2.1

When $k=2$, according to the definition of convexity,

$$\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{C}, \forall \boldsymbol{\theta_1}, \boldsymbol{\theta_2} \in \mathbb{R}, \boldsymbol{\theta_i} \geq \mathbf{0}, \boldsymbol{\theta_1}+\boldsymbol{\theta_2}=\mathbf{1} \Longrightarrow \boldsymbol{\theta_1} \boldsymbol{x_1}+\boldsymbol{\theta_2} \boldsymbol{x_2} \in \boldsymbol{C}$$

Assuming $k=t$,the following equation holds:

$$\forall \boldsymbol{x_1}, \boldsymbol{x_2} \ldots \boldsymbol{x_t} \in \boldsymbol{C}, \forall \boldsymbol{\theta_1}, \boldsymbol{\theta_2} \ldots \boldsymbol{\theta_t} \in \mathbb{R}, \boldsymbol{\theta_i} \geq \mathbf{0}, \boldsymbol{\theta_1}+\boldsymbol{\theta_2}+\ldots+\boldsymbol{\theta_t}=\mathbf{1}$$

$$\Longrightarrow \theta_1 x_1+\theta_2 x_2+\ldots+\theta_t x_t \in C(1)$$

On the basis of $k=t$, prove that $k=t+1$ alse holds:

C is the convex set , obviously $\exists \theta^* \in \mathbb{R}, 0 \leq \theta_t^* \leq \theta_t$ and $\boldsymbol{x_t^*}, \boldsymbol{x_{t+1}} \in C$

$$\boldsymbol{x_t}=\frac{\boldsymbol{\theta_t^*}}{\boldsymbol{\theta_t}} \boldsymbol{x_t^*}+(\mathbf{1}-\frac{\boldsymbol{\theta_t^*}}{\boldsymbol{\theta_t}}) \boldsymbol{x_{t+1}}$$

so $(1)=\boldsymbol{\theta_1} \boldsymbol{x_1}+\boldsymbol{\theta_2} \boldsymbol{x_2}+\ldots+\boldsymbol{\theta_t^*} \boldsymbol{x_t^*}+(\boldsymbol{\theta_t}-\boldsymbol{\theta_t^*}) \boldsymbol{x_{t+1}} \in \boldsymbol{C}(2)$

Let $\boldsymbol{\theta_t}-\boldsymbol{\theta_t^*}=\boldsymbol{\theta_{t+1}}$

$$(2)=\boldsymbol{\theta_1} \boldsymbol{x_1}+\boldsymbol{\theta_2} \boldsymbol{x_2}+\ldots+\boldsymbol{\theta_t^*} \boldsymbol{x_t^*}+\boldsymbol{\theta_{t+1}} \boldsymbol{x_{t+1}} \in \boldsymbol{C}$$

So on the basis of $k=t$, when $k=t+1$ the following equation also holds:

$$\forall \boldsymbol{x_1}, \boldsymbol{x_2} \ldots \boldsymbol{x_t}, \boldsymbol{x_{t+1}} \in \boldsymbol{C}, \forall \boldsymbol{\theta_1}, \boldsymbol{\theta_2} \ldots \boldsymbol{\theta_t}, \boldsymbol{\theta_{t+1}} \in \mathbb{R}, \boldsymbol{\theta_i} \geq \mathbf{0}, \boldsymbol{\theta_1}+\boldsymbol{\theta_2}+\ldots+\boldsymbol{\theta_t}+\boldsymbol{\theta_{t+1}}$$

$$\Longrightarrow \boldsymbol{\theta_1} \boldsymbol{x_1}+\boldsymbol{\theta_2} \boldsymbol{x_2}+\ldots+\boldsymbol{\theta_t} \boldsymbol{x_t}+\boldsymbol{\theta_{t+1}} \boldsymbol{x_{t+1}} \in \boldsymbol{C}$$

The proof is over.

2.3

C is midpoint convex. Prove $\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{C}, \forall \boldsymbol{\theta} \in [\mathbf{0}, \mathbf{1}], \boldsymbol{\theta} \boldsymbol{x_1}+(\mathbf{1}-\boldsymbol{\theta}) \boldsymbol{x_2} \in C$

According to the definition of midpoint convex, $\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in \boldsymbol{C}, \dfrac{x_1+x_2}{2} \in C$

So $\dfrac{x_1+\frac{x_1+x_2}{2}}{2} \in C, \dfrac{x_2+\frac{x_1+x_2}{2}}{2} \in C$

By iteratively using this property, it can be obtained that

$$(\textstyle\sum_{i=1}^{k} \mathbf{2}^{-i} \boldsymbol{c_i}) \boldsymbol{x_1}+(\mathbf{1}-\textstyle\sum_{i=1}^{k} \mathbf{2}^{-i} \boldsymbol{c_i}) \boldsymbol{x_2} \in \boldsymbol{C}, \boldsymbol{c_i}=\{\mathbf{0}, \mathbf{1}\}$$

Let $\boldsymbol{\theta_k = \sum_{i=1}^{k} 2^{-i} C_i}$

$\boldsymbol{\theta_k x_1 + (1-\theta_k) x_2} \in C$

$\forall \boldsymbol{\theta \in [0,1], \exists c_1, c_2, ..., c_k \in \{0,1\}, lim_{k \to \infty} \theta_k = \theta}$

Since $\boldsymbol{C}$ is closed

So $lim_{k \longrightarrow \infty} \boldsymbol{\theta_k x_1 + (1-\theta_k) x_2} = \boldsymbol{\theta x_1 + (1-\theta) x_2} \in C$

The proof is over.

2.5

$\forall \boldsymbol{x_1 \in \{x \in R^n | a^T x = b_1\}, \forall x_2 \in \{x \in R^n | a^T x = b_2\}}$

$$\begin{aligned} D &=<| \ \boldsymbol{(x_2 - x_1)} \ , \frac{\boldsymbol{a}}{||\boldsymbol{a}||_2}| \\ &= \frac{\boldsymbol{1}}{||\boldsymbol{a}||_2} | < \boldsymbol{(x_2 - x_1), a} | > \\ &= \frac{\boldsymbol{1}}{||\boldsymbol{a}||_2} |\boldsymbol{a^T (x_2 - x_1)}| \\ &= \frac{\boldsymbol{1}}{||\boldsymbol{a}||_2} |\boldsymbol{b_1 - b_2}| \end{aligned}$$

2.8

(a)

(b)$\boldsymbol{S = \{x \in} \mathbb{R}^n \boldsymbol{| x \succeq 0, Ax = b\}, A = (1^T, (a_1, ..., a_n), (a_1^2, ..., a_n^2))^T}$

$\boldsymbol{b = (1, b_1, b_2)^T}$

(c)

$x^T y \leq ||x||_2 ||y||_2 = ||x||_2$

Since $x^T y \leq 1$,So $||x||_2 \leq 1$,which means the set is the intersection of the unit ball and the nonnegative orthant $\mathbb{R}_+^n$

The set contains the intersection of infinite halfspaces, so it's not a polyhedron.

(d)

$x^Ty \le |x^Ty| = |\sum_{i=1}^{n} x_i y_i| \le \sum_{i=1}^{n} |x_i||y_i| \le max_i|x_i|$

since $x^Ty \le 1$,so $max_i|x_i| \le 1$

so $1 \succeq x \succeq 0$,which means

$x_i \ge 0, i = 1, ..., n$

$x_i \le 1, i = 1, ...n$

So the set is a polyhedron.

2.12

(a) A slab obviously is a polyhedron, so it's a convex set.

(b) A rectangle is a polyhedron, so it's a convex set.

(c) A wedge is a polyhedron, so it's convex set.

(d) $\forall \boldsymbol{y} \in \boldsymbol{C}, ||\boldsymbol{x} - \boldsymbol{x_0}||_2 \le ||\boldsymbol{x} - \boldsymbol{y}||_2$

$$(\boldsymbol{x} - \boldsymbol{x_0})^{\boldsymbol{T}}(\boldsymbol{x} - \boldsymbol{x_0}) \le (\boldsymbol{x} - \boldsymbol{y})^{\boldsymbol{T}}(\boldsymbol{x} - \boldsymbol{y})$$

$$(\boldsymbol{y} - \boldsymbol{x_0})^{\boldsymbol{T}}\boldsymbol{x} \le \frac{||\boldsymbol{y}||_2||\boldsymbol{x_0}||_2}{\boldsymbol{2}}$$

It's an intersection of halfspaces, so it's a convex set.

(e) The set is not a convex set:

Let $\boldsymbol{S} = \{\boldsymbol{x}|\boldsymbol{1} \le ||\boldsymbol{x}||_2 \le \boldsymbol{4}, \boldsymbol{x} \in \mathbb{R}^2\}, \boldsymbol{T} = \{\boldsymbol{x}|||\boldsymbol{x}||_2 \le \boldsymbol{1}, \boldsymbol{x} \in \mathbb{R}^2\}$

Obviously, $\boldsymbol{x_1} = (2,0)^T, \boldsymbol{x_2}(-2,0)^T \in \{\boldsymbol{x}|\boldsymbol{dist}(\boldsymbol{x},\boldsymbol{S}) \le \boldsymbol{dist}(\boldsymbol{x},\boldsymbol{T})\}$

Let $\theta = \frac{1}{2}, \boldsymbol{\theta x_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{x_2} = (\boldsymbol{0},\boldsymbol{0})^{\boldsymbol{T}} \notin \{\boldsymbol{x}|\boldsymbol{dist}(\boldsymbol{x},\boldsymbol{S}) \le \boldsymbol{dist}(\boldsymbol{x},\boldsymbol{T})\}$

So the set is not a convex set.

(f)

$\{x|x + S_2 \subseteq S_1\} = \cap_{y \in S_2}\{x|x + y \in S_1\} = \cap_{y \in S_2}(S_1 - y)$

$\forall y \in S_2, (S_1 - y)$ obvious is a convex set

The intersetcion of convex set is still convex.

(g)

$||x-a||_2 \leq \theta||x-b||_2$

$(x-a)^T(x-a) \leq \theta(x-b)^T(x-b)$

$x^Tx - \dfrac{2(a-\theta b)^Tx}{1-\theta} \leq \dfrac{\theta b^Tb - a^Ta}{1-\theta}$

$x^Tx - \dfrac{2(a-\theta b)^Tx}{1-\theta} + \dfrac{(a-\theta b)^T(a-\theta b)}{(1-\theta)^2} \leq \dfrac{\theta(a-b)^T(a-b)}{(1-\theta)^2}$

$||x - \dfrac{(a-\theta b)}{1-\theta}||_2 \leq \dfrac{\sqrt{\theta}||a-b||_2}{1-\theta}$

The set is a ball, so it's a convex set.

2.10

(a)

$$f(\boldsymbol{x}) = \boldsymbol{x^T A x} + \boldsymbol{b^T x} + c$$

$$\forall \boldsymbol{x_1, x_2} \in \boldsymbol{C}, \boldsymbol{f(x_1) \leq 0, f(x_2) \leq 0}$$

$\forall \theta \in [0,1]$

$$\boldsymbol{f(\theta x_1 + (1-\theta)x_2) = (\theta x_1 + (1-\theta)x_2)^T A(\theta x_1 + (1-\theta)x_2)}$$
$$\boldsymbol{+b^T(\theta x_1 + (1-\theta)x_2)} + c$$

Let $g(\theta) = \boldsymbol{f(\theta x_1 + (1-\theta)x_2)}$

So

$$g(\theta) = \boldsymbol{(x_1 - x_2)^T A(x_1 - x_2)\theta^2 + (x_1 - x_2)^T(2Ax + b)\theta}$$
$$\boldsymbol{+(x_2^T A x_2 + b^T x_2 + c)}$$

Since $\boldsymbol{A} \succeq 0$

So $\boldsymbol{(x_1 - x_2)^T A(x_1 - x_2) \geq 0}$

$$g(0) = \boldsymbol{x_2^T A x_2 + b^T x_2 + c} = f(\boldsymbol{x_2}) \leq 0$$

$$g(1) = \boldsymbol{x_1^T A x_1 + b^T x_1} + c = f(\boldsymbol{x_1}) \leq 0$$

So $g(\theta) \leq 0$ holds for $\forall \theta \in [0,1]$

So C is convex if $\boldsymbol{A} \succeq 0$

The converse is false:

if $\boldsymbol{A} \preceq 0, \boldsymbol{b} = \boldsymbol{0}, c < 0$

$\boldsymbol{C} = \mathbb{R}^{\boldsymbol{n}}$ is convex

(b)

$\forall \boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R} | \boldsymbol{g^T x} + \boldsymbol{h} = \boldsymbol{0}\}, \boldsymbol{x} = \boldsymbol{y} - \frac{\boldsymbol{h}}{||g||_2}\boldsymbol{g}$ where $\boldsymbol{g^T y} = \boldsymbol{0}$

If $\boldsymbol{x} \in \boldsymbol{C}$,$(\boldsymbol{y} - \frac{\boldsymbol{h}}{||g||_2}\boldsymbol{g})^{\boldsymbol{T}}\boldsymbol{A}(\boldsymbol{y} - \frac{\boldsymbol{h}}{||g||_2}\boldsymbol{g}) + \boldsymbol{b^T}(\boldsymbol{y} - \frac{\boldsymbol{h}}{||g||_2}\boldsymbol{g}) + c \leq 0$

So $\boldsymbol{y^T A y} + (\boldsymbol{b} - \frac{\boldsymbol{h}}{||g||_2^2}\boldsymbol{Ag})^{\boldsymbol{T}}\boldsymbol{y} + (\frac{\boldsymbol{h^2}}{||g||_2^4}\boldsymbol{g^T A g} - \frac{\boldsymbol{h}}{||g||_2^2}\boldsymbol{b^T g}) + c \leq 0$

Let $d(\boldsymbol{y}) = \boldsymbol{y^T A y} + (\boldsymbol{b} - \frac{\boldsymbol{h}}{||g||_2^2}\boldsymbol{Ag})^{\boldsymbol{T}}\boldsymbol{y} + (\frac{\boldsymbol{h^2}}{||g||_2^4}\boldsymbol{g^T A g} - \frac{\boldsymbol{h}}{||g||_2^2}\boldsymbol{b^T g}) + c$

$\boldsymbol{D} = \{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}} | d(\boldsymbol{y}) \leq 0\}$

It can be known from affine transformation that $\boldsymbol{D}$ is convex for $\boldsymbol{C}$ is convex.

$\forall \boldsymbol{y_1}, \boldsymbol{y_2} \in \boldsymbol{D}, \boldsymbol{d}(\boldsymbol{y_1}) \leq \boldsymbol{0}, \boldsymbol{d}(\boldsymbol{y_2}) \leq \boldsymbol{0}, \boldsymbol{g^T y_1} = \boldsymbol{0}, \boldsymbol{g^T y_2} = \boldsymbol{0}$

$\forall \theta \in [0,1]$

$$d(\boldsymbol{\theta y_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{y_2}) =$$
$$(\boldsymbol{y_1} - \boldsymbol{y_2})^{\boldsymbol{T}}\boldsymbol{A}(\boldsymbol{y_1} - \boldsymbol{y_2})\boldsymbol{\theta^2} + (\boldsymbol{y_1} - \boldsymbol{y_2})^{\boldsymbol{T}}(\boldsymbol{2Ay_2} + \boldsymbol{b} - \frac{\boldsymbol{2h}}{||g||_2^2}\boldsymbol{Ag}) + \boldsymbol{d}(\boldsymbol{y_2})$$

Let $G(\theta) = d(\boldsymbol{\theta y_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{y_2})$

Since $\exists \lambda \in R, \boldsymbol{A} + \boldsymbol{\lambda g g^T} \succeq \boldsymbol{0}$

So $(\boldsymbol{y_1} - \boldsymbol{y_2})^{\boldsymbol{T}}\boldsymbol{A}(\boldsymbol{y_1} - \boldsymbol{y_2}) = (\boldsymbol{y_1} - \boldsymbol{y_2})^{\boldsymbol{T}}(\boldsymbol{A} + \boldsymbol{\lambda g g^T})(\boldsymbol{y_1} - \boldsymbol{y_2}) \geq \boldsymbol{0}$

$G(0) = d(y_1) \leq 0$

$G(1) = d(y_2) \leq 0$

So $G(\theta) \leq 0$,holds for $\forall \theta \in [0,1]$

So $\boldsymbol{D}$ is convex.

The converse if false:

Let $\boldsymbol{A} \preceq 0, b = 0, c < 0$

So $\boldsymbol{C} = \mathbb{R}^n$

$\boldsymbol{C} \cap \{x \in \mathbb{R} | g^T x + h = 0\}$ is convex

But $\boldsymbol{A} + \boldsymbol{\lambda g g^T}$ is not neccessary.

2.16

$$\forall (\boldsymbol{x_1}, \boldsymbol{y_{11}} + \boldsymbol{y_{21}}), (\boldsymbol{x_2}, \boldsymbol{y_{12}} + \boldsymbol{y_{22}}) \in \boldsymbol{S}$$

$$(\boldsymbol{x_1}, \boldsymbol{y_{11}}), (\boldsymbol{x_2}, \boldsymbol{y_{12}}) \in \boldsymbol{S_1}, (\boldsymbol{x_1}, \boldsymbol{y_{21}}), (\boldsymbol{x_2}, \boldsymbol{y_{22}}) \in \boldsymbol{S_2}$$

Since $\boldsymbol{S_1}$、$\boldsymbol{S_2}$ is convex set.

$\forall \boldsymbol{\theta} \in [\boldsymbol{0}, \boldsymbol{1}], \boldsymbol{\theta}(\boldsymbol{x_1}, \boldsymbol{y_{11}}) + (\boldsymbol{1} - \boldsymbol{\theta})(\boldsymbol{x_2}, \boldsymbol{y_{12}}) = (\boldsymbol{\theta x_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{x_2}, \boldsymbol{\theta y_{11}} + (\boldsymbol{1} - \boldsymbol{\theta})$

$$\boldsymbol{\theta}(\boldsymbol{x_1}, \boldsymbol{y_{21}}) + (\boldsymbol{1} - \boldsymbol{\theta})(\boldsymbol{x_2}, \boldsymbol{y_{22}}) = (\boldsymbol{\theta x_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{x_2}, \boldsymbol{\theta y_{21}} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{y_{22}}) \in \boldsymbol{S_2}$$

So

$$(\boldsymbol{\theta x_1} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{x_2}, \boldsymbol{\theta}(\boldsymbol{y_{11}} + \boldsymbol{y_{21}}) + (\boldsymbol{1} - \boldsymbol{\theta})(\boldsymbol{y_{12}} + \boldsymbol{y_{22}})) \in \boldsymbol{S}$$

$$\boldsymbol{\theta}(\boldsymbol{x_1}, \boldsymbol{y_{11}} + \boldsymbol{y_{21}}) + (\boldsymbol{1} - \boldsymbol{\theta})(\boldsymbol{x_2}, \boldsymbol{y_{21}} + \boldsymbol{y_{22}}) \in \boldsymbol{S}$$

So $\boldsymbol{S}$ is convex set.

2.19

(a)

$$\boldsymbol{g^T} \frac{\boldsymbol{Ax} + \boldsymbol{b}}{\boldsymbol{c^T x} + d} \leq h \Longrightarrow (\boldsymbol{A^T g} - \boldsymbol{hc})^{\boldsymbol{T}} \boldsymbol{x} \leq hd - \boldsymbol{g^T b}$$

$$f^{-1}(\boldsymbol{C}) = \{x | (\boldsymbol{A^T g} - \boldsymbol{hc})^{\boldsymbol{T}} \boldsymbol{x} \leq hd - \boldsymbol{g^T b}, \boldsymbol{c^T x} + d > 0\}$$

(b)

$$\frac{\boldsymbol{G}(\boldsymbol{Ax} + \boldsymbol{b})}{\boldsymbol{c^T x} + d} \succeq h \Longrightarrow \boldsymbol{GAx} + \boldsymbol{Gb} \succeq \boldsymbol{hc^T x} + \boldsymbol{h}d \Longrightarrow (\boldsymbol{GA} - \boldsymbol{hx^T}) \succeq d\boldsymbol{h} - \boldsymbol{Gb}$$

$$f^{-1}(\boldsymbol{C}) = \{x | (\boldsymbol{GA} - \boldsymbol{hx^T}) \succeq d\boldsymbol{h} - \boldsymbol{Gb}, \boldsymbol{c^T x} + d > 0\}$$

(c)

$$(\frac{\boldsymbol{Ax}+\boldsymbol{b}}{\boldsymbol{c^T x}+d})^T \boldsymbol{P}^{-1}(\frac{\boldsymbol{Ax}+\boldsymbol{b}}{\boldsymbol{c^T x}+d}) \leq 1$$

$$\boldsymbol{(x^T A^T + b^T) P^{-1} (AX + b)} \leq \boldsymbol{(c^T x + d)^2}$$

$$\boldsymbol{x^T (A^T P^{-1} A - cc^T) x + 2(A^T Pb - dc)^T x + b^T P^{-1} b} - d^2 \leq 0$$

$$f^{-1}(\boldsymbol{C}) = \{x|\boldsymbol{x^T (A^T P^{-1} A - cc^T) x + 2(A^T Pb - dc)^T x + b^T P^{-1} b} - d^2 \leq 0,$$

$$\boldsymbol{c^T x} + d > 0\}$$

(d)

Let $\boldsymbol{A = (a_1^T, ..., a_n^T)^T, a_i = (a_{i1}, ...a_{in})^T, b = (b_1, ..., b_n)^T, c = (c_1, ..., c_n)^T}$

$\boldsymbol{x = (x_1, ..., x_n)^T}$

So

$$(\frac{\boldsymbol{a_1^T x + b_1}}{\boldsymbol{c^T x} + d})\boldsymbol{A_1} + ... + (\frac{\boldsymbol{a_n^T x + b_n}}{\boldsymbol{c^T x} + d})\boldsymbol{A_n} \preceq \boldsymbol{B}$$

$$\boldsymbol{(a_1^T x + b_1) A_1 + ... + (a_n^T x + b_n) A_n \preceq B(c^T x + d)}$$

$$(\sum_{i=1}^{n} a_{1i}\boldsymbol{A_i} - c_1\boldsymbol{B})x_1 + ... + \sum_{i=1}^{n} a_{ni}\boldsymbol{A_i} - c_n\boldsymbol{B})x_1 \preceq d\boldsymbol{B} - \sum_{i=1}^{n} b_i\boldsymbol{A_i}$$

$$f^{-1}(\boldsymbol{C}) = \{x|(\sum_{i=1}^{n} a_{1i}\boldsymbol{A_i} - c_1\boldsymbol{B})x_1 + ... + \sum_{i=1}^{n} a_{ni}\boldsymbol{A_i} - c_n\boldsymbol{B})x_1 \preceq d\boldsymbol{B} - \sum_{i=1}^{n} b_i\boldsymbol{A_i}$$

$, \boldsymbol{c^T x} + d > 0\}$

3.5

$f(x)$ is convex, so $\forall s \in \mathbb{R}, f(sx)$ is convex.

Since the nonnegative weighted sum is convexity preserving,

$\int_0^1 f(sx)ds = \sum f(sx)\Delta s$ is also convex.

$F(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(\frac{t}{x}x)d(\frac{t}{x})$

So $F(x)$ is convex.

3.6

(a)When epi f is a halfspace

$\boldsymbol{a} = (\boldsymbol{a_0}, a_{n+1})^T$

$\boldsymbol{a^T}(\boldsymbol{x}, t) \leq b \Longrightarrow \boldsymbol{a_0^T x} \leq b - t a_{n+1}$

$\Longrightarrow (\frac{-\boldsymbol{a_0}}{\boldsymbol{a_{n+1}}})^T \boldsymbol{x} + \frac{\boldsymbol{b}}{\boldsymbol{a_{n+1}}} \leq t$

So $\boldsymbol{f}(\boldsymbol{x}) = (\frac{-\boldsymbol{a_0}}{\boldsymbol{a_{n+1}}})^T \boldsymbol{x} + \frac{\boldsymbol{b}}{\boldsymbol{a_{n+1}}}$

The $f(x)$ should be affine.

(b)When epi f is a convex cone:

$\forall(\boldsymbol{x}, t) \in epif, \forall\theta \geq 0, \theta(\boldsymbol{x}, t) \in epif$

$f(\theta\boldsymbol{x}) \leq \theta t$

Given that $f(\boldsymbol{x}) \leq t \Longrightarrow \theta f(\boldsymbol{x}) \leq \theta t$ with $\forall(\boldsymbol{x}, t) \in epif, \forall\theta \geq 0$

So the only feasible case for epi f to be a cone is that $\forall\theta \geq 0, f(\theta\boldsymbol{x}) = \theta f(\boldsymbol{x})$

Thus, when epi f is a convex cone,$f(\boldsymbol{x})$, must be positively homogeneous and convex.

(c)When epi f is a polyhedron,$\boldsymbol{A}(\boldsymbol{x}, t) \preceq \boldsymbol{b}, \boldsymbol{C}(\boldsymbol{x}, t) = \boldsymbol{d}$

$\boldsymbol{A} = ((\boldsymbol{a_1^T}, a_1^*), ..., (\boldsymbol{a_n^T}, a_n^*))^T, \boldsymbol{b} = (b_1, ..., b_n)^T$

$\boldsymbol{A}$ must satisfy $a_i^* < 0$

$\boldsymbol{C} = ((\boldsymbol{c_1^T}, c_1^*), ..., (\boldsymbol{c_n^T}, c_n^*))^T, \boldsymbol{d} = (d_1, ..., d_n)^T$

$\boldsymbol{C}$ must satisfy $c_i^* = 0$

So $\boldsymbol{A}(\boldsymbol{x}, t) \preceq \boldsymbol{b} \Longrightarrow \boldsymbol{a_i^T x} + a_i^* t \leq b_i \Longrightarrow (-\frac{a_i}{a_i^*})^T \boldsymbol{x} + \frac{b_i}{a_i^*} \leq t \Longrightarrow max_i[(-\frac{a_i}{a_i^*})^T \boldsymbol{x} + \frac{b_i}{a_i^*}] \leq t$ $f(\boldsymbol{x}) = max_i[(-\frac{a_i}{a_i^*})^T \boldsymbol{x} + \frac{b_i}{a_i^*}]$

$\boldsymbol{C}(\boldsymbol{x}, t) = \boldsymbol{d} \Longrightarrow \boldsymbol{c_i^T x} = d_i \Longrightarrow domf = \{x | \boldsymbol{cx} = \boldsymbol{d}\}$

So when epi f is a polyhedron,f(x) should be piecewise affine.

3.8

1.Consider the case firstly, $f : \mathbb{R} \longrightarrow \mathbb{R}$

(1). $f^{''}(x) \geq 0 \Longrightarrow f$ is convex

$\int_x^y f^{''}(z)(y-z)dz = \int_x^y (y-z)df^{'}(z) = [f^{'}(z)(y-z)]_x^y + \int_x^y f^{'}(z)dz = -f^{'}(x)(y-x) + f(y) - f(x)$

Since $f(x)^{''} \geq 0$

So $-f^{'}(x)(y-x) + f(y) - f(x) \geq 0 \Longrightarrow f(y) \geq f(x) + f^{'}(x)(y-x) \Longrightarrow f(x)$is convex

(2). f is convex $\Longrightarrow f^{''}(x) \geq 0$

Since $f(x)$ is convex

So $f(x) - f(y) \geq f^{'}(y)(x-y)$ and $f(y) - f(x) \geq f^{'}(x)(y-x)$

$\Longrightarrow f^{'}(x)(y-x) \leq f(y) - f(x) \leq f^{'}(y)(y-x)$

$$\Longrightarrow \frac{f^{'}(y) - f^{'}(x)}{y-x} \geq 0$$

$$f^{''}(x) = lim_{y \longrightarrow x} \frac{f^{'}(y) - f^{'}(x)}{y-x} \geq 0$$

2. Consider the case $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$f(\boldsymbol{x})$ is convex if and only if $g(t) = f(\boldsymbol{x}+t\boldsymbol{v}), domg = \{t \in R | \boldsymbol{x} + t\boldsymbol{v} \in domf\}$ is convex

$g^{''}(t) = \boldsymbol{v^T} \bigtriangledown^2 f(\boldsymbol{x} + t\boldsymbol{v})\boldsymbol{v}$

As proved before, $g(t)$ is convex if and only if $g^{''}(t) \geq 0$

And $g^{''}(t) \geq 0$ if and only if $\bigtriangledown^2 f(\boldsymbol{x} + t\boldsymbol{v}) \succeq 0$

Thus $f(\boldsymbol{x})$ is convex if and only if $domf$ if convex and $\bigtriangledown^2 f(\boldsymbol{x}) \succeq 0$

3.11

Since $f(\boldsymbol{x})$ is convex

$f(\boldsymbol{y}) - f(\boldsymbol{x}) \geq \triangledown f(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x})$ and $f(\boldsymbol{x}) - f(\boldsymbol{y}) \geq \triangledown f(\boldsymbol{y})^T(\boldsymbol{x}-\boldsymbol{y})$

So $\triangledown f(\boldsymbol{y})^T(\boldsymbol{x}-\boldsymbol{y}) \leq f(\boldsymbol{x}) - f(\boldsymbol{y}) \leq \triangledown f(\boldsymbol{x})^T(\boldsymbol{x}-\boldsymbol{y})$

So $(\triangledown f(\boldsymbol{x})^T - \triangledown f(\boldsymbol{y})^T)(\boldsymbol{x}-\boldsymbol{y}) \geq 0$

Thus $\triangledown f$ is monotone.

The converse is false:

Consider $\psi(\boldsymbol{x}) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}$

$(\psi(\boldsymbol{x}) - \psi(\boldsymbol{y}))^T(\boldsymbol{x}-\boldsymbol{y}) = ((x_1-y_1) - \frac{1}{2}(x_2-y_2))^2 + \frac{3}{4}(x_2-y_2)^2 \geq 0$

But $\psi(\boldsymbol{x})$ can not be the gradient of any differntiable convex $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Since $\psi(\boldsymbol{x})$ is differentiable, which means that $f(\boldsymbol{x})$ is twice differentiable,

there is no such function satisfying $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 0 = -1$

3.12

Since $f(\boldsymbol{x})$ is convex,$epif$ is convex

Since $g(\boldsymbol{x})$ is concave,$hypo\ g$ is convex

Since $epi\ f \cap hypo\ g$ =$\emptyset$

there exists $\boldsymbol{a} \neq 0, b > 0$

$\boldsymbol{a^T x} + bt \geq c, t \longrightarrow f(x)^+$ and $\boldsymbol{a^T x} + bg(\boldsymbol{x}) \leq c$

So $g(\boldsymbol{x}) \leq (-\dfrac{\boldsymbol{a}}{b})^T\boldsymbol{x} + \dfrac{c}{b} \leq t$

Thus, there exists $h(\boldsymbol{x}) = (-\dfrac{\boldsymbol{a}}{b})^T\boldsymbol{x} + \dfrac{c}{b}$ satisfying $g(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq f(\boldsymbol{x})$

3.15

(a)

$lim_{\alpha \longrightarrow 0}\mu_\alpha(x) = lim_{\alpha \longrightarrow 0}\frac{x^\alpha - 1}{\alpha} = lim_{\alpha \longrightarrow 0}\frac{x^\alpha logx}{1} = logx = \mu_0(x)$

(b)

$$\mu_\alpha^{'}(x) = \begin{cases} \frac{1}{x} & \alpha = 0 \\ x^{\alpha-1} & \alpha \neq 0 \end{cases} = x^{\alpha-1} > 0$$

Thus $\mu_\alpha(x)$ is monotone increasing.

$$\mu_\alpha^{''}(x) = (\alpha - 1)x^{\alpha-2} \leq 0$$

Thus $\mu_\alpha(x)$ is concave.

$$\mu_\alpha(1) = \begin{cases} log1 & \alpha = 0 \\ \frac{1^\alpha - 1}{\alpha} & \alpha \neq 0 \end{cases} = 0$$

3.16

(a)

$$f^{'}(x) = e^x > 0, f^{''}(x) = e^x > 0$$

Thus $f(x)$ is convex.

$$\forall x, y \in domf, \forall \theta \in [0, 1]$$

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

Since $f(x)$ is monotone increasing,

Thus $f(\theta x + (1-\theta)y) \leq \begin{cases} f(x) & x \geq y \\ f(y) & x < y \end{cases}$

$$\Longrightarrow f(\theta x + (1-\theta)y) \leq max\{f(x), f(y)\}$$

Thus $f(x)$ is quasiconvex

Obviously, $f(\theta x + (1-\theta)y) \geq min\{f(x), f(y)\}$

Thus $f(x)$ is quasiconcave

(b)

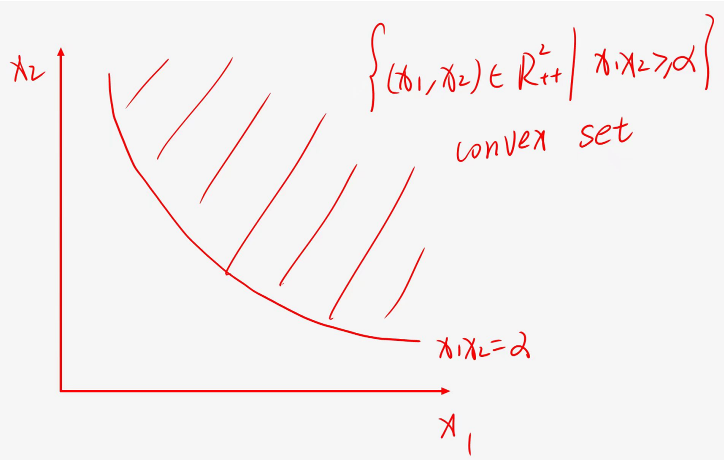
$$\bigtriangledown f(x_1, x_2) = (x_2, x_1)^T$$

$$\bigtriangledown^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The Hessian of f is neither positive semidefinite nor negative semidefinite.

Thus f is neither convex nor concave.

As the following graph, f is quasiconcave:



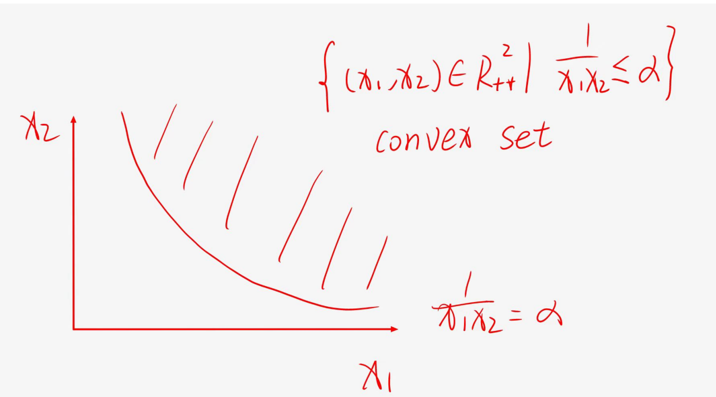
(c)

$$\bigtriangledown f(x_1, x_2) = -\frac{1}{x_1x_2}(\frac{1}{x_1}, \frac{1}{x_2})^T$$

$$\bigtriangledown^2 f(x_1, x_2) = \frac{1}{x_1x_2}\begin{bmatrix} \frac{1}{x_1^2} & \frac{1}{x_1x_2} \\ \frac{1}{x_1x_2} & \frac{1}{x_2^2} \end{bmatrix} \succeq 0$$

Thus f is convex

As the following graph, f is quasiconvex:



(d)

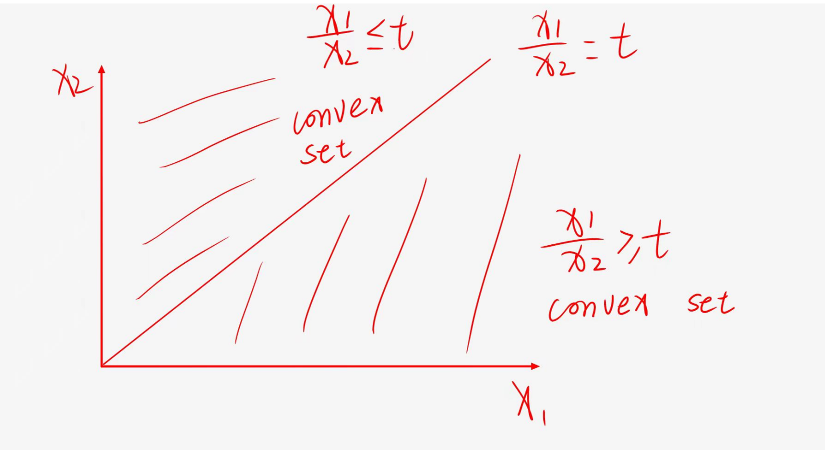
$$\bigtriangledown f(x_1, x_2) = \frac{1}{x_2}(1, -\frac{x_1}{x_2})^T$$

$$\nabla^2 f(x_1, x_2) = \frac{1}{x_2^2}\begin{bmatrix} 0 & -1 \\ -1 & \frac{2x_1}{x_2} \end{bmatrix}$$

The Hessian of f is neither positive semidefinite nor negative semidefinite.

Thus f is neither convex nor concave.

As the following graph, f is quasicconvex and quasiconcave:



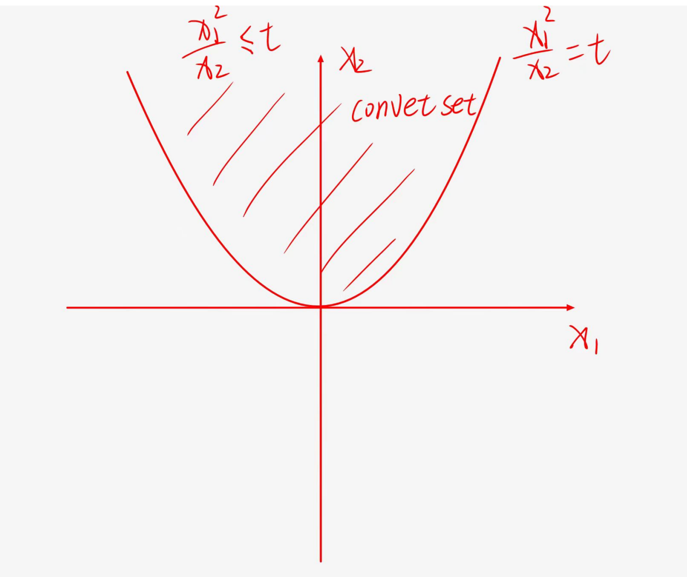
(e)

$$\nabla f(x_1, x_2) = \frac{x_1}{x_2}(2, -\frac{x_1}{x_2})^T$$

$$\nabla^2 f(x_1, x_2) = \frac{2}{x_2}\begin{bmatrix} 1 & -\frac{x_1}{x_2} \\ -\frac{x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix} \succeq 0$$

Thus f is convex

As the following graph, f is quasiconvex :



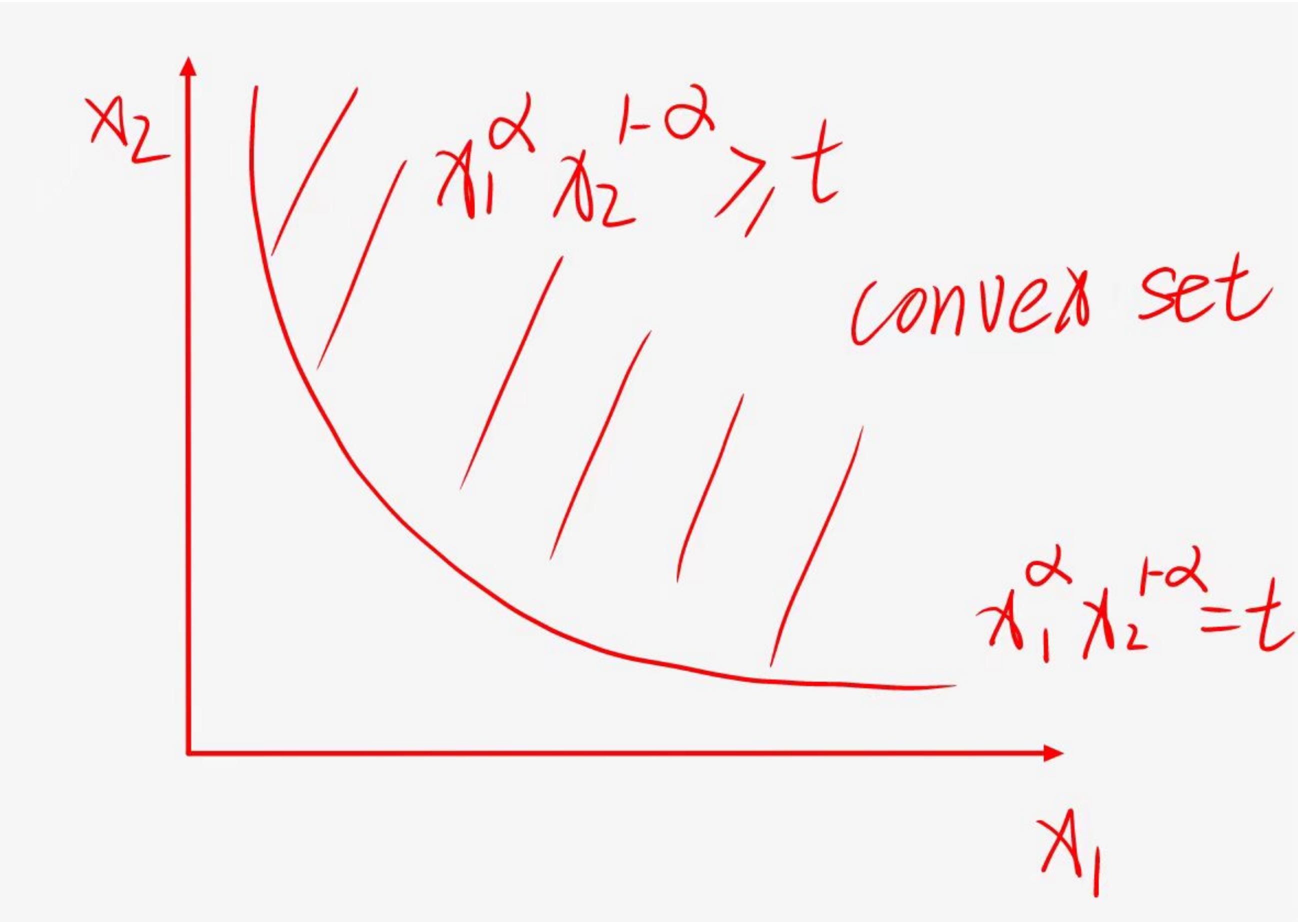
(f)

$$\bigtriangledown f(x_1, x_2) = (\frac{x_1}{x_2})^{\alpha-1}(\alpha, (1-\alpha)\frac{x_1}{x_2})$$

$$\bigtriangledown^2 f(x_1, x_2) = \alpha(\alpha-1)\frac{x_1^{\alpha-2}}{x_2^{\alpha-1}} \begin{bmatrix} 1 & -\frac{x_1}{x_2} \\ -\frac{x_1}{x_2} & \frac{x_1^2}{x_2} \end{bmatrix} \preceq 0$$

Thus f is concave.

As the following graph, f is quasiconcave :



3.18 Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

(a) $f(X) = \mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(a) Let $g(t) = f(X+tV)$

$= tr((X+tV)^{-1})$

$= tr(X^{-\frac{1}{2}}(I+tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})^{-1}X^{-\frac{1}{2}})$

$= tr(X^{-1}(I+tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})^{-1})$

$X^{-\frac{1}{2}}VX^{-\frac{1}{2}} = Q\Lambda Q^T$ eigenvalue decomposition

so $g(t) = tr(X^{-1}Q(I+t\Lambda)^{-1}Q^T)$

$= tr(Q^TX^{-1}Q(I+t\Lambda)^{-1})$

Since $X \in S^n_{++}$, Q is orthonormal $Q^TX^{-1}Q \in S^n_{++}$

Let λ_i' be the eigenvalue of $Q^TX^{-1}Q$, $\lambda_i' > 0$

Let λ_i be the eigenvalue of Λ

Since $X+tV \in S^n_{++}$, $Q^TX^{-1}Q \in S^n_{++}$

so $1+t\lambda_i > 0$

so $g(t) = \sum_{i=1}^{n} \frac{\lambda_i'}{1+t\lambda_i}$ is the nonnegative weighted sum of a group of convex functions

Thus $g(t)$ is convex and further $f(X)$ is convex

(b). Let $g(t) = f(X+tV)$

$= (\det(X+tV))^{\frac{1}{n}} = (\det X \det(I+tX^{-\frac{1}{2}}VX^{-\frac{1}{2}}))^{\frac{1}{n}}$

$X^{-\frac{1}{2}}VX^{-\frac{1}{2}} = Q\Lambda Q^T$ eigenvalue decomposition

So $g(t) = (\det X)^{\frac{1}{n}}(\det(I+t\Lambda))^{\frac{1}{n}}$

$= (\det X)^{\frac{1}{n}}(\prod_{i=1}^{n}(1+t\lambda_i))^{\frac{1}{n}}$ (λ_i is the eigenvalue of Λ)

$(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ is the geometric mean which is concave

since $x = \lambda t + 1$ is affine

so $g(t)$ is concave and further $f(X)$ is concave

3.19 *Nonnegative weighted sums and integrals.*

(a) Show that $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of x, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of x. (You can use the fact that $f(x) = \sum_{i=1}^k x_{[i]}$ is convex on $\mathbf{R}^n$.)

$f(x) = \sum_{i=1}^{r} a_i x_{[i]} = a_r \sum_{i=1}^{r} x_{[i]} + \sum_{i=1}^{r-1} [(a_i - a_{i+1}) \sum_{j=1}^{i} x_{[j]}]$

which is the nonnegative weighted sum of a group of convex functions $\sum_{i=1}^{k} x_{[i]}$. thus f is convex

(b) Let $T(x,\omega)$ denote the trigonometric polynomial

$$T(x,\omega) = x_1 + x_2\cos\omega + x_3\cos 2\omega + \cdots + x_n\cos(n-1)\omega.$$

Show that the function

$$f(x) = -\int_0^{2\pi} \log T(x,\omega)\, d\omega$$

is convex on $\{x \in \mathbf{R}^n \mid T(x,\omega) > 0,\ 0 \leq \omega \leq 2\pi\}$.

$h(x) = -\log x$ is convex

$T(x,w) = a^Tx$ is affine

$a = (1, \cos w, \ldots, \cos(n-1)w)^T$

So $-\log T(x,w) = h(T(x,w))$ is convex in x

$f(x) = \int_0^{2\pi} -\log(T(x,w))dw$

$= \sum -\log(T(x,w))$ is convex for the nonnegtive weighted sum is convexity preserving

Composition with an affine function. Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \|Ax - b\|$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

$h(x) = \|x\|$ is a norm and convex

Thus $f(x) = h(Ax-b)$ is convex

(b) $f(x) = -(\det(A_0 + x_1A_1 + \cdots + x_nA_n))^{1/m}$, on $\{x \mid A_0 + x_1A_1 + \cdots + x_nA_n \succ 0\}$, where $A_i \in \mathbf{S}^m$.

As is proved in 3.18 (b)

$g(X) = -(\det X)^{\frac{1}{m}}$ is convex on S^n_{++}

Thus $f(X) = g(Ax + A_0)$ is convex

(c) $f(X) = \mathbf{tr}\left(A_0 + x_1A_1 + \cdots + x_nA_n\right)^{-1}$, on $\{x \mid A_0 + x_1A_1 + \cdots + x_nA_n \succ 0\}$, where $A_i \in \mathbf{S}^m$. (Use the fact that $\mathbf{tr}(X^{-1})$ is convex on $\mathbf{S}^m_{++}$; see exercise 3.18.)

As proved in 3.18 (a)

$g(x) = tr(X^{-1})$ is convex in S^n_{++}

Thus $f(x) = g(Ax + A_0)$ is convex

3.21

(a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

As proved in 3.20 (a)

$f_i(x) = \|A^{(i)}x - b^{(i)}\|$ is convex

Thus $f(x) = \max\{f_1(x), \ldots f_k(x)\}$ is convex

(b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

$g(X) = \sum_{i=1}^{n} |x_i|$ is a norm and convex

$f(X) = \sum_{i=1}^{r} |x|_{[i]} = \max\{|x_{i_1}| + \cdots + |x_{i_r}|\}$

($x_{i_1}, \ldots, x_{i_r}$ are r arbitrary elements of x)

$f(x)$ is the pointwise maximum of C_n^r convex functions

Thus $f(x)$ is convex

3.22 *Composition rules.* Show that the following functions are convex.

(a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^Tx+b_i}))$ on $\mathbf{dom}\, f = \{x \mid \sum_{i=1}^m e^{a_i^Tx+b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^n e^{y_i})$ is convex.

$g(x) = \log(\sum_{i=1}^{n} e^{x_i})$ is convex as proved before

So $-g(Ax+b) = -\log(\sum_{i=1}^{m} e^{a_i^Tx+b_i})$ is concave

$h(x) = -\log x$ is convex and nonincreasing

Thus $f(x) = h(-g(Ax+b))$ is convex

(b) $f(x,u,v) = -\sqrt{uv - x^Tx}$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^Tx,\ u,\ v > 0\}$. Use the fact that x^Tx/u is convex in (x,u) for $u > 0$, and that $-\sqrt{x_1x_2}$ is convex on $\mathbf{R}^2_{++}$.

$h(x_1, x_2) = -\sqrt{x_1x_2}$ is convex and nonincreasing on R^2_{++}

$g(x,u) = \frac{x^Tx}{u}$ is convex in (x,u) for $u>0$

$v - g(x,u)$ is concave

Thus $f(x,u,v) = h(u, v-g(x,u)) = -\sqrt{uv - x^Tx}$ is convex

(c) $f(x,u,v) = -\log(uv - x^Tx)$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^Tx,\ u,\ v > 0\}$.

$f(x) = -\log(u(v - \frac{x^Tx}{u}))$

$= -\log u - \log(v - \frac{x^Tx}{u})$

As is proved in (b), $g(x,u,v) = v - \frac{x^Tx}{u}$ is concave.

$h(x) = -\log x$ is convex and nonincreasing

Thus $f(x,u,v) = -\log u + h(g(x,u,v))$ is convex

(d) $f(x,t) = -(t^p - \|x\|_p^p)^{1/p}$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t \geq \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23), and that $-x^{1/p}y^{1-1/p}$ is convex on $\mathbf{R}^2_+$ (see exercise 3.16).

$h(x,y) = -x^{\frac{1}{p}}y^{1-\frac{1}{p}}$ is convex and nonincreasing on R^2_+

$g(x,u) = \frac{\|x\|_p^p}{t^{p-1}}$ is convex in (x,u) for $u>0$

So $u - g(x,u)$ is concave

Thus $f(x,t) = h(t - g(x,t), t)$

$= -(t^p - \|x\|_p^p)^{\frac{1}{p}}$ is convex

(e) $f(x,t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23).

$f(x,t) = -\log(t^{p-1}(t - \frac{\|x\|_p^p}{t^{p-1}}))$

$= -(p-1)\log t - \log(t - \frac{\|x\|_p^p}{t^{p-1}})$

As is proved in (d)

$g(x,t) = t - \frac{\|x\|_p^p}{t^{p-1}}$ is concave

$h(x) = -\log x$ is convex and nonincreasing

Thus $f(x,t) = -(p-1)\log t + h(g(x,t))$ is convex

3.23 *Perspective of a function.*

(a) Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$.

$f(x)$ is the perspective function of $\|x\|_p^p$

Thus $f(x)$ is convex

(b) Show that

$$f(x) = \frac{\|Ax + b\|_2^2}{c^Tx + d}$$

is convex on $\{x \mid c^Tx + d > 0\}$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$ and $d \in \mathbf{R}$.

Let $g(x,t) = \frac{x^Tx}{t}$

$\nabla^2 g(x,t) = 2\begin{bmatrix} \frac{I}{t} & \frac{-x}{t^2} \\ \frac{-x^T}{t^2} & \frac{x^Tx}{t^3} \end{bmatrix}$

$\forall (z,u)^T \in R_{n+1}$

$(z,u)\nabla^2 g(x,t)(z,u)^T$

$= 2(z_1, \ldots, z_n, u)\begin{pmatrix} \frac{1}{t} & & & \frac{-x_1}{t^2} \\ & \frac{1}{t} & & \vdots \\ & & \ddots \quad \frac{1}{t} & \frac{-x_n}{t^2} \\ \frac{-x_1}{t^2} & \frac{-x_2}{t^2} & \cdots \quad \frac{-x_n}{t^2} & \frac{x^Tx}{t^3} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ u \end{pmatrix}$

$= 2\left(\frac{z_1}{t} - \frac{x_1u}{t^2}, \ldots, \frac{z_n}{t} - \frac{x_nu}{t^2}, \frac{-x_1z_1}{t^2} - \frac{x_2z_2}{t^2} - \cdots \frac{x_nz_n}{t^2} + u\frac{x^Tx}{t^3}\right)\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ u \end{pmatrix}$

$= 2\left[\frac{1}{t}z^Tz + \frac{u^2}{t^3}x^Tx - \frac{2u}{t^2}(x^Tz)\right]$

$= \frac{2}{t^3}(tz - ux)^T(tz - ux) \geq 0$

Thus $\nabla^2 g(x,t) \succeq 0$, $g(x,t)$ is convex.

Let $(x,t) = (Ax+b, c^Tx+d)$, an affine transformation

So $f(x) = g(Ax+b, c^Tx+d)$ is also convex.