

# Important Definitions for CS1231S

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Semester 1, AY23/24 (Prof Aaron Tan)

# Appendix A (Properties of Real Numbers)

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- F1. Commutative Laws** For all real numbers  $a$  and  $b$ ,  $a+b = b+a$  and  $ab = ba$ .
- F2. Associative Laws** For all real numbers  $a$ ,  $b$  and  $c$ ,  $(a+b)+c = a+(b+c)$  and  $(ab)c = a(bc)$ .
- F3. Distributive Laws** For all real numbers  $a$ ,  $b$  and  $c$ ,  $a(b+c) = ab+ac$  and  $(b+c)a = ba+bc$ .
- F4. Existence of Identity Elements** There exists two distinct real numbers, denoted  $0$  and  $1$ , such that for every real number  $a$ ,  $0+a = a+0$  and  $1 \cdot a = a \cdot 1$ .
- F5. Existence of Additive Inverses** For every real number  $a$ , there is a real number, denoted  $-a$  and called the **additive inverse** of  $a$ , such that  $a+(-a) = (-a)+a = 0$ .
- F6. Existence of Reciprocals** For every real number  $a \neq 0$ , there is a real number, denoted  $1/a$  or  $a^{-1}$ , called the **reciprocal** of  $a$ , such that  $a \cdot (\frac{1}{a}) = (\frac{1}{a}) \cdot a = 1$ .
- T1. Cancellation Law for Addition** If  $a+b = a+c$ , then  $b = c$ . (In particular, this shows that the number  $0$  of Axiom F4 is unique.)
- T2. Possibility of Subtraction** Given  $a$  and  $b$ , there is exactly one  $x$  such that  $a+x = b$ . This  $x$  is denoted by  $b-a$ . In particular,  $0-a$  is the additive inverse of  $a$ ,  $-a$ .
- T3.**  $b-a = b+(-a)$ .
- T4.**  $-(-a) = a$ .
- T5.**  $a(b-c) = ab-ac$ .
- T6.**  $0 \cdot a = a \cdot 0 = 0$ .
- T7. Cancellation Law for Multiplication** If  $ab = bc$  and  $a \neq 0$ , then  $b = c$ . (In particular, this shows that the number  $1$  of Axiom F4 is unique.)
- T8. Possibility of Division** Given  $a$  and  $b$  with  $a \neq 0$ , there is exactly one  $x$  such that  $ax = b$ . This  $x$  is denoted by  $b/a$  and is called the **quotient** of  $b$  and  $a$ . In particular,  $1/a$  is the reciprocal of  $a$ .
- T9.** If  $a \neq 0$ , then  $b/a = b \cdot a^{-1}$ .
- T10.** If  $a \neq 0$ , then  $(a^{-1})^{-1} = a$ .
- T11. Zero Product Property** If  $ab = 0$ , then  $a = 0$  or  $b = 0$ .
- T12. Rule for Multiplication with Negative Signs**  
 $(-a)b = a(-b) = -(ab)$ , and  $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ .
- T13. Equivalent Fractions Property**  $\frac{a}{b} = \frac{ac}{bc}$ , if  $b \neq 0$  and  $c \neq 0$ .
- T14. Rule for Addition of Fractions**  $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ , if  $b \neq 0$  and  $d \neq 0$ .
- T15. Rule for Multiplication of Fractions**  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ , if  $b \neq 0$  and  $d \neq 0$ .
- T16. Rule for Division of Fractions**  $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ , if  $b \neq 0$ ,  $c \neq 0$  and  $d \neq 0$ .
- T17. Trichotomy Law** For arbitrary real numbers  $a$  and  $b$ , exactly one of these three relations  $a < b$ ,  $b > a$  or  $a = b$  holds.
- T18. Transitive Law** If  $a < b$  and  $b < c$ , then  $a < c$ .
- T19.** If  $a < b$ , then  $a+c < b+c$ .
- T20.** If  $a < b$  and  $c > 0$ , then  $ac < bc$ .
- T21.** If  $a \neq 0$ , then  $a^2 > 0$ .
- T22.**  $1 > 0$ .
- T23.** If  $a < b$  and  $c < 0$ , then  $ac > bc$ .
- T24.** If  $a < b$ , then  $-a > -b$ . In particular, if  $a < 0$ , then  $-a > 0$ .
- T25.** If  $ab > 0$ , then both  $a$  and  $b$  are positive or both are negative.
- T26.** If  $a < c$  and  $b < d$ , then  $a+b < c+d$ .
- T27.** If  $0 < a < c$  and  $0 < b < d$ , then  $0 < ab < cd$ .
- Ord1.** For any real numbers  $a$  and  $b$ , if  $a$  and  $b$  are positive, so are  $a+b$  and  $ab$ .
- Ord2.** For every real number  $a \neq 0$ , either  $a$  is positive or  $-a$  is positive but not both.
- Ord3.** The number  $0$  is not positive.
- Definition** Given real numbers  $a$  and  $b$ ,  $a < b$  means  $b+(-a)$  is positive.  $b > a$  means  $a < b$ .  $a \leq b$  means  $a < b$  or  $a = b$ .  $b \geq a$  means  $a \leq b$ . If  $a < 0$ , we say that  $a$  is **negative**. If  $a \geq 0$ , we say that  $a$  is **non-negative**.
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- Note:** Whenever you are proving a universal statement using an arbitrary particular, you should quote **WLOG** (Without Loss Of Generality). This means that the proof for the special case can be easily applied to all other cases.

# Definitions

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**Divisibility** If  $n, d \in \mathbb{Z}$  and  $d \neq 0$ ,  $d|n \Leftrightarrow \exists k \in \mathbb{Z}$  such that  $n = dk$ .

**Rational Numbers**  $r$  is rational  $\Leftrightarrow \exists a, b \in \mathbb{Z}$  s.t.  $r = \frac{a}{b}$  and  $b \neq 0$ .

**Fraction in lowest term** A fraction  $\frac{a}{b}$  where  $b \neq 0$  is said to be in **lowest terms** if the largest integer that divides both  $a$  and  $b$  is 1.

**Prime and Composite** An integer  $n$  is **prime** iff  $n > 1$  and for all positive integers  $r$  and  $s$ , if  $n = rs$ , then either  $r$  or  $s$  equals  $n$ . An integer  $n$  is **composite** iff  $n > 1$  and  $n = rs$  for some integers  $r$  and  $s$  with  $1 < r < n$  and  $1 < s < n$ . In symbols,

**$n$  is prime:**  $(n > 1) \wedge \forall r, s \in \mathbb{Z}^+, (n = rs \rightarrow (r = 1 \wedge s = n) \vee (r = n \wedge s = 1))$ .

**$n$  is composite:**  $\exists r, s \in \mathbb{Z}^+ (n = rs \wedge (1 < r < n) \wedge (1 < s < n))$ .

## Compound Statements

**2.1.1 Statement** A **statement** (or **proposition**) is a sentence that is true or false, but not both.

**2.1.2 Negation** If  $p$  is a variable, the **negation** of  $p$  is "not  $p$ " or it is not the case that  $p$ " and is denoted  $\sim p$ .

**2.1.3 Conjunction** If  $p$  and  $q$  are statement variables, the conjunction of  $p$  and  $q$  is " $p$  and  $q$ ", denoted  $p \wedge q$ .

**2.1.4 Disjunction** If  $p$  and  $q$  are statement variables, the disjunction of  $p$  and  $q$  is " $p$  or  $q$ ", denoted  $p \vee q$ .

**2.1.5 Statement Form** A statement form (or propositional form) is an expression made up of statement variables and logical connectives that becomes a statement when actual statements are substituted for the component statement variables.

**2.1.6 Logical Equivalence** Two statement forms are called logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms  $P$  and  $Q$  is denoted by  $P \equiv Q$ .

**2.1.7 Tautology** A tautology is a statement form that is **always true** regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement**.

**2.1.8 Contradiction** A contradiction is a statement form that is **always false** regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a **contradictory statement**.

**2.2.1 Conditional** If  $p$  and  $q$  are statement variables, the conditional of  $q$  by  $p$  is "if  $p$  then  $q$ " or " $p$  implies  $q$ ", denoted  $p \rightarrow q$ . It is false when  $p$  is true and  $q$  is false; otherwise it is true. We called  $p$  the **hypothesis** (or **antecedent**) of the conditional and  $q$  the **conclusion** (or **consequent**).

**2.2.2 Contrapositive** The contrapositive of a conditional statement of the form "if  $p$  then  $q$ " is "if  $\sim q$  then  $\sim p$ ". Symbolically, the contrapositive of  $p \rightarrow q$  is  $\sim q \rightarrow \sim p$ .

**2.2.3 Converse** The **converse** of a conditional statement "if  $p$  then  $q$ " is "if  $q$  then  $p$ ". Symbolically, the converse of  $p \rightarrow q$  is  $q \rightarrow p$ .

**2.2.4 Inverse** The **inverse** of a conditional statement "if  $p$  then  $q$ " is "if  $\sim p$  then  $\sim q$ ". Symbolically, the inverse of  $p \rightarrow q$  is  $\sim p \rightarrow \sim q$ .

Note that  $p \rightarrow q \not\equiv q \rightarrow p$ .

**2.2.5 Only If** If  $p$  and  $q$  are statements, " $p$  only if  $q$ " means "if not  $q$  then not  $p$ " or  $\sim q \rightarrow \sim p$ . Or, equivalently, "if  $p$  then  $q$ " or " $p \rightarrow q$ ".

**2.2.6 Biconditional** Given statement variables  $p$  and  $q$ , the **biconditional** of  $p$  and  $q$  is " $p$  if, and only if,  $q$ " and is denoted  $p \leftrightarrow q$ . It is true if both  $p$  and  $q$  have the same truth values and is false if  $p$  and  $q$  have opposite truth values. The words *if and only if* are sometimes abbreviated as *iff*.

**2.2.7 Necessary and Sufficient Conditions** If  $r$  and  $s$  are statements,  $r$  is a sufficient condition for  $s$  means "if  $r$  then  $s$ " or  $r \rightarrow s$ , and " $r$  is a necessary condition for  $s$ " means "if  $s$  then  $r$ " or  $s \rightarrow r$ .  $r$  is a necessary and sufficient condition for  $s$  means " $r$  if and only if  $s$ " or  $r \leftrightarrow s$ .

**2.3.1 Argument** An **argument** (**argument form**) is a sequence of statements (statement forms). All statements in an argument (argument form), except for the final one, are called **premises** (or **assumptions** or **hypothesis**). The final statement (statement form) is called the **conclusion**. The symbol  $\bullet$ , which is read “therefore”, is normally placed just before the conclusion. To say that an argument form is valid means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true.

**2.3.2 Sound and Unsound Argument** An argument is called **sound** if, and only if, it is valid and all its premises are true. An argument that is not sound is called **unsound**.

## Quantified Statements

**3.1.1 Predicate** A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

“Domain” may also be known as “domain of discourse”, “universe of discourse”, “universal set”, or simply “universe”.

**3.1.2 Truth Set** If  $P(x)$  is a predicate and  $x$  has a domain  $D$ , the **truth set** is the set of all elements of  $D$  that make  $P(x)$  true when they are substituted for  $x$ . The truth set for  $P(x)$  is denoted as  $\{x \in D | P(x)\}$ .

**3.1.3 Universal Statement** Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ . A **universal statement** is a statement of the form “ $\forall x \in D, Q(x)$ ”. It is defined to be true iff  $Q(x)$  is **true for every**  $x$  in  $D$ . It is defined false iff  $Q(x)$  is **false for at least one**  $x$  in  $D$ . A value for  $x$  for which  $Q(x)$  is false is called a **counterexample**.

**3.1.4 Existential Statement** Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ . An **existential statement** is a statement of the form “ $\exists x \in D, Q(x)$ ”. It is defined to be true iff  $Q(x)$  is **true for at least one**  $x$  in  $D$ . It is defined false iff  $Q(x)$  is **false for all**  $x$  in  $D$ .

The  $\exists!$  is used to denote “there exists a unique” or “there is one and only one”.

**3.2.1 Contrapositive, converse, inverse** Consider a statement of the form:  $\forall x \in D (P(x) \rightarrow Q(x))$ .

1. It’s **contrapositive** is:  $\forall x \in D (\sim Q(x) \rightarrow \sim P(x))$ .

2. It’s **converse** is:  $\forall x \in D (Q(x) \rightarrow P(x))$ .

3. It’s **inverse** is:  $\forall x \in D (\sim P(x) \rightarrow \sim Q(x))$ .

**3.2.2 Necessary and Sufficient conditions, Only if** “ $\forall x, r(x)$  is a **sufficient condition** for  $s(x)$ ” means  $\forall x (r(x) \rightarrow s(x))$ .

“ $\forall x, r(x)$  is a **necessary condition** for  $s(x)$ ” means  $\forall x (\sim r(x) \rightarrow \sim s(x))$  or equivalently, “ $\forall x (s(x) \rightarrow r(x))$ ”.

“ $\forall x, r(x)$  **only if**  $s(x)$ ” means  $\forall x (\sim s(x) \rightarrow \sim r(x))$  or equivalently, “ $\forall x (r(x) \rightarrow s(x))$ ”.

**Universal Modus Ponens**  $\forall x (P(x) \rightarrow Q(x)). \quad P(a)$  for a particular  $a. \quad \bullet Q(a)$ .

**Universal Modus Tollens**  $\forall x (P(x) \rightarrow Q(x)). \quad \sim Q(a)$  for a particular  $a. \quad \bullet \sim P(a)$ .

**3.4.1 Valid Argument Form** To say that an **argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An argument is called **valid** if, and only if, its form is valid.

**Converse Error (Quantified Form)**  $\forall x (P(x) \rightarrow Q(x)). \quad Q(a)$  for a particular  $a. \quad \bullet P(a)$ .

**Inverse Error (Quantified Form)**  $\forall x (P(x) \rightarrow Q(x)). \quad \sim P(a)$  for a particular  $a. \quad \bullet \sim Q(a)$ .

**Universal Transitivity**  $\forall x (P(x) \rightarrow Q(x)). \quad \forall x (Q(x) \rightarrow R(x)). \quad \bullet \forall x (P(x) \rightarrow R(x))$ .

**Additional Notes** (from Tutorial 2)

Equivalent expressions:  $\forall x \in D, P(X) \equiv \forall x ((x \in D) \wedge P(X))$ .

Well-formed formulas (wff): **true** and **false** are wffs. A propositional variable (e.g.  $x, p$ ) is a wff. A predicate name followed by a list of variables (e.g.  $P(x), Q(x, y)$ ), which is called an *atomic formula*, is a wff. If  $A, B$  and  $C$  are wffs, then so are  $\sim A, (A \wedge B), (A \vee B), (A \rightarrow B)$  and  $(A \leftrightarrow B)$ . If  $x$  is a propositional variable and  $A$  is a wff, then so are  $\forall x A$  and  $\exists x A$ .

Scope of quantifiers / bound variables / use of parentheses:

The *scope* of a quantifier is the range in the formula where the quantifier “engages in”. It is put right after the quantifier and is usually in parentheses.

Example:  $\forall x \exists y P(x, y)$  - both  $x$  and  $y$  are bound. However,

$\forall x (\exists y P(x, y) \vee Q(x, y))$  - in  $Q(x, y)$ ,  $x$  is bound but  $y$  is free as the  $\exists y$  quantifier applies only to  $P(x, y)$ .

If you want the  $y$  in  $Q(x, y)$  to be bound as well, you have to put parentheses over the entire formula, i.e.  $\exists y (P(x, y) \vee Q(x, y))$ , in which case you can just remove the outermost parentheses and it just becomes  $\forall x \exists y (P(x, y) \vee Q(x, y))$ .

Tip for negating quantified statements: if you need to negate nested quantifiers, just flip each of the quantifier symbols ( $\forall$  to  $\exists$  and vice versa) and apply the negation to the inner predicate, then apply De Morgan’s laws from there

## Sets

**Set-Roster Notation** A set may be specified by writing all of its elements between braces. Examples:  $\{1, 2, 3\}$ ,  $\{1, 2, 3, \dots, 100\}$ ,  $\{1, 2, 3, \dots\}$ . (The symbol  $\dots$  is called an ellipsis and is read “and so forth”).

**Membership of a Set (Notation:  $\in$ )** If  $S$  is a set, the notation  $x \in S$  means that  $s$  is an element of  $S$ . ( $x \notin S$  means  $x$  is not an element of  $S$ .)

**Cardinality of a Set (Notation:  $|S|$ )** The cardinality of a set  $S$ , denoted as  $|S|$ , is the size of the set, that is, the number of elements in  $S$ .

**Set Builder Notation** Let  $U$  be a set and  $P(x)$  be a predicate over  $U$ . Then the set of all elements  $x \in U$  such that  $P(x)$  is true is denoted as  $\{x \in U : P(x)\}$  or  $\{x \in U | P(x)\}$  which reads as “the set of all  $x$  in  $U$  such that  $P(x)$  is true”.

**Replacement Notation** Let  $A$  be a set and  $t(x)$  be a term in a variable  $x$ . Then the set of all objects of the form  $t(x)$  where  $x$  ranges over the elements of  $A$  is denoted  $\{t(x) : x \in A\}$  or  $\{t(x) | x \in A\}$  which is read as “the set of all  $t(x)$ ” where  $x \in A$ .

**Subset and superset** Let  $A$  and  $B$  be sets.  $A$  is a **subset** of  $B$ , written  $A \subseteq B$ , iff every element of  $A$  is also an element of  $B$ . Symbolically,  $A \subseteq B$  iff  $\forall x (x \in A \Rightarrow x \in B)$ . Another way of saying “ $A$  is a subset of  $B$ ” is “ $A$  is contained in  $B$ ”. If  $A \subseteq B$ , we may also write  $B \supseteq A$  which reads as “ $B$  is contained in  $A$ ” or “ $B$  includes  $A$ ” or “ $B$  is a superset of  $A$ ”.

**Proper Subset** Let  $A$  and  $B$  be sets.  $A$  is a **proper subset** of  $B$ , denoted  $A \subset B$ , iff  $A \subseteq B$  and  $A \neq B$ . In this case, we may say that the inclusion of  $A$  in  $B$  is proper or strict.

**Ordered Pair** An **ordered pair** is an expression of the form  $(x, y)$ . Two ordered pairs  $(a, b)$  and  $(c, d)$  are equal iff  $a = c$  and  $b = d$ . Symbolically,  $(a, b) = (c, d) \Rightarrow (a = c) \wedge (b = d)$ .

**Cartesian Product** Given sets  $A$  and  $B$ , the **Cartesian product** of  $A$  and  $B$ , denoted  $\mathbf{A} \times \mathbf{B}$  and read “ $A$  cross  $B$ ”, is the set of all ordered pairs  $(a, b)$  where  $a$  is in  $A$  and  $b$  is in  $B$ . Symbolically,  $A \times B = \{(a, b) : a \in A \wedge b \in B\}$ .

**Set Equality** Given sets  $A$  and  $B$ ,  $A$  equals  $B$ , written  $A = B$  iff every element of  $A$  is in  $B$  and every element of  $B$  is in  $A$ . Symbolically,  $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$ . (Alternative definition:  $A = B \Leftrightarrow \forall x (x \in A \Leftrightarrow x \in B)$ ).

**Universal set / Universe of Discourse** The context or domain of the problem.

**Union** The **union** of  $A$  and  $B$ , denoted  $\mathbf{A} \cup \mathbf{B}$ , is the set of all elements that are in at least one of  $A$  or  $B$ . Symbolically,  $A \cup B = \{x \in U : x \in A \vee x \in B\}$ .

**Intersection** The **intersection** of  $A$  and  $B$ , denoted  $\mathbf{A} \cap \mathbf{B}$ , is the set of all elements that are common to both  $A$  and  $B$ . Symbolically,  $A \cap B = \{x \in U : x \in A \wedge x \in B\}$ .

**Difference** The **difference** of  $B$  minus  $A$  (or **relative complement** of  $A$  in  $B$ ), denoted  $\mathbf{B} - \mathbf{A}$ , or  $\mathbf{B} \setminus \mathbf{A}$ , is the set of all elements that are in  $B$  and not  $A$ . Symbolically,  $B \setminus A = \{x \in U : x \in B \wedge x \notin A\}$ .

**Complement** The complement of  $A$ , denoted  $\overline{A}$ , is the set of all elements in  $U$  that are not in  $A$ . Symbolically,  $\overline{A} = \{x \in U \mid x \notin A\}$ .

**Unions and Intersections of an Indexed Collection of Sets** Given sets  $A_0, A_1, A_2, \dots$  that are subsets of a universal set  $U$  and a given nonnegative integer  $n$ ,

$$\bigcup_{i=0}^n A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \dots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^n A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}$$

**Disjoint** Two sets are **disjoint** iff they have no elements in common. Symbolically:  $A$  and  $B$  are disjoint iff  $A \cap B = \emptyset$ .

**Mutually disjoint** Sets  $A_1, A_2, A_3, \dots$  are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) iff no two sets  $A_i$  and  $A_j$  with distinct subscripts have any elements in common, i.e. for all  $i, j = 1, 2, 3, \dots$   $A_i \cap A_j = \emptyset$  wherever  $i \neq j$ .

**Power Set** Given a set  $A$ , the **power set** of  $A$ , denoted  $P(A)$ , is the set of all subsets of  $A$ . (symbol for power set is  $\wp$ )

**Ordered  $n$ -tuples** Let  $n \in \mathbb{Z}^+$  and let  $x_1, x_2, \dots, x_n$  be (not necessarily distinct) elements. An **ordered  $n$ -tuple** is an expression of the form  $(x_1, x_2, \dots, x_n)$ . Equality of two ordered  $n$ -tuples:  $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \Leftrightarrow x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$ .

**Cartesian product** Given sets  $A_1, A_2, \dots, A_n$ , the **Cartesian product** of  $x_1, x_2, \dots, x_n$ , denoted  $A_1 \times A_2 \times \dots \times A_n$ , is the set of all ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ .

$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) : a_1 \in A_1 \wedge a_2 \in A_2 \wedge \dots \wedge a_n \in A_n\}$ . If  $A$  is a set, then  $A^n = A \times A \times \dots \times A$ .

**Procedural Versions of Set Definitions** Let  $X$  and  $Y$  be subsets of a universal set  $U$  and suppose  $a$  and  $b$  are elements of  $U$ .

1.  $a \in X \cup Y \Leftrightarrow a \in X \vee a \in Y$ .
2.  $a \in X \cap Y \Leftrightarrow a \in X \wedge a \in Y$ .
3.  $a \in X - Y \Leftrightarrow a \in X \wedge a \notin Y$ .
4.  $a \in \overline{X} \Leftrightarrow a \notin X$ .
5.  $(a, b) \in X \times Y \Leftrightarrow a \in X \wedge b \in Y$ .

Note: In a context where  $U$  is the universal set (so that implicitly means  $U \supseteq X$ ), the complement of  $X$ , denoted  $\overline{X}$  or  $X^c$ , is defined by  $\overline{X} = U \setminus X$ .

## Relations

**Relation** Let  $A$  and  $B$  be sets. A (binary) **relation from  $A$  to  $B$**  is a subset of  $A \times B$ .  $x$  is related to  $y$  by  $R$ , or  $x$  is related to  $y$ , written  $x R y$ , iff  $(x, y) \in R$ .

**Domain, Co-Domain, Range** Let  $A$  and  $B$  be sets and  $R$  be a relation from  $A$  to  $B$ . The **domain of  $R$** ,  $Dom(R)$ , is the set  $\{a \in A : a R b \text{ for some } b \in B\}$ . The **co-domain of  $R$** ,  $coDom(R)$ , is the set  $B$ . The **range of  $R$** ,  $Range(R)$ , is the set  $\{b \in B : a R b \text{ for some } a \in A\}$ .

**Inverse of a Relation** Let  $R$  be a relation from  $A$  to  $B$ . Define the **inverse relation**  $R^{-1}$  from  $B$  to  $A$  as follows:  $R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}$ .

**Relation on a Set** A **relation on a set  $A$**  is a relation from  $A$  to  $A$ . In other words, a relation on set  $A$  is a subset of  $A \times A$ . (The arrow diagram can be modified such that it becomes a **directed graph**).

**Composition of Relations** Let  $A, B$  and  $C$  be sets. Let  $R \subseteq A \times B$  be a relation. Let  $S \subseteq B \times C$  be a relation. The **composition of  $R$  with  $S$** , denoted  $S \circ R$ , is the relation from  $A$  to  $C$  such that:  $\forall x \in A, \forall z \in C (x S \circ R z \Leftrightarrow (\exists y \in B (x R y \wedge y S z)))$ .

**Proposition: Composition is Associative (Lecture 6 Slide 18)** Let  $A, B, C, D$  be sets. Let  $R \subseteq A \times B, S \subseteq B \times C$  and  $T \subseteq C \times D$  be relations.  $T \circ (S \circ R) = (T \circ S) \circ R = T \circ S \circ R$ .

**Proposition: Inverse of Composition (Lecture 6 Slide 18)** Let  $A, B$  and  $C$  be sets. Let  $R \subseteq A \times B$  and  $S \subseteq B \times C$  be relations. Then  $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ .

**$n$ -ary Relation** Given  $n$  sets  $A_1, A_2, \dots, A_n$ , an  **$n$ -ary relation**  $R$  on  $A_1 \times A_2 \times \dots \times A_n$  is a subset of  $A_1 \times A_2 \times \dots \times A_n$ . The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary**, **ternary** and **quaternary relations** respectively.

**Reflexivity, Symmetry, Transitivity** Let  $R$  be a relation on a set  $A$ .

1.  $R$  is **reflexive** iff  $\forall x \in A(xRx)$ .
2.  $R$  is **symmetric** iff  $\forall x, y \in A(xRy \rightarrow yRx)$ .
3.  $R$  is **transitive** iff  $\forall x, y, z \in A(xRy \wedge yRz \rightarrow xRz)$ .

Note: for transitivity, if either of the premises are false,  $R$  is transitive as the argument is vacuously true. Reflexivity, symmetry and transitivity are **properties of a relation**, not properties of members of the set. You say that a relation is reflexive or not reflexive, while an element is related or not related to itself.

**Transitive Closure** Let  $A$  be a set and  $R$  a relation on  $A$ . The transitive closure of  $R$  is the relation  $R^t$  on  $A$  that satisfies the following three properties:

1.  $R^t$  is transitive.
2.  $R \subseteq R^t$ .
3. If  $S$  is any other transitive relation that contains  $R$  then  $R^t \subseteq S$ .

**Reflexive Closure (Tutorial 5 Q5)** The reflexive closure  $S$  of a relation  $R$  on a set  $A$  is obtained by adding  $(a, a)$  to  $R$  for each  $a \in A$ . Symbolically,  $S = R \cup \{(x, x) : x \in X\}$ .

**Partition**  $\mathcal{C}$  is a **partition** of a set  $A$  if the following hold:

1.  $\mathcal{C}$  is a set of which all elements are non-empty subsets of  $A$ , i.e.,  $\emptyset \neq S \subseteq A$  for all  $S \in \mathcal{C}$ .
2. Every element of  $A$  is in exactly one element of  $\mathcal{C}$ , i.e.,  $\forall x \in A \exists S \in \mathcal{C}(x \in S)$  and  $\forall x \in A \exists S_1, S_2, \in \mathcal{C}(x \in S_1 \wedge x \in S_2 \rightarrow S_1 = S_2)$ .

(In simpler terms:  $\mathcal{C}$  is a partition of set  $A$  if  $\mathcal{C}$  is a set of all elements which are nonempty subsets of  $A$ , and every element of  $A$  is in exactly one component of  $\mathcal{C}$ ).

Elements of a partition are called **components** of the partition.

**Partition (shorter definition)** A **partition** of set  $A$  is a set  $\mathcal{C}$  of non-empty subsets of  $A$  such that  $\forall x \in A \exists! S \in \mathcal{C}(x \in S)$ .

**Relation Induced by a Partition** Given a partition  $\mathcal{C}$  of a set  $A$ , the relation  $R$  **induced by the partition** is defined on  $A$  as follows:  $\forall x, y \in A, xRy \Leftrightarrow \exists$  a component  $S$  of  $\mathcal{C}$  s.t.  $x, y \in S$ .

**Equivalence Relation** Let  $A$  be a set and  $R$  a relation on  $A$ .  $R$  is an **equivalence relation** iff  $R$  is reflexive, symmetric and transitive. Note: the symbol  $\sim$  is commonly used to denote an equivalence relation.

**Equivalence Class** Suppose  $A$  is a set and  $\sim$  is an equivalence relation on  $A$ . For each  $a \in A$ , the **equivalence class** of  $a$ , denoted  $[a]$  and called the **class of  $a$**  for short, is the set of all elements  $x \in A$  s.t.  $a$  is  $\sim$ -related to  $x$ . Symbolically,  $[a]_{\sim} = \{x \in A : a \sim x\}$ . The procedural definition is:  $\forall x \in A(x \in [a]_{\sim} \Leftrightarrow a \sim x)$ .

**Proof (Tutorial 4 Q9(a)):** If  $x \in S \in \mathcal{C}$ , then  $[x] = S$ . (If  $x$  is an element of a component  $S$  which is an element of a partition, then the equivalence class of  $x$  is  $S$ .)

Tip: think of classes as “school buses” - two students are in the same equivalence class if they are in the same “school bus”.

**Congruence** Let  $a, b \in \mathbb{Z}$  and  $n \in \mathbb{Z}^+$ . Then  $a$  is congruent to  $b$  modulo  $n$  iff  $a - b = nk$  for some  $k \in \mathbb{Z}$ . In other words,  $n|(a - b)$ . In this case, we write  $a \equiv b \pmod{n}$ .

**Proposition (Lecture 6 Slide 54)** Congruence-mod  $n$  is an equivalence relation on  $\mathbb{Z}$  for every  $n \in \mathbb{Z}^+$ .

**Set of equivalence classes** Let  $A$  be a set and  $\sim$  be an equivalence relation on  $A$ . Denote by  $A/\sim$  the set of all equivalence classes with respect to  $\sim$ , i.e.,  $A/\sim = \{x_{\sim} : x \in A\}$ . We may read  $A/\sim$  as “the quotient of  $A$  by  $\sim$ ”.

**Proof (Tutorial 4 Q9(b)):**  $A/\sim = \mathcal{C}$  (The set of equivalence classes of  $A$  is a partition of  $A$ .)

**Antisymmetry** Let  $R$  be a relation on a set  $A$ .  $R$  is **antisymmetric** iff  $\forall x, y \in A(xRy \wedge yRx \rightarrow x = y)$ .

**Asymmetry (Tutorial 5 Q6)** Let  $R$  be a binary relation on a set  $A$ .  $R$  is **asymmetric** iff  $\forall x, y \in A(xRy \rightarrow y \not R x)$ .

**Tutorial 5 Q6(c)** All asymmetric relations are antisymmetric.

**Partial Order Relations** Let  $R$  be a relation on a set  $A$ . Then  $R$  is a **partial order relation** (or simply **partial order**) iff  $R$  is reflexive, antisymmetric and transitive.

Note: the symbol  $\preceq$  is often used to refer to a general partial order, and the notation  $x \preceq y$  is read as “ $x$  is curly less than or equal to  $y$ ”.

**Proof (Tutorial 5 Q3):** Binary relation  $\subseteq$  on  $P(A)$  is a partial order.

**Partially Ordered Sets** A set  $A$  is called a **partially ordered set** (or **poset**) with respect to a partial order relation  $R$  on  $A$ , denoted by  $(A, R)$ .

**Hasse Diagram** Let  $\preceq$  be a partial order on a set  $A$ . A **Hasse diagram** of  $\preceq$  satisfies the following condition for all distinct  $x, y, m \in A$ : If  $x \preceq y$  and no  $m \in A$  is such that  $x \preceq m \preceq y$ , then  $x$  is placed below  $y$  with a line joining them, else no line joins  $x$  and  $y$ .

(Tip: to obtain a Hasse Diagram, start with a directed graph of the relation, placing vertices on the page so that all arrows point upwards. Then **eliminate** 1. the loops at all the vertices, 2. all arrows whose existence is implied by the transitive property, and 3. the direction indicators on the arrows.)

**Comparability** Suppose  $\preceq$  is a partial order relation on a set  $A$ . Elements  $a$  and  $b$  of  $A$  are said to be **comparable** iff either  $a \preceq b$  or  $b \preceq a$ . Otherwise,  $a$  and  $b$  are **noncomparable**.

**Compatible (Tutorial 5 Q7)** Elements  $a, b$  are **compatible** iff there exists  $c \in A$  such that  $a \preceq c$  and  $b \preceq c$ .

**Maximal/Minimal/Largest/Smallest Element**

1.  $c$  is a **maximal element** of  $A$  iff  $\forall x \in A$ , either  $x \preceq c$ , or  $x$  and  $c$  are not comparable. Alternatively,  $c$  is a maximal element of  $A$  iff  $\forall x \in A (c \preceq x \rightarrow c = x)$ .
2.  $c$  is a **minimal element** of  $A$  iff  $\forall x \in A$ , either  $c \preceq x$ , or  $x$  and  $c$  are not comparable. Alternatively,  $c$  is a minimal element of  $A$  iff  $\forall x \in A (x \preceq c \rightarrow c = x)$ .
3.  $c$  is the **largest element** of  $A$  iff  $\forall x \in A (x \preceq c)$ .
4.  $c$  is the **smallest element** of  $A$  iff  $\forall x \in A (c \preceq x)$ .

Note: Alternative terms: Largest element = greatest element = maximum; smallest element = least element = minimum.

**Proposition (Lecture 6 Slide 83)** Consider a partial order  $\preceq$  on a set  $A$ . Any smallest element is minimal. (Likewise, any largest element is maximal.)

**Total Order Relations** If  $R$  is a partial order relation on a set  $A$ , and for any two elements  $x$  and  $y$  in  $A$ , either  $xRy$  or  $yRx$ , then  $R$  is a **total order relation** (or simply **total order**) on  $A$ . In other words,  $R$  is a total order iff  $R$  is a partial order and  $\forall x, y \in A (xRy \vee yRx)$ .

**Linearization of a partial order** Let  $\preceq$  be a partial order on a set  $A$ . A **linearization** of  $\preceq$  is a total order  $\preceq^*$  on  $A$  such that  $\forall x, y \in A (x \preceq y \rightarrow x \preceq^* y)$ .

**Well-Ordered Set** Let  $\preceq$  be a total order on a set  $A$ .  $A$  is **well-ordered** iff every non-empty subset of  $A$  contains a smallest element. Symbolically,  $\forall S \in P(A), S \neq \emptyset \rightarrow (\exists x \in S \forall y \in S (x \preceq y))$ .

**Tutorial 5 Discussion Q1** Let  $R$  be a binary relation on a non-empty set  $A$ . If  $R = \emptyset$ , then  $R$  is not reflexive, but it is symmetric and transitive (vacuously true).

## Functions

**Function** A function  $f$  from a set  $X$  to a set  $Y$ , denoted  $f : X \rightarrow Y$ , is a relation satisfying the following properties:

- (F1)  $\forall x \in X \exists y \in Y (x, y) \in f$
- (F2)  $\forall x \in X \forall y_1, y_2 \in Y ((x, y_1) \in f \wedge (x, y_2) \in f) \rightarrow y_1 = y_2$
- (F3)  $\forall x_1, x_2 \in X (x_1 = x_2 \rightarrow f(x_1) = f(x_2))$

**Function (alternative definition)** Let  $f$  be a relation on sets  $X$  and  $Y$ , i.e.  $f \subseteq X \times Y$ . Then  $f$  is a function from  $X$  to  $Y$ , denoted  $f : X \rightarrow Y$ , iff  $\forall x \in X \exists! y \in Y (x, y) \in f$ . Informally, a function from  $X$  and  $Y$  is an assignment of each element of  $X$  to **exactly one element** of  $Y$ .

**Another view of function** Let  $f : X \rightarrow Y$  be the type signature of function.  $\forall x \in X \exists y \in Y, \{y\} = \{b \mid (x, b) \in f\}$ .



**Argument, image, preimage, input, output** Let  $f : X \rightarrow Y$  be a function. We write  $f(x) = y$  iff  $(x, y) \in f$ . We say that “ $f$  sends/maps  $x$  to  $y$ ” and we may also write  $x \rightarrow y$  or  $f : x \mapsto y$ . Also,  $x$  is called the **argument** of  $f$ .  $f(x)$  is read “ $f$  of  $x$ ” or “the **output** of  $f$  for the **input**  $x$ ”, or “the value of  $f$  at  $x$ ”, or “the **image** of  $x$  under  $f$ ”. If  $f(x) = y$ , then  $x$  is a **preimage** of  $y$ .

**Setwise image and preimage** Let  $f : X \rightarrow Y$  be a function from set  $X$  and set  $Y$  and  $f : P(X) \rightarrow P(Y)$

- If  $A \subseteq X$ , then let  $f(A) = \{f(x) : x \in A\}$ .
- If  $B \subseteq Y$ , then let  $f^{-1}(B) = \{x \in X : f(x) \in B\}$ .

We call  $f(A)$  the **(setwise) image** of  $A$ , and  $f^{-1}(B)$  the **(setwise) preimage** of  $B$  under  $f$ .

**Domain, Co-domain, Range** Let  $f : X \rightarrow Y$  be a function from set  $A$  to set  $B$ .

- $A$  is the **domain** of  $f$  and  $B$  the **co-domain** of  $f$ .
- The **range** of  $f$  is the (setwise) image of  $A$  under  $f$ :  $\{b \in B : b = f(x) \text{ for some } a \in A\}$ .

**Sequence (of infinite length)** A sequence  $a_0, a_1, a_2, \dots$  can be represented by a function  $a$  whose domain is  $\mathbb{Z}_{\geq 0}$  that satisfies  $a(n) = a_n$  for every  $n \in \mathbb{Z}_{\geq 0}$ .

**Fibonacci Sequence** The **Fibonacci Sequence**  $F_0, F_1, F_2, \dots$  is defined by setting, for each  $n \in \mathbb{Z}_{\geq 0}$ ,  $F_0 = 0$  and  $F_1 = 1$  and  $F_{n+2} = F_{n+1} + F_n$ .

**String (of finite length)** Let  $A$  be a set. A **string** or word over  $A$  is an expression of the form  $a_0 a_1 a_2 \dots a_{l-1}$  where  $l \in \mathbb{Z}_{\geq 0}$  and  $a_0 a_1 a_2 \dots a_{l-1} \in A$ . Here  $l$  is called the **length** of the string. The **empty string**  $\varepsilon$  is the string of length 0.

**Equality of Sequences** Given two sequences  $a_0, a_1, a_2, \dots$  and  $b_0, b_1, b_2, \dots$  defined by the functions  $a(n) = a_n$  and  $b(n) = b_n$  respectively for every  $n \in \mathbb{Z}_{\geq 0}$ , we say that the two sequences are equal if and only if  $a(n) = b(n)$  for every  $n \in \mathbb{Z}_{\geq 0}$ .

**Equality of Strings** Given two strings  $s_1 = a_0 a_1 a_2 \dots a_{l-1}$  and  $s_2 = b_0 b_1 b_2 \dots b_{l-1}$  where  $l \in \mathbb{Z}_{\geq 0}$ , we say that  $s_1 = s_2$  if and only if  $a_i = b_i$  for all  $i \in \{0, 1, 2, \dots, l-1\}$ .

**Injection (one-to-one function)** A function  $f : X \rightarrow Y$  is **injective** (or **one-to-one**) iff  $\forall x_1, x_2 \in X (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$ , or equivalently (contrapositive),  $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$ . An injective function is called an **injection**. Informally, every element in the codomain must have **at most one arrow** going into it.

**Surjective (onto function)** A function  $f : X \rightarrow Y$  is **surjective** (or **onto**) iff  $\forall y \in Y \exists x \in X (y = f(x))$ . Every element in the co-domain has at least one preimage. So, range = co-domain. A surjective function is called a **surjection**. Informally, every element in the codomain must have **at least one arrow** going into it.

**Bijection (one-to-one correspondence)** A function  $f : X \rightarrow Y$  is **bijective** iff  $f$  is injective and surjective, i.e.  $\forall y \in Y \exists x \in X (y = f(x))$ . A bijective function is called a **bijection** or **one-to-one correspondence**. Informally, every element in the codomain must have **exactly one arrow** going into it.

**Inverse Function** Let  $f : X \rightarrow Y$ . Then  $g : Y \rightarrow X$  is an **inverse** of  $f$  iff  $\forall x \in X \forall y \in Y (y = f(x) \Leftrightarrow x = g(y))$ .

**Proposition (Lecture 7 Slide 39)** If  $g_1$  and  $g_2$  are inverses of  $f : X \rightarrow Y$ , then  $g_1 = g_2$ .

**Composition of Functions** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be functions. Define a new function  $g \circ f : X \rightarrow Z$  as follows:  $(g \circ f)(x) = g(f(x)) \forall x \in X$ , where  $g \circ f$  is read “ $g$  circle  $f$ ” and  $g(f(x))$  is read “ $g$  of  $f$  of  $x$ ”. The function  $g \circ f$  is called the **composition** of  $f$  and  $g$ .

**Addition and Multiplication on  $\mathbb{Z}_n$**  Define addition  $+$  and multiplication  $\cdot$  on  $\mathbb{Z}_n$  as follows: whenever  $[x], [y] \in \mathbb{Z}_n$ ,  $[x] + [y] = [x + y]$  and  $[x] \cdot [y] = [x \cdot y]$ .

**Proposition: Addition on  $\mathbb{Z}_n$  is well defined (Lecture 7 Slide 63)** For all  $n \in \mathbb{Z}^+$  and all  $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$ ,  $([x_1], [y_1]) = ([x_2], [y_2]) \rightarrow [x_1] + [y_1] = [x_2] + [y_2]$ .

**General Well-Defined Function Property**  $\forall x_1, x_2 \in X, \forall f : X \rightarrow Y, x_1 = x_2 \rightarrow f(x_1) = f(x_2)$ .

**Well-Defined Property w.r.t Equiv Relation  $\sim$**   $\forall x_1, x_2 \in X, \forall f : X \rightarrow Y, x_1 \sim x_2 \rightarrow f(x_1) \sim f(x_2)$ .

**Well-Defined Property w.r.t Equiv Class  $[x]$**   $\forall x_1, x_2 \in X, \forall f : X \rightarrow Y, [x_1] = [x_2] \rightarrow [f(x_1)] = [f(x_2)]$ .

## Mathematical Induction

**Sequences and Terms** A **sequence** is an ordered set with members called **terms**. Usually, the terms are numbers. A sequence may have infinite terms. General form:  $a_m, a_{m+1}, a_{m+2}, \dots, a_n$  where  $m \leq n$ . The  $k$  in  $a_k$  is called a **subscript** or **index**. Infinite sequence:  $a_m, a_{m+1}, a_{m+2}, \dots$

An **explicit** formula for a sequence is a rule that shows how the values of  $a_k$  depend on  $k$ . E.g.  $a_k = \frac{k}{k+1}$  for all integers  $k \geq 1$ .  $a_1, a_2, a_3, \dots$  Dots are element separators.

### Sequence Comprehension

$\{f(k) : k \in S\} : P(B)$   $f : S \rightarrow B$  Order not important, duplicates discarded

$[f(k) : k \in [n..m]] : Seq(B)$   $f : Z \rightarrow B$  Order important, duplicates are kept

$[f(k) : k \in [n..]] : Seq(B)$  Infinite sequences

**Summation** If  $m$  and  $n$  are integers,  $m \leq n$ , the symbol

$$\sum_{k=m}^n a_k$$

is the **sum** of all the terms  $a_m, a_{m+1}, a_{m+2}, \dots, a_n$  (a.k.a summation notation). We say that  $a_m + a_{m+1} + a_{m+2} + \dots + a_n$  is the **expanded** form of the sum, and we write  $\sum_{k=m}^n a_k = a_m + a_{m+1} + a_{m+2} + \dots + a_n$ . We call  $k$  the **index** of the summation,  $m$  the **lower limit** of the summation and  $n$  the **upper limit** of the summation.

Recursive definition of summation: If  $m$  is any integer, then

$$\sum_{k=m}^m a_k = a_m \text{ and } \sum_{k=m}^n a_k = \left( \sum_{k=m}^{n-1} a_k \right) + a_n$$

By convention, an **empty** sum (e.g.  $\sum_{k=m}^n a_k$  where  $m > n$ ) is equal to the additive identity **0**.

**Telescoping sums** are finite sums in which pairs of consecutive terms **cancel each other out**, leaving only the **initial** and **final terms**. E.g.  $\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$ . (Note that  $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ )

**Product Notation** If  $m$  and  $n$  are integers,  $m \leq n$ , the symbol

$$\prod_{k=m}^n a_k$$

is the **product** of all the terms  $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ . We write  $\prod_{k=m}^n a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \dots \cdot a_n$ .

Recursive definition for product notation: If  $m$  is any integer, then

$$\prod_{k=m}^m a_k = a_m \text{ and } \prod_{k=m}^n a_k = \left( \prod_{k=m}^{n-1} a_k \right) \cdot a_n$$

**Arithmetic Sequence** A sequence  $a_0, a_1, a_2, \dots$  is called an **arithmetic sequence** (or **arithmetic progression**) iff there is a constant  $d$  such that  $a_k = a_{k-1} + d$  for all integers  $k \geq 1$ . It follows that  $a_n = a_0 + dn$  for all integers  $n \geq 0$ .  $d$  is the **common difference**,  $a_0$  is the **initial value**.

Summing an arithmetic sequence of  $n$  terms:

$$\sum_{k=0}^{n-1} a_k = \frac{n}{2}(2a_0 + (n-1)d)$$

**Geometric Sequence** A sequence  $a_0, a_1, a_2, \dots$  is called a **geometric sequence** (or **geometric progression**) iff there is a constant  $r$  such that  $a_k = ra_{k-1}$  for all integers  $k \geq 1$ . It follows that  $a_n = a_0 r^n$  for all integers  $n \geq 0$ .  $r$  is the **common ratio**,  $a_0$  is the **initial value**.

Summing a geometric sequence of  $n$  terms:

$$\sum_{k=0}^{n-1} a_k = a_0 \left( \frac{1-r^n}{1-r} \right)$$

**Principal of Mathematical Induction (PMI)** Let  $P(n)$  be a property that is defined for integers  $n$ , and let  $a$  be a fixed integer. Suppose the following 2 statements are true:

1. (basis step)  $P(a)$  is true.
2. (inductive step) For all integers  $k \geq a$ , if  $P(k)$  is true then  $P(k + 1)$  is true.

Then the statement “for all integers  $n \geq a, P(n)$ ” is true.

Note: the basis step need not be  $P(1)$ ; it can be  $P(a)$  where  $a$  is a fixed integer.

**Closed Form** If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis (...) or a summation symbol ( $\sum$ ), we say that it is written in **closed form**. E.g.  $\frac{n(n+1)}{2}$  is the closed form formula for  $1 + 2 + 3 + \dots + n$ .

**Well-Ordering Principle for the Integers** Every nonempty subset of  $\mathbb{Z}_{\geq 0}$  has a smallest element.

**Well-Ordering Principle for Non-Negative Integers** Every nonempty subset of  $\mathbb{Z}_{\geq 0}$  has a smallest element.

Take note that the well-ordering principle applies only to **integers** and **non-empty subsets**. That means it does not apply for (and is not violated by) **real numbers** or **non-empty sets**.

**Recurrence Relation** A **recurrence relation** for a sequence  $a_0, a_1, a_2, \dots$  is a formula that relates each term  $a_k$  to certain of its predecessors  $a_{k-1}, a_{k-2}, \dots, a_{k-i}$ , where  $i$  is an integer with  $k - i \geq 0$ . If  $i$  is a fixed integer, the **initial conditions** for such a recurrent relation specify the values of  $a_0, a_1, a_2, \dots, a_{i-1}$ . If  $i$  depends on  $k$ , the initial conditions specify the values of  $a_0, a_1, a_2, \dots, a_m$ , where  $m$  is an integer with  $m \geq 0$ .

Recursive definition of Fibonacci:  $F_n = F_{n-1} + F_{n-2}$  for  $n > 1$ .

Recursive definition of factorial:  $n! = n \cdot (n - 1)!$  for  $n \geq 1$ .

Recursive definition of power:  $a^n = a^{n-1} \cdot a$  for  $n \geq 1$ .

**Recursively Defined Sets** Let  $S$  be a finite set with at least one element. A **string over  $S$**  is a finite sequence of elements from  $S$ . The elements of  $S$  are called **characters** of the string, and the **length** of a string is the number of characters it contains. The **null string over  $S$**  is defined to be the “string” with no characters. It is usually denoted  $\epsilon$  and is said to have length 0.

$$S = \{c_1, \dots, c_n\}$$

$$Str(S) ::= \epsilon \mid c.Str(S) \text{ s.t. } c \in S$$

**Recursive Definition of a set  $S$**

- |                    |  |
|--------------------|--|
| Base clause:       | Specify that certain elements, called <b>founders</b> , are in $S$ : if $c$ is a founder, then $c \in S$ .   |
| Recursion clause:  | Specify certain functions, called <b>constructors</b> , under which the set $S$ is closed: if $f$ is a constructor and $x \in S$ , then $f(x) \in S$ . |
| Minimality Clause: | Membership for $S$ can always be demonstrated by (infinitely many) successive applications of the clauses above.                                       |

**Structural Induction over  $S$**  To prove that  $\forall x \in S P(x)$  is true, where  $P(x)$  is a proposition, it suffices to:

- |                 |  |
|-----------------|--|
| Base clause:    | Show that $P(c)$ is true for every founder $c$ ; and   |
| Induction step: | Show that $\forall x \in S (P(x) \rightarrow P(f(x)))$ is true for every constructor $f$ . In words, if all the founders satisfy a property $P$ , and $P$ is preserved by all constructors, then all elements of $S$ satisfy $P$ . |

**Induction vs Co-Induction** Inductive proofs are proof based on how data are *constructed*. Co-inductive proofs are proofs based on how data are *decomposed*. Think: A property holds by *induction* if there is **good reason** for it to hold. A property holds by *co-induction* if there is **no good reason** for it not to hold.

**Cardinality**

**Pigeonhole Principle** Let  $A$  and  $B$  be **finite** sets. If there is an injection  $f : A \rightarrow B$ , then  $|A| \leq |B|$ . Contrapositive: Let  $m, n \in \mathbb{Z}_+$  with  $m > n$ . If  $m$  pigeons are put into  $n$  pigeonholes, then there must be (at least) one pigeonhole with (at least) two pigeons.

**Dual Pigeonhole Principles** Let  $A$  and  $B$  be **finite** sets. If there is an surjection  $f : A \rightarrow B$ , then  $|A| \geq |B|$ . Contrapositive: Let  $m, n \in \mathbb{Z}_+$  with  $m < n$ . If  $m$  pigeons are put into  $n$  pigeonholes, then there must be (at least) one pigeonhole with no pigeons.

**Finite set and Infinite set** Let  $\mathbb{Z}_n = \{1, 2, 3, \dots, n\}$ , the set of positive integers from 1 to  $n$ . A set  $S$  is said to be **finite** iff  $S$  is empty, or there exists a bijection from  $S$  to  $\mathbb{Z}_n$  for some  $n \in \mathbb{Z}^+$ . A set  $S$  is said to be **infinite** if it is not finite.

**IMPORTANT!!!** CS1231S defines  $\mathbb{N}$  to include 0

**Cardinality** The **cardinality** of a finite set  $S$ , denoted  $|S|$ , is

- (i) 0 if  $S = \emptyset$ , or
- (ii)  $n$  if  $f : S \rightarrow \mathbb{Z}_n$  is a bijection.

**Same Cardinality (Cantor)** Given any two sets  $A$  and  $B$ .  $A$  is said to have the same cardinality as  $B$ , written as  $|A| = |B|$ , iff there is a bijection  $f : A \rightarrow B$ .

Note: For a finite set  $A$ , any proper subset  $B$  of  $A$  will have  $|B| < |A|$ . But this is **not true** for infinite sets. For example,  $2\mathbb{Z}$  is a proper subset of  $\mathbb{Z}$ , but  $|2\mathbb{Z}| = |\mathbb{Z}|$ .

**Cardinal Numbers** Define  $\aleph_0 = |\mathbb{Z}^+|$ .  $\aleph$  is pronounced “aleph”. This is the first cardinal number.

**Countably Infinite** A set  $S$  is said to be **countably infinite** (or,  $S$  has the cardinality of natural numbers) iff  $|S| = \aleph_0$ .

**Countable set and Uncountable Set** A set is said to be **countable** iff it is finite or countably infinite. A set is said to be **uncountable** if it is not countable.

Note: to prove a set is countable, it either has to be a finite set or countably infinite. To prove that a set is countably infinite, its cardinality must be the same as  $\aleph_0$ . In other words, it must have a one-to-one mapping from the set of all positive integers to its domain.

Extra tip: most of the time, proving that something is uncountable can be done with Proposition 9.2 or Lemma 9.2 and showing that a sequence does not contain every element via a diagonalization process.

## Counting

**Sample Space** A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

**Notation** For a finite set  $A$ ,  $|A|$  denotes the number of elements in  $A$ .

**Equally Likely Probability Formula** If  $S$  is a finite sample space in which all outcomes are equally likely and  $E$  is an event in  $S$ , then the probability of  $E$ , denoted  $P(E)$ , is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The number of outcomes in } S} = \frac{|E|}{|S|}$$

**r-permutation** An **r-permutation** of a set of  $n$  elements is an ordered selection of  $r$  elements taken from the set. The number of r-permutations of a set of  $n$  elements is denoted  $P(n, r)$ . Other notations:  ${}_nP_r$ ,  $P_n^r$ .

**Formula for the Probability of the Complement of an Event** If  $S$  is a finite sample space and  $A$  is an event in  $S$ , then  $P(\overline{A}) = 1 - P(A)$ .

**Pigeonhole Principle (PHP)** A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.

**Generalized Pigeonhole Principle** For any function  $f$  from a finite set  $X$  with  $n$  elements to a finite set  $Y$  with  $m$  elements and for any positive integer  $k$ , if  $k < n/m$ , then there is some  $y \in Y$  such that  $y$  is the image of at least  $k + 1$  distinct elements of  $X$ .

**Generalized Pigeonhole Principle (Contrapositive Form)** For any function  $f$  from a finite set  $X$  with  $n$  elements to a finite set  $Y$  with  $m$  elements and for any positive integer  $k$ , if for each  $y \in Y$ ,  $f^{-1}(\{y\})$  has at most  $k$  elements, then  $X$  has at most  $km$  elements; in other words,  $n \leq km$ .

**r-combination** Let  $n$  and  $r$  be non-negative integers with  $r \leq n$ . An **r-combination** of a set of  $n$  elements is a subset of  $r$  of the  $n$  elements.  $\binom{n}{r}$ , read “ $n$  choose  $r$ ”, denotes the number of subsets of size  $r$  (r-combinations) that can be chosen from a set of  $n$  elements. Other symbols used are  $C(n, r)$ ,  ${}_nC_r$ ,  $C_{n,r}$ , or  ${}^nC_r$ .

**Multiset** An **r-combination with repetition allowed**, or multiset of size  $r$ , chosen from a set  $X$  of  $n$  elements is an unordered selection of elements taken from  $X$  with repetition allowed. If  $X = \{x_1, x_2, \dots, x_n\}$ , we write an r-combination with repetition allowed as  $[x_{i_1}, x_{i_2}, \dots, x_{i_r}]$  where each  $x_{i_j}$  is in  $X$  and some of the  $x_{i_j}$  may equal each other.

# Theorems, Lemmas & Corollaries

**Theorem 2.1.1 Logical Equivalences** Given any statement variables  $p, q$  and  $r$ , a tautology is **true** and a contradiction is **false**:

1	Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative Laws	$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity Laws	$p \wedge \mathbf{true} \equiv p$	$p \vee \mathbf{false} \equiv p$
5	Negation Laws	$p \vee \sim p \equiv \mathbf{true}$	$p \wedge \sim p \equiv \mathbf{false}$
6	Double Negation Law	$\sim(\sim p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \mathbf{true} \equiv \mathbf{true}$	$p \wedge \mathbf{false} \equiv \mathbf{false}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of <b>true</b> and <b>false</b>	$\sim \mathbf{true} \equiv \mathbf{false}$	$\sim \mathbf{false} \equiv \mathbf{true}$

**Implication Law**  $p \rightarrow q \equiv \sim p \vee q$

**Table 2.3.1 Rules of Inference** (Quote the rules if you use them in proofs)

Rule of Inference		Rule of Inference	
Modus Ponens	$p \rightarrow q \quad p \quad \bullet q$	Elimination	$p \vee q \quad \sim q \quad \bullet p$
Modus Tollens	$p \rightarrow q \quad \sim q \quad \bullet \sim p$	Transitivity	$p \rightarrow q \quad q \rightarrow r \quad \bullet p \rightarrow r$
Generalization	$p \quad \bullet p \vee q$	Proof by Division into Cases	$p \vee q \quad p \rightarrow r \quad q \rightarrow r \quad \bullet r$
Specialization	$p \wedge q \quad \bullet p$	Contradiction Rule	$\sim p \rightarrow \mathbf{false} \quad \bullet p$
Conjunction	$p \quad q \quad \bullet p \wedge q$		

**Theorem 3.2.1 Negation of Universal Statement** The **negation** of a statement of the form  $\forall x \in D, P(x)$  is logically equivalent to a statement of the form  $\exists x \in D$  such that  $\sim P(x)$ . Symbolically,  $\sim(\forall x \in D, P(x)) \equiv \exists x \in D$  such that  $\sim P(x)$ .

**Theorem 3.2.2 Negation of an Existential Statement** The **negation** of a statement of the form  $\exists x \in D, P(x)$  is logically equivalent to a statement of the form  $\forall x \in D$  such that  $\sim P(x)$ . Symbolically,  $\sim(\exists x \in D, P(x)) \equiv \forall x \in D$  such that  $\sim P(x)$ .

**Rules of Inference (Quantified Statements)**

Rule of Inference	Name
$\forall x \in DP(x) \quad \therefore P(a) \text{ if } a \in D$	Universal instantiation
$P(a) \text{ for every } a \in D \quad \therefore \forall x \in DP(x)$	Universal generalization
$\exists x \in DP(x) \quad \therefore P(a) \text{ for some } a \in D$	Existential instantiation
$P(a) \text{ for some } a \in D \quad \therefore \exists x \in DP(x)$	Existential generalization

**Theorem 4.2.1 (5th: 4.3.1)** Every integer is a rational number.

**Theorem 4.2.2 (5th: 4.3.2)** The sum of any two rational numbers is rational.

**Corollary 4.2.3 (5th: 4.2.3)** The double of a rational number is rational.

**Theorem 4.3.1 (5th: 4.4.1) A Positive Divisor of a Positive Integer:** For all positive integers  $a$  and  $b$ , if  $a|b$ , then  $a \leq b$ .

**Theorem 4.3.2 (5th: 4.4.2) Divisors of 1:** The only divisors of 1 are 1 and -1.

**Theorem 4.3.3 (5th: 4.4.3) Transitivity of Divisibility:** For all integers  $a, b$  and  $c$ , if  $a|b$  and  $b|c$ , then  $a|c$ .

**Theorem 4.4.1 The Quotient-Remainder Theorem** Given any integer  $n$  and a positive integer  $d$ , there exists unique integers  $q$  and  $r$  such that  $n = dq + r$  and  $0 \leq r < d$ .

**Theorem 4.6.1 (5th: 4.7.1)** There is no greatest integer.

**Theorem 4.6.4 (5th: 4.7.4)** For all integers  $n$ , if  $n^2$  is even then  $n$  is even.

**Proof (Tutorial 1 Q10)** The product of any two odd integers is an odd integer.

**Proof (Tutorial 1 Q11)**  $n^2$  is odd if and only if  $n$  is odd.

**Proof (Tutorial 2 Q4(a))** Integers are not closed under division.

**Proof (Tutorial 2 Q4(b))** Rational numbers are closed under addition.

**Proof (Tutorial 2 Q4(c))** Rational numbers are not closed under division.

**Proof (Tutorial 2 Q8)**  $\forall x \in \mathbb{R}((x^2 > x) \rightarrow (x < 0) \vee (x > 1))$ .

**Proof (Tutorial 2 Q11)** If  $n$  is a product of two positive integers  $a$  and  $b$ , then  $a \leq n^{1/2}$  or  $b \leq n^{1/2}$ .

**Theorem 4.7.1 (5th: 4.8.1)**  $\sqrt{2}$  is irrational.

**Theorem 5.1.1** If  $a_m, a_{m+1}, a_{m+2}, \dots$  and  $b_m, b_{m+1}, b_{m+2}, \dots$  are sequences of real numbers and  $c$  is any real number, then the following equations hold for any integer  $n \geq m$ :

1.  $\sum_{k=m}^n a_k + \sum_{k=m}^n b_k = \sum_{k=m}^n (a_k + b_k)$
2.  $c \cdot \sum_{k=m}^n a_k = \sum_{k=m}^n c \cdot a_k$
3.  $(\prod_{k=m}^n a_k) \cdot (\prod_{k=m}^n b_k) = (\prod_{k=m}^n (a_k \cdot b_k))$

**Theorem 5.2.2 (5th: 5.2.1) Sum of first  $n$  Integers** For all integers  $n \geq 1$ ,  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .

**Theorem 5.2.3 (5th: 5.2.2) Sum of a Geometric Sequence** For any real number  $r \neq 1$ , and any integers  $n \geq 0$ ,

$$\sum_{i=0}^n r^i = \frac{r^{n+1} - 1}{r - 1}$$

**Proposition 5.3.1 (5th: 5.3.2)** For all integers  $n \geq 0$ ,  $2^{2n} - 1$  is divisible by 3.

**Proposition 5.3.2 (5th: 5.3.3)** For all integers  $n \geq 3$ ,  $2n + 1 < 2^n$ .

**Proof (Lecture 8 Slide 39)** For  $n \in \mathbb{Z}^+$ , any  $2^n \times 2^n$  board with one square removed can be tiled by L-trominoes.

**Proof (Lecture 8 Slide 45)** Any integer  $> 1$  is divisible by a prime number.

**Theorem 6.2.1 Subset Relations**

1. **Inclusion of Intersection:** For all sets  $A$  and  $B$ , (a)  $A \cap B \subseteq A$  (b)  $A \cap B \subseteq B$ .
2. **Inclusion in Union:** For all sets  $A$  and  $B$ , (a)  $A \subseteq A \cup B$  (b)  $B \subseteq A \cup B$ .
3. **Transitive Property Of Subsets:** For all sets  $A$ ,  $B$  and  $C$ ,  $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$ .

**Theorem 6.2.2 Set Identities** Let all sets referred to below be subsets of a universal set  $U$ .

1	Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
2	Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
3	Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4	Identity Laws	$A \cup \emptyset = A$	$A \cap U = A$
5	Complement Laws	$A \cup \bar{A} = U$	$A \cap \bar{A} = \emptyset$
6	Double Complement Law	$\overline{\bar{A}} = A$	
7	Idempotent Laws	$A \cup A = A$	$A \cap A = A$
8	Universal Bound Laws	$A \cup U = U$	$A \cap \emptyset = \emptyset$
9	De Morgan's Laws	$\overline{A \cup B} = \bar{A} \cap \bar{B}$	$\overline{A \cap B} = \bar{A} \cup \bar{B}$
10	Absorption Laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
11	Complements of $U$ and $\emptyset$	$\overline{\bar{U}} = \emptyset$	$\bar{\emptyset} = U$
12	Set Difference Law	$A \setminus B = A \cap \bar{B}$	

**Theorem 6.2.4** An empty set is a **subset** of every set, i.e.  $\emptyset \subseteq A$  for all sets  $A$ .

Note: a set with exactly one element is called a **singleton**.

**Theorem: Cardinality of a Power Set of a Finite Set** Let  $A$  be a finite set where  $|A| = n$ , then  $|P(A)| = 2^n$ .

**Theorem 6.3.1** Suppose  $A$  is a finite set with  $n$  elements, then  $P(A)$  has  $2^n$  elements. In other words,  $|P(A)| = 2^{|A|}$ .

**Theorem 8.3.1 Relation Induced by a Partition** Let  $A$  be a set with a partition and let  $R$  be the relation induced by the partition. Then  $R$  is reflexive, symmetric, and transitive.

**Lemma Rel.1 Equivalence Classes** Let  $\sim$  be an equivalence relation on a set  $A$ . The following are equivalent for all  $x, y \in A$ .  
(i)  $x \sim y$       (ii)  $[x] = [y]$       (iii)  $[x] \cap [y] \neq \emptyset$ .

**Theorem 8.3.4 The Partition Induced by an Equivalence Relation** If  $A$  is a set and  $R$  is an equivalence relation on  $A$ , then the distinct equivalence classes of  $R$  form a partition of  $A$ ; that is, the union of the equivalence classes is all of  $A$ , and the intersection of any two distinct classes is empty.

**Theorem Rel.2 Equivalence classes form a partition** Let  $\sim$  be an equivalence relation on a set  $A$ . Then  $A/\sim$  is a partition of  $A$ .

**Theorem 7.1.1 Function Equality** Two functions  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are equal, i.e.  $f = g$ , iff (i)  $A = C$ , and (ii)  $f(x) = g(x) \forall x \in A$ .

**Theorem 7.2.3** If  $f : X \rightarrow Y$  is a bijection, then  $f^{-1} : Y \rightarrow X$  is also a bijection. In other words,  $f : X \rightarrow Y$  is bijective iff  $f$  has an inverse.

**Theorem 7.3.1 Composition with an Identity Function** If  $f$  is a function from set  $X$  to set  $Y$ , and  $id_x$  is the identity function on  $X$ , and  $id_y$  is the identity function on  $Y$ , then  $f \circ id_x = f$  and  $id_y \circ f = f$ .

**Theorem 7.3.2 Composition of a Function with its Inverse** If  $f : X \rightarrow Y$  is a bijection with the inverse function  $f^{-1} : Y \rightarrow X$ , then  $f^{-1} \circ f = id_x$  and  $f \circ f^{-1} = id_y$ .

**Theorem: Associativity of Function Composition** Let  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ . Then  $(h \circ g) \circ f = h \circ (g \circ f)$ . Function composition is associative.

**Theorem 7.3.3** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both injective, then  $g \circ f$  is injective.

**Theorem 7.3.4** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both surjective, then  $g \circ f$  is surjective.

**Theorem: Equality of Cardinality of Finite Sets** Let  $A$  and  $B$  be any finite sets. iff there is a bijection  $f : A \rightarrow B$ .

**Theorem 7.4.1 Properties of Cardinality** The same-cardinality relation is an equivalence relation. For all sets  $A$ ,  $B$  and  $C$ :

**Reflexive:**  $|A| = |A|$

**Symmetric:**  $|A| = |B| \rightarrow |B| = |A|$

**Transitive:**  $(|A| = |B|) \wedge (|B| = |C|) \rightarrow |A| = |C|$ .

**Theorem:  $\mathbb{Z}^+ \times \mathbb{Z}^+$  is countable**

**Theorem (Cartesian Product)** If sets  $A$  and  $B$  are both countably infinite, then so is  $A \times B$ .

**Corollary (General Cartesian Product)** Given  $n \geq 2$  countably infinite sets  $A_1, A_2, \dots, A_n$  the Cartesian product  $A_1 \times A_2 \times \dots \times A_n$  is also countably infinite.

**Theorem: Unions (Lecture 9 Slide 30)** The union of countably many countable sets is countable. That is, if  $A_1, A_2, \dots$  are all countable sets, then so is

$$\bigcup_{i=1}^{\infty} A_i$$

**Proposition 9.1** An infinite set  $B$  is countable if and only if there is a sequence  $b_0, b_1, b_2, \dots \in B$  in which every element of  $B$  appears exactly once.

(Definition of sequence) A **sequence**  $a_0, a_1, a_2, \dots$  can be represented by a function  $a$  whose domain is  $\mathbb{Z}_{\geq 0}$  that satisfies  $a(n) = a_n$  for every  $n \in \mathbb{Z}_{\geq 0}$ .



**Lemma 9.2: Countability via Sequence** An infinite set  $B$  is countable if and only if there is a sequence  $b_0, b_1, b_2 \dots$  in which every element of  $B$  appears.

**Theorem 7.4.2 (Cantor)** The set of real numbers between 0 and 1,  $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ , is uncountable. To prove that a set is uncountable means proving that there is no possibility of a bijection from that set to  $\mathbb{Z}^+$ .

**Theorem 7.4.3** Any subset of any countable set is countable.

**Corollary 7.4.4 (Contrapositive of Theorem 7.4.3)** Any set with an uncountable subset is uncountable.

Corollary 7.4.4 implies that  $\mathbb{R}$  is uncountable since  $(0, 1) \subseteq \mathbb{R}$  and  $(0, 1)$  is uncountable.

**Proposition 9.3** Every infinite set has a countably infinite subset.

**Lemma 9.4: Union of Countably Infinite Sets** Let  $A$  and  $B$  be countably infinite sets. Then  $A \cup B$  is countable.

**Theorem 9.1.1 The Number of Elements in a List** If  $m$  and  $n$  are integers and  $m \leq n$ , then there are  $n - m + 1$  integers from  $m$  to  $n$  inclusive.

**Theorem 9.2.1 The Multiplication/Product Rule** If an operation consists of  $k$  steps and the first step can be performed in  $n_1$  ways, the second step can be performed in  $n_2$  ways (regardless of how the first step was performed), the  $k^{th}$  step can be performed in  $n_k$  ways, (regardless of how the preceding steps were performed), then the entire operation can be performed in  $n_1 \times n_2 \times n_3 \times \dots n_k$  ways.

**Theorem 9.2.2 Permutations** The number of permutations of a set with  $n$  ( $n \geq 1$ ) elements is  $n!$ .

**Theorem 9.2.3 r-permutations from a set of  $n$  elements** If  $n$  and  $r$  are integers and  $1 \leq r \leq n$ , then the number of  $r$ -permutations of a set of  $n$  elements is given by the formula  $P(n, r) = n(n-1)(n-2)\dots(n-r+1)$ . Or equivalently,  $P(n, r) = \frac{n!}{(n-r)!}$ .

**Theorem 9.3.1 The Addition/Sum Rule** Suppose a finite set  $A$  equals the union of  $k$  distinct mutually disjoint subsets  $A_1, A_2, \dots, A_k$ . Then  $|A| = |A_1| + |A_2| + \dots + |A_k|$ .

**Theorem 9.3.2 The Difference Rule** If  $A$  is a finite set and  $B \subseteq A$ , then  $|A \setminus B| = |A| - |B|$ .

Derivation from Addition Rule: If  $B \subseteq A$ , then the two sets  $B$  and  $A \setminus B$  have no elements in common and  $B \cup (A \setminus B) = A$ . Hence, by addition rule,  $|B| + |A \setminus B| = |A|$ . Subtracting  $|B|$  from both sides gives the equation  $|A \setminus B| = |A| - |B|$ .

**Theorem 9.3.3 The Inclusion/Exclusion Rule for 2 or 3 Sets** If  $A$ ,  $B$ , and  $C$  are any finite sets, then  $|A \cup B| = |A| + |B| - |A \cap B|$  and  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ .

**Theorem 9.5.1 Formula for  $\binom{n}{r}$**  The number of subsets of size  $r$  (or  $r$ -combinations) that can be chosen from a set of  $n$  elements,  $\binom{n}{r}$ , is given by the formula  $\binom{n}{r} = \frac{P(n, r)}{r!}$  or equivalently,  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$  where  $n$  and  $r$  are non-negative integers with  $r \leq n$ .

**Theorem 9.5.2 Permutations with sets of indistinguishable objects** Suppose a collection consists of  $n$  objects of which  $n_1$  are of type 1 and are indistinguishable from each other,  $n_2$  are of type 2 and are indistinguishable from each other,  $\dots$   $n_k$  are of type  $k$  and are indistinguishable from each other, and suppose that  $n_1 + n_2 + \dots + n_k = n$ . Then the number of distinguishable permutations of the  $n$  objects is

$$\binom{n}{r} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots \binom{n-n_1-n_2-\dots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!n_3!\dots n_k!}$$

**Theorem 9.6.1 Number of r-combinations with Repetition Allowed** The number of  $r$ -combination with repetition allowed (multisets of size  $r$ ) that can be selected from a set of  $n$  elements is:  $\binom{r+n-1}{r}$ . This equals the number of ways  $r$  objects can be selected from  $n$  categories of objects with repetitions allowed.

**Formula to Use**

**Theorem 9.7.1 Pascal's Formula** Let  $n$  and  $r$  be positive integers,  $r \leq n$ . Then  $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$ .

**Theorem 9.7.2 Binomial Theorem** Given any real numbers  $a$  and  $b$  and any non-negative integer  $n$ ,

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

**Theorem 9.9.1 Bayes' Theorem** Suppose that a sample space  $S$  is a union of mutually disjoint events  $B_1, B_2, B_3, \dots, B_n$ . Suppose  $A$  is an event in  $S$ , and suppose  $A$  and all the  $B_i$  have non-zero probabilities. If  $k$  is an integer with  $1 \leq k \leq n$ , then

$$P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + \cdots + P(A|B_n) \cdot P(B_n)}$$

## Examples of Proofs (For reference)

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**Prove that the product of two consecutive odd numbers is always odd.**

1. Let  $a$  and  $b$  be the two consecutive odd numbers.
  - 1.1 WLOG, assume that  $a < b$ , hence  $b = a + 2$ .
  - 1.2 Now,  $a = 2k + 1$  for some integer  $k$  (by definition of odd numbers).
  - 1.3 Similarly,  $b = a + 2 = 2k + 3$ .
  - 1.4 Therefore,  $ab = (2k + 1)(2k + 3) = (4k^2 + 6k) + (2k + 3) = 4k^2 + 8k + 3 = 2(2k^2 + 4k + 1) + 1$  (by basic algebra).
  - 1.5 Let  $m = (2k^2 + 4k + 1)$ , which is an integer (by closure of integers under  $\times$  and  $+$ ).
  - 1.6 Then  $ab = 2m + 1$ , which is odd (by definition of odd numbers).
2. Therefore, the product of two consecutive odd numbers is always odd.

**Prove that the following statement is false: The product of two irrational numbers is always irrational.**

1. Let them two irrational numbers be  $\sqrt{2}$  and  $\sqrt{8}$ .
  - 1.1 Then  $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ , which is a rational number (by basic algebra).
2. Therefore, the statement “the product of two irrational numbers is always irrational” is false.

**Note:** One counter-example is sufficient.

**Prove that the difference of two consecutive squares between 30 and 100 is odd. (Proof by exhaustion / brute force)**

1. The squares between 30 and 100 are 36, 49, 64 and 81.
  - 1.1 Case 1:  $49 - 36 = 13$  which is odd.
  - 1.2 Case 2:  $64 - 49 = 15$  which is odd.
  - 1.3 Case 3:  $81 - 64 = 17$  which is odd.
2. Therefore, the difference of two consecutive squares between 30 and 100 is odd.

**Prove that the difference of two consecutive squares is always odd. (Proof by deduction / direct proof)**

1. Let the numbers be  $n$  and  $n + 1$ .
  - 1.1  $(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$  (by basic algebra).
  - 1.2  $2n + 1$  is odd (by definition of odd numbers).
2. Therefore, the difference of two consecutive squares is odd.

**Prove Theorem 4.7.1(5th: 4.8.1)  $\sqrt{2}$  is irrational. (Proof by contradiction)**

**Proposition 4.6.4(5th: 4.7.4)** For all integers  $n$ , if  $n^2$  is even then  $n$  is even.

1. Suppose not, that is,  $\sqrt{2}$  is rational.
  - 1.1 Then  $\exists a, b \in \mathbb{Z}, b \neq 0$  s.t.  $\sqrt{2} = \frac{a}{b}$  (by definition of rational numbers).
  - 1.2 Convert  $\frac{a}{b}$  into its lowest term  $\frac{m}{n}$ .
  - 1.3  $m^2 = 2n^2$  (by basic algebra).
  - 1.4 Hence  $m^2$  is even (by definition of even number, as  $n^2$  is an integer by closure).
  - 1.5 Hence  $m$  is even (by Proposition 4.6.4).
  - 1.6 Let  $m = 2k$ ; substituting into 1.3:  $4k^2 = 2n^2$ , or  $n^2 = 2k^2$ .
  - 1.7 Hence  $n^2$  is even (by definition of even number).
  - 1.8 Hence  $n$  is even (by Proposition 4.6.4).
  - 1.9 So both  $m$  and  $n$  are even, but this contradicts that  $\frac{m}{n}$  is in its lowest term.
2. Therefore, the assumption that  $\sqrt{2}$  is rational is false.
3. Hence  $\sqrt{2}$  is irrational.

**Note:** To prove a statement  $S$  by contradiction, you first assume that  $\sim S$  is true. Based on this, you use known facts and theorems to arrive at a logical contradiction. Since every step of your argument thus far is logically correct, the problem must lie in your initial assumption (that  $\sim S$  is true). Thus you conclude that  $\sim S$  is false, that is,  $S$  is true.

**Prove that there exist irrational numbers  $p$  and  $q$  such that  $p^q$  is rational.**

1. From Theorem 4.7.1,  $\sqrt{2}$  is irrational.
2. Consider  $\sqrt{2}^{\sqrt{2}}$ . It is either rational or irrational.
3. Case 1:  $\sqrt{2}^{\sqrt{2}}$  is rational.
  - 3.1 Let  $p = q = \sqrt{2}$ , and we are done.
4. Case 2:  $\sqrt{2}^{\sqrt{2}}$  is irrational.
  - 4.1 Let  $p = \sqrt{2}^{\sqrt{2}}$ , and  $q = \sqrt{2}$ .
  - 4.2 Now  $p$  is irrational (by assumption), so is  $q$  (by Theorem 4.7.1).
  - 4.3 Consider  $p^q = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} \times \sqrt{2}} = (\sqrt{2})^2 = 2$  (by basic algebra).
  - 4.4 Clearly 2 is rational.
5. In either case, we have found the required  $p$  and  $q$ .

# Mathematical Induction Proofs

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Prove that the sum of the first  $n$  integers is  $\frac{n(n+1)}{2}$

1. Let  $P(n) \equiv (1 + 2 + \cdots + n = \frac{n(n+1)}{2})$ ,  $\forall n \in \mathbb{Z}^+$ .
2. **Basis step:**  $1 = \frac{1(1+1)}{2}$ , therefore  $P(1)$  is true.
3. Assume  $P(k)$  is true for some  $k \geq 1$ . That is,  $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$
4. **Inductive Step:** (to show  $P(k+1)$  is true)
  - 4.1  $1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2}$
  - 4.2 Therefore  $P(k+1)$  is true.
5. Therefore,  $P(n)$  is true for  $n \in \mathbb{Z}^+$ .