

Chebyshev polynomials

Chebyshev polynomials of the first kind are solutions to differential equation:

$$(1 - x^2)y'' - xy' + n^2y = 0. \quad (1)$$

They are defined by recursion formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{with} \quad T_0(x) = 0 \quad \text{and} \quad T_1(x) = x. \quad (2)$$

On an interval $[-1, 1]$ they form a complete basis set, similar to $\cos(x)$ and $\sin(x)$ for $x \in [-\pi, \pi]$. Of course one can expand these intervals to arbitrary intervals $[-a, a]$ by a simple transformation $x \rightarrow x/a$. Basis polynomials $T_n(x)$ are orthogonal with respect to the following scalar product:

$$\langle T_m(x) | T_n(x) \rangle = \int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

or, written for arbitrary interval $[-a, a]$:

$$\langle T_m(x/a) | T_n(x/a) \rangle = \int_{-a}^a \frac{T_m(x/a)T_n(x/a)}{\sqrt{1-(x/a)^2}} \frac{dx}{a} = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

One can now expand \sin and \cos functions into series on an interval $[-\pi, \pi]$ using transformation $x \rightarrow x/\pi$:

$$\cos(x) = \sum_n a_n T_n\left(\frac{x}{\pi}\right). \quad (3)$$

Multiplying both sides by $T_m(\frac{x}{\pi})/\sqrt{1-(x/\pi)^2}$ and integrating over the interval $[-\pi, \pi]$ yields for $m \neq 0$:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(x)T_m(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}} dx = \frac{\pi}{2} a_m \quad (4)$$

and for $m = 0$:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(x)T_0(\frac{x}{\pi})}{\sqrt{1-(x/\pi)^2}} dx = \pi a_0. \quad (5)$$

To simplify integral calculation we use relation between $T_n(x)$ and $\cos(x)$:

$$T_n(\cos \theta) = \cos n\theta. \quad (6)$$

Using $x/\pi = \cos(z)$ in equations leads to:

$$a_m = \frac{2}{\pi^2} \int_{-1}^1 \frac{\cos(\pi \cos z) \cos(mz)}{\sqrt{1-\cos^2 z}} d(\pi \cos z) = -\frac{2}{\pi} \int_{\pi}^0 \cos(\pi \cos z) \cos(mz) dz, \quad (7)$$

where we took into account that $\cos z = -1$ when $z = \pi$ and 1 when $z = 0$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi^2} \int_{-1}^1 \frac{\cos(\pi \cos z)}{\sqrt{1-\cos^2 z}} d(\pi \cos z) = -\frac{1}{\pi} \int_{\pi}^0 \cos(\pi \cos z) dz. \quad (8)$$

Chebyshev polynomial can be divided into even and odd polynomials. For even n polynomial is even and for odd n polynomial is odd. Thus the integral $\int \cos(x)T_n(x\pi)$ will be zero for all odd n . Only even n polynomials will participate in expansion of $\cos(x)$ into a series using Chebyshev polynomials. The opposite happens when we expand $\sin(x)$ into such series. Therefore only odd n polynomials will participate in series expansion. For $m \neq 0$ we get:

$$a_m = \frac{2}{\pi} \int_0^\pi \sin(\pi \cos z) \cos(mz) dz, \quad (9)$$

where we took into account that $\cos z = -1$ when $z = \pi$ and 1 when $z = -\pi$. For $m = 0$ we get:

$$a_0 = \frac{1}{\pi} \int_0^\pi \sin(\pi \cos z) dz. \quad (10)$$

1 Legendre polynomials

Legendre polynomials are solutions to a differential equation:

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx} + n(n+1)P_n(x) \right] = 0. \quad (11)$$

They can also be obtained from a Taylor series:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \quad (12)$$

Legendre polynomials are defined by equation:

$$P_0 = 1 \quad P_1 = x \quad P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x). \quad (13)$$

They form an orthonormal basis set on the interval $[-1, 1]$ with respect to the following dot product:

$$\langle P_i, P_j \rangle = \int_{-1}^1 P_i(x)P_j(x)dx = \begin{cases} 0 & i \neq j \\ \frac{2}{2i+1} & \text{otherwise} \end{cases}$$

2 Laguerre polynomials

Laguerre polynomials are a solution to the following differential equation:

$$xy'' + (1-x)y' + ny = 0, \quad (14)$$

where n is non-negative integer. The generating function for these polynomials is:

$$\frac{1}{1-t} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x). \quad (15)$$

The polynomials can be calculated using derivation:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n) \quad (16)$$

or recursive relation:

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}. \quad (17)$$

They are orthonormal with respect to the following scalar product:

$$\langle L_k(x), L_j(x) \rangle = \int_0^\infty L_k(x)L_j(x)e^{-x}dx = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

3 Tan approximation

For numerical tangens calculation is best to use integrals involving Legendre polynomials:

$$I(n, a) = \int_{t=0}^{t=1} P_{2n}(t) \cos(at) dt = N(n, a) \sin(a) + M(n, a) \cos(a) \quad (18)$$

and

$$J(n, a) = \int_{t=0}^{t=1} P_{2n+1}(t) \sin(at) dt = K(n, a) \sin(a) + L(n, a) \cos(a), \quad (19)$$

where M,N,K,L are polynomials. Their power depends on parameter n . P are Legendre polynomials. As $n \rightarrow \infty$ and a stays small, integrals go to zero and we get the following approximations:

$$\tan(a) \approx \frac{M(n, a)}{N(n, a)} = C(n, a) \quad \text{and} \quad \tan(a) \approx \frac{L(n, a)}{K(n, a)} = S(n, a). \quad (20)$$

Using the above expression we can obtain for increasing n the following expressions:

$$C(1, a) = \frac{3a}{3 - a^2} \quad (21)$$

$$S(1, a) = \frac{15a - a^3}{15 - 6a^2} \quad (22)$$

$$C(2, a) = \frac{105a - 10a^3}{105 - 45a^2 + a^4} \quad (23)$$

$$S(2, a) = \frac{945a - 105a^3 + a^5}{945 - 420a^2 + 15a^4} \quad (24)$$

$$C(3, a) = \frac{10395a - 1260a^3 + 21a^5}{10395 - 4725a^2 + 210a^4 - a^6} \quad (25)$$

$$S(3, a) = \frac{135135a - 17325a^3 + 378a^5 - a^7}{135135 - 62370a^2 + 3150a^4 - 28a^6} \quad (26)$$

$$C(4, a) = \frac{2027025a - 270270a^3 + 6930a^5 - 36a^7}{2027025 - 945945a^2 + 51975a^4 - 630a^6 + a^8} \quad (27)$$

$$S(4, a) = \frac{34459425a - 4729725a^3 + 1351355a^5 - 990a^7 + a^9}{34459425 - 16216200a^2 + 945945a^4 - 138600a^6 + 45a^8} \quad (28)$$

$$S(5, a) = \frac{3604742431983t - 600790405215t^3 + 30039519510t^5 - 715224510t^7 + 9930635t^9 - 88179t^{11}}{33(29393t^{10} - 2708355t^8 + 151714290t^6 - 4551442350t^4 + 54617309565t^2 - 109234619151)} \quad (29)$$

This library implements S(4,a) routine to calculate tangens.