x)$ are orthogonal with respect to the following scalar product:

$$\langle T_m(x)|T_n(x)\rangle = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\mathrm{d}x = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

or, written for arbitrary interval $[-a, a]$:

$$\langle T_m(x/a)|T_n(x/a)\rangle = \int_{-a}^{a} \frac{T_m(x/a)T_n(x/a)}{\sqrt{1-(x/a)^2}}\frac{\mathrm{d}x}{a} = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

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One can now expand sin and cos functions into series on an interval $[-\pi, \pi]$ using transformation $x \to x/\pi$:

$$cos(x) = \sum_n a_n T_n\left(\frac{x}{\pi}\right). \tag{3}$$

Multiplying both sides by $T_m(\frac{x}{\pi})/\sqrt{1-(x/\pi)^2}$ and integrating over the interval $[-\pi, \pi]$ yields for $m \neq 0$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(x)T_m(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}}\mathrm{d}x = \frac{\pi}{2}a_m \tag{4}$$

and for $m = 0$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(x)T_0(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}}\mathrm{d}x = \pi a_0. \tag{5}$$

To simplify integral calculation we use relation between $T_n(x)$ and $\cos(x)$:

$$T_n(\cos\theta) = \cos n\theta. \tag{6}$$

Using $x/\pi = cos(z)$ in equations leads to:

$$a_m = \frac{2}{\pi^2}\int_{-1}^{1} \frac{\cos(\pi\cos z)\cos(mz)}{\sqrt{1-\cos^2 z}}\mathrm{d}(\pi cos z) = -\frac{2}{\pi}\int_{\pi}^{0} \cos(\pi\cos z)\cos(mz)\mathrm{d}z, \tag{7}$$

where we took into account that $\cos z = -1$ when $z = \pi$ and 1 when $z = -\pi$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi^2}\int_{-1}^{1} \frac{\cos(\pi\cos z)}{\sqrt{1-\cos^2 z}}\mathrm{d}(\pi cos\theta) = -\frac{1}{\pi}\int_{\pi}^{0} \cos(\pi\cos z)\mathrm{d}z. \tag{8}$$

Chebyshev polynomial can be divided into even and odd polynomials. For even n polynomial is even and for odd n polynomial is odd. Thus the integral $\int cos(x)T_n(x\pi)$ will be zeo for all odd n. Only even n polynomials will participate in expansion of $\cos(x)$ into a series using Chebyshev polynomials. The opposite happens when we expand $sin(x)$ into such series. Therefore only odd n polynomials will participate in series expansion. For $m \neq 0$ we get:

$$a_m = \frac{2}{\pi} \int_0^\pi \sin(\pi \cos z) \cos(mz) \mathrm{d}z, \tag{9}$$

where we took into account that $\cos z = -1$ when $z = \pi$ and 1 when $z = -\pi$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi} \int_0^\pi \sin(\pi \cos z) \mathrm{d}z. \tag{10}$$