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For 2026 Examinations

MATHEMATICS & STATISTICS (FOR COMMERCE)

Board's Question Paper of July 2025 Included

Solution on QR Code

STD. XII

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MATHEMATICS AND STATISTICS

(Commerce)

EVALUATION PLAN

1. (a) Practical examination : 20 marks

   (b) Theory/Written examination : 80 marks

   Total : 100 marks

2. **Scheme for the conduct of Practical Examination :**

   There will be Practical Examination based on topics in Part I and Part II of the textbooks for 20 marks.

   Distribution of Marks :

1. Journal	5 marks
2. Problem solving (Three Problems out of four practical problems each of 5 marks)	15 marks
Total	**20 marks**

3. **Question Paper Pattern for the Theory/Written Examination :**

   (a) For Mathematics and Statistics (Commerce), there will be one question paper divided into two sections, viz. Section–I and Section–II. Students should write the answers of both sections in the same answer book.

   Section – I : 40 marks

   Section – II : 40 marks

   Total : 80 marks

(b) Each section will have three main questions as follows :

SECTION – I

Question No.	Question Type		Marks
Q. 1. (A)	6 Multiple Choice Questions (MCQ)	(1 mark each)	06
(B)	3 True/False type Questions	(1 mark each)	03
(C)	3 Fill in the blanks type Questions	(1 mark each)	03
Q. 2. (A)	Solve any 2 out of 3	(3 marks each)	06
(B)	Solve any 2 out of 3	(4 marks each)	08
Q. 3. (A)	Solve any 2 out of 3	(3 marks each)	06
(B)	Solve any 1 out of 2	(4 marks each)	04
(C)	Solve any 1 out of 2 (Activity)	(4 marks each)	04

SECTION – II

Question No.	Question Type		Marks
Q. 4. (A)	6 Multiple Choice Questions (MCQ)	(1 mark each)	06
(B)	3 True/False type Questions	(1 mark each)	03
(C)	3 Fill in the blanks type Questions	(1 mark each)	03
Q. 5. (A)	Solve any 2 out of 3	(3 marks each)	06
(B)	Solve any 2 out of 3	(4 marks each)	08
Q. 6. (A)	Solve any 2 out of 3	(3 marks each)	06
(B)	Solve any 1 out of 2	(4 marks each)	04
(C)	Solve any 1 out of 2 (Activity)	(4 marks each)	04

4. Chapter-wise distribution of marks in the Question Paper :

SECTION – I

Sr. No.	Chapters	Marks with Options
1.	Mathematical Logic	08
2.	Matrices	08
3.	Differentiation	07
4.	Applications of Derivatives	09
5.	Integration	07
6.	Definite Integration	05
7.	Applications of Definite Integration	04
8.	Differential Equations and Applications	10
	Total Marks	58

SECTION – II

Sr. No.	Chapters	Marks with Options
1.	Commission, Brokerage and Discount	06
2.	Insurance and Annuity	04
3.	Linear Regression	08
4.	Time Series	07
5.	Index Numbers	07
6.	Linear Programming	06
7.	Assignment Problem and Sequencing	09
8.	Probability Distribution	11
	Total Marks	58

CONTENTS

Section 1

BOARD'S QUESTION PAPER : JULY 2024

(With Full Solution and Marking Scheme)

Time : 3 Hours] **[Max. Marks : 80**

General Instructions :

(i) All questions are compulsory.

(ii) There are 6 questions divided into two sections.

(iii) Write answers of Section – I and Section – II in the same answer book.

(iv) Use of logarithmic tables is allowed. Use of calculator is not allowed.

(v) For LPP and Time series, graph paper is not necessary. Only rough sketch of graph is expected.

(vi) Start answer to each question on a new page.

(vii) For each objective type of questions (i.e. Q. 1 and Q. 4), only the first attempt will be considered for evaluation.

SECTION – I

Q. 1. (A) Select and write the correct answer of the following multiple choice type of questions : **[6]**

(i) Which of the following sentences is a statement in logic :

(a) He is a good actor.

(b) Did you eat lunch yet?

(c) Every real number is a complex number.

(d) Bring the motor car here. **(1)**

(ii) If $y = 2x^2 + \log 2 + 5$, then $\frac{dy}{dx} =$ ....

(a) x (b) $4x$ (c) $2x + \log 2$ (d) $-4x$ **(1)**

(iii) If $x = 2at^2$, $y = 4at$, then $\frac{dy}{dx} =$ ....

(a) $-\frac{1}{2at^2}$ (b) $\frac{1}{2at^3}$ (c) $\frac{1}{t}$ (d) $\frac{1}{4at^3}$ **(1)**

(iv) The equation of tangent to the curve $y = x^2 + 4x + 1$ at P$(-1, -2)$ is

(a) $2x - y = 0$ (b) $x + 2y + 5 = 0$

(c) $2x + 4 = 3y$ (d) $5x + y = 1$ **(1)**

(v) $\int_{-2}^{3} \frac{dx}{x+5} =$ ....

(a) $-\log\left(\frac{8}{3}\right)$ (b) $3\log\left(\frac{3}{8}\right)$

(c) $\log\left(\frac{8}{3}\right)$ (d) $-2\log\left(\frac{3}{8}\right)$ **(1)**

(vi) The order and degree of the differential equation

$\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+8=0$ are

(a) order $=2$, degree $=2$ (b) order $=1$, degree $=2$

(c) order $=1$, degree $=1$ (d) order $=2$, degree $=1$ **(1)**

Answers :

(i) (c) Every real number is a complex number.

(ii) (b) $4x$ **(iii)** (c) $\frac{1}{t}$ **(iv)** (a) $2x-y=0$

(v) (c) $\log\left(\frac{8}{3}\right)$ **(vi)** (d) order $=2$, degree $=1$.

(1 mark for each correct answer)

(B) State whether the following statements are *True* or *False* : **[3]**

(i) Every identity matrix is a scalar matrix. **(1)**

(ii) The rate of change of demand (x) of a commodity w.r.t. its price (y) is $\frac{dy}{dx}$. **(1)**

(iii) The integrating factor of $\frac{dy}{dx}-y=x$ is e^x. **(1)**

Answers :

(i) True **(ii)** False **(iii)** False. *(1 mark for each correct answer)*

(C) Fill in the following blanks : **[3]**

(i) If $y=x\log x$, then $\frac{d^2y}{dx^2}=$ .... **(1)**

(ii) If the marginal revenue $R_m=40$ and elasticity of demand η is 5, then the average revenue R_A will be .... **(1)**

(iii) Area of the region bounded by $y=x^4$, $x=1$, $x=5$ and the X-axis will be .... **(1)**

Answers :

(i) $\frac{1}{x}$ (ii) 50 (iii) $\frac{3124}{5}$ sq units. ***(1 mark for each correct answer)***

Q. 2. (A) Attempt *any TWO* of the following questions : **[6]**

(i) Write converse, inverse and contrapositive of the following statement :

'If the train reaches on time, then I can catch the connecting flight.' **(3)**

Solution : Let p : The train reaches on time.

q : I can catch the connecting flight.

Then the symbolic form of the given statement is $p \rightarrow q$.

Converse : $q \rightarrow p$ is the converse of $p \rightarrow q$.

i.e. If I can catch the connecting flight, then the train reaches on time.

(1 mark)

Inverse : $\sim p \rightarrow \sim q$ is the inverse of $p \rightarrow q$.

i.e. If the train does not reach on time, then I cannot catch the connecting flight. ***(1 mark)***

Contrapositive : $\sim q \rightarrow \sim p$ is the contrapositive of $p \rightarrow q$.

i.e. If I cannot catch the connecting flight, then the train does not reach on time. ***(1 mark)***

(ii) If $A=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}$ and if $(A+B)^2=A^2+B^2$, find values of a and b. **(3)**

Solution : $(A+B)^2=A^2+B^2$

$\therefore (A+B)(A+B)=A^2+B^2$

$\therefore A^2+AB+BA+B^2=A^2+B^2$

$\therefore AB+BA=0$

$\therefore AB=-BA$ ***(1 mark)***

$\therefore \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}=-\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

$\therefore \begin{bmatrix} 2-2 & a+2b \\ -2+2 & -a-2b \end{bmatrix}=-\begin{bmatrix} 2-a & 4-2a \\ -1-b & -2-2b \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & a+2b \\ 0 & -a-2b \end{bmatrix}=\begin{bmatrix} a-2 & 2a-4 \\ 1+b & 2+2b \end{bmatrix}$ ***(1 mark)***

By the equality of matrices, we get

$0 = a - 2$ ... (1)

$0 = 1 + b$ ... (2)

$a + 2b = 2a - 4$ ... (3)

$-a - 2b = 2 + 2b$ ... (4)

From equations (1) and (2), we get

$a = 2$ and $b = -1$

The values of a and b satisfy equations (3) and (4) also.

Hence, $a = 2$ and $b = -1$. ***(1 mark)***

(iii) Find $\frac{dy}{dx}$, if $y = (x)^x + (a)^x$. **(3)**

Solution : $y = x^x + a^x$

Let $u = x^x$

Then $\log u = \log x^x = x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \cdot \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$ ... (1) ***(1 mark)***

Now, $y = u + a^x$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx}(a^x)$$ ***(1 mark)***

$$= x^x(1 + \log x) + a^x \cdot \log a$$ ... [By (1)] ***(1 mark)***

(B) Attempt *any TWO* of the following questions : **[8]**

(i) Evaluate : $\int \frac{x}{4x^4 - 20x^2 - 3} dx$. **(4)**

Solution : Let $I = \int \frac{x}{4x^4 - 20x^2 - 3} dx$

Put $x^2 = t$ $\quad \therefore 2x\,dx = dt$

$\therefore x\,dx = \dfrac{dt}{2}$ **(1 mark)**

$$\therefore I = \int \frac{1}{4t^2 - 20t - 3} \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \times \frac{1}{4} \int \frac{1}{t^2 - 5t - \frac{3}{4}}\, dt$$

$$= \frac{1}{8} \int \frac{1}{\left(t^2 - 5t + \frac{25}{4}\right) - \frac{25}{4} - \frac{3}{4}}\, dt$$

$$= \frac{1}{8} \int \frac{1}{\left(t - \frac{5}{2}\right)^2 - (\sqrt{7})^2}\, dt$$ **(1 mark)**

$$= \frac{1}{8} \times \frac{1}{2\sqrt{7}} \log \left| \frac{t - \frac{5}{2} - \sqrt{7}}{t - \frac{5}{2} + \sqrt{7}} \right| + c$$ **(1 mark)**

$$= \frac{1}{16\sqrt{7}} \log \left| \frac{2t - 5 - 2\sqrt{7}}{2t - 5 + 2\sqrt{7}} \right| + c$$

$$= \frac{1}{16\sqrt{7}} \log \left| \frac{2x^2 - 5 - 2\sqrt{7}}{2x^2 - 5 + 2\sqrt{7}} \right| + c.$$ **(1 mark)**

(ii) Evaluate : $\int_1^3 \log x\,dx.$ **(4)**

Solution : $\int_1^3 \log x\,dx = \int_1^3 (\log x) \cdot 1\,dx$

$$= [(\log x) \int 1\,dx]_1^3 - \int_1^3 \left[\frac{d}{dx}(\log x) \int 1\,dx \right] dx$$ **(1 mark)**

$$= [(\log x)x]_1^3 - \int_1^3 \frac{1}{x} \times x\,dx$$ **(1 mark)**

$$= (3 \log 3 - \log 1) - \int_1^3 1\,dx$$

$= 3 \log 3 - [x]_1^3$ ... [$\because \log 1 = 0$] ***(1 mark)***

$= \log 3^3 - (3 - 1)$

$= \log 27 - 2.$ ***(1 mark)***

(iii) In a certain culture of bacteria, their rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours. **(4)**

Solution : Let x be the number of bacteria in the culture at time t.

Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = kx$, where k is a constant

$\therefore \frac{dx}{x} = k\,dt$

On integrating, we get

$\int \frac{dx}{x} = k \int dt$ ***(1 mark)***

$\therefore \log x = kt + c$

Initially, i.e. when $t = 0$, let $x = x_0$

$\therefore \log x_0 = k \times 0 + c \qquad \therefore c = \log x_0$

$\therefore \log x = kt + \log x_0 \qquad \therefore \log x - \log x_0 = kt$

$\therefore \log\left(\frac{x}{x_0}\right) = kt$ ... (1) ***(1 mark)***

Since the number doubles in 4 hours, i.e. when $t = 4$, $x = 2x_0$

$\therefore \log\left(\frac{2x_0}{x_0}\right) = 4k \qquad \therefore k = \frac{1}{4}\log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = \frac{t}{4}\log 2$ ***(1 mark)***

When $t = 12$, we get

$$\log\left(\frac{x}{x_0}\right) = \frac{12}{4}\log 2 = 3\log 2$$

$$\therefore \log\left(\frac{x}{x_0}\right) = \log 8$$

$$\therefore \frac{x}{x_0} = 8 \qquad \therefore x = 8x_0$$

Hence, the number of bacteria will be 8 times the original number in 12 hours. ***(1 mark)***

Q. 3. (A) Attempt *any TWO* of the following questions : [6]

(i) A metal wire of 36 cm length is bent to form a rectangle. Find its dimensions, when its area is maximum. (3)

Solution : Let x cm and y cm be the length and breadth of the rectangle.

Then its perimeter is $2(x+y) = 36$

$\therefore x+y = 18 \qquad \therefore y = 18 - x$

Area of the rectangle $= xy = x(18-x)$

Let $f(x) = x(18-x) = 18x - x^2$ ***(1 mark)***

Then $f'(x) = \dfrac{d}{dx}(18x - x^2) = 18 \times 1 - 2x = 18 - 2x$

and $f''(x) = \dfrac{d}{dx}(18 - 2x) = 0 - 2 \times 1 = -2$ ***(1 mark)***

Now, $f'(x) = 0$, if $18 - 2x = 0$, i.e. if $x = 9$

and $f''(9) = -2 < 0$

$\therefore$ by the second derivative test, f has maximum value at $x = 9$.

When $x = 9$, $y = 18 - 9 = 9$

Hence, the rectangle is a square of side 9 cm. ***(1 mark)***

(ii) Evaluate : $\displaystyle\int \frac{2x+1}{x(x-1)(x-4)}\,dx$. (3)

Solution : Let $I = \displaystyle\int \frac{2x+1}{x(x-1)(x-4)}\,dx$

Let $\dfrac{2x+1}{x(x-1)(x-4)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x-4}$

$\therefore 2x+1 = A(x-1)(x-4) + Bx(x-4) + Cx(x-1)$ ... (1) ***(1 mark)***

Put $x=0$ in (1), we get

$$2(0)+1=A(-1)(-4)+B(0)(-4)+C(0)(-1)$$

$\therefore\ 1=4A \qquad \therefore\ A=\frac{1}{4}$

Put $x-1=0$, i.e. $x=1$ in (1), we get

$$2(1)+1=A(0)(-3)+B(1)(-3)+C(1)(0)$$

$\therefore\ 3=-3B \qquad \therefore\ B=-1$

Put $x-4=0$, i.e. $x=4$ in (1), we get

$$2(4)+1=A(3)(0)+B(4)(0)+C(4)(3)$$

$\therefore\ 9=12C \qquad \therefore\ C=\frac{3}{4}$

$$\therefore\ \frac{2x+1}{x(x-1)(x-4)}=\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}$$ *(1 mark)*

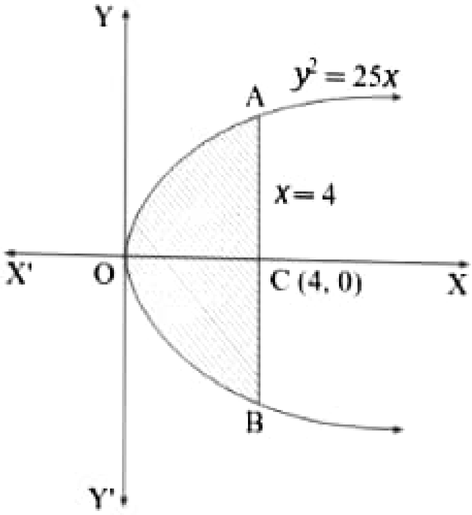
$$\therefore\ I=\int\left[\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}\right]dx$$

$$=\frac{1}{4}\int\frac{1}{x}\,dx-\int\frac{1}{x-1}\,dx+\frac{3}{4}\int\frac{1}{x-4}\,dx$$

$$=\frac{1}{4}\log|x|-\log|x-1|+\frac{3}{4}\log|x-4|+c.$$ *(1 mark)*

(iii) Find the area of the region bounded by $y^2=25x$ and the line $x=4$. **(3)**

Solution :



Required area = area of the region OABO = 2(area of the region OACO)

$= 2\int_0^4 y\,dx$, where $y^2 = 25x$, i.e. $y = 5\sqrt{x}$

$= 2\int_0^4 5\sqrt{x}\,dx = 10\int_0^4 x^{\frac{1}{2}}dx$ ***(1 mark)***

$= 10\left[\dfrac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^4 = \dfrac{20}{3}\left[x^{\frac{3}{2}}\right]_0^4$ ***(1 mark)***

$= \dfrac{20}{3}[8-0] = \dfrac{160}{3}$ sq units. ***(1 mark)***

(B) Attempt *any ONE* of the following questions : **[4]**

(i) Using the truth table, verify : $\sim(p \to \sim q) \equiv p \wedge \sim(\sim q) \equiv p \wedge q$. **(4)**

Solution :

1	2	3	4	5	6	7	8
p	q	$\sim q$	$p \to \sim q$	$\sim(p \to \sim q)$	$\sim(\sim q)$	$p \wedge \sim(\sim q)$	$p \wedge q$
T	T	F	F	T	T	T	T
T	F	T	T	F	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	F	F	F	F

The entries in columns 5, 7 and 8 are identical.

$\therefore \sim(p \to \sim q) \equiv p \wedge \sim(\sim q) \equiv p \wedge q$.

(Column 5 : 1 mark, Column 7 : 1 mark, Column 8 : 1 mark
Conclusion : 1 mark)

(ii) Solve the following equations by method of inversion :

$x + y + z = 1,\ x - y + z = 2,\ x + y - z = 3.$ **(4)**

Solution : The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

This is of the form AX = B, where

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$ ***(1 mark)***

Let us find A^{-1}.

$$|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=1(1-1)-1(-1-1)+1(1+1)$$
$$=0+2+2=4\neq 0$$

$\therefore$ A^{-1} exists.

We write $AA^{-1}=I$

$$\therefore \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

By R_2-R_1 and R_3-R_1, we get

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

By $\left(-\frac{1}{2}\right)R_2$ and $\left(-\frac{1}{2}\right)R_3$, we get

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

By R_1-R_2, we get

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

By R_1-R_3, we get

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}$$ ***(1 mark)***

Now, premultiply $AX=B$ by A^{-1}, we get

$A^{-1}(AX)=A^{-1}B$

$\therefore (A^{-1}A)X=A^{-1}B$

$\therefore IX=A^{-1}B$

$$\therefore X=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$ ***(1 mark)***

$$\therefore \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0+2+3\\ 1-2+0\\ 1+0-3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}5\\ -1\\ -2\end{bmatrix}$$

$$\therefore \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\frac{5}{2}\\ -\frac{1}{2}\\ -1\end{bmatrix}$$

By equality of matrices, $x=\frac{5}{2}$, $y=-\frac{1}{2}$, $z=-1$ is the required solution.

(1 mark)

(C) Attempt *any ONE* of the following questions (Activity) : [4]

(i) The cost C for producing x articles is given as $C=x^3-16x^2+47x$. For what values of x, the average cost is decreasing?

Given : $C=x^3-16x^2+47x$

Average cost $C_A=\frac{C}{x}$

$\therefore C_A=\boxed{}$

Differentiating w.r.t. x, we get

$\frac{d}{dx}(C_A)=\boxed{}$

We know that, C_A is decreasing, if

$\frac{d}{dx}(C_A)\boxed{}0$

$\therefore 2x - 16 < 0 \qquad \therefore 2x < 16$

$\therefore x < \boxed{}$

$\therefore$ average cost is decreasing for $x \in (0, 8)$. **(4)**

Solution : Given : $C = x^3 - 16x^2 + 47x$

Average cost $C_A = \frac{C}{x} = \frac{x^3 - 16x^2 + 47x}{x}$

$\therefore C_A = \boxed{x^2 - 16x + 47}$ ***(1 mark)***

Differentiating w.r.t. x, we get

$\frac{d}{dx}(C_A) = \frac{d}{dx}(x^2 - 16x + 47) = 2x - 16 \times 1 + 0$

$\therefore \frac{d}{dx}(C_A) = \boxed{2x - 16}$ ***(1 mark)***

We know that, C_A is decreasing, if

$\frac{d}{dx}(C_A) \boxed{<} 0$ ***(1 mark)***

$\therefore 2x - 16 < 0 \qquad \therefore 2x < 16$

$\therefore x < \boxed{8}$ ***(1 mark)***

$\therefore$ average cost is decreasing for $x \in (0, 8)$.

(ii) Solve the differential equation : $y - x\frac{dy}{dx} = 0$.

Given equation is $y - x\frac{dy}{dx} = 0$

Separating the variables, we get

$$\frac{dx}{\boxed{}} = \frac{dy}{\boxed{}}$$

Integrating, we get

$$\int \frac{dx}{\boxed{}} = \int \frac{dy}{\boxed{}} + c$$

$\therefore \log x = \boxed{} + c$

$\therefore \log x - \log y = \log c_1$, where $c = \log c_1$

$\therefore \log\left(\frac{x}{y}\right) = \log c_1$

$\therefore \frac{x}{\boxed{}} = c_1$

Hence, the required solution is $x = c_1 y$. **(4)**

Solution : Given equation is $y - x\frac{dy}{dx} = 0$

i.e. $x\frac{dy}{dx} = y$

Separating the variables, we get

$\therefore \frac{dx}{\boxed{x}} = \frac{dy}{\boxed{y}}$ ***(1 mark)***

Integrating, we get

$\int \frac{dx}{\boxed{x}} = \int \frac{dy}{\boxed{y}} + c$ ***(1 mark)***

$\therefore \log x = \boxed{\log y} + c$ ***(1 mark)***

$\therefore \log x - \log y = \log c_1$, where $c = \log c_1$

$\therefore \log\left(\frac{x}{y}\right) = \log c_1$

$\therefore \frac{x}{\boxed{y}} = c_1$ ***(1 mark)***

Hence, the required solution is $x = c_1 y$.

SECTION – II

Q. 4. (A) Select and write the correct answer of the following multiple choice type of questions : **[6]**

(I) The date on which the period of the bill expires is called

(a) legal due date (b) grace date

(c) nominal due date (d) date of drawing **(1)**

(II) A person insured a property of ₹ 4,00,000. The rate of premium is ₹ 35 per thousand p.a. The amount of annual premium is

(a) ₹ 14,000 (b) ₹ 24,000 (c) ₹ 34,000 (d) ₹ 15,000 **(1)**

(iii) Paasche's Price Index Number is given by

(a) $\frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \times 100$ (b) $\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times 100$

(c) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ **(1)**

(iv) If jobs I, II, III have processing times as 8, 6, 5 on machine M_1 and 8, 3, 4 on machine M_2 in the order $M_1 - M_2$, then the optimal sequence is

(a) I, II, III (b) I, III, II (c) II, I, III (d) III, II, I **(1)**

(v) If $E(X) = 4$ and X follows Poisson's distribution, then $V(X) =$ ....

(a) 2 (b) -2 (c) 4 (d) -4 **(1)**

(vi) Three coins are tossed simultaneously. X is the number of heads. Then the expected value of X is

(a) 1 (b) 1.5 (c) 1.9 (d) 1.017 **(1)**

Answers :

(i) (c) nominal due date **(ii)** (a) ₹14,000

(iii) (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ **(iv)** (b) I, III, II

(v) (c) 4 **(vi)** (b) 1.5. ***(1 mark for each correct answer)***

(B) State whether the following statements are *True* or *False* : **[3]**

(i) In the regression of Y on X, X is the independent variable and Y is the dependent variable. **(1)**

(ii) The region represented by the inequalities $x \leqslant 0, y \leqslant 0$ lies in the first quadrant. **(1)**

(iii) In an assignment problem, if the number of columns are greater than number of rows, then a dummy column is added. **(1)**

Answers :

(i) True **(ii)** False **(iii)** False. ***(1 mark for each correct answer)***

(C) Fill in the following blanks : **[3]**

(i) If an agent charges 12% commission on the sales of ₹ 48,000, then his total commission is ₹ .... **(1)**

(ii) The optimal value of the objective function is attained at the .... points of feasible region. (1)

(iii) Given p.d.f. of a continuous random variable X is

$$f(x) = \frac{x}{8}, \quad \text{for } 0 < x < 4$$
$$= 0, \quad \text{otherwise.}$$

Then $P(1 < X < 2) =$ .... (1)

Answers :

(i) ₹ 5760 **(ii)** corner **(iii)** $\frac{3}{16}$. ***(1 mark for each correct answer)***

Q. 5. (A) Attempt *any TWO* of the following questions : [6]

(i) Find the rate of interest compounded annually, if an immediate annuity of ₹ 20,000 per year amounts to ₹ 41,000 in 2 years. (3)

Solution : Here, $C =$ ₹ 20,000, $A =$ ₹ 41,000, $n = 2$.

$$A = \frac{C}{i}[(1+i)^n - 1]$$ ***(1 mark)***

$$\therefore 41000 = \frac{20000}{i}[(1+i)^2 - 1]$$

$$\therefore \frac{41000}{20000} = \frac{1}{i}[(1+i)^2 - 1]$$

$$\therefore 2.05 = \frac{1}{i}(1 + 2i + i^2 - 1)$$

$$\therefore 2.05 = \frac{1}{i}(2i + i^2)$$

$$\therefore 2.05 = 2 + i$$

$$\therefore i = 0.05$$ ***(1 mark)***

But $i = \frac{r}{100}$

$$\therefore 0.05 = \frac{r}{100}$$

$$\therefore r = 0.05 \times 100 = 5\%$$

Hence, rate of interest is 5%. ***(1 mark)***

(ii) Find the Value Index Number using Simple Aggregate Method for the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	30	22	40	18
B	40	16	60	12
C	10	38	15	24
D	50	12	60	16
E	20	28	25	36

(3)

Solution : Here, p_0 = Price in base year, p_1 = Price in current year, q_0 = Quantity of base year and q_1 = Quantity of current year.

Commodity	Base Year		Current Year		p_0q_0	p_1q_1
	p_0	q_0	p_1	q_1		
A	30	22	40	18	660	720
B	40	16	60	12	640	720
C	10	38	15	24	380	360
D	50	12	60	16	600	960
E	20	28	25	36	560	900
Total					$\Sigma p_0q_0 = 2840$	$\Sigma p_1q_1 = 3660$

(1 mark)

Value Index Number by Simple Aggregate Method :

$$V_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \times 100$$ *(1 mark)*

$$= \frac{3660}{2840} \times 100 = 1.2887 \times 100 = 128.87$$

Hence, Value Index Number is 128.87. *(1 mark)*

(iii) Five jobs must pass through a lathe and a surface grinder, in that order. The processing times in hours are shown below. Determine the optimal sequence of the jobs. Also, find the total elapsed time :

Jobs	I	II	III	IV	V
Lathe	4	1	5	2	5
Surface grinder	3	2	4	3	6

(3)

Solution :

Jobs	**Time (in hours)**	
	Lathe (A)	**Surface grinder (B)**
I	4	3
II	1	2
III	5	4
IV	2	3
V	5	6

Here, Min. (A, B) = 1, which corresponds to A.

Therefore, job II is processed first.

II				

The problem now reduces to jobs I, III, IV, V.

Here, Min. (A, B) = 2, which corresponds to A.

Therefore, job IV is processed next to job II.

II	IV			

The problem now reduces to jobs I, III, V.

Here, Min. (A, B) = 3, which corresponds to B.

Therefore, job I is processed at the last.

II	IV			I

The problem now reduces to jobs III and V.

Here, Min. (A, B) = 4, which corresponds to B.

Therefore, job III is processed at the last next to job I.

II	IV		III	I

Now, job V is processed next to job IV and the optional sequence of jobs as follows :

II	IV	V	III	I

(1 mark)

Total elapsed time is obtained as follows :

Jobs Sequence	Lathe (A)		Surface grinder (B)	
	Time in	Time out	Time in	Time out
II	0	1	1	3
IV	1	3	3	6
V	3	8	8	14
III	8	13	14	18
I	13	17	18	21

(1 mark)

Total elapsed time T = 21 hours. *(1 mark)*

(B) Attempt *any TWO* of the following questions : **[8]**

(i) A bill was drawn on 14th April for ₹ 7000 and was discounted on 6th July at 5% p.a. The banker paid ₹ 6930 for the bills. Find the period of the bill. **(4)**

Solution : Face Value (FV) or SD = ₹ 7000, $r = 5\%$.

Cash Value (CV) = ₹ 6930

BD = SD − CV = 7000 − 6930 = ₹ 70 *(1 mark)*

$$\text{Also, BD} = \frac{\text{SD} \times n \times r}{100}$$

$$\therefore 70 = \frac{7000 \times n \times 5}{100}$$

$$\therefore 70 = 350n$$

$$\therefore n = \frac{70}{350} = \frac{1}{5} \text{ years}$$

$$\therefore n = \frac{1}{5} \times 365 = 73 \text{ days}$$ *(1 mark)*

To find the legal due date, 73 days are to be counted from the date of discounting, i.e. 6th July.

July	Aug.	Sept.	Total
25	31	17	73

(1 mark)

Hence, the legal due date is 17th September

∴ nominal due date is 14th September

Now, date of drawing is 14^{th} April.

Hence, the period of the bill is from 14^{th} April to 14^{th} September,

i.e. 5 months. ***(1 mark)***

(ii) The following table gives the production of steel (in millions of tons) for years 1976 to 1986 :

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Fit a trend line to the above data by the method of least squares.

(4)

Solution : Here, $n = 11$. We transform year t to u by taking $u = t - 1981$.

We construct the following table for calculation :

Year t	**Production** x_t	$u = t - 1981$	u^2	ux_t
1976	0	−5	25	0
1977	4	−4	16	−16
1978	4	−3	9	−12
1979	2	−2	4	−4
1980	6	−1	1	−6
1981	8	0	0	0
1982	5	1	1	5
1983	9	2	4	18
1984	4	3	9	12
1985	10	4	16	40
1986	10	5	25	50
Total	$\Sigma x_t = 62$	$\Sigma u = 0$	$\Sigma u^2 = 110$	125 −38 $\Sigma ux_t = 87$

(1 mark)

The equation of trend line is $x_t = a' + b'u$

The normal equations are

$\Sigma x_t = na' + b' \Sigma u$ ... (1)

$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2$ ... (2) ***(1 mark)***

Here, $n = 11$, $\Sigma x_t = 62$, $\Sigma u = 0$, $\Sigma u^2 = 110$, $\Sigma u x_t = 87$

Putting these values in normal equations, we get

$62 = 11a' + b'(0)$ ... (3)

$87 = a'(0) + b'(110)$ ... (4)

From equation (3), we get

$$a' = \frac{62}{11} = 5.6364$$

From equation (4), we get

$$b' = \frac{87}{110} = 0.7909$$ ***(1 mark)***

Putting $a' = 5.6364$ and $b' = 0.7909$ in the equation

$x_t = a' + b'u$, we get, the equation of trend line as

$x_t = 5.6364 + 0.7909\,u$ ***(1 mark)***

(iii) Solve the following LPP by graphical method :

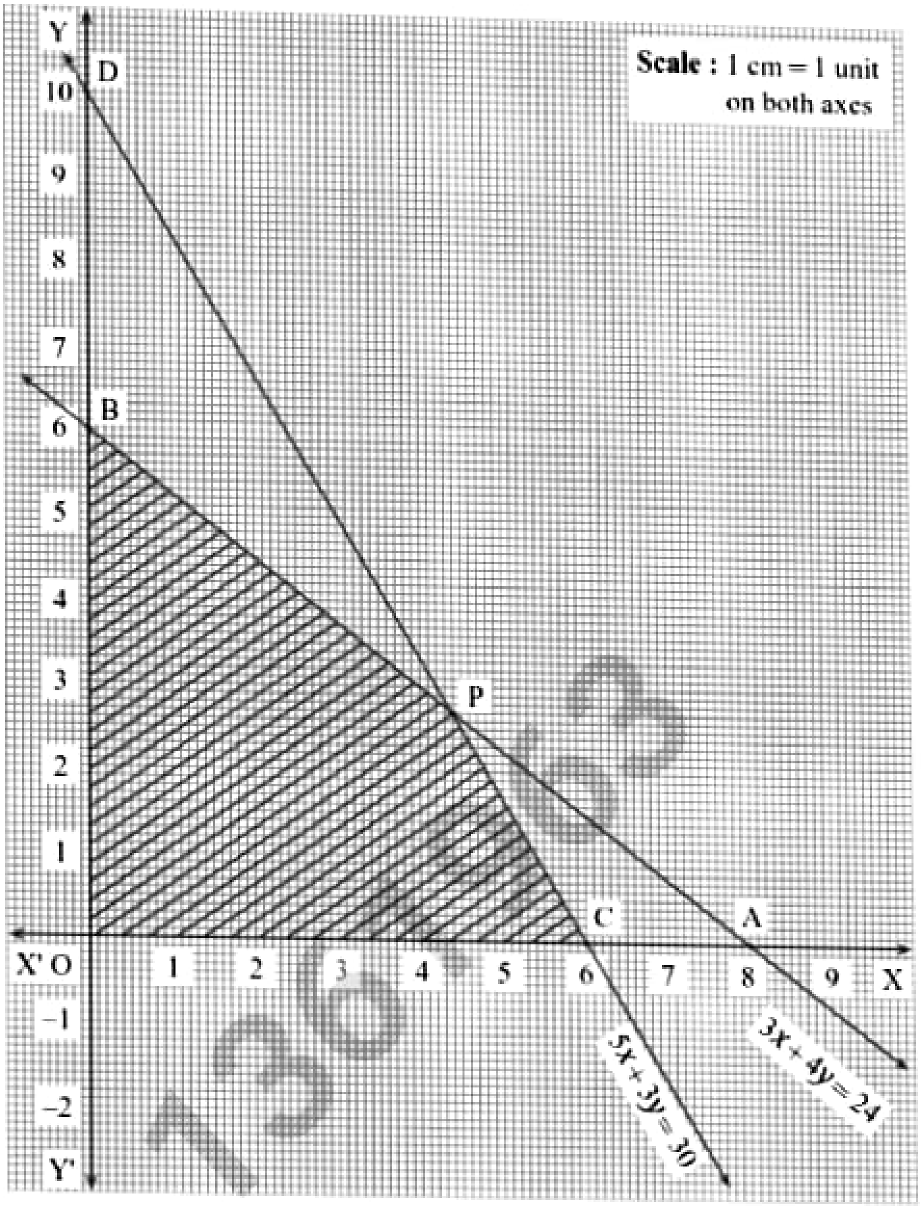
Maximize : $z = 7x + 11y$, subject to :

$3x + 4y \leqslant 24$, $5x + 3y \leqslant 30$, $x \geqslant 0$, $y \geqslant 0$. **(4)**

Solution : First we draw the lines AB and CD whose equations are $3x + 4y = 24$ and $5x + 3y = 30$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x + 4y = 24$	A(8, 0)	B(0, 6)	$\leqslant$	origin side of the line AB
CD	$5x + 3y = 30$	C(6, 0)	D(0, 10)	$\leqslant$	origin side of the line CD

(1 mark)



(1 mark)

The feasible region is OCPBO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), C (6, 0), P and B (0, 6).

The vertex P is the point of intersection of the lines

$$3x+4y=24 \quad \dots (1)$$

and $$5x+3y=30 \quad \dots (2)$$

Multiplying equation (1) by 3 and equation (2) by 4, we get

$$9x+12y=72$$

and $$20x+12y=120$$

On subtracting, we get

$11x=48 \qquad \therefore x=\dfrac{48}{11}$

Substituting $x=\frac{48}{11}$ in equation (2), we get

$$5\left(\frac{48}{11}\right)+3y=30$$

$$\therefore 3y=30-\frac{240}{11}=\frac{90}{11}$$

$$\therefore y=\frac{30}{11}$$

$\therefore$ P is $\left(\frac{48}{11}, \frac{30}{11}\right)$ ***(1 mark)***

The values of the objective function $z=7x+11y$ at these corner points are

$z(O)=7(0)+11(0)=0+0=0$

$z(C)=7(6)+11(0)=42+0=42$

$z(P)=7\left(\frac{48}{11}\right)+11\left(\frac{30}{11}\right)=\frac{336}{11}+\frac{330}{11}=\frac{666}{11}=60.54$

$z(B)=7(0)+11(6)=0+66=66$

$\therefore$ z has maximum value 66, when $x=0$ and $y=6$. ***(1 mark)***

Q. 6. (A) Attempt *any TWO* of the following questions : **[6]**

(i) For a bivariate data, $\bar{x}=53, \bar{y}=28, b_{yx}=-1.2, b_{xy}=-0.3$.

Find : (a) Correlation coefficient between x and y.

(b) Estimate y for $x=50$. **(3)**

Solution : Given : $\bar{x}=53, \bar{y}=28, b_{yx}=-1.2, b_{xy}=-0.3$

(a) **Correlation coefficient between X and Y :**

$$r=\pm\sqrt{b_{yx}\cdot b_{xy}}$$

$$=\pm\sqrt{(-1.2)(-0.3)}$$

$$=\pm\sqrt{0.36}$$

$\therefore r=-0.6$ ... [$\because$ b_{yx} and b_{xy} are negative] ***(1 mark)***

(b) **Estimation of Y for $X=50$:**

Regression equation of Y on X is,

$y=a+b_{yx}\cdot x$, where $b_{yx}=-1.2$

$\therefore a=\bar{y}-b_{yx}\cdot\bar{x}=28-(-1.2)53$

$=28+63.6=91.6$

$\therefore y=91.6-1.2x$

i.e. $y = -1.2x + 91.6$ *(1 mark)*

Put $x = 50$

$\therefore y = -1.2(50) + 91.6$

$\therefore y = -60 + 91.6 \qquad \therefore y = 31.6.$ *(1 mark)*

(ii) Given that $\Sigma p_0 q_0 = 220$, $\Sigma p_0 q_1 = 380$, $\Sigma p_1 q_1 = 350$ and Marshall-Edgeworth's Price Index Number is 150, find Laspeyre's Price Index Number. **(3)**

Solution : Given : $\Sigma p_0 q_0 = 220$, $\Sigma p_0 q_1 = 380$, $\Sigma p_1 q_1 = 350$.

$P_{01}(\text{M-E}) = 150$, $P_{01}(\text{L}) = ?$

We have

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$ *(1 mark)*

$$\therefore 150 = \frac{\Sigma p_1 q_0 + 350}{220 + 380} \times 100$$

$$\therefore 150 = \frac{\Sigma p_1 q_0 + 350}{600} \times 100$$

$\therefore 150 \times 6 = \Sigma p_1 q_0 + 350$

$\therefore 900 - 350 = \Sigma p_1 q_0$

$\therefore \Sigma p_1 q_0 = 550$ *(1 mark)*

$$\text{Now, } P_{01}(\text{L}) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{550}{220} \times 100$$

$$= 2.5 \times 100 = 250$$

Hence, Laspeyre's Price Index Number is 250. *(1 mark)*

(iii) The following data gives the production of bleaching powder (in '000 tonnes) for the years 1962 to 1972 :

Year	1962	1963	1964	1965	1966	1967
Production	0	0	1	1	4	2
Year	1968	1969	1970	1971	1972	
Production	4	9	7	10	8	

Obtain the trend values for the above data using 5-yearly moving averages. **(3)**

Solution : We construct the following table to obtain 5-yearly moving averages for the given data :

Year t	Production (in '000 tonnes) x_t	5-yearly moving total	5-yearly moving averages (Trend value)
1962	0	–	–
1963	0	–	–
1964	1	6	1.2
1965	1	8	1.6
1966	4	12	2.4
1967	2	20	4.0
1968	4	26	5.2
1969	9	32	6.4
1970	7	38	7.6
1971	10	–	–
1972	8	–	–

(Column 3 : 2 marks, Column 4 : 1 mark)

(B) Attempt *any ONE* of the following questions : [4]

(i) Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost matrix is given below :

Machines	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	–	5	4
M_3	–	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule. (4)

Solution : As the number of machines is less than the number of places, the problem is unbalanced. It can be balanced by introducing a dummy machine M_5 with zero cost.

As M_2 cannot be placed at C and M_3 cannot be placed at A, a very high cost say ∞ is assigned to the corresponding element.

Machines	Places				
	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
M_5	0	0	0	0	0

(1 mark)

Step 1 : Minimum element of each row is subtracted from every element of that row :

Machines	Places				
	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

(1 mark)

Step 2 : Subtract the smallest element in each column from every element of it. New assignment matrix is obtained as above, because each column in it contains one zero.

Step 3 : Draw minimum number of vertical and horizontal lines to cover all zeros.

Machines	Places				
	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

Step 4 : Since the number of lines covering zeros is 5 and is equal to order of matrix 5, the optimal solution has reached. Optimal assignment can be made as follows :

Machines	Places				
	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

(1 mark)

The following optimal solution is obtained :

Machines	Places	Cost (in ₹)
M_1	A	4
M_2	B	4
M_3	E	2
M_4	D	2
M_5	C	0

Total cost $=$ ₹ 12. ***(1 mark)***

(ii) There are 10% defective items in a large bulk of items. What is the probability that a sample of 4 items will include not more than one defective item? **(4)**

Solution : Here, $n=4$, $X=$ Defective item

$p=$ Probability of defective item

$$=10\%=\frac{10}{100}=0.1$$

$\therefore q=1-p=1-0.1=0.9$ ***(1 mark)***

Now, $X \sim B(n, p)$ with $n=4$, $p=0.1$ and $q=0.9$

$\therefore P(X=x)={}^nC_x p^x q^{n-x}={}^4C_x(0.1)^x(0.9)^{4-x}$ ***(1 mark)***

P(Not more than one defective)

$=P(X \leqslant 1)$

$=P(X=0)+P(X=1)$

$={}^4C_0(0.1)^0(0.9)^4+{}^4C_1(0.1)(0.9)^3$ ***(1 mark)***

$$= 1 \times (0.9)^4 + 4 \times 0.1 \times (0.9)^3$$

$$= (0.9)^3 [0.9 + 0.4]$$

$$= (0.9)^3 (1.3) = 1.3 (0.9)^3.$$ ***(1 mark)***

(C) Attempt *any ONE* of the following questions (Activity) : [4]

(i) The equations of the two regression lines are $3x + 2y - 26 = 0$ and $6x + y - 31 = 0$. Obtain the correlation coefficient between x and y.

To find correlation coefficient, we have to find the regression coefficients b_{yx} and b_{xy}.

Let $3x + 2y = 26$ be equation of the line of regression of y on x.

This gives $y = \boxed{}\, x + 13$

$\therefore b_{yx} = -\frac{3}{2}$

Now, consider $6x + y = 31$ as equation of the line of regression of x on y.

This can be written as $x = \boxed{}\, y + \frac{31}{6}$

$\therefore b_{yx} = -\frac{1}{6}$

Now, $r^2 = \boxed{} = 0.25$

As both b_{yx} and b_{xy} are negative,

$\therefore r = \boxed{}$ **(4)**

Solution : To find correlation coefficient, we have to find the regression coefficient b_{yx} and b_{xy}.

Let $3x + 2y = 26$ be equation of the line of regression of y on x.

This gives $y = \boxed{-\frac{3}{2}}\, x + 13$ ***(1 mark)***

$\therefore b_{yx} = -\frac{3}{2}$

Now, consider $6x + y = 31$ as equation of the line of regression of x on y.

This can be written as $x = \boxed{-\frac{1}{6}}\, y + \frac{31}{6}$ ***(1 mark)***

$\therefore b_{xy} = -\frac{1}{6}$

Now, $r^2 = \boxed{b_{yx} \cdot b_{xy}} = \left(-\frac{3}{2}\right)\left(-\frac{1}{6}\right) = 0.25$ **(1 mark)**

As both b_{yx} and b_{xy} are negative, $r = \boxed{-0.5}$. **(1 mark)**

(ii) The probability distribution of X is as follows :

x	0	1	2	3	4
$P(X=x)$	0.1	k	$2k$	$2k$	k

Find : (a) k (b) $P(X<2)$ (c) $P(1 \leqslant X < 4)$ (d) $F(2)$.

The table gives a probability distribution.

$\therefore \Sigma p_i = 1 \qquad \therefore 0.1 + k + 2k + 2k + k = 1$

(a) $k = \boxed{}$

(b) $P(X<2) = P(X=0) + P(X=1) = \boxed{}$

(c) $P(1 \leqslant X < 4) = P(1) + P(2) + P(3) = \boxed{}$

(d) $F(2) = P(X \leqslant 2) = P(0) + P(1) + P(2) = \boxed{}$ **(4)**

Solution : The table gives a probability distribution.

$\therefore \Sigma p_i = 1$

$\therefore 0.1 + k + 2k + 2k + k = 1$

$\therefore 6k = 1 - 0.1 = 0.9$

(a) $\therefore k = \frac{0.9}{6} = \boxed{0.15}$ **(1 mark)**

(b) $P(X<2) = P(X=0) + P(X=1)$

$= 0.1 + k = 0.1 + 0.15 = \boxed{0.25}$ **(1 mark)**

(c) $P(1 \leqslant X < 4) = P(1) + P(2) + P(3)$

$= k + 2k + 2k = 5k = 5(0.15) = \boxed{0.75}$ **(1 mark)**

(d) $F(2) = P(X \leqslant 2)$

$= P(0) + P(1) + P(2)$

$= 0.1 + k + 2k = 0.1 + 3k$

$= 0.1 + 3(0.15) = 0.1 + 0.45 = \boxed{0.55}$. **(1 mark)**

Section 2 MOST LIKELY QUESTION SETS

SECTION – I

Question Set 1 MATHEMATICAL LOGIC

(Marks with option : 08)

1.1 STATEMENT

1. By a statement in logic, we mean a declarative (assertive) sentence, which is either true or false, but not both. This fact is known as *law of excluded middle.*
2. Statements are denoted by small letters like $p, q, r, \dots$.
3. The truth or falsity of a statement is called the *truth value* of the statement.
   If a statement is true, its truth value is denoted by T.
   If a statement is false, its truth value is denoted by F.
4. A simple statement is a statement which is not in any way a combination of two or more statements.

Examples for Practice | 3 marks each

State which of the following sentences are statements. In case of statement, write down the truth value :

1. (1) The square of a real number is negative.
(2) The sum of interior angles of a triangle is 180°.
(3) $x+4=8$.

2. (1) The number π is an irrational number.
(2) If $a+b<7$, where $a \geqslant 0$, $b \geqslant 0$, then $a<7$, $b<7$.
(3) May god bless you.

3. (1) $x^2-y^2=(x+y)(x-y)$ for all $x, y \in R$.
(2) Sum of opposite angles in a cyclic rectangle is 180°.
(3) I hate you!

4. (1) Sum of two odd numbers is odd.
(2) Two coplanar lines are either parallel or intersecting.
(3) Can you speak in English?

Answers

1. (1) Statement, F (2) Statement, T (3) Not a statement
2. (1) Statement, T (2) Statement, T (3) Not a statement
3. (1) Statement, T (2) Statement, T (3) Not a statement
4. (1) Statement, F (2) Statement, T (3) Not a statement.

1.2 LOGICAL CONNECTIVES

1. A combination of two or more simple statements is called a *compound statement*. The simple statements whose combination is a compound statement are called the *components* or *constituents* of the compound statement.
2. The words or phrases which connect two or more simple statements are called *logical connectives*.

Sr. No.	Connective	Symbol	Name of the compound statement
(1)	and	$\wedge$	conjunction
(2)	or	$\vee$	disjunction
(3)	not	$\sim$	negation
(4)	If ... then	$\rightarrow$ or $\Rightarrow$	conditional or implication
(5)	if and only if or iff	$\leftrightarrow$ or $\Leftrightarrow$	biconditional or double implication or equivalence

Note : The connective 'not' operates on a single statement.

Examples for Practice | **3 marks each**

1. Express the following statements in symbolic form :

(1) Mango is a fruit but potato is a vegetable.

(2) An angle is a right angle and its measure is 90°.

(3) Even though it is not cloudy, it is still raining.

2. Write the following statements in symbolic form :

(1) If triangle is equilateral, then it is equiangular.

(2) It is not true that $\sqrt{2}$ is a rational number.

(3) 4 is an odd number if and only if 3 is not a prime factor of 6.

3. Write the following statements in symbolic form :

(1) Stock prices are high if and only if stocks are rising.

(2) If a quadrilateral is a square, than it is not a rhombus.

(3) ABCD is a parallelogram but it is not a quadrilateral.

4. If p : He swims, q : Water is warm. Give the verbal statements for the following symbolic statements :

(1) $p \leftrightarrow \sim q$ (2) $q \rightarrow p$ (3) $q \wedge \sim p$

5. Write the truth value of negation of each of the following statements :

(1) π is an irrational number.

(2) Every equilateral triangle is an isosceles triangle.

(3) For every $x \in N$, $x+3<8$.

Answers

1. (1) $p \wedge q$ (2) $p \wedge q$ (3) $\sim p \wedge q$.
2. (1) $p \rightarrow q$ (2) $\sim p$ (3) $p \leftrightarrow \sim q$.
3. (1) $p \leftrightarrow q$ (2) $p \rightarrow \sim q$ (3) $p \wedge \sim q$.
4. (1) He swims if and only if water is not warm.
   (2) If water is warm, then he swims.
   (3) Water is warm and he does not swim.
5. (1) F (2) F (3) T.

1.3 TRUTH TABLES

The truth value of a compound statement depends on the truth values of its component statements :

A table which contains the truth values of a given compound statement for all the possible combinations of the truth values of the component statements, is called *truth table* for that compound statement.

(1) Conjunction :

The truth value of $p \wedge q$ is T, if both p and q have the truth value T. In all other cases, $p \wedge q$ is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(2) Disjunction :

The truth value of $p \vee q$ is F, if both p and q have the truth value F. In all other cases, $p \vee q$ has truth value T, i.e. $p \vee q$ is true, if at least one of p and q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(3) Implication or Conditional :

The truth value of $p \to q$ is F, if p has truth value T and q has the truth value F. In all other cases, $p \to q$ has the truth value T.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(4) Double Implication or Biconditional :

The truth value of $p \leftrightarrow q$ is T, if both p and q have the same truth values. In other cases, the truth value of $p \leftrightarrow q$ is F.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(5) Negation :

If p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

p	$\sim p$
T	F
F	T

Notes : (1) The components of the implication $p \to q$ are p and q. The first component p is called *antecedent* or the *hypothesis* and the second component q is called *consequent* or *conclusion*.

(2) For the implication $p \to q$, the *converse* is defined as $q \to p$, the *inverse* is defined as $\sim p \to \sim q$ and the *contrapositive* is defined as $\sim q \to \sim p$.

(3) For the implication $p \to q$, all the following statements have the same meaning :

(i) p implies q.
(ii) If p then q.
(iii) q whenever p.
(iv) p only if q.
(v) p is sufficient for q.
(vi) q is necessary for p.
(vii) Sufficient condition for q is p.
(viii) q follows from p.

(4) For the double implication $p \leftrightarrow q$, all the following statements have the same meaning :

(i) p if and only if q.
(ii) p implies q and q implies p.
(iii) p implies and implied by q.
(iv) p is necessary and sufficient for q.
(v) q is necessary and sufficient for p.
(vi) p is equivalent to q.

(5) $p \leftrightarrow q$ means $p \to q$ and $q \to p$. Hence, $p \leftrightarrow q$ is the conjunction of $p \to q$ and $q \to p$.

Solved Examples	3 marks each

Ex. 1. Express the following statements in symbolic form and determine the truth value of each statement :

(1) Neither 27 is a prime number nor divisible by 4.

(2) Fixed deposit scheme gives fixed return or share market gives uncertain returns.

(3) A quadratic equation has two distinct roots or 6 has three prime factors.

Solution :

(1) Let p : 27 is a prime number.

q : 27 is divisible by 4.

Then the symbolic form of the given statement is $\sim p \wedge \sim q$.

The truth values of both p and q are F.

$\therefore$ the truth value of $\sim p \wedge \sim q$ is T ... $[\sim F \wedge \sim F \equiv T \wedge T \equiv T]$

---

(2) Let p : Fixed deposit scheme gives fixed return.

q : Share market gives uncertain returns.

Then the symbolic form of the given statement is $p \vee q$.

The truth values of both p and q are T.

$\therefore$ the truth value of $p \vee q$ is T ... $[T \vee T \equiv T]$

---

(3) Let p : A quadratic equation has two distinct roots.

q : 6 has three prime factors.

Then the symbolic form of the given statement is $p \vee q$.

The truth values of both p and q are F.

$\therefore$ the truth value of $p \vee q$ is F ... $[F \vee F \equiv F]$

Ex. 2. Express the following statements in symbolic form and determine the truth value of each statement :

(1) If a joint venture is a temporary partnership, then discount on purchase is credited to the supplier.

(2) It is not true that $2+3=6$ or $12+3=5$.

(3) Every accountant is free to apply his own accounting rules if and only if machinery is an asset.

Solution :

(1) Let p : Joint venture is a temporary partnership.

q : Discount on purchases is credited to supplier.

Then the symbolic form of the given statement is $p \rightarrow q$.

The truth values of p and q are T and F respectively.

$\therefore$ the truth value of $p \to q$ is F ... [$\mathrm{T} \to \mathrm{F} \equiv \mathrm{F}$]

---

(2) Let p : $2+3=6$.

q : $12+3=5$.

Then the symbolic form of the given statement is $\sim(p \vee q)$.

The truth values of both p and q are F.

$\therefore$ the truth value of $\sim(p \vee q)$ is T ... [$\sim(\mathrm{F} \vee \mathrm{F}) \equiv \sim \mathrm{F} \equiv \mathrm{T}$]

---

(3) Let p : Every accountant is free to apply his own accounting rules.

q : Machinery is an asset.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

The truth values of p and q are F and T respectively.

$\therefore$ the truth value of $p \leftrightarrow q$ is F ... [$\mathrm{F} \leftrightarrow \mathrm{T} \equiv \mathrm{F}$]

Ex. 3. Prepare the truth tables for the following statement patterns :

(1) $[(p \wedge \sim q) \leftrightarrow (q \to p)$ ***(Sept. '21; March '23)*** **(2)** $(p \wedge r) \to (p \vee \sim q)$.

Solution :

(1)

p	q	$\sim q$	$p \wedge \sim q$	$q \to p$	$(p \wedge \sim q) \leftrightarrow (q \to p)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	F	T	F

---

(2)

p	q	r	$\sim q$	$p \wedge r$	$p \vee \sim q$	$(p \wedge r) \to (p \vee \sim q)$
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

Ex. 4. If p and q are true and r and s are false, find the truth value of each of the following statements :

(1) $(p \rightarrow q) \leftrightarrow \sim(p \vee q)$

(2) $\sim[(\sim p \vee s) \wedge (\sim q \wedge r)]$

(3) $\sim[p \vee (r \wedge s)] \wedge \sim[(r \wedge \sim s) \wedge q]$.

Solution :

(1) $(p \rightarrow q) \leftrightarrow \sim(p \vee q) \equiv (T \rightarrow T) \leftrightarrow \sim(T \vee T)$

$\equiv T \leftrightarrow \sim T$

$\equiv T \leftrightarrow F \equiv F$

Hence, the truth value of the given statement is **false.**

---

(2) $\sim[(\sim p \vee s) \wedge (\sim q \wedge r)] \equiv \sim[(\sim T \vee F) \wedge (\sim T \wedge F)]$

$\equiv \sim[(F \vee F) \wedge (F \wedge F)]$

$\equiv \sim(F \wedge F)$

$\equiv \sim F \equiv T$

Hence, the truth value of the given statement is **true.**

---

(3) $\sim[p \vee (r \wedge s)] \wedge \sim[(r \wedge \sim s) \wedge q]$

$\equiv \sim[T \vee (F \wedge F] \wedge \sim[(F \wedge \sim F) \wedge T]$

$\equiv \sim[T \vee F] \wedge \sim[(F \wedge T) \wedge T]$

$\equiv \sim T \wedge \sim[F \wedge T]$

$\equiv F \wedge \sim F$

$\equiv F \wedge T \equiv F$

Hence, the truth value of the given statement is **false.**

Ex. 5. Write the converse, inverse and contrapositive of the following statement :

'If a triangle is equilateral, then it is equiangular. ***(March '25)***

Solution : Let p : A triangle is equilateral.

q : It is equiangular.

Then the symbolic form of the given statement is $p \rightarrow q$.

Converse : $q \rightarrow p$ is the converse of $p \rightarrow q$.

i.e. If a triangle is equiangular, then it is equilateral.

Inverse : $\sim p \rightarrow \sim q$ is the inverse of $p \rightarrow q$.

i.e. If a triangle is not equilateral, then it is not equiangular.

Contrapositive : $\sim q \rightarrow \sim p$ is the contrapositive of $p \rightarrow q$.

i.e. If a triangle is not equiangular, then it is not equilateral.

Examples for Practice | 3 marks each

1. Determine the truth values of the following statements :

(1) 2 is a prime number and 3 is a rational number.

(2) If $9 > 1$, then $x^2 - 2x + 1 = 0$ for $x = 1$.

(3) $3 + 5 = 8$ if and only if $3 + 2 = 7$.

2. Assuming the following statements

p **: Stock prices are high**

q **: Stocks are rising**

to be true, find the truth values of the following :

(1) Stock prices are high and stocks are rising if and only if stock prices are high.

(2) If stock prices are high, then stocks are not rising.

(3) It is false that stocks are rising and stock prices are high.

3. Express the given statements in symbolic form and determine the truth value of each statement :

(1) If 4 is an odd number, then 6 is divisible by 3.

(2) The sun rises in the west if and only if $4 + 3 = 10$.

(3) 3 is a prime number and odd number.

4. Prepare the truth tables for the following statement patterns :

(1) $[(p \rightarrow q) \vee p] \rightarrow p$

(2) $\sim p \wedge [(p \vee \sim q) \wedge q]$

(3) $(p \vee r) \rightarrow \sim (q \wedge r)$

(4) $\sim (p \vee q) \rightarrow \sim (p \wedge q)$.

5. If p, q, r are statements with truth values T, T, F respectively, determine the truth values of the following :

(1) $(p \wedge q) \rightarrow \sim p$

(2) $p \leftrightarrow (q \rightarrow \sim p)$

(3) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(4) $\sim (p \wedge q) \rightarrow \sim (q \wedge p)$

(5) $\sim [(p \rightarrow q) \leftrightarrow (p \wedge \sim q)]$.

6. If p and q are true and r and s are false, find the truth value of each of the following statement patterns :

(1) $\sim [(p \wedge \sim s) \vee (q \wedge \sim r)]$

(2) $[p \wedge (q \wedge r)] \vee [(p \vee q) \wedge (\sim r \vee s)]$

(3) $[(p \vee s) \rightarrow r] \vee \sim [\sim (p \rightarrow q) \vee s]$

(4) $[(p \rightarrow q) \rightarrow (q \rightarrow r)] \rightarrow (r \rightarrow s)$.

7. **If the statements p and q are false and the statements r and s are true, find the truth value of :**

$[(\sim p \wedge q) \wedge \sim r] \vee [(q \rightarrow p) \rightarrow (\sim s \vee r)]$.

8. **Write the converse, inverse and contrapositive of the following statements :**

(1) If $2+5=10$, then $4+10=20$. *(March '22)*

(2) If he studies, then he will go to college. *(July '22)*

(3) If Sanjay studies, then he will go to college. *(July '23)*

(4) If the train reaches on time, than I can catch the connecting flight. *(July '24)*

Answers

1. (1) $p \wedge q$, T (2) $p \rightarrow q$, T (3) $p \leftrightarrow q$, F.
2. (1) $(p \wedge q) \leftrightarrow p$, T (2) $p \rightarrow \sim q$, F (3) $\sim (q \wedge p)$, F.
3. (1) $p \rightarrow q$, T (2) $p \leftrightarrow q$, T (3) $p \wedge q$, T.
4. (1) TTFF (2) FFFF (3) FTTTFTTT (4) TTTT.
5. (1) F (2) F (3) F (4) T (5) T.
6. (1) F (2) T (3) T (4) T. 7. T.
8. (1) **Converse :** If $4+10=20$, then $2+5=10$.

   Inverse : If $2+5 \neq 10$, then $4+10 \neq 20$.

   Contrapositive : If $4+10 \neq 20$, then $2+5 \neq 10$.

   (2) **Converse :** If he will go to college, then he studies.

   Inverse : If he does not study, then he will not go to college.

   Contrapositive : If he will not go to college, then he does not study.

   (3) **Converse :** If Sanjay go to college, then he studies.

   Inverse : If Sanjay does not study, then he will not go to college.

   Contrapositive : If Sanjay will not go to college, then he does not study.

   (4) **Converse :** If I can catch the connecting flight, then the train reaches on time.

   Inverse : If the train does not reach on time, then I can not catch the connecting flight.

   Contrapositive : If I can not catch the connecting flight, then the train does not reach on time.

1.4 LOGICAL EQUIVALENCE

Two statement patterns S_1 and S_2 are said to be *logically equivalent* (or *equivalent* or *equal*), if they have identical truth values in their truth tables for each combination of the truth values of their components.
In this case, we write $S_1 \equiv S_2$.

Solved Examples	3 or 4 marks each

Ex. 6. Using truth tables, prove the following equivalences :

(1) $p \rightarrow q \equiv \sim q \rightarrow \sim p \equiv \sim p \vee q$

(2) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

(3) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. *(March '25)*

Solution :

(1)

1	2	3	4	5	6	7
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$\sim p \vee q$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

The entries in columns 5, 6 and 7 are identical.

$\therefore p \rightarrow q \equiv \sim q \rightarrow \sim p \equiv \sim p \vee q$.

(2)

1	2	3	4	5	6
p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

The entries in columns 3 and 6 are identical.

$\therefore p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

(3)

1	2	3	4	5	6	7	8
p	q	r	$q\wedge r$	$p\vee(q\wedge r)$	$p\vee q$	$p\vee r$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

The entries in columns 5 and 8 are identical.

$\therefore p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$.

Ex. 7. Without using truth table, show that

(1) $p\leftrightarrow q\equiv(p\wedge q)\vee(\sim p\wedge\sim q)$

(2) $(p\vee q)\rightarrow r\equiv(p\rightarrow r)\wedge(q\rightarrow r)$.

Solution :

(1) LHS $=p\leftrightarrow q$

$\equiv(p\rightarrow q)\wedge(q\rightarrow p)$

$\equiv(\sim p\vee q)\wedge(\sim q\vee p)$ ... (Conditional Law)

$\equiv[\sim p\wedge(\sim q\vee p)]\vee[q\wedge(\sim q\vee p)]$ ... (Distributive Law)

$\equiv[(\sim p\wedge\sim q)\vee(\sim p\wedge p)]\vee[(q\wedge\sim q)\vee(q\wedge p)]$ ... (Distributive Law)

$\equiv[(\sim p\wedge\sim q)\vee c]\vee[c\vee(q\wedge p)]$ ... (Complement Law)

$\equiv(\sim p\wedge\sim q)\vee(q\wedge p)$ ... (Identity Law)

$\equiv(\sim p\wedge\sim q)\vee(p\wedge q)$ ... (Commutative Law)

$\equiv(p\wedge q)\vee(\sim p\wedge\sim q)$ ... (Commutative Law)

$\equiv$ RHS.

---

(2) LHS $=(p\vee q)\rightarrow r$

$\equiv\sim(p\vee q)\vee r$ ... (Conditional Law)

$\equiv(\sim p\wedge\sim q)\vee r$ ... (De Morgan's Law)

$\equiv(\sim p\vee r)\wedge(\sim q\vee r)$ ... (Distributive Law)

$\equiv(p\rightarrow r)\wedge(q\rightarrow r)$ ... (Conditional Law)

$\equiv$ RHS.

Ex. 8. Consider the following statements :

(a) If D is dog, then D is very good.

(b) If D is very good, then D is dog.

(c) If D is not very good, then D is not a dog.

(d) If D is not a dog, then D is not very good.

Identify the pairs of statements having the same meaning. Justify.

(March '24)

Solution : Let p : D is dog and q : D is very good.

Then the given statements in the symbolic form are :

(a) $p \to q$ (b) $q \to p$ (c) $\sim q \to \sim p$ (d) $\sim p \to \sim q$.

				(a)	(b)	(c)	(d)
p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim q \to \sim p$	$\sim p \to \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

The entries in columns (a) and (c) are identical. Hence, these statements are equivalent.

$\therefore$ the statements (a) and (c) have the same meaning.

Similarly, the entries in columns (b) and (d) are identical. Hence, these statements are equivalent. $\therefore$ the statements (b) and (d) have the same meaning.

Examples for Practice	**3 or 4 marks each**

1. Using truth tables, prove the following equivalences :

(1) $\sim(p \to \sim q) \equiv p \wedge \sim(\sim q) \equiv p \wedge q$ *(July '24)*

(2) $p \leftrightarrow q \equiv \sim[(p \vee q) \wedge \sim(p \wedge q)$

(3) $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

(4) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

(5) $[\sim(p \vee q) \vee (p \vee q)] \wedge r \equiv r$

(6) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ *(July '23)*

2. Prove that the following pairs of statement patterns are equivalent :

(1) $\sim p \wedge q$ and $(p \vee q) \wedge \sim p$

(2) $p \wedge (\sim p \vee q)$ and $p \wedge q$

(3) $p\lor(q\lor r)$ and $(p\lor q)\lor(p\lor r)$

(4) $p\land(\sim q\lor r)$ and $\sim[p\to(q\land\sim r)]$

3. Without using truth table, prove that

(1) $\sim(p\lor q)\lor(\sim p\land q)\equiv\sim p$

(2) $[p\land(q\lor r)\lor[\sim r\land\sim q\land p]\equiv p$

(3) $(p\land q)\lor(p\land\sim q)\lor(\sim p\land\sim q)\equiv p\lor\sim q$.

1.5 TAUTOLOGY, CONTRADICTION AND CONTINGENCY

1. A statement which is true for all the possible combinations of the truth values of its component statements, is called a **tautology.**
2. A statement which is false for all the possible combinations of the truth values of its component statements, is called a **contradiction.**

   It is obvious that the negation of a tautology is a contradiction and the negation of contradiction is a tautology.
3. A statement which is neither a tautology nor a contradiction, is called a **contingency.**

Solved Examples	**3 or 4 marks each**

Ex. 9. Examine, whether each of the following statement pattern is a tautology or a contradiction or a contingency :

(1) $(\sim p\lor\sim q)\leftrightarrow\sim(p\land q)$

(2) $(p\land\sim q)\to(\sim p\land\sim q)$

(3) $[(\sim p\land q)\land(q\land r)]\land(\sim q)$ ***(March '22)***

Solution :

(1)

p	q	$\sim p$	$\sim q$	$\sim p\lor\sim q$	$p\land q$	$\sim(p\land q)$	$(\sim p\lor\sim q)\leftrightarrow\sim(p\land q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

All the entries in the last column of the above truth table are T.

$\therefore (\sim p\lor\sim q)\leftrightarrow\sim(p\land q)$ is a **tautology.**

(2)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge \sim q$	$(p \wedge \sim q) \rightarrow (\sim p \wedge \sim q)$
T	T	F	F	F	F	T
T	F	F	T	T	F	F
F	T	T	F	F	F	T
F	F	T	T	F	T	T

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore (p \wedge \sim q) \rightarrow (\sim p \wedge \sim q)$ is a **contingency.**

(3)

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge q$	$q \wedge r$	$(\sim p \wedge q) \wedge (q \wedge r)$	$[(\sim p \wedge q) \wedge (q \wedge r)] \wedge (\sim q)$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	F	F	F	F
T	F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T	F
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	F	F	F

All the entries in the last column of the above truth table are F.

$\therefore [(\sim p \wedge q) \wedge (q \wedge r)] \wedge (\sim q)$ is a **contradiction.**

Examples for Practice	**3 or 4 marks each**

1. Examine, whether each of the following statement pattern is a tautology or a contradiction or a contingency :

(1) $\sim p \rightarrow (p \rightarrow \sim q)$ *(March '24)*

(2) $(p \wedge \sim q) \leftrightarrow (q \rightarrow p)$ *(Sept. '21)*

(3) $[(p \rightarrow q) \wedge (\sim q)] \rightarrow (\sim p)$

(4) $(p \rightarrow q) \leftrightarrow (\sim p \wedge q)$

(5) $[\sim (p \wedge q) \rightarrow p] \leftrightarrow (\sim p \wedge \sim q)$

(6) $[\sim (p \wedge q) \wedge (q \wedge r)] \vee (\sim q)$

(7) $[p \to (\sim q \vee r)] \leftrightarrow \sim [p \to (q \to r)]$

(8) $p \to (q \vee r)$ *(July '22)*

2. Prove that each of the following statement pattern is a tautology :

(1) $(\sim p \to \sim q) \wedge (\sim q \to \sim p)$ (2) $(\sim p \wedge \sim q) \to (p \to q)$

(3) $(p \wedge r) \to (p \vee \sim q)$

3. Prove that each of the following statement pattern is a contradiction :

(1) $\sim p \wedge [(p \vee \sim q) \wedge q]$ (2) $(p \to q) \wedge (p \wedge \sim q)$

(3) $(\sim p \wedge \sim q) \wedge (q \wedge r)$

4. Prove that each of the following statement pattern is a contingency :

(1) $(p \to q) \leftrightarrow (\sim p \wedge q)$ (2) $p \wedge [(p \to \sim q) \to q]$

(3) $(p \to q) \wedge (p \to r)$.

Answers

1. **Tautology :** (1), (3) **Contradiction :** (7)
   Contingency : (2), (4), (5), (6), (8).

1.6 QUANTIFIERS AND QUANTIFIED STATEMENTS

Quantifiers : A sentence contains one or more variables is called an **open sentence.**

An open sentence become true or false statement, when we replace the variable by some specific value from a given set.

e.g. Consider an open sentence $x + 5 = 8$.

If $x = 3$, then this sentence becomes true statement and if $x \neq 3$, then this sentence becomes false statement. The phrases quantify the variable(s) in open sentences are called **quantifiers.** There are two types of quantifiers :

(i) Universal quantifier : The quantifier 'for all' or 'for every' is called universal quantifier and is denoted by $\forall$.

(ii) Existential quantifier : The quantifier 'for some' or 'for one' or 'there exists at least one' or 'there exists' is called existential quantifier and is denoted by $\exists$.

Quantified statement : An open sentence with a quantifier becomes a statement. Such statement is called a quantified statement.

Solved Examples	3 marks each

Ex. 10. Use qualifiers to convert each of the following open sentences defined on N, into a true statement :

(1) $x^2+3x-10=0$ (2) $3x-4<9$ (3) $n^2 \geqslant 1$.

Solution :

(1) $\exists\, x \in N$, such that $x^2+3x-10=0$ is a true statement

($x=2 \in N$ satisfy $x^2+3x-10=0$)

(2) $\exists\, x \in N$, such that $3x-4<9$ is a true statement.

($x=1, 2, 3, 4 \in N$ satisfy $3x-4<9$)

(3) $\forall\, n \in N$, $n^2 \geqslant 1$ is a true statement.

(All $n \in N$ satisfy $n^2 \geqslant 1$)

Ex. 11. If $B=\{2, 3, 5, 6, 7\}$, determine the truth value of each of the following :

(1) $\forall\, x \in B$, x is a prime number. (2) $\exists\, n \in B$, such that $n+6>12$.

(3) $\exists\, n \in B$, such that $2n+2<4$.

Solution :

(1) $x=6 \in B$ does not satisfy x is a prime number.

So, the given statement is **false**, hence its truth value is F.

(2) Clearly $n=7 \in B$ satisfies $n+6>12$.

So, the given statement is **true**, hence its truth value is T.

(3) No element $n \in B$ satisfy $2n+2<4$.

So, the given statement is **false**, hence its truth value is F.

Examples for Practice	3 marks each

1. If $A=\{2, 3, 4, 5, 6, 7, 8\}$, determine the truth value of each of the following :

(1) $\exists x \in A$, such that $x+5=8$. (2) $\forall x \in A$, $x+1 \leqslant 10$.

(3) $\forall x \in A$, $x+5>8$. (4) $\exists x \in A$, such that x^2+1 is even.

(5) $\forall x \in A$, such that $x-2$ is a rational number.

2. Use quantifiers to convert each of the following open sentences defined on N, into a true statement :

(1) $2n-1=5$ (2) $y+4>6$ (3) $3y-2 \leqslant 9$ (4) $x^2>0$.

Answers

1. (1) T (2) T (3) F (4) T (5) T.

2. (1) $\exists n \in N$, such that $2n - 1 = 5$ $(n = 3)$

(2) $\exists y \in N$, such that $y + 4 > 6$ $(y = 3, 4, 5, \ldots)$

(3) $\exists y \in N$, such that $3y - 2 \leqslant 9$ $(y = 1, 2, 3)$

(4) $\forall x \in N$, such that $x^2 > 0$.

1.7 DUALITY

Two compound statements S_1 and S_2 are said to be **duals** of each other, if one can be obtained from the other by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$.

The connectives $\wedge$ and $\vee$ are called duals of each other.

e.g.

Consider the following results :

$$\sim (p \wedge q) \equiv \sim p \vee \sim q \qquad \ldots (1)$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q \qquad \ldots (2)$$

From these results, we observe that a statement pattern involving a conjunction ($\wedge$) can be expressed as a disjunction ($\vee$) and vice versa.

If we replace ($\wedge$ by $\vee$) and ($\vee$ by $\wedge$) in (1), we get, (2) and by the same replacement, we get, (1) from (2).

This is referred to as *duality* of conjunction and disjunction. Statement patterns (1) and (2) are called *duals* of each other. Results (1) and (2) are known as *De Morgan's Laws*.

Note : If a compound statement S contains t (tautology) or c (contradiction), then the dual of S is obtained by replacing t by c, c by t, $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Solved Examples	1 mark each

Ex. 12. Write the dual of each of the following :

(1) $\sim (p \wedge q) \vee (\sim q \wedge \sim p)$ **(2)** $[(p \wedge q) \vee r] \wedge [(q \wedge r) \vee s]$

(3) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ **(4)** $(p \wedge t) \vee (c \wedge \sim q)$.

Solution : The duals are given by :

(1) $\sim (p \vee q) \wedge (\sim q \vee \sim p)$ **(2)** $[(p \vee q) \wedge r] \vee [(q \vee r) \wedge s]$

(3) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ **(4)** $(p \vee c) \wedge (t \vee \sim q)$.

Ex. 13. Write the dual statement of each of the following compound statements :

(1) All natural numbers are integers or rational numbers.

(2) 13 is a prime number and India is a democratic country.

(3) Some roses are red and all lilies are white.

Solution : The duals are given by

(1) All natural numbers are integers and rational numbers.

(2) 13 is a prime number or India is a democratic country.

(3) Some roses are red or all lilies are white.

Examples for Practice	1 mark each

1. Write the duals of each of the following :

(1) $(p \wedge q) \vee \sim q$ (2) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim r)]$

(3) $(\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

(4) $p \vee (p \wedge q) \equiv p$

(5) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(6) $p \vee (q \vee r) \equiv \sim[(p \wedge q) \vee (r \vee s)]$

(7) $(p \vee q) \wedge t$

(8) $c \vee (p \wedge q)$.

2. Write the dual statement of each of the following compound statements :

(1) Karina is very good or everybody likes her.

(2) He is tall and handsome.

(3) Leela speaks Marathi or English.

(4) 2 is even number or 9 is a perfect square.

Answers

1. (1) $(p \vee q) \wedge \sim q$

(2) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim r)]$

(3) $(\sim p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim q)$

(4) $p \wedge (p \vee q) \equiv p$

(5) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(6) $p \wedge (q \wedge r) \equiv \sim[(p \vee q) \wedge (r \wedge s)]$

(7) $(p \wedge q) \vee c$ (8) $t \wedge (p \vee q)$.

2. (1) Karina is very good and everybody likes her.

(2) He is tall or handsome.

(3) Leela speaks Marathi and English.

(4) 2 is even number and 9 is a perfect square.

1.8 NEGATION OF A COMPOUND STATEMENT

Negation of a simple statement is obtained by inserting 'not' at the appropriate place in the statement.

e.g. The negation of 'Ram is a good boy' is 'Ram is not a good boy'.

Now, we will study the negations of the compound statements involving conjunction, disjunction, conditional and biconditional.

1. Negation of Conjunction : The negation of conjunction is denoted by $\sim(p \wedge q)$. Using truth table, we show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

1	2	3	4	5	6	7
p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The entries in the columns 4 and 7 are identical.

$\therefore \sim(p \wedge q) \equiv \sim p \vee \sim q$.

Thus, **the negation of the conjunction of two statements is the disjunction of their negations.**

2. Negation of Disjunction : The negation of disjunction is denoted by $\sim(p \vee q)$. Using truth table, we show that $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

1	2	3	4	5	6	7
p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The entries in the columns 4 and 7 are identical.

$\therefore \sim(p \vee q) \equiv \sim p \wedge \sim q$.

Thus, **the negation of the disjunction of two statements is the conjunction of their negations.**

---

3. **Negation of Negation :** The negation of negation of a statement is the statement itself.

   i.e. $\sim(\sim p) \equiv p$

1	2	3
p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

   The entries in columns 1 and 3 are identical.

   $\therefore \sim(\sim p) \equiv p$.

   Thus, **the negation of negation of a statement is statement itself.**

---

4. **Negation of Implication (Conditional) :** The negation of the conditional $p \rightarrow q$ is denoted by $\sim(p \rightarrow q)$. Using truth tables, we show that $\sim(p \rightarrow q) \equiv p \wedge \sim q$.

1	2	3	4	5	6
p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

   The entries in the columns 4 and 6 are identical.

   $\therefore \sim(p \rightarrow q) \equiv p \wedge \sim q$.

---

5. **Negation of Biconditional (Double Implication) :** The negation of the biconditional $p \leftrightarrow q$ is denoted by $\sim(p \leftrightarrow q)$. Using truth tables, we show that $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

1	2	3	4	5	6	7	8	9
p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$p \wedge \sim q$	$q \wedge \sim p$	$(p \wedge \sim q) \vee (q \wedge \sim p)$
T	T	F	F	T	F	F	F	F
T	F	F	T	F	T	T	F	T
F	T	T	F	F	T	F	T	T
F	F	T	T	T	F	F	F	F

The entries in the columns 6 and 9 are identical.

$\therefore \sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

6. **Negation of Quantified Statement :** The negation of the quantified statement is obtained by replacing the word 'all' by 'some', 'for every' by 'there exists', and vice versa.

Solved Examples	**3 marks each**

Ex. 14. Write the negation of each of the following statements :

(1) All girls are sincere.

(2) Some continuous functions are differentiable. *(March '23)*

(3) $\forall n \in N, n+1>0$. *(Sept. '21)*

(4) $\exists n \in N$, such that (n^2+2) is odd number. *(March '23)*

Solution : The negations of the given statements are :

(1) Some girls are not sincere.

(2) All continuous functions are not differentiable.

(3) $\exists n \in N$, such that $n+1 \leqslant 0$.

(4) $\forall n \in N$, (n^2+2) is not an odd number.

Ex. 15. Write the negations of the following :

(1) 7 is a prime number and Taj Mahal is in Agra.

(2) A triangle is an equilateral if and only if it is an equiangular triangle.

(3) $2+3<6$ or $\sqrt{2}$ is an irrational number.

(4) The number is neither odd nor even.

(5) All the stars are shining if it is night.

Solution :

(1) Let p : 7 is prime number.

q : Taj Mahal is in Agra.

Then the symbolic form of the given statement is $p \wedge q$.

Since $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'7 is not a prime number or Taj Mahal is not in Agra.'

(2) Let p : A triangle is an equilateral.

q : It is an equiangular triangle.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

Since $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$, the negation of given statement is :

'A triangle is an equilateral and it is not equiangular triangle or the triangle is an equiangular and it is not equilateral triangle.'

(3) Let p : $2+3<6$

q : $\sqrt{2}$ is an irrational number.

Then the symbolic form of the given statement is $p \vee q$.

Since $\sim(p \vee q) \equiv \sim p \wedge \sim q$, the negation of given statement is :

'$2+3 \nless 6$ and $\sqrt{2}$ is not an irrational number.'

---

(4) Let p : The number is odd.

q : The number is even.

$\therefore$ the symbolic form of given statement is $\sim p \wedge \sim q$.

Since $\sim(\sim p \wedge \sim q) \equiv \sim(\sim p) \vee \sim(\sim q) \equiv p \vee q$.

the negation of given statement is :

'The number is either odd or even.'

---

(5) The given statement can be written as :

If it is night, then all the stars are shining.

Let p : It is night.

q : All the stars are shining.

Then the symbolic form of the given statement is $p \to q$.

Since $\sim(p \to q) \equiv p \wedge \sim q$, the negation of given statement is :

'It is night and all the stars are not shining.'

Ex. 16. Using the rules of negation, write the negations of the following :

(1) $(p \to r) \wedge q$ **(2) $(p \to q) \vee (p \to r)$** ***(March '23)***

(3) $(p \leftrightarrow q) \vee (\sim q \to \sim r)$.

Solution :

(1) The negation of $(p \to r) \wedge q$ is :

$\sim[(p \to r) \wedge q] \equiv \sim(p \to r) \vee (\sim q)$ ... [Negation of conjunction]

$\equiv (p \wedge \sim r) \vee (\sim q)$ ... [Negation of implication]

---

(2) The negation of $(p \to q) \vee (p \to r)$ is :

$\sim[(p \to q) \vee (p \to r)] \equiv \sim(p \to q) \wedge \sim(p \to r)$ ... [Negation of disjunction]

$\equiv (p \wedge \sim q) \wedge (p \wedge \sim r)$ ... [Negation of implication]

---

(3) The negation of $(p \leftrightarrow q) \vee (\sim q \to \sim r)$ is :

$\sim[(p \leftrightarrow q) \vee (\sim q \to \sim r)] \equiv \sim(p \leftrightarrow q) \wedge \sim(\sim q \to \sim r)$

... [Negation of disjunction]

$\equiv [(p \wedge \sim q) \vee (q \wedge \sim p)] \wedge [\sim q \wedge \sim(\sim r)]$

... [Negation of biconditional and implication]

$\equiv [(p \wedge \sim q) \vee (q \wedge \sim p)] \wedge (\sim q \wedge r)$ ... [Negation of negation]

Examples for Practice	3 marks each

1. Write the negation of each of the following statements :

(1) All politicians are honest.

(2) $\sqrt{5}$ is an irrational number. *(Sept. '21)*

(3) It is false that Nagpur is the capital of Maharashtra.

(4) $2+3 \neq 5$.

(5) Some bureaucrats are efficient.

(6) $\exists x \in A$, such that $x+5<11$.

(7) $\forall n \in N, n+3>9$.

2. Write the negations of the following statements :

(1) $3+4<8$ and $\sqrt{3}$ is an irrational number.

(2) $10>5$ and $3<8$.

(3) The crop will be destroyed if there is a flood.

(4) $\exists x \in A$, such that $x+5<11$.

(5) The number is an odd number if and only if it is not divisible by 2.

(6) If a quadrilateral is a rectangle, then it is a parallelogram.

3. Using the rules of negation, write the negations of the following statements and justify :

(1) $p \vee \sim q$ *(Sept. '21)*

(2) $(\sim p \wedge r) \vee (p \wedge \sim r)$

(3) $(\sim p \wedge q) \vee (p \wedge \sim q)$

(4) $(p \rightarrow q) \wedge r$

(5) $(p \vee \sim r) \wedge \sim q$

(6) $(p \vee \sim q) \rightarrow (p \wedge \sim q)$.

Answers

1. (1) Some politicians are not honest.

(2) $\sqrt{5}$ is not an irrational number.

(3) Nagpur is the capital of Maharashtra.

(4) $2+3=5$.

(5) All bureaucrats are not efficient.

(6) $\forall x \in A, x+5 \geqslant 11$.

(7) $\exists n \in N$, such that $n+3 \leqslant 9$.

2. (1) $3+4 \geqslant 8$ or $\sqrt{3}$ is not an irrational number.

(2) $10 \leqslant 5$ or $3 \geqslant 8$.

(3) The crop will not be destroyed and there is a flood.

(4) $\forall x \in A,\ x+5 \geqslant 11$.

(5) The number is an odd number and it is divisible by 2 or the number is not divisible by 2 and it is not an odd number.

(6) A quadrilateral is a rectangle and it is not a parallelogram.

3. (1) $\sim p \wedge q$ (2) $(p \vee \sim r) \wedge (\sim p \vee r)$ (3) $p \leftrightarrow q$

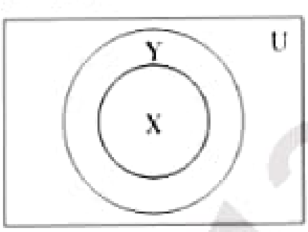
(4) $(p \wedge \sim q) \vee (\sim r)$ (5) $(\sim p \wedge r) \vee q$

(6) $(p \vee \sim q) \wedge (\sim p \vee q)$.

1.9 VENN DIAGRAM

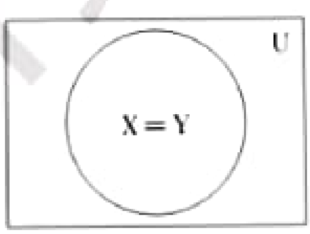
We consider the Venn diagrams representing the truth of the following statements :

(1) All X's are Y's :



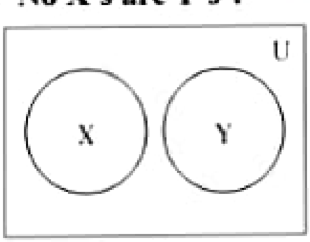
$X \cap Y = X \neq \phi$

OR



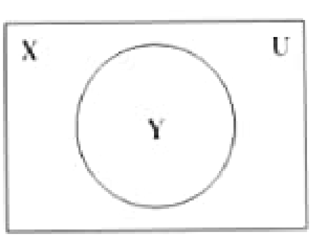
$X = Y$

(2) No X's are Y's :



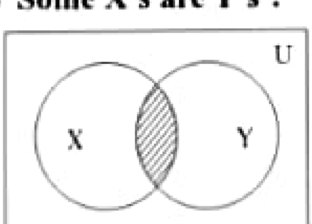
$X \cap Y = \phi$

OR



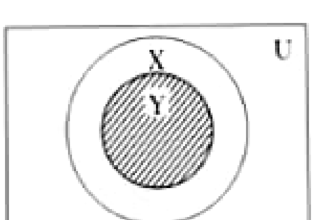
$X \cap Y = \phi$

(3) Some X's are Y's :



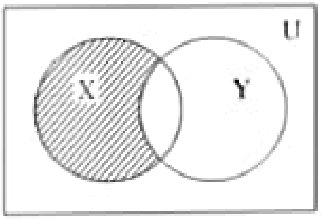
$X \cap Y \neq \phi$

OR



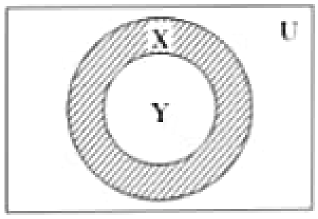
$X \cap Y \neq \phi$

(4) Some X's are not Y's :



$X - Y \neq \phi$

OR



$X - Y \neq \phi$

Solved Examples	3 marks each

Ex. 17. Express the truth of each of the following statements by Venn diagrams :

(1) Some hardworking students are obedient.

(2) No circle is a rectangle.

(3) All teachers are scholars and scholars are teachers.

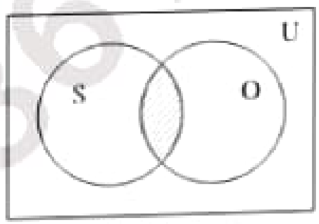
Solution :

(1) Let U : set of all students

S : set of all hardworking students

O : set of all obedient students.

Then the Venn diagram represents the truth of the given statement is as below :



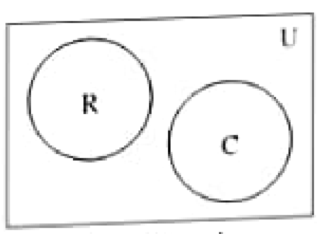
$S \cap O \neq \phi$

(2) Let U : set of closed geometrical figures in plane

R : set of all rectangles

C : set of all circles.

Then the Venn diagram represents the truth of the given statement is as below :



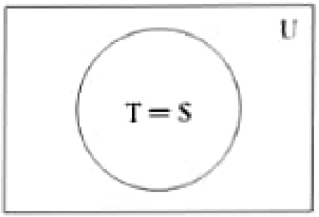
$R \cap C = \phi$

(3) Let U : set of all human beings

T : set of all teachers

S : set of all scholars.

Then the Venn diagram represents the truth of the given statement is as below :



Ex. 18. Express the truth of each of the following statements by Venn diagram :

(1) Equilateral triangles are isosceles.

(2) Some persons are not politician.

(3) Some non-resident Indians are not rich.

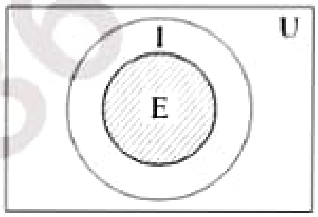
Solution :

(1) Let U : set of all triangles

E : set of all equilateral triangles

I : set of all isosceles triangles.

Then the Venn diagram represents the truth of the given statement is as below :



$E \subset I$

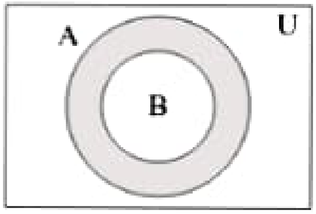
---

(2) Let U : set of all human beings

A : set of all persons

B : set of all politicians.

Then the Venn diagram represents the truth of the given statement is as follows :



$A - B \neq \phi$

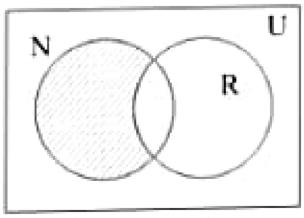
---

(3) Let U : set of all human beings

N : set of all non-resident Indians

R : set of all rich people.

Then the Venn diagram represents the truth of the given statement is as below :



$N - R \neq \phi$

Examples for Practice	3 marks each

1. Express the truth of each of the following statements by Venn diagram :

(1) All students are hardworking.

(2) No child is an adult.

(3) Some rectangles are squares.

2. Draw Venn diagrams to represent the truth of the following statements :

(1) Every rectangle is a parallelogram.

(2) Some teachers are scholars.

(3) No cooperative industry is proprietary firm.

3. Represent the following statements by Venn diagrams :

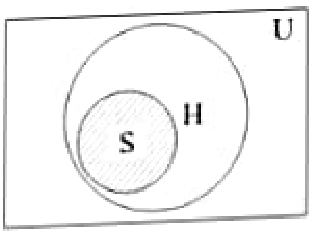
(1) Some share brokers are chartered accountants.

(2) No straight line is a circle.

(3) If n is a prime number and $n \neq 2$, then it is odd.

Answers

1. (1)

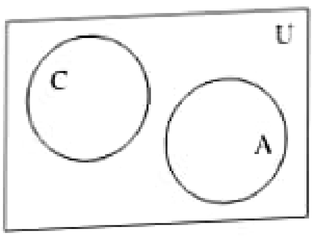


U : set of all human beings

H : set of all hardworking persons

S : set of all students.

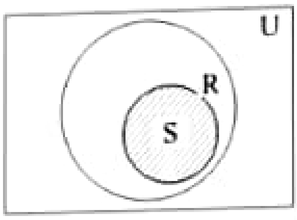
(2)



U : set of all human beings

C : set of all children

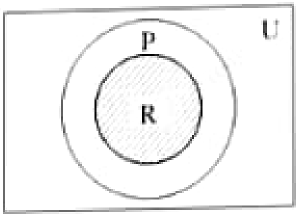
A : set of all adults.

(3) 

U : set of all quadrilaterals

R : set of all rectangles

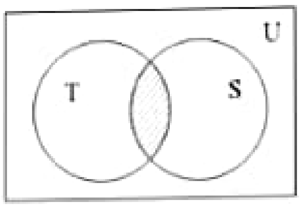
S : set of all squares.

2. (1) 

U : set of all quadrilaterals

R : set of all rectangles

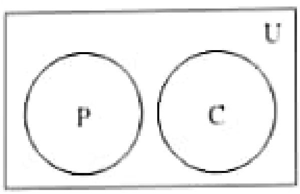
P : set of all parallelograms.

(2) 

U : set of all human beings

T : set of all teachers

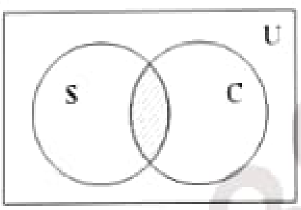
S : set of all scholars.

(3) 

U : set of all industries

C : set of all cooperative industries

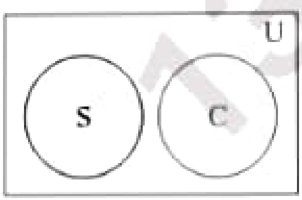
P : set of all proprietary firms.

3. (1) 

U : set of all human beings

S : set of all share brokers

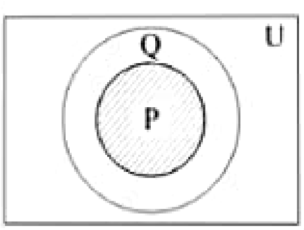
C : set of all chartered accountants.

(2) 

U : set of all geometrical figures

S : set of all straight lines

C : set of all circles.

(3) 

U : set of all real numbers

P : set of all prime numbers n, where $n \neq 2$

Q : set of all odd numbers.

ACTIVITIES	**4 marks each**

1. Let p : The train reaches on time.

q : I can catch the connecting flight.

Therefore, the symbolic form of the statement :

'If the train reaches on time, then I can catch the connecting flight' is $p \rightarrow q$.

Then converse is $q \to p$, i.e. $\square$

Inverse is $\square$, i.e. $\square$

Contrapositive is $\square$, i.e. $\square$.

Solution : Let p : The train reaches on time.

q : I can catch the connecting flight.

Therefore, the symbolic form of the statement :

' If the train reaches on time, then I can catch the connecting flight ' is $p \to q$.

Then converse is $q \to p$, i.e. $\boxed{\text{If I can catch the connecting flight, then the train reaches on time.}}$

Inverse is $\boxed{\sim p \to \sim q}$, i.e. $\boxed{\text{If the train does not reach on time, then I cannot catch the connecting flight.}}$

Contrapositive is $\boxed{\sim q \to \sim p}$, i.e. $\boxed{\text{If I cannot catch the connecting flight, then the train does not reach on time.}}$

---

2. Complete truth table for $\sim [p \vee (\sim q)] \equiv \sim p \wedge q$. Justify it.

1	2	3	4	5	6	7
p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim [p \vee (\sim q)]$	$\sim p \wedge q$
T	T	F	F	T	F	$\square$
T	F	F	T	$\square$	$\square$	F
F	T	T	F	$\square$	$\square$	$\square$
F	F	T	T	$\square$	$\square$	$\square$

Justification : ....

Solution :

1	2	3	4	5	6	7
p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim [p \vee (\sim q)]$	$\sim p \wedge q$
T	T	F	F	T	F	$\boxed{\text{F}}$
T	F	F	T	$\boxed{\text{T}}$	$\boxed{\text{F}}$	F
F	T	T	F	$\boxed{\text{F}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$
F	F	T	T	$\boxed{\text{T}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$

Justification : The entries in columns 6 and 7 are identical.

$\therefore \sim[p \vee (\sim q)] \equiv \sim p \wedge q.$

---

3. You have given following statements :

p **: $9 \times 5 = 45$**

q **: Pune is in Maharashtra.**

r **: 3 is smallest prime number.**

Then write truth values by activity :

(i) $(p \wedge q) \wedge r \equiv (T \wedge T) \wedge F \equiv \boxed{}$

(ii) $\sim(p \wedge r) \equiv \boxed{}$

(iii) $p \rightarrow q \equiv \boxed{}$

(iv) $p \rightarrow r \equiv \boxed{}$

Solution : The truth values of p, q, r are T,T,F respectively.

(i) $(p \wedge q) \wedge r \equiv (T \wedge T) \wedge F \equiv T \wedge F \equiv \boxed{F}$

(ii) $\sim(p \wedge r) \equiv \sim(T \wedge F) \equiv \sim F \equiv \boxed{T}$

(iii) $p \rightarrow q \equiv T \rightarrow T \equiv \boxed{T}$

(iv) $p \rightarrow r \equiv T \rightarrow F \equiv \boxed{F}$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. If p : He is intelligent, q : He is strong are statements in logic, then symbolic form of statement : 'It is false that, he is intelligent or strong' is

(a) $\sim p \vee \sim q$ (b) $\sim(p \wedge q)$

(c) $\sim(p \vee q)$ (d) $p \vee \sim q$ *(July '22; March '25)*

2. The conditional statement $p \rightarrow q$ is equivalent to

(a) $p \vee \sim q$ (b) $\sim p \vee q$

(c) $\sim p \wedge q$ (d) $q \rightarrow p$ *(Sept. '21)*

3. The statement $p \rightarrow q$ is equivalent to

(a) $p \rightarrow \sim q$ (b) $\sim p \vee q$

(c) $\sim p \rightarrow \sim q$ (d) $p \vee \sim q$ *(July '22)*

4. Which of the following is not a statement?

(a) Smoking is injurious to health (b) $2+2=4$

(c) 2 is the only even prime number (d) Come here ***(March '24)***

5. Which of the following sentences is a statement is logic?

(a) He is good actor (b) Did you eat lunch yet?

(c) Every real number is a complex number (d) Bring the motor car here

(July '24)

6. The statement $(\sim p \wedge q) \vee \sim q$ is

(a) $p \vee q$ (b) $p \wedge q$ (c) $\sim(p \vee q)$ (d) $\sim(p \wedge q)$

7. If $p \vee q$ is true and $p \wedge q$ is false, then which of the following is not true?

(a) $p \vee q$ (b) $p \leftrightarrow q$ (c) $\sim p \vee \sim q$ (d) $q \vee \sim q$

8. The converse of contrapositive of $\sim p \rightarrow q$ is

(a) $q \rightarrow p$ (b) $\sim q \rightarrow p$ (c) $p \rightarrow \sim q$ (d) $\sim q \rightarrow \sim p$

9. The negation of $p \wedge (q \rightarrow r)$ is

(a) $(\sim p) \vee (q \wedge \sim r)$ (b) $(\sim p) \vee (q \vee \sim r)$

(c) $(\sim p) \vee (q \rightarrow r)$ (d) $(\sim p) \rightarrow (q \wedge r)$

10. The dual statement $(p \vee q) \wedge (r \vee s)$ is

(a) $(p \wedge q) \wedge (r \wedge s)$ (b) $(p \wedge q) \vee (r \wedge s)$

(c) $(p \vee q) \vee (r \vee s)$ (d) $(r \vee s) \wedge (p \vee q)$ ***(March '23)***

11. The negation of the proposition 'If 2 is prime, then 3 is odd' is

(a) If 2 is not prime, then 3 is not odd.

(b) 2 is prime and 3 is not odd.

(c) 2 is not prime and 3 is odd.

(d) If 2 is not prime, then 3 is odd. ***(July '23)***

Answers

1. (c) $\sim(p \vee q)$
2. (b) $\sim p \vee q$
3. (b) $\sim p \vee q$
4. (d) Come here
5. (c) Every real number is a complex number
6. (d) $\sim(p \wedge q)$
7. (b) $p \leftrightarrow q$
8. (c) $p \rightarrow \sim q$
9. (a) $(\sim p) \vee (q \wedge \sim r)$
10. (b) $(p \wedge q) \vee (r \wedge s)$
11. (b) 2 is prime and 3 is not odd.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. $x^2 = 25$ is a true statement.
2. The negation of $10 + 20 = 30$ is, it is false that $10 + 20 \neq 30$.
3. Dual of $(p \wedge \sim q) \vee t$ is $(p \vee \sim q) \vee c$.
4. The negation of $p \wedge (q \rightarrow r)$ is $p \vee (q \wedge \sim r)$.
5. $p \vee [\sim (p \wedge q)]$ is a tautology.
6. $p \wedge q$ has truth value F. if both p and q have truth value F.
7. $p \wedge t \equiv p$. *(July '23)*

Answers

1. False 2. False 3. False 4. False 5. True 6. False 7. True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. If $p \vee q$ is true, then truth value of $\sim p \wedge \sim q$ is .... *(March '22)*
2. Truth value of $2 + 3 = 5$ if and only if $-3 > -9$ is ....
3. $p \leftrightarrow q$ is false when p and q have .... truth values.
4. The dual of $(p \rightarrow \sim q) \vee q$ is ....
5. The statement pattern $(p \wedge q) \wedge (\sim q)$ is a ....
6. Converse of the statement $q \rightarrow p$ is .... *(March '23)*

Answers

1. false 2. true 3. different 4. $(\sim p \wedge \sim q) \wedge q$ 5. contradiction
6. $p \rightarrow q$.

Question Set 2

MATRICES

(Marks with option : 08)

2.1 ALGEBRA OF MATRICES

Solved Examples	3 marks each

Ex. 1. If $A=\begin{bmatrix}1 & -2\\ 3 & -5\\ -6 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & -2\\ 4 & 2\\ 1 & 5\end{bmatrix}$ and $C=\begin{bmatrix}2 & 4\\ -1 & -4\\ -3 & 6\end{bmatrix}$,

find the matrix X such that $3A-4B+5X=C$.

Solution : $3A-4B+5X=C$

$\therefore\ 5X=C-3A+4B$

$$=\begin{bmatrix}2 & 4\\ -1 & -4\\ -3 & 6\end{bmatrix}-3\begin{bmatrix}1 & -2\\ 3 & -5\\ -6 & 0\end{bmatrix}+4\begin{bmatrix}-1 & -2\\ 4 & 2\\ 1 & 5\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4\\ -1 & -4\\ -3 & 6\end{bmatrix}-\begin{bmatrix}3 & -6\\ 9 & -15\\ -18 & 0\end{bmatrix}+\begin{bmatrix}-4 & -8\\ 16 & 8\\ 4 & 20\end{bmatrix}$$

$$=\begin{bmatrix}2-3+(-4) & 4-(-6)-8\\ -1-9+16 & -4-(-15)+8\\ -3-(-18)+4 & 6-0+20\end{bmatrix}$$

$$=\begin{bmatrix}-5 & 2\\ 6 & 19\\ 19 & 26\end{bmatrix}$$

$$\therefore\ X=\frac{1}{5}\begin{bmatrix}-5 & 2\\ 6 & 19\\ 19 & 26\end{bmatrix}=\begin{bmatrix}-1 & \frac{2}{5}\\ \frac{6}{5} & \frac{19}{5}\\ \frac{19}{5} & \frac{26}{5}\end{bmatrix}$$

Ex. 2. Find x, y, z if $\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Solution : Let $A = \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$

Since A is skew-symmetric matrix, $a_{ij} = -a_{ji}$ for all i and j.

$\therefore a_{13} = -a_{31}$, $a_{12} = -a_{21}$ and $a_{23} = -a_{32}$

$\therefore x = -\frac{3}{2}$, $-5i = -y$ and $z = -(-\sqrt{2})$

$\therefore x = -\frac{3}{2}, y = 5i$ and $z = \sqrt{2}$.

Ex. 3. Solve the following equations for X and Y, if

$$3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

Solution :

$$3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \dots (1)$$

$$X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \quad \dots (2)$$

Multiplying (1) by 3, we get

$$9X - 3Y = 3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \quad \dots (3)$$

Subtracting (2) from (3), we get

$$8X = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0 & -3-(-1) \\ -3-0 & 3-(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\therefore X = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Substituting the value of X in (1), we get

$$3\begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix} - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore Y=\begin{bmatrix}\frac{9}{8} & -\frac{3}{4}\\ -\frac{9}{8} & \frac{3}{2}\end{bmatrix}-\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{9}{8}-1 & -\frac{3}{4}-(-1)\\ -\frac{9}{8}-(-1) & \frac{3}{2}-1\end{bmatrix}=\begin{bmatrix}\frac{1}{8} & \frac{1}{4}\\ -\frac{1}{8} & \frac{1}{2}\end{bmatrix}$$

Hence, $X=\begin{bmatrix}\frac{3}{8} & -\frac{1}{4}\\ -\frac{3}{8} & \frac{1}{2}\end{bmatrix}$ and $Y=\begin{bmatrix}\frac{1}{8} & \frac{1}{4}\\ -\frac{1}{8} & \frac{1}{2}\end{bmatrix}$.

Ex. 4. If $A=\begin{bmatrix}1 & 3 & 3\\ 3 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$, find A^2-5A. What is your conclusion?

Solution :

$$A^2=A\cdot A=\begin{bmatrix}1 & 3 & 3\\ 3 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 3\\ 3 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+9+9 & 3+3+9 & 3+9+3\\ 3+3+9 & 9+1+9 & 9+3+3\\ 3+9+3 & 9+3+3 & 9+9+1\end{bmatrix}=\begin{bmatrix}19 & 15 & 15\\ 15 & 19 & 15\\ 15 & 15 & 19\end{bmatrix}$$

$$\therefore A^2-5A=\begin{bmatrix}19 & 15 & 15\\ 15 & 19 & 15\\ 15 & 15 & 19\end{bmatrix}-5\begin{bmatrix}1 & 3 & 3\\ 3 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$$

$$=\begin{bmatrix}19 & 15 & 15\\ 15 & 19 & 15\\ 15 & 15 & 19\end{bmatrix}-\begin{bmatrix}5 & 15 & 15\\ 15 & 5 & 15\\ 15 & 15 & 5\end{bmatrix}$$

$$=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

Since all the non-diagonal elements are zero and diagonal elements are same, A^2-5A is a scalar matrix.

Ex. 5. If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$, find k so that $A^2-8A-kI=0$, where I is a 2×2 unit matrix and 0 is null matrix of order 2. ***(Sept. '21)***

Solution :

$$A^2=A\cdot A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$$

$$=\begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$\therefore A^2-8A-kI=\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}-8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}-k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}-\begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}-\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$=\begin{bmatrix} 1-8-k & 0-0-0 \\ -8+8-0 & 49-56-k \end{bmatrix}$$

$$=\begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix}$$

But $A^2-8A-kI=0$

$$\therefore \begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices $-k-7=0$ $\quad \therefore k=-7.$

Ex. 6. Find x, y, z if $\left\{5\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$ ***(March '25)***

Solution :

$$\left\{5\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

$$\therefore \left\{\begin{bmatrix} 0 & 5 \\ 5 & 0 \\ 5 & 5 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} -2 & 4 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} -4+4 \\ 4+2 \\ 8+2 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix}=\begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

By equality of matrices

$0=x-1$, $6=y+1$ and $10=2z$

$\therefore x=1, y=5$ and $z=5$.

Ex. 7. If $A = \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}$ and $(A+B)(A-B) = A^2 - B^2$, find a and b.

Solution : $(A+B)(A-B) = A^2 - B^2$

$\therefore A^2 - AB + BA - B^2 = A^2 - B^2$

$\therefore -AB + BA = 0$

$\therefore AB = BA$

$\therefore \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$

$\therefore \begin{bmatrix} -3+2b & -3a+0 \\ 2+4b & 2a+0 \end{bmatrix} = \begin{bmatrix} -3+2a & 2+4a \\ -3b+0 & 2b+0 \end{bmatrix}$

$\therefore \begin{bmatrix} -3+2b & -3a \\ 2+4b & 2a \end{bmatrix} = \begin{bmatrix} -3+2a & 2+4a \\ -3b & 2b \end{bmatrix}$

By equality of matrices

$-3+2b = -3+2a$ ... (1)

$-3a = 2+4a$ ... (2)

$2+4b = -3b$ ... (3)

$2a = 2b$ ... (4)

From (2), $7a = -2$ $\therefore a = -\frac{2}{7}$

From (3), $7b = -2$ $\therefore b = -\frac{2}{7}$

These values of a and b also satisfy equations (1) and (4).

Hence, $a = -\frac{2}{7}$ and $b = -\frac{2}{7}$.

Examples for Practice	**3 marks each**

1. Find k if the matrix $\begin{bmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$ is a singular matrix.

2. If $A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$, show that matrix AB is non-singular.

3. Find a, b, c, if $\begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

4. If $X+Y=\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$ and $X-2Y=\begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$, then find X and Y.

5. If $A+I=\begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}$, find the product $(A+I)(A-I)$.

6. If $A=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, prove that $A^2-5A+7I=0$, where I is 2×2 units matrix and 0 is zero matrix of order 2. *(July '23)*

7. If $A=\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k so that $A^2-kA+2I=0$, where I is a 2×2 identity matrix and 0 is null matrix of order 2.

8. If $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that A^2-4A is a scalar matrix.

9. If $A=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}$ and if $(A+B)^2=A^2+B^2$, find values of a and b. *(July '24)*

10. (i) Find x, y, z, if $\left\{5\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} x+1 \\ y-1 \\ 3z \end{bmatrix}$ *(March'22)*

(ii) Find x and y, if $\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}-\begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$. *(July '22)*

Answers

1. $k=\frac{49}{8}$ 3. $a=-7, b=5, c=3$

4. $X=\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{5}{3} \\ -\frac{2}{3} & -2 \end{bmatrix}$, $Y=\begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$ 5. $\begin{bmatrix} 9 & 6 & 4 \\ 15 & 32 & -2 \\ 35 & -7 & 29 \end{bmatrix}$

7. $k=1$ 9. $a=2, b=-1$ 10. (i) $x=-1, y=7, z=\frac{10}{3}$ (ii) $x=19, y=12$.

2.2 PROPERTIES OF TRANSPOSE OF A MATRIX

Solved Examples | **3 marks each**

Ex. 8. If $A=\begin{bmatrix}7 & 3 & 0\\0 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & -2 & 3\\2 & 1 & -4\end{bmatrix}$, **then find** A^T+4B^T. ***(March '24)***

Solution :

$A=\begin{bmatrix}7 & 3 & 0\\0 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & -2 & 3\\2 & 1 & -4\end{bmatrix}$

$\therefore A^T=\begin{bmatrix}7 & 0\\3 & 4\\0 & -2\end{bmatrix}$, $B^T=\begin{bmatrix}0 & 2\\-2 & 1\\3 & -4\end{bmatrix}$

$$\therefore A^T+4B^T=\begin{bmatrix}7 & 0\\3 & 4\\0 & -2\end{bmatrix}+4\begin{bmatrix}0 & 2\\-2 & 1\\3 & -4\end{bmatrix}=\begin{bmatrix}7 & 0\\3 & 4\\0 & -2\end{bmatrix}+\begin{bmatrix}0 & 8\\-8 & 4\\12 & -16\end{bmatrix}$$

$$=\begin{bmatrix}7+0 & 0+8\\3-8 & 4+4\\0+12 & -2-16\end{bmatrix}=\begin{bmatrix}7 & 8\\-5 & 8\\12 & -18\end{bmatrix}.$$

Ex. 9. If $A=\begin{bmatrix}1 & 0 & 1\\3 & 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -4\\3 & 5 & -2\end{bmatrix}$ **and** $C=\begin{bmatrix}0 & 2 & 3\\-1 & -1 & 0\end{bmatrix}$, **verify that**

$(A+2B+3C)^T=A^T+2B^T+3C^T$.

Solution :

$$A+2B+3C=\begin{bmatrix}1 & 0 & 1\\3 & 1 & 2\end{bmatrix}+2\begin{bmatrix}2 & 1 & -4\\3 & 5 & -2\end{bmatrix}+3\begin{bmatrix}0 & 2 & 3\\-1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 1\\3 & 1 & 2\end{bmatrix}+\begin{bmatrix}4 & 2 & -8\\6 & 10 & -4\end{bmatrix}+\begin{bmatrix}0 & 6 & 9\\-3 & -3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1+4+0 & 0+2+6 & 1-8+9\\3+6-3 & 1+10-3 & 2-4+0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 8 & 2\\6 & 8 & -2\end{bmatrix}$$

$$\therefore (A+2B+3C)^T=\begin{bmatrix}5 & 6\\8 & 8\\2 & -2\end{bmatrix} \quad \text{... (1)}$$

$A^T=\begin{bmatrix}1 & 3\\0 & 1\\1 & 2\end{bmatrix}$, $B^T=\begin{bmatrix}2 & 3\\1 & 5\\-4 & -2\end{bmatrix}$, $C^T=\begin{bmatrix}0 & -1\\2 & -1\\3 & 0\end{bmatrix}$

$$\therefore\ A^T + 2B^T + 3C^T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \\ 1 & 5 \\ -4 & -2 \end{bmatrix} + 3\begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 2 & 10 \\ -8 & -4 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -3 \\ 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+6-3 \\ 0+2+6 & 1+10-3 \\ 1-8+9 & 2-4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 8 & 8 \\ 2 & -2 \end{bmatrix} \qquad \dots (2)$$

From (1) and (2), $(A+2B+3C)^T = A^T + 2B^T + 3C^T$.

Ex. 10. Express the matrix $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

Solution :

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$$\therefore\ A + A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & 3-2 & -1-4 \\ -2+3 & -2-2 & 1-5 \\ -4-1 & -5+1 & 2+2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

This is a symmetric matrix.

$$\text{Also, } A - A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & 3-(-2) & -1-(-4) \\ -2-3 & -2-(-2) & 1-(-5) \\ -4-(-1) & -5-1 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

This is a skew-symmetric matrix.

Now, $A=\frac{1}{2}\left(A+A^{T}\right)+\frac{1}{2}\left(A-A^{T}\right)$

$$\therefore A=\frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}.$$

Ex. 11. If $A=\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$, verify that $(AB)^{T}=B^{T}A^{T}$.

Solution :

$$A=\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}, B=\begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{T}=\begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}, B^{T}=\begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$\therefore AB=\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0-2 & 6+1 & -8-1 \\ 0-4 & 9+2 & -12-2 \\ 0+2 & 12-1 & -16+1 \end{bmatrix}=\begin{bmatrix} -2 & 7 & -9 \\ -4 & 11 & -14 \\ 2 & 11 & -15 \end{bmatrix}$$

$$\therefore (AB)^{T}=\begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \qquad \text{... (1)}$$

$$B^{T}A^{T}=\begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}=\begin{bmatrix} 0-2 & 0-4 & 0+2 \\ 6+1 & 9+2 & 12-1 \\ -8-1 & -12-2 & -16+1 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \qquad \text{... (2)}$$

From (1) and (2), $(AB)^{T}=B^{T}A^{T}$.

Examples for Practice	**3 marks each**

1. If $A=[a_{ij}]_{3\times 3}$, where $a_{ij}=2(i-j)$. Find A and A^{T}. State whether A and A^{T} both are symmetric or skew-symmetric matrices.

2. If $A=\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$, then find C^{T} such that $3A-2B+C=I$, where I is the unit matrix of order 2.

3. If $A=\begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$, verify

(i) $(A+2B^T)^T=A^T+2B$

(ii) $(3A-5B^T)^T=3A^T-5B$.

4. If $A=\begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$, prove that $(A+B^T)^T=A^T+B$.

5. Express the matrix $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

6. If $A=\begin{bmatrix} 2 & -4 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$, then show that $(BA)^T=A^T\cdot B^T$.

Answers

1. A and A^T are both skew-symmetric matrices.

2. $\begin{bmatrix} -16 & 14 \\ -6 & -10 \end{bmatrix}$ 5. $\begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & -5 \end{bmatrix}+\begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$.

2.3 INVERSE OF A MATRIX

Solved Examples	3 or 4 marks each

Ex. 12. Find the inverse of the matrix $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by elementary transformations.

Solution : Let $A=\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Then $|A|=\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}=6-5=1\neq 0$ $\quad\therefore A^{-1}$ exists.

We write $AA^{-1}=I$

$\therefore \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}A^{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $R_1\leftrightarrow R_2$, we get, $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}A^{-1}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

By $R_2 - 2R_1$, we get, $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

By $(-1)\, R_2$, we get, $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

By $R_1 - 3R_2$, we get, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Note : A^{-1} can also be obtained by using column transformations taking $A^{-1}A = I$.

Ex. 13. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$. *(Sept. '21)*

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{vmatrix} = 1(7-20) - 2(7-10) + 3(4-2)$
$= -13 + 6 + 6 = -1 \neq 0$

$\therefore A^{-1}$ exists.

We write $AA^{-1} = I$

$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - R_1$ and $R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $(-1)R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

By $R_1 - 7R_3$ and $R_2 + 2R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}.$$

Ex. 14. Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformations.

Solution : Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1(2-6) - 0 + 1(0-2) = -4 - 2 = -6 \neq 0$$

$\therefore A^{-1}$ exists.

We write $A^{-1}A = I$

$$\therefore A^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_3 - C_1$, we get

$$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\left(\frac{1}{2}\right)C_2$, we get

$$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_3 - 3C_2$, we get

$$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{3}\right)C_3$, we get

$$A^{-1}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & \frac{1}{3}\\0 & \frac{1}{2} & \frac{1}{2}\\0 & 0 & -\frac{1}{3}\end{bmatrix}$$

By $C_1 - C_3$ and $C_2 - C_3$, we get

$$A^{-1}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{1}{3}\\-\frac{1}{2} & 0 & \frac{1}{2}\\\frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{6}\begin{bmatrix}4 & -2 & 2\\-3 & 0 & 3\\2 & 2 & -2\end{bmatrix}$$

Examples for Practice	3 or 4 marks each

1. Find the inverses of the following matrices (if they exist) :

(1) $\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$ (2) $\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ (3) $\begin{bmatrix}1 & 3\\2 & 5\end{bmatrix}$ (4) $\begin{bmatrix}2 & 1\\7 & 4\end{bmatrix}$.

2. Find the inverses of the following matrices (if they exist) :

(1) $\begin{bmatrix}2 & 0 & -1\\5 & 1 & 0\\0 & 1 & 3\end{bmatrix}$ (2) $\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

(3) $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix}$ (4) $\begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix}$.

3. Find the inverse of $A=\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$ by elementary column transformations.

4. If $A=\begin{bmatrix}1 & 0 & 1\\0 & 2 & 3\\1 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$, then find matrix X such that $XA = B$.

Answers

1. (1) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ (2) $\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

2. (1) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ 4. $\frac{1}{6}\begin{bmatrix} 4 & 4 & 2 \\ 11 & 8 & -5 \\ 10 & 10 & 2 \end{bmatrix}$.

2.4 INVERSE BY ADJOINT METHOD

Adjoint of a Matrix : The adjoint of a matrix $A=[a_{ij}]$ is the transpose of the cofactor matrix. It is denoted by 'adj A'.

i.e. if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Inverse by adjoint method : If $A=[a_{ij}]_{m\times m}$ is a non-singular square matrix, i.e. $|A| \neq 0$, then its inverse exists and it is given as $A^{-1}=\frac{1}{|A|}$ (adj A).

Solved Examples	**3 or 4 marks each**

Ex. 15. Find the inverse of the matrix $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$ by the adjoint method. ***(July '22)***

Solution :

Let $A=\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

Then $|A|=\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix}=-3-(-2)=-1 \neq 0$

$\therefore A^{-1}$ exists.

First we have to find the cofactor matrix

$=[A_{ij}]_{2\times 2}$, where $A_{ij}=(-1)^{i+j} M_{ij}$

Now, $A_{11}=(-1)^{1+1}M_{11}=-1$

$A_{12}=(-1)^{1+2}M_{12}=-2$

$A_{21}=(-1)^{2+1}M_{21}=-(-1)=1$

$A_{22}=(-1)^{2+2}M_{22}=3$

$\therefore$ the cofactor matrix $=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$

$\therefore$ adj $A=\begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}(\text{adj } A)=\frac{1}{-1}\begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1}=\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$.

Ex. 16. Find the inverse of $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$ by adjoint method. ***(March '24)***

Solution : Let $A=\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$

Then $|A|=\begin{vmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{vmatrix}=3(35-16)-1(10-8)+5(4-7)$

$=57-2-15=40\neq 0$

$\therefore A^{-1}$ exists.

First, we have to find the cofactor matrix

$=[A_{ij}]_{3\times 3}$, where $A_{ij}=(-1)^{i+j}M_{ij}$

Now, $A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix} 7 & 8 \\ 2 & 5 \end{vmatrix}=35-16=19$

$A_{12}=(-1)^{1+2}M_{12}=-\begin{vmatrix} 2 & 8 \\ 1 & 5 \end{vmatrix}=-(10-8)=-2$

$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix} 2 & 7 \\ 1 & 2 \end{vmatrix}=4-7=-3$

$A_{21}=(-1)^{2+1}M_{21}=-\begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix}=-(5-10)=5$

$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix} 3 & 5 \\ 1 & 5 \end{vmatrix}=15-5=10$

$A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}=-(6-1)=-5$

$$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 1 & 5 \\ 7 & 8 \end{vmatrix} = 8 - 35 = -27$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = -(24 - 10) = -14$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix} = 21 - 2 = 19$$

$\therefore$ the cofactor matrix

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 19 & -2 & -3 \\ 5 & 10 & -5 \\ -27 & -14 & 19 \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} 19 & 5 & -27 \\ -2 & 10 & -14 \\ -3 & -5 & 19 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$\therefore A^{-1} = \frac{1}{40}\begin{bmatrix} 19 & 5 & -27 \\ -2 & 10 & -14 \\ -3 & -5 & 19 \end{bmatrix}.$$

Examples for Practice	**3 or 4 marks each**

1. Find the adjoints of the following matrices :

(1) $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$.

2. Find the inverses of the following matrices by adjoint method :

(1) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$ (3) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

3. Find the inverses of the following matrices by adjoint method :

(1) $\begin{bmatrix} -3 & -1 & 1 \\ 0 & 0 & 1 \\ -15 & 6 & -6 \end{bmatrix}$ ***(March '22)*** (2) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 7 & -6 & -2 \\ -18 & 16 & 5 \\ -10 & 9 & 3 \end{bmatrix}$ (4) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$.

Answers

1. (1) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$ (2) $\begin{bmatrix} -3 & -1 & -11 \\ -12 & 3 & -9 \\ 6 & 2 & 1 \end{bmatrix}$.

2. (1) $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$ (2) $\frac{1}{18}\begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}$ (3) $\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

3. (1) $\frac{1}{33}\begin{bmatrix} -6 & 0 & -1 \\ -15 & 33 & 3 \\ 0 & 33 & 0 \end{bmatrix}$ (2) $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 3 & 0 & 2 \\ 4 & 1 & 1 \\ -2 & -3 & 4 \end{bmatrix}$ (4) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

2.5 SIMULTANEOUS EQUATIONS

1. Reduction Method

First, we write the given linear equations in the matrix form : $AX = B$.

In reduction method, we obtain $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$ in the first column of the coefficient matrix A by using elementary row transformations. Then we obtain another '0' in R_2 or R_3 by using R_2 and R_3 only.

The same row transformations are performed simultaneously on the matrix B.

Now, we can multiply the matrices in LHS and write the equivalent equations which can be easily solved.

Solved Examples	3 or 4 marks each

Ex. 17. Express the following equations in matrix form and solve them by the method of reduction :

$x + 3y = 2,\ 3x + 5y = 4.$

Solution : The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

By $R_2 - 3R_1$, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+3y \\ 0-4y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

By equality of matrices

$$x + 3y = 2 \quad \text{... (1)}$$

$$-4y = -2 \quad \text{... (2)}$$

From (2), $y = \frac{1}{2}$

Substituting $y = \frac{1}{2}$ in (1), we get

$$x + \frac{3}{2} = 2 \qquad \therefore\ x = 2 - \frac{3}{2} = \frac{1}{2}$$

Hence, $x = \frac{1}{2}, y = \frac{1}{2}$ is the required solution.

Ex. 18. Express the following equations in matrix form and solve them by method of reduction :

$x + 2y + z = 8$, $2x + 3y - z = 11$, $3x - y - 2z = 5$. ***(March '23)***

Solution : The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 5 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -19 \end{bmatrix}$$

By $R_3 - 7R_2$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+2y+z \\ 0-y-3z \\ 0+0+16z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

By equality of matrices

$$x + 2y + z = 8 \quad \text{... (1)}$$

$-y-3z=-5$ ... (2)

$16z=16$ ... (3)

From (3), $z=1$

Substituting $z=1$ in (2), we get, $-y-3=-5$ $\therefore y=2$

Substituting $y=2$, $z=1$ in (1), we get

$x+4+1=8$ $\therefore x=3$

Hence, $x=3$, $y=2$, $z=1$ is the required solution.

Ex. 19. The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number, we get 11. By adding first and third numbers, we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.

Solution : Let the three numbers be x, y and z.

According to the given condition,

$x+y+z=6$

$3z+y=11$, i.e. $y+3z=11$

and $x+z=2y$, i.e. $x-2y+z=0$

Hence, the system of linear equations is

$x+y+z=6$, $y+3z=11$, $x-2y+z=0$

These equations can be written in matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

By R_3-R_1, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+y+z \\ 0+y+3z \\ 0-3y+0 \end{bmatrix}=\begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

By equality of matrices

$x+y+z=6$ ... (1)

$y+3z=11$ ... (2)

$-3y=-6$ ... (3)

From (3), $y=2$

Substituting $y=2$ in (2), we get

$2+3z=11 \qquad 3z=9 \qquad \therefore\ z=3$

Substituting $y=2$, $z=3$ in (1), we get

$x+2+3=6 \qquad \therefore\ x=1$

$\therefore\ x=1, y=2, z=3$

Hence, the required numbers are 1, 2 and 3.

Examples for Practice	**3 or 4 marks each**

1. Solve the following equations by the method of reduction :

(1) $2x-y=-2$, $3x+4y=3$

(2) $2x+y=5$, $3x+5y=-3$

(3) $x-y+z=1$, $2x-y=1$, $3x+3y-4z=2$

(4) $x+y+z=1$, $2x+3y+2z=2$, $x+y+2z=4$ *(July '23)*

2. The sum of the cost of one Economics book, one Cooperation book and one Account book is ₹ 420. The total cost of an Economic book, 2 Cooperation books and an Account book is ₹ 480. Also, the total cost of an Economic book, 3 Cooperation books and 2 Account books is ₹ 600. Find the cost of each book.

3. If three numbers are added, their sum is 2. If 2 times the second number is subtracted from the sum of first and third numbers, we get 8 and if three times the first number is added to the sum of second and third numbers, we get 4. Find the numbers using matrices.

Answers

1. (1) $-\frac{5}{11}, \frac{12}{11}$ (2) 4, -3 (3) 1, 1, 1 (4) -2, 0, 3

2. Cost of each Economics book is ₹ 300, each cooperation book is ₹ 60 and each Account book is ₹ 60.

3. 1, -2, 3.

2. Inversion Method

First, we write the given linear equations in the matrix form as $AX=B$.

If the solution of the given equations exists, then the matrix A is non-singular, i.e. $|A| \neq 0$.

$\therefore A^{-1}$ exists.

Now, premultiply $AX = B$ by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore (A^{-1}A)X = A^{-1}B$

$\therefore IX = A^{-1}B \qquad \therefore X = A^{-1}B$

By comparing both the sides, we can find x, y, z

Solved Examples	4 marks each

Ex. 20. Solve the following equations by the method of inversion :

$2x + 3y = 5, 6x - 2y = 4.$

Solution : The given equations can be written in the matrix form as :

$$\begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

This is of the form $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Let us find A^{-1}.

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix} = -4 - 18 = -22 \neq 0$$

$\therefore A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By $\left(\frac{1}{2}\right)R_1$, we get, $\begin{bmatrix} 1 & \frac{3}{2} \\ 6 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

By $R_2 - 6R_1$, we get, $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -11 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -3 & 1 \end{bmatrix}$

By $\left(-\frac{1}{11}\right)R_2$, we get, $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

By $R_1 - \frac{3}{2}R_2$, we get, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{22} & \frac{3}{22} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{22}\begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}$

Now, premultiply $AX = B$ by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore (A^{-1}A)X = A^{-1}B$

$\therefore IX = A^{-1}B$

$\therefore X = \frac{1}{22}\begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 10+12 \\ 30-8 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 22 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By equality of matrices, $x = 1, y = 1$ in the required solution.

Ex. 21. Solve the following equations by the method of inversion :
$2x - y + z = 1, x + 2y + 3z = 8, 3x + y - 4z = 1.$ ***(March '25)***

Solution : The given equations can be written in matrix form as :

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

This is of the form $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

Let us find A^{-1}.

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix} = 2(-8-3) + 1(-4-9) + 1(1-6)$$
$$= -22 - 13 - 5 = -40 \neq 0$$

$\therefore A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{5}\right)R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$ and $R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{8}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

By $R_1 - R_3$ and $R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{11}{40} & \frac{3}{40} & \frac{1}{8} \\ -\frac{13}{40} & \frac{11}{40} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{40}\begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

Now, premultiply $AX = B$ by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore (A^{-1}A)X = A^{-1}B$

$\therefore IX = A^{-1}B$

$$\therefore X = \frac{1}{40}\begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 11+24+5 \\ -13+88+5 \\ 5+40-5 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ 40 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

By equality of matrices $x = 1, y = 2, z = 1$ is the required solution.

Examples for Practice	4 marks each

Solve the following equations by the method of inversion :

1. $2x + y = 5,\ 3x + 5y = -3$
2. $x + 2y = 2,\ 2x + 3y = 3$
3. $x + y + z = 1,\ x - y + z = 2,\ x + y - z = 3$ *(July '24)*
4. $x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2$
5. $x + y - z = 2,\ x - 2y + z = 3,\ 2x - y - 3z = -1$

Answers

1. $4, -3$ **2.** $0, 1$ **3.** $\frac{5}{2}, -\frac{1}{2}, -1$ **4.** $2, -1, 1$ **5.** $3, 1, 2$.

ACTIVITIES	4 marks each

1. If $A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$, **complete the following activity to find** $(AB)^{-1}$ **by adjoint method.**

$$AB = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$\therefore AB = \boxed{\begin{bmatrix} \quad \end{bmatrix}}$

$\therefore |AB| = \boxed{\quad}$

$M_{11} = -2,\ A_{11} = (-1)^{1+1}(-2) = -2$

$M_{12} = -3,\ A_{12} = (-1)^{1+2}(-3) = 3$

$M_{21} = 4,\ A_{21} = (-1)^{2+1}(4) = -4$

$M_{22} = 3,\ A_{22} = (-1)^{2+2}(3) = 3$

Cofactor matrix $[A_{ij}] = \begin{bmatrix} -2 & 3 \\ -4 & 3 \end{bmatrix}$

$\therefore$ adj (AB) = $\boxed{\begin{bmatrix} \quad & \quad \end{bmatrix}}$

$(AB)^{-1} = \frac{1}{|AB|}[\text{adj}(AB)]$

$\therefore (AB)^{-1} = \boxed{\quad}$

(March '23)

Solution : $AB = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 4-3+2 & 8+0-4 \\ -1-2+0 & -2+0-0 \end{bmatrix}$

$\therefore AB = \boxed{\begin{bmatrix} 3 & 4 \\ -3 & -2 \end{bmatrix}}$

$\therefore |AB| = \begin{vmatrix} 3 & 4 \\ -3 & -2 \end{vmatrix} = -6 + 12 = \boxed{6}$

$M_{11} = -2,\ A_{11} = (-1)^{1+1}(-2) = -2$

$M_{12} = -3,\ A_{12} = (-1)^{1+2}(-3) = 3$

$M_{21} = 4,\ A_{21} = (-1)^{2+1}(4) = -4$

$M_{22} = 3,\ A_{22} = (-1)^{2+2}(3) = 3$

Cofactor matrix $[A_{ij}] = \begin{bmatrix} -2 & 3 \\ -4 & 3 \end{bmatrix}$

$\therefore$ adj (AB) = $\boxed{\begin{bmatrix} -2 & -4 \\ 3 & 3 \end{bmatrix}}$

$(AB)^{-1} = \frac{1}{|AB|}[\text{adj}(AB)]$

$\therefore (AB)^{-1} = \boxed{\frac{1}{6}\begin{bmatrix} -2 & -4 \\ 3 & 3 \end{bmatrix}}.$

2. **Solve the following equations by inversion method by completing the following activity :**

$x+2y=1,\ 2x-3y=4$.

$$A=\begin{bmatrix}1 & 2\\ 2 & -3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ B=\begin{bmatrix}1\\ 4\end{bmatrix}$$

Given equations can be written as $AX=B$.

Premultiplying by A^{-1}, we get

$A^{-1}(AX)=A^{-1}B$

$\therefore \boxed{}\, X=A^{-1}B$

$\therefore X=A^{-1}B$

First, we find the inverse of A by row transformations.

We write $AA^{-1}=I$

$$\therefore \begin{bmatrix}1 & 2\\ 2 & -3\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$\therefore A^{-1}=\boxed{}$

$$X=A^{-1}B=\frac{1}{7}\begin{bmatrix}3 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}1\\ 4\end{bmatrix}$$

$\therefore X=\frac{1}{7}\boxed{}$

$\therefore \begin{bmatrix}x\\ y\end{bmatrix}=\boxed{}$

$\therefore x=\frac{11}{7},\ y=-\frac{2}{7}$

Hence, the solution of given linear equations is $x=\frac{11}{7},\ y=-\frac{2}{7}$.

Solution : $A=\begin{bmatrix}1 & 2\\ 2 & -3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix},\ B=\begin{bmatrix}1\\ 4\end{bmatrix}$

Given equations can be written as $AX=B$.

Premultiplying by A^{-1}, we get

$A^{-1}(AX)=A^{-1}B$

$\therefore \boxed{(A^{-1}A)}\, X=A^{-1}B$

$\therefore IX=A^{-1}B \qquad \therefore X=A^{-1}B$

First, we find the inverse of A by row transformations.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$, we get, $\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

By $\left(-\frac{1}{7}\right) R_2$, we get, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$

By $R_1 - 2R_2$, we get, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$

$$\therefore A^{-1} = \boxed{\frac{1}{7}\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}}$$

$$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 3+8 \\ 2-4 \end{bmatrix} = \frac{1}{7}\boxed{\begin{bmatrix} 11 \\ -2 \end{bmatrix}}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \boxed{\begin{bmatrix} \frac{11}{7} \\ -\frac{2}{7} \end{bmatrix}} \qquad \therefore x = \frac{11}{7}, y = -\frac{2}{7}$$

Hence, the solution of given linear equations is $x = \frac{11}{7}, y = -\frac{2}{7}$.

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3. Express the following equations in matrix form and solve them by the method of reduction :

$x + 3y + 3z = 12, x + 4y + 4z = 15, x + 3y + 4z = 13.$

The matrix form of given equations is

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{}$$

By $R_2 - R_1$ and $R_3 - R_1$, we get

$$\boxed{}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

By equality of matrices, we get

$x+3y+3z=12$ ... (1)

$\boxed{}=3$ ... (2)

$z=1$

By solving these equations, we get

$x=\boxed{}, y=2, z=1.$

Solution : The matrix form of given equations is

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\boxed{\begin{bmatrix} 12 \\ 15 \\ 13 \end{bmatrix}}$$

By R_2-R_1 and R_3-R_1, we get

$$\boxed{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+3y+3z \\ 0+y+z \\ 0+0+z \end{bmatrix}=\begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

By equality of matrices, we get

$x+3y+3z=12$ ... (1)

$\boxed{y+z}=3$ ... (2)

$z=1$

Substituting $z=1$ in (2), we get

$y+1=3 \quad \therefore y=2$

Substituting $y=2, z=1$ in (1), we get

$x+6+3=12 \quad \therefore x=3$

$\therefore x=\boxed{3}, y=2, z=1.$

MULTIPLE CHOICE QUESTIONS | 1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. If $A=\begin{bmatrix} 2 & 3 \\ a & 6 \end{bmatrix}$ is a singular matrix, then $a=$......... . ***(March '22)***

(a) 6 (b) -5 (c) 3 (d) 4

2. If A is a non-singular matrix, then $\det(A^{-1}) = \text{.........}$

(a) 1 (b) 0 (c) det (A) (d) $\dfrac{1}{\det(A)}$ ***(July '23)***

3. The matrix $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is

(a) identity matrix (b) scalar matrix
(c) null matrix (d) column matrix ***(July '22)***

4. If $A^2 + mA + nI = 0$ and $n \neq 0$, $|A| \neq 0$, then $A^{-1} = \text{.........}$

(a) $-\dfrac{1}{m}(A + nI)$ (b) $-\dfrac{1}{n}(A + mI)$
(c) $-\dfrac{1}{n}(I + mA)$ (d) $A + mnI$

5. If A is a 2×2 matrix such that $A\,(\text{adj } A) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, then $|A| = \text{.........}$

(a) 0 (b) 5 (c) 10 (d) 25

6. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $A^{-1} = \text{.........}$

(a) $\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}$

7. If the matrix $\begin{bmatrix} 6 & -5 & 1 \\ 4 & 2 & -1 \\ 14 & -1 & k \end{bmatrix}$ has no inverse, then

(a) $k = 1$ (b) $k = -1$ (c) $k = 0$ (d) $k = 2$

8. Adjoint of $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is

(a) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

Answers

1. (d) 4 **2.** (d) $\dfrac{1}{\det(A)}$ **3.** (b) scalar matrix **4.** (b) $-\dfrac{1}{n}(A + mI)$

5. (b) 5 **6.** (b) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ **7.** (b) $k = -1$ **8.** (a) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The product of two non-zero matrices cannot be a zero matrix. *(Sept. '21)*
2. If A and B are conformable for the product AB, then $(AB)^T = A^T B^T$. *(July '22)*
3. If AB and BA both exist, then AB = BA.
4. Every identity matrix is a scalar matrix. *(July '24)*
5. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$, then AB is a singular matrix.
6. If A is a matrix and k is a constant, then $(kA)^T = kA^T$. *(March '25)*

Answers

1. False 2. False 3. False 4. True 5. False 6. True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. Matrix $\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ p & 4 & 0 \end{bmatrix}$ is a skew-symmetric matrix, then value of p is ....
2. If $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$, then cofactor of a_{12} is ....
3. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ x & 2 \end{bmatrix}$, then $x =$ ....
4. If A is a square matrix such that $A(\text{adj } A) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$, then $|A| =$ ....
5. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I$, then $k =$ ....
6. If $A = \begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix, then $a + b + c =$ ....

Answers

1. -1 2. -2 3. -1 4. 20 5. 1 6. 1.

Question Set 3

DIFFERENTIATION

(Marks with option : 07)

Remember :

1. If $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ exists, then this limit is called the derivative of f and it is denoted by $f'(x)$.

Finding the derivative of a given function by using the above definition is called finding its derivatives from *first principle.*

2. $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$, if it exists, is called derivative of f at $x=a$ and is denoted by $f'(a)$.

3. Derivatives of Standard Functions :

Functions	**Derivatives** $\frac{dy}{dx}$ or $f'(x)$
x^n	nx^{n-1}
k	0
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$

Functions	**Derivatives** $\frac{dy}{dx}$ or $f'(x)$
a^x	$a^x \cdot \log a$
e^x	e^x
$\log x$	$\frac{1}{x}$

In derivatives, if there is a function of x in place of x, use the same formula and multiply the result by the derivative of the function.

e.g. (1) $\dfrac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1} \cdot f'(x)$

(2) $\dfrac{d}{dx}[a^{f(x)}] = a^{f(x)} \cdot \log a \cdot f'(x)$

(3) $\dfrac{d}{dx}\{\log [f(x)]\} = \dfrac{1}{f(x)} \times f'(x)$.

3.1 DERIVATIVE OF A COMPOSITE FUNCTION

Theorem 1 : If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x, then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$.

Note : If y is a differentiable function of u_1, u_i is a differentiable function of u_{i+1}, $i=1, 2, 3, ..., n-1$ and u_n is a differentiable function of x, then

$$\frac{dy}{dx}=\frac{dy}{du_1}\times\frac{du_1}{du_2}\times\frac{du_2}{du_3}\times...\times\frac{du_n}{dx}.$$

Solved Examples	3 marks each

Ex. 1. If $y=\sqrt[3]{(3x^2+8x-7)^5}$, find $\frac{dy}{dx}$.

Solution : Given : $y=\sqrt[3]{(3x^2+8x-7)^5}=(3x^2+8x-7)^{\frac{5}{3}}$

Let $u=3x^2+8x-7$

Then $y=u^{\frac{5}{3}}$

$\therefore \frac{dy}{du}=\frac{d}{du}(u^{\frac{5}{3}})=\frac{5}{3}u^{\frac{2}{3}}=\frac{5}{3}(3x^2+8x-7)^{\frac{2}{3}}$

and $\frac{du}{dx}=\frac{d}{dx}(3x^2+8x-7)=3\times 2x+8\times 1-0=6x+8$

$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{5}{3}(3x^2+8x-7)^{\frac{2}{3}}(6x+8)$.

Ex. 2. Find $\frac{dy}{dx}$, if $y=\log(ax^2+bx+c)$.

Solution : Given : $y=\log(ax^2+bx+c)$

Let $u=ax^2+bx+c$

Then $y=\log u$

$\therefore \frac{dy}{du}=\frac{d}{du}(\log u)=\frac{1}{u}=\frac{1}{ax^2+bx+c}$

and $\frac{du}{dx}=\frac{d}{dx}(ax^2+bx+c)=a\frac{d}{dx}(x^2)+b\frac{d}{dx}(x)+\frac{d}{dx}(c)$

$=a\times 2x+b\times 1\times 0=2ax+b$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{ax^2+bx+c}\times(2ax+b)=\frac{2ax+b}{ax^2+bx+c}.$$

Ex. 3. Find $\frac{dy}{dx}$, if

(1) $y=e^{(\log x+6)}$ (2) $y=5^{(x+\log x)}$.

Solution :

(1) Given : $y=e^{(\log x+6)}$

Let $u=\log x+6$

Then $y=e^u$

$$\therefore \frac{dy}{du}=\frac{d}{du}(e^u)=e^u=e^{(\log x+6)}$$

$$\text{and } \frac{du}{dx}=\frac{d}{dx}(\log x+6)=\frac{1}{x}+0=\frac{1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=e^{(\log x+6)}\times\frac{1}{x}=\frac{e^{(\log x+6)}}{x}.$$

(2) Given : $y=5^{(x+\log x)}$

Let $u=x+\log x$

Then $y=5^u$

$$\therefore \frac{dy}{du}=\frac{d}{du}(5^u)=5^u\cdot\log 5=5^{(x+\log x)}\cdot\log 5$$

$$\text{and } \frac{du}{dx}=\frac{d}{dx}(x+\log x)=1+\frac{1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=5^{(x+\log x)}\cdot\log 5\cdot\left(1+\frac{1}{x}\right).$$

Examples for Practice	3 marks each

Find $\frac{dy}{dx}$, if

1. $y=\sqrt{x+\frac{1}{x}}$ **2.** $y=(5x^3-4x^2-8x)^9$

3. $y=\sqrt[3]{a^2+x^2}$

4. $y=\log(\log x)$

5. $y=\log(4x^2+3x-1)$

6. $y=\log\sqrt{x^2+4}$

7. $y=a^{(1+\log x)}$

8. $y=7^{x+\frac{1}{x}}$

9. $y=e^{5x^2-2x+4}$

10. $y=3^{(\sqrt{x}+2)}$.

Answers

1. $\frac{1}{2}\left(x+\frac{1}{x}\right)^{-\frac{1}{2}}\left(1-\frac{1}{x^2}\right)$

2. $9(5x^3-4x^2-8x)^8(15x^2-8x-8)$

3. $\frac{2x}{3}(a^2+x^2)^{-\frac{2}{3}}$

4. $\frac{1}{x\log x}$

5. $\frac{8x+3}{4x^2+3x-1}$

6. $\frac{x}{x^2+4}$

7. $\frac{a^{(1+\log x)}\cdot\log a}{x}$

8. $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

9. $(10x-2)\,e^{5x^2-2x+4}$

10. $\frac{3^{(\sqrt{x}+2)}\cdot\log 3}{2\sqrt{x}}$

3.2 DERIVATIVE OF INVERSE FUNCTION

Theorem 2 : If $y=f(x)$ is a differentiable function of x such that the inverse function $x=f^{-1}(y)$ exists, then x is a differentiable function of y and $\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}$, $\frac{dy}{dx}\neq 0$.

Solved Examples	3 marks each

Ex. 4. Find the rate of change of demand (x) of a commodity with respect to price (y), if $y=5+x^2e^{-x}+2x$.

Solution : Given : $y=5+x^2e^{-x}+2x$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(5+x^2e^{-x}+2x)$$

$$=\frac{d}{dx}(5)+x^2\frac{d}{dx}(e^{-x})+e^{-x}\frac{d}{dx}(x^2)+2\frac{d}{dx}(x)$$

$$= 0 + x^2 \cdot e^{-x}(-1) + e^{-x} \times 2x + 2 \times 1$$

$$= -x^2 e^{-x} + 2xe^{-x} + 2 = 2 + x(2-x)\, e^{-x}$$

By derivative of inverse function

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{2 + x(2-x)e^{-x}}$$

Hence, the rate of change of demand (x) with respect to price (y)

$$= \frac{dx}{dy} = \frac{1}{2 + x(2-x)\, e^{-x}}.$$

Ex. 5. Find the marginal demand of a commodity where demand is x and price is y and $y = \frac{5x+9}{2x-10}$.

Solution : Given : $y = \frac{5x+9}{2x-10}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{5x+9}{2x-10}\right)$$

$$= \frac{(2x-10) \cdot \frac{d}{dx}(5x+9) - (5x+9) \cdot \frac{d}{dx}(2x-10)}{(2x-10)^2}$$

$$= \frac{(2x-10)(5 \times 1 + 0) - (5x+9)(2 \times 1 - 0)}{(2x-10)^2}$$

$$= \frac{10x - 50 - 10x - 18}{(2x-10)^2} = -\frac{68}{(2x-10)^2}$$

By the derivative of inverse function

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{(2x-10)^2}{68}$$

Hence, marginal demand $= \frac{dx}{dy} = -\frac{(2x-10)^2}{68}$.

Examples for Practice	**3 marks each**

1. Find the rate of change of demand (x) of a commodity with respect to its price (y), if

(1) $y = 20 + 15x + x^2$ (2) $y = 25x + \log(1 + x^2)$ (3) $y = \frac{3x+7}{2x^2+5}$.

2. **Find the marginal demand of a commodity where demand is x and price is y, if**

(1) $y = xe^{-x} + 7$ (2) $y = \frac{x+2}{x^2+1}$ (3) $y = \sqrt[3]{x-2}$.

Answers

1. (1) $\frac{1}{15+2x}$ (2) $\frac{1+x^2}{25x^2+2x+25}$ (3) $-\frac{(2x^2+5)^2}{(6x^2+28x-15)}$.

2. (1) $\frac{e^x}{1-x}$ (2) $\frac{(x^2+1)^2}{1-4x-x^2}$ (3) $3(x-2)^{\frac{2}{3}}$.

3.3 LOGARITHMIC DIFFERENTIATION

When we want to find the derivative of a function which is expressed as :

(i) a product of a number of functions or

(ii) a quotient of functions or

(iii) of the form $[f(x)]^{g(x)}$.

then it is convenient to find the derivative of the logarithm of the function. Hence, this method of finding the derivative of a function is known as **logarithmic differentiation**.

Solved Examples	3 or 4 marks each

Ex. 6. Find $\frac{dy}{dx}$, if

(1) $y = x^{e^x}$ *(March '24)* **(2) $y = 2^{x^x}$.**

Solution :

(1) $y = x^{e^x}$

$\therefore \log y = \log x^{e^x} = e^x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(e^x \cdot \log x) = e^x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \frac{1}{x} + (\log x)(e^x)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{e^x}{x} + e^x \cdot \log x\right] = x^{e^x} \cdot e^x\left[\frac{1}{x} + \log x\right].$$

(2) Given : $y = 2^{x^x}$

Let $u = x^x$

Then $y = 2^u$

$\therefore \frac{dy}{du} = \frac{d}{du}(2^u) = 2^u \cdot \log 2 = 2^{x^x} \cdot \log 2$ ... (1)

Now, $u = x^x$

$\therefore \log u = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x\frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) = x \times \frac{1}{x} + (\log x) \times 1$$

$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$ ... (2)

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= 2^{x^x} \cdot \log 2 \cdot x^x(1 + \log x)$ ... [By (1) and (2)]

$= 2^{x^x} \cdot x^x(\log 2)(1 + \log x).$

Ex. 7. Find $\frac{dy}{dx}$, if

(1) $y = (3 + x)^x$ **(2)** $y = \sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}$.

Solution :

(1) $y = (3 + x)^x$

$\therefore \log y = \log (3 + x)^x = x \log (3 + x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}[x \log(3 + x)]$$

$$= x\frac{d}{dx}[\log(3 + x)] + \log(3 + x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{3+x} \cdot \frac{d}{dx}(3 + x) + \log(3 + x) \times 1$$

$$= \frac{x}{3+x} \times (0 + 1) + \log(3 + x)$$

$$\therefore \frac{dy}{dx}=y\left[\frac{x}{3+x}+\log(3+x)\right]$$

$$=(3+x)^x\left[\frac{x}{3+x}-\log(3+x)\right].$$

---

(2) $y=\sqrt{\dfrac{(2x+3)^5}{(3x-1)^3(5x-2)}}$

$$\therefore \log y=\log\left[\frac{(2x+3)^5}{(3x-1)^3(5x-2)}\right]^{\frac{1}{2}}=\frac{1}{2}\log\left[\frac{(2x+3)^5}{(3x-1)^3(5x-2)}\right]$$

$$=\frac{1}{2}[\log(2x+3)^5-\log(3x-1)^3-\log(5x-2)]$$

$$=\frac{1}{2}[5\log(2x+3)-3\log(3x-1)-\log(5x-2)]$$

$$=\frac{5}{2}\log(2x+3)-\frac{3}{2}\log(3x-1)-\frac{1}{2}\log(5x-2)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{5}{2}\frac{d}{dx}[\log(2x+3)]-\frac{3}{2}\frac{d}{dx}[\log(3x-1)]-\frac{1}{2}\frac{d}{dx}[\log(5x-2)]$$

$$=\frac{5}{2}\times\frac{1}{2x+3}\cdot\frac{d}{dx}(2x+3)-\frac{3}{2}\times\frac{1}{3x-1}\cdot\frac{d}{dx}(3x-1)-\frac{1}{2}\times\frac{1}{5x-2}\cdot\frac{d}{dx}(5x-2)$$

$$=\frac{5}{2(2x+3)}\times(2\times1+0)-\frac{3}{2(3x-1)}\times(3\times1-0)-\frac{1}{2(5x-2)}\times(5\times1-0)$$

$$\therefore \frac{dy}{dx}=y\left[\frac{5}{2x+3}-\frac{9}{2(3x-1)}-\frac{5}{2(5x-2)}\right]$$

$$=\sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}\left[\frac{5}{2x+3}-\frac{9}{2(3x-1)}-\frac{5}{2(5x-2)}\right].$$

Ex. 8. Differentiate the following w.r.t. x :

(1) $x^x+(\log x)^x$ **(2)** $x^x+a^x+x^a+a^a$.

Solution :

(1) Let $y=x^x+(\log x)^x$

Put $u=x^x$ and $v=(\log x)^x$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \dots (1)$$

Take $u = x^x$

$\therefore \log u = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x) = x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \qquad \dots (2)$$

Also, $v = (\log x)^x$

$\therefore \log v = \log (\log x)^x = x \log (\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[x \log (\log x)] = x \cdot \frac{d}{dx}[\log (\log x)] + [\log (\log x)] \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) + [\log (\log x)] \times 1$$

$$= x \times \frac{1}{\log x} \times \frac{1}{x} + \log (\log x)$$

$$= (\log x)^x \left[\frac{1}{\log x} + \log (\log x)\right] \qquad \dots (3)$$

$$\therefore \frac{dv}{dx} = v\left[\frac{1}{\log x} + \log (\log x)\right] = (\log x)^x \left[\frac{1}{\log x} + \log (\log x)\right]$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + (\log x)^x \left[\frac{1}{\log x} + \log (\log x)\right].$$

---

(2) Let $y = x^x + a^x + x^a + a^a$

Let $u = x^x$

Then $\log u = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\log x)=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times 1$$

$$\therefore \frac{du}{dx}=u(1+\log x)=x^x(1+\log x) \qquad \dots (1)$$

Now, $y=u+a^x+x^a+a^a$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{d}{dx}(a^x)+\frac{d}{dx}(x^a)+\frac{d}{dx}(a^a)$$

$$=x^x(1+\log x)+a^x\cdot\log a+ax^{a-1}+0 \qquad \dots \text{[By (1)]}$$

$$\therefore \frac{dy}{dx}=x^x(1+\log x)+a^x\cdot\log a+ax^{a-1}.$$

Examples for Practice	**3 or 4 marks each**

Differentiate the following w.r.t. x :

1. x^x
2. e^{x^x}
3. $(2x+5)^x$
4. $\left(1+\frac{1}{x}\right)^x$
5. $\sqrt[3]{\frac{3x-1}{(2x+3)(5-x)^2}}$
6. $\frac{(3x^2-1)\sqrt{1+x^2}}{x^3}$
7. $x^x+(7x-1)^x$
8. $x^{\sqrt{x}}+(\sqrt{x})^x$
9. $(\log x)^x+x^{\log x}$
10. x^x+a^x ***(July '24; March '25)***

Answers

1. $x^x(1+\log x)$
2. $e^{x^x}\cdot x^x(1+\log x)$
3. $(2x+5)^x\left[\log(2x+5)+\frac{2x}{2x+5}\right]$
4. $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$
5. $\sqrt[3]{\frac{3x-1}{(2x+3)(5-x)^2}}\left[\frac{1}{3x-1}-\frac{2}{3(2x+3)}+\frac{2}{3(5-x)}\right]$

6. $\dfrac{(3x^2-1)\sqrt{1+x^2}}{x^3}\left[\dfrac{6x}{3x^2-1}+\dfrac{x}{1+x^2}-\dfrac{3}{x}\right]$

7. $x^x(1+\log x)+(7x-1)^x\left[\dfrac{7x}{7x-1}+\log(7x-1)\right]$

8. $x^{\sqrt{x}}\left[\dfrac{2+\log x}{2\sqrt{x}}\right]+(\sqrt{x})^x\left[\dfrac{1+\log x}{2}\right]$

9. $(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]+x^{\log x}\left[\dfrac{2\log x}{x}\right]$

10. $x^x(1+\log x)+a^x\log a$.

3.4 DERIVATIVE OF AN IMPLICIT FUNCTION

Solved Examples	3 or 4 marks each

Ex. 9. Find $\dfrac{dy}{dx}$, if

(1) $x^3+y^2+xy=7$ (2) $x^3y^3=x^2-y^2$ *(July '22)* (3) $xy=\log(xy)$.

Solution :

(1) $x^3+y^2+xy=7$

Differentiating both sides w.r.t. x, we get

$$3x^2+2y\cdot\frac{dy}{dx}+x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)=0$$

$$\therefore\ 3x^2+2y\frac{dy}{dx}+x\frac{dy}{dx}+y\times 1=0$$

$$\therefore\ (2y+x)\frac{dy}{dx}=-3x^2-y$$

$$\therefore\ \frac{dy}{dx}=-\frac{(y+3x^2)}{2y+x}.$$

---

(2) $x^3y^3=x^2-y^2$

Differentiating both sides w.r.t. x, we get

$$x^3\cdot\frac{d}{dx}(y^3)+y^3\cdot\frac{d}{dx}(x^3)=2x-2y\frac{dy}{dx}$$

$$\therefore x^3 \times 3y^2\frac{dy}{dx} + y^3 \times 3x^2 = 2x - 2y\frac{dy}{dx}$$

$$\therefore (3x^3y^2 + 2y)\frac{dy}{dx} = 2x - 3x^2y^3$$

$$\therefore y(2 + 3x^3y)\frac{dy}{dx} = x(2 - 3xy^3)$$

$$\therefore \frac{dy}{dx} = \frac{x(2 - 3xy^3)}{y(2 + 3x^3y)}.$$

---

(3) $xy = \log(xy)$

$\therefore xy = \log x + \log y$

Differentiating both sides w.r.t. x, we get

$$x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore x \cdot \frac{dy}{dx} + y \times 1 = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \left(x - \frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x} - y$$

$$\therefore \left(\frac{xy - 1}{y}\right)\frac{dy}{dx} = \frac{1 - xy}{x} = \frac{-(xy - 1)}{x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}.$$

Ex. 10. If $\log(x + y) = \log(xy) + a$, then show that $\frac{dy}{dx} = -\frac{y^2}{x^2}$.

Solution : $\log(x + y) = \log(xy) + a$

$\therefore \log(x + y) = \log x + \log y + a$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x + y} \cdot \frac{d}{dx}(x + y) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore \frac{1}{x + y} \cdot \left(1 + \frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \frac{1}{x+y}+\frac{1}{x+y}\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$$

$$\therefore \left(\frac{1}{x+y}-\frac{1}{y}\right)\frac{dy}{dx}=\frac{1}{x}-\frac{1}{x+y}$$

$$\therefore \left[\frac{y-x-y}{y(x+y)}\right]\frac{dy}{dx}=\frac{x+y-x}{x(x+y)}$$

$$\therefore \left[\frac{-x}{y(x+y)}\right]\frac{dy}{dx}=\frac{y}{x(x+y)}$$

$$\therefore -\frac{x}{y}\cdot\frac{dy}{dx}=\frac{y}{x}$$

$$\therefore \frac{dy}{dx}=-\frac{y^2}{x^2}.$$

Ex. 11. If $x^5\cdot y^7=(x+y)^{12}$, then show that $\frac{dy}{dx}=\frac{y}{x}$. *(March '23)*

Solution : $x^5\cdot y^7=(x+y)^{12}$

$\therefore \log(x^5\cdot y^7)=\log(x+y)^{12}$

$\therefore \log x^5+\log y^7=\log(x+y)^{12}$

$\therefore 5\log x+7\log y=12\log(x+y)$

Differentiating both sides w.r.t. x, we get

$$5\times\frac{1}{x}+7\times\frac{1}{y}\cdot\frac{dy}{dx}=12\times\frac{1}{x+y}\cdot\frac{d}{dx}(x+y)$$

$$\therefore \frac{5}{x}+\frac{7}{y}\cdot\frac{dy}{dx}=\frac{12}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\therefore \frac{5}{x}+\frac{7}{y}\frac{dy}{dx}=\frac{12}{x+y}+\frac{12}{x+y}\cdot\frac{dy}{dx}$$

$$\therefore \left(\frac{7}{y}-\frac{12}{x+y}\right)\frac{dy}{dx}=\frac{12}{x+y}-\frac{5}{x}$$

$$\therefore \left[\frac{7x+7y-12y}{y(x+y)}\right]\frac{dy}{dx}=\frac{12x-5x-5y}{x(x+y)}$$

$$\therefore \left[\frac{7x-5y}{y(x+y)}\right]\frac{dy}{dx}=\frac{7x-5y}{x(x+y)}$$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}$

$\therefore \frac{dy}{dx}=\frac{y}{x}.$

Ex. 12. If $e^x+e^y=e^{(x+y)}$, then show that $\frac{dy}{dx}=-e^{y-x}$.

Solution : $e^x+e^y=e^{(x+y)}$ ... (1)

Differentiating both sides w.r.t. x, we get

$$e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}\cdot\frac{d}{dx}(x+y)$$

$$\therefore\ e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\therefore\ e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}+e^{(x+y)}\cdot\frac{dy}{dx}$$

$$\therefore\ [e^y-e^{(x+y)}]\frac{dy}{dx}=e^{(x+y)}-e^x$$

$$\therefore\ (e^y-e^x-e^y)\frac{dy}{dx}=e^x+e^y-e^x \quad \text{... [By (1)]}$$

$$\therefore\ -e^x\cdot\frac{dy}{dx}=e^y$$

$$\therefore\ \frac{dy}{dx}=-\frac{e^y}{e^x}=-e^{y-x}.$$

Examples for Practice	3 or 4 marks each

1. Find $\frac{dy}{dx}$, if

(1) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (2) $x^3+y^3+4x^3y=0$

(3) $y=x^3+3xy^2+3x^2y$ (4) $y\cdot e^x+x\cdot e^y=1.$

2. If $x^m\cdot y^n=(x+y)^{m+n}$, then show that $\frac{dy}{dx}=\frac{y}{x}$.

3. If $x^7\cdot y^9=(x+y)^{16}$, then show that $\frac{dy}{dx}=\frac{y}{x}$. ***(Sept. '21; July '23)***

4. If $x^y = e^{(x-y)}$, then show that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

5. If $x^y = y^x$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{x\log y - y}{y\log x - x}\right)$.

Answers

1. (1) $-\sqrt{\dfrac{y}{x}}$ (2) $-\dfrac{3x^2(1+4y)}{3y^2+4x^3}$ (3) $-\dfrac{3(x^2+y^2+2xy)}{6xy+3x^2-1}$ (4) $-\dfrac{e^y+ye^x}{e^x+xe^y}$.

3.5 DERIVATIVE OF A PARAMETRIC FUNCTION

Theorem 3 : If $x=f(t)$ and $y=g(t)$ are differentiable functions of t such that y is a function of x, then y is differentiable function of x and

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right), \text{ if } \frac{dx}{dt} \neq 0.$$

Note : If $x=f(t)$ and $y=g(t)$ are differentiable functions of t, then

$$\frac{dx}{dt} = f'(t) \quad \text{and} \quad \frac{dy}{dt} = g'(t) \qquad \therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{g'(t)}{f'(t)}.$$

Solved Examples	3 marks each

Ex. 13. Find $\dfrac{dy}{dx}$, if

(1) $x=2at^2, y=at^4$ (2) $x=\sqrt{1+u^2}, y=\log(1+u^2)$ ***(March '22)***

(3) $x=e^{3t}, y=e^{(4t+5)}$ ***(March '24)***

Solution :

(1) $x=2at^2,\ y=at^4$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a\frac{d}{dt}(t^2) = 2a\times 2t = 4at$$

and $$\frac{dy}{dt} = \frac{d}{dt}(at^4) = a\frac{d}{dt}(t^4) = a\times 4t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{4at^3}{4at} = t^2.$$

(2) $x = \sqrt{1+u^2}$, $y = \log(1+u^2)$

Differentiating x and y w.r.t. u, we get

$$\frac{dx}{du} = \frac{d}{du}(\sqrt{1+u^2}) = \frac{1}{2\sqrt{1+u^2}} \cdot \frac{d}{du}(1+u^2)$$

$$= \frac{1}{2\sqrt{1+u^2}} \times (0+2u) = \frac{u}{\sqrt{1+u^2}}$$

and $\frac{dy}{du} = \frac{d}{du}[\log(1+u^2)] = \frac{1}{1+u^2} \cdot \frac{d}{du}(1+u^2)$

$$= \frac{1}{1+u^2} \times (0+2u) = \frac{2u}{1+u^2}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/du)}{(dx/du)} = \frac{\left(\frac{2u}{1+u^2}\right)}{\left(\frac{u}{\sqrt{1+u^2}}\right)}$$

$$= \frac{2u}{1+u^2} \times \frac{\sqrt{1+u^2}}{u} = \frac{2}{\sqrt{1+u^2}}.$$

---

(3) $x = e^{3t}$, $y = e^{(4t+5)}$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(e^{3t}) = e^{3t} \cdot \frac{d}{dt}(3t)$$

$$= e^{3t} \times 3 = 3e^{3t}$$

and $\frac{dy}{dt} = \frac{d}{dt}[e^{(4t+5)}] = e^{(4t+5)} \cdot \frac{d}{dt}(4t+5)$

$$= e^{(4t+5)} \times (4 \times 1 + 0) = 4e^{(4t+5)}$$

$$\therefore \frac{dy}{dt} = \frac{(dy/dt)}{(dx/dt)} = \frac{4e^{(4t+5)}}{3e^{3t}} = \frac{4}{3} \cdot e^{(t+5)}.$$

Ex. 14. Differentiate $\log(1+x^2)$ with respect to a^x.

Solution : Let $u = \log(1+x^2)$ and $v = a^x$

Then we want to find $\frac{du}{dv}$.

Differentiating u and v w.r.t. x, we get

$$\frac{du}{dx}=\frac{d}{dx}[\log(1+x^2)]=\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$=\frac{1}{1+x^2}\times(0+2x)=\frac{2x}{1+x^2}$$

and $\dfrac{dv}{dx}=\dfrac{d}{dx}(a^x)=a^x\cdot\log a$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(\dfrac{2x}{1+x^2}\right)}{a^x\cdot\log a}=\frac{2x}{(1+x^2)\cdot a^x\log a}.$$

Ex. 15. If $x=\dfrac{4t}{1+t^2}, y=3\left(\dfrac{1-t^2}{1+t^2}\right)$**, then show that** $\dfrac{dy}{dx}=-\dfrac{9x}{4y}$. ***(March '25)***

Solution : $x=\dfrac{4t}{1+t^2},\ y=3\left(\dfrac{1-t^2}{1+t^2}\right)$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{4t}{1+t^2}\right)=\frac{(1+t^2)\cdot\dfrac{d}{dt}(4t)-4t\cdot\dfrac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$=\frac{(1+t^2)(4)-4t(0+2t)}{(1+t^2)^2}$$

$$=\frac{4+4t^2-8t^2}{(1+t^2)^2}=\frac{4-4t^2}{(1+t^2)^2}=\frac{4(1-t^2)}{(1+t^2)^2}$$

and $\dfrac{dy}{dt}=3\dfrac{d}{dt}\left(\dfrac{1-t^2}{1+t^2}\right)=3\left[\dfrac{(1+t^2)\cdot\dfrac{d}{dt}(1-t^2)-(1-t^2)\cdot\dfrac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$

$$=3\left[\frac{(1+t^2)(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}\right]$$

$$=3\left[\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}\right]$$

$$=-\frac{12t}{(1+t^2)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left[-\dfrac{12t}{(1+t^2)^2}\right]}{\left[\dfrac{4(1-t^2)}{(1+t^2)^2}\right]}$$

$$\therefore \frac{dy}{dx} = -\frac{3t}{1-t^2} \quad \text{... (1)}$$

$$-\frac{9x}{4y} = -\frac{9}{4} \cdot \frac{\left(\frac{4t}{1+t^2}\right)}{3\left(\frac{1-t^2}{1+t^2}\right)} = -\frac{3t}{1-t^2} \quad \text{... (2)}$$

From (1) and (2)

$$\frac{dy}{dx} = -\frac{9x}{4y}.$$

Examples for Practice	3 marks each

1. Find $\frac{dy}{dx}$, if

(1) $x = at^2, y = 2at$ (2) $x = e^{2t}, y = e^{\sqrt{t}}$

(3) $x = \left(u + \frac{1}{u}\right)^2, y = 2^{\left(u+\frac{1}{u}\right)}$ (4) $x = a\left(t - \frac{1}{t}\right), y = a\left(t + \frac{1}{t}\right)$.

2. If $x = t \cdot \log t, y = t^t$, then show that $\frac{dy}{dx} - y = 0$. *(July '23)*

3. Differentiate :

(1) $\log t$ w.r.t. $\log(1+t^2)$ (2) 5^x w.r.t. $\log x$ (3) $e^{(4x+5)}$ w.r.t. 10^{4x}.

Answers

1. (1) $\frac{1}{t}$ (2) $\frac{e^{\sqrt{t}}}{4\sqrt{t} \cdot e^{2t}}$ (3) $\frac{y \log 2}{2\sqrt{x}}$ (4) -1.

3. (1) $\frac{1+t^2}{2t^2}$ (2) $x \cdot 5^x \cdot \log 5$ (3) $\frac{e^{(4x+5)}}{10^{4x} \cdot \log 10}$.

3.6 SECOND ORDER DERIVATIVE

If $y = f(x)$ is derivable function of x, then $\frac{dy}{dx} = f'(x)$ is called *first order derivative* of y w.r.t. x and $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)]$ is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$.

Solved Examples	3 marks each

Ex. 16. Find $\frac{d^2y}{dx^2}$, if

(1) $y=\sqrt{x}$ (2) $y=\log x$.

Solution :

(1) $y=\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right)=\frac{1}{2}\frac{d}{dx}(x^{-\frac{1}{2}})$$

$$=\frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}=-\frac{1}{4}x^{-\frac{3}{2}}.$$

---

(2) $y=\log x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(x^{-1})=-1x^{-2}=-\frac{1}{x^2}.$$

Ex. 17. Find $\frac{d^2y}{dx^2}$, if $y=2at$, $x=at^2$.

Solution : $x=at^2$, $y=2at$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(at^2)=a\frac{d}{dt}(t^2)=a\times 2t=2at \qquad \text{... (1)}$$

and $\frac{dy}{dt}=\frac{d}{dt}(2at)=2a\frac{d}{dt}(t)=2a\times 1=2a$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx}$$

$$= -\frac{1}{t^2} \times \frac{1}{\left(\frac{dx}{dt}\right)} = -\frac{1}{t^2} \times \frac{1}{2at} \qquad \text{... [By (1)]}$$

$$= -\frac{1}{2at^3}.$$

Ex. 18. If $ax^2 + 2hxy + by^2 = 0$, then show that $\frac{d^2y}{dx^2} = 0$. *(March '22)*

Solution : $ax^2 + 2hxy + by^2 = 0$ ... (1)

$\therefore ax^2 + hxy + hxy + by^2 = 0$

$\therefore x(ax + hy) + y(hx + by) = 0$

$\therefore x(ax + hy) = -y(hx + by)$

$$\therefore \frac{ax + hy}{hx + by} = -\frac{y}{x} \qquad \text{... (2)}$$

Differentiating (1) w.r.t. x, we get

$$a \times 2x + 2h\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] + b \times 2y\frac{dy}{dx} = 0$$

$$\therefore 2ax + 2hx\frac{dy}{dx} + 2hy \times 1 + 2by\frac{dy}{dx} = 0$$

$$\therefore (2hx + 2by)\frac{dy}{dx} = -2ax - 2hy$$

$$\therefore \frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx + by)} = -\left(\frac{ax + hy}{hx + by}\right)$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \qquad \text{... [By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = \frac{x\left(\frac{y}{x}\right) - y \times 1}{x^2} \qquad \text{... }\left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= \frac{y - y}{x^2} = \frac{0}{x^2} \qquad \therefore \frac{d^2y}{dx^2} = 0.$$

Examples for Practice	3 marks each

1. Find $\dfrac{d^2y}{dx^2}$, if

(1) $y=\dfrac{1}{x^2}$ (2) $y=\dfrac{1}{x\sqrt{x}}$ (3) $y=e^{4x}$ (4) $y=(\log x)^2$ (5) $y=e^{\log x}$.

2. Find $\dfrac{d^2y}{dx^2}$, if $y=x^2\cdot e^x$.

3. If $x^m\cdot y^n=(x+y)^{m+n}$, then show that $\dfrac{d^2y}{dx^2}=0$.

Answers

1. (1) $\dfrac{6}{x^4}$ (2) $\dfrac{15}{4}x^{-\frac{7}{2}}$ (3) $16e^{4x}$ (4) $\dfrac{2(1-\log x)}{x^2}$ (5) 0.

2. $e^x(x^2+4x+2)$.

ACTIVITIES	4 marks each

1. Complete the following activity :

Given : $y=(6x^4-5x^3+2x+3)^5$

Let $u=6x^4-5x^3+2x+3$

Then $y=\boxed{\quad\quad}$

$\therefore \dfrac{dy}{du}=\boxed{\quad}$

and $\dfrac{du}{dx}=\boxed{\quad\quad}$

By chain rule

$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$ $\quad\therefore \dfrac{dy}{dx}=\boxed{\quad\quad}$

Solution :

Given : $y=(6x^4-5x^3+2x+3)^5$

Let $u=6x^4-5x^3+2x+3$

Then $y=\boxed{u^5}$

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^5) = \boxed{5u^4}$$

$$\text{and } \frac{du}{dx} = \frac{d}{dx}(6x^4 - 5x^3 + 2x + 3)$$

$$= 6 \times 4x^3 - 5 \times 3x^2 + 2 \times 1 + 0$$

$$\therefore \frac{du}{dx} = \boxed{24x^3 - 15x^2 + 2}$$

By Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times (24x^3 - 15x^2 + 2)$$

$$\therefore \frac{dy}{dx} = \boxed{5(6x^4 - 5x^3 + 2x + 3)^4 (24x^3 - 15x^2 + 2)}$$

---

2. If $y = x^{\log x} + 10^x$, find $\frac{dy}{dx}$ by completing the following activity :

$y = x^{\log x} + 10^x$

Let $u = x^{\log x}$, $v = 10^x$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{... (1)}$$

Now, $u = x^{\log x}$

$\therefore \log u = \log x^{\log x} = (\log x)(\log x) = (\log x)^2$

Differentiating w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(\log x)^2 = \boxed{}$$

$$\therefore \frac{du}{dx} = \boxed{} \qquad \text{... (2)}$$

Also, $v = 10^x$

$$\therefore \frac{dv}{dx} = \boxed{} \qquad \text{... (3)}$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = \boxed{}$$

Solution : $y = x^{\log x} + 10^x$

Let $u = x^{\log x}$, $v = 10^x$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{... (1)}$$

Now, $u = x^{\log x}$

$\therefore \log u = \log x^{\log x} = (\log x)(\log x) = (\log x)^2$

Differentiating w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(\log x)^2 = 2 \log x \cdot \frac{d}{dx}(\log x)$$

$$= 2 \log x \times \frac{1}{x} = \boxed{\frac{2 \log x}{x}}$$

$$\therefore \frac{du}{dx} = u\left[\frac{2 \log x}{x}\right] = \boxed{x^{\log x}\left(\frac{2 \log x}{x}\right)} \quad \text{... (2)}$$

Also, $v = 10^x$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(10^x) = \boxed{10^x \cdot \log 10} \quad \text{... (3)}$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = \boxed{x^{\log x}\left(\frac{2 \log x}{x}\right) + 10^x \cdot \log 10}.$$

---

3. Complete the following activity :

$x = e^t, y = e^{\sqrt{t}}$.

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \boxed{}$$

$$\text{and } \frac{dy}{dt} = \boxed{}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\boxed{}}{e^t} = \boxed{}$$

Solution : $x = e^t, y = e^{\sqrt{t}}$.

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(e^t) = \boxed{e^t}$$

$$\text{and } \frac{dy}{dt} = \frac{d}{dt}\left(e^{\sqrt{t}}\right) = e^{\sqrt{t}} \cdot \frac{d}{dt}(\sqrt{t})$$

$$= e^{\sqrt{t}} \times \frac{1}{2\sqrt{t}} = \boxed{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}$$

$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\boxed{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}}{e^t}=\boxed{\frac{e^{\sqrt{t}}}{2\sqrt{t}\cdot e^t}}$$

MULTIPLE CHOICE QUESTIONS	**1 mark each**

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. If $y=2x^2+a^2+2^2$, then $\frac{dy}{dx}=$ ....

(a) $4x$ (b) $4x+2a$ (c) $4x+4$ (d) $2x$

(March '23; July '23)

2. If $y=2x^2+\log 2+5$, then $\frac{dy}{dx}=$ ....

(a) x (b) $4x$ (c) $2x+\log 2$ (d) $-4x$ *(July '24)*

3. If $y=e^{\log x}$, then $\frac{dy}{dx}=$ ....

(a) $\frac{e^{\log x}}{x}$ (b) $\frac{1}{x}$ (c) 0 (d) $\frac{1}{2}$ *(July '22)*

4. If $y=\log\left(\frac{e^x}{x^2}\right)$, then $\frac{dy}{dx}=$ ....

(a) $\frac{2-x}{x}$ (b) $\frac{x-2}{x}$ (c) $\frac{e-x}{ex}$ (d) $\frac{x-e}{ex}$ *(March '24)*

5. If $x=2at^2$, $y=4at$, then $\frac{dy}{dx}=$ ....

(a) $\frac{1}{t}$ (b) $-\frac{1}{2at^2}$ (c) $\frac{1}{2at^3}$ (d) $\frac{1}{4at^3}$ *(July '22-'24)*

6. If $y=x\cdot\log x$, then $\frac{dy}{dx}=$ ....

(a) 1 (b) $\frac{1}{x}$ (c) $\log x$ (d) $1+\log x$

(March '23)

7. If $x^4\cdot y^5=(x+y)^{(m+1)}$ and $\frac{dy}{dx}=\frac{y}{x}$, then $m=$ ....

(a) 8 (b) 4 (c) 5 (d) 20

8. If $x=\dfrac{e^t+e^{-t}}{2}$ and $y=\dfrac{e^t-e^{-t}}{2}$, then $\dfrac{dy}{dx}=$ ....

(a) $-\dfrac{y}{x}$ (b) $\dfrac{y}{x}$ (c) $-\dfrac{x}{y}$ (d) $\dfrac{x}{y}$

9. If $x^y=e^{x-y}$, then $\dfrac{dy}{dx}=$ ....

(a) $\dfrac{1+x}{1+\log x}$ (b) $\dfrac{1-\log x}{1+\log x}$ (c) $\dfrac{\log x}{(1+\log x)^2}$ (d) $\dfrac{1-x}{1+\log x}$

10. If $xy=1+\log y$, then $\dfrac{dy}{dx}=$ ....

(a) $\dfrac{y}{\log y}$ (b) $-\dfrac{y^2}{\log y}$ (c) $\dfrac{y^2}{\log y}$ (d) $-\dfrac{y}{\log y}$

Answers

1. (a) $4x$ 2. (b) $4x$ 3. (a) $\dfrac{e^{\log x}}{x}$ 4. (b) $\dfrac{x-2}{x}$ 5. (a) $\dfrac{1}{t}$

6. (d) $1+\log x$ 7. (a) 8 8. (d) $\dfrac{x}{y}$ 9. (c) $\dfrac{\log x}{(1+\log x)^2}$ 10. (b) $-\dfrac{y^2}{\log y}$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The derivative of $\log_a x$, where a is a constant, is $\dfrac{1}{x\log a}$.
2. The derivative of $f(x)=a^x$ is $x\cdot a^{x-1}$, where a is a constant. *(March '23)*
3. If $x^m\cdot y^n=(x+y)^{m+n}$, then $\dfrac{dy}{dx}=\dfrac{x}{y}$.
4. If $x=2at$, $y=2at^2$, then $\dfrac{dy}{dx}=\dfrac{1}{2t}$.
5. If $y=e^{\log x}$, then $\dfrac{d^2y}{dx^2}=0$.

6. The rate of change of demand (x) of a commodity w.r.t. its price (y) is $\frac{dy}{dx}$.

(July '24)

Answers

1. True **2.** False **3.** False **4.** False **5.** True **6.** False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. If $y = \log_x a = 0$, then $\frac{dy}{dx} = \text{.........}$.
2. If $0 = \log (xy) + a$, then $\frac{dy}{dx} = \text{.........}$.
3. If $y = x \cdot \log x$, then $\frac{dy}{dx} = \text{.........}$. ***(July '22)***
4. If $y = (\log x)^2$, then $\frac{dy}{dx} = \text{.........}$. ***(March '23)***
5. The derivative of $\log (\log x)$ w.r.t. $\log x$ is ....
6. If $y = x \cdot \log x$, then $\frac{d^2y}{dx^2} = \text{.........}$. ***(July '24)***

Answers

1. $-\frac{\log a}{x(\log x)^2}$ **2.** $-\frac{y}{x}$ **3.** $1 + \log x$ **4.** $\frac{2 \log x}{x}$ **5.** $\frac{1}{\log x}$ **6.** $\frac{1}{x}$.

Question Set 4

APPLICATIONS OF DERIVATIVES

(Marks with option : 09)

4.1 GEOMETRICAL APPLICATION

Remember :

1. Let $y=f(x)$ be any curve and $P(a, f(a))$ be any point on it, then the slope of the tangent to the curve at the point P is $f'(a)$. It is also called **gradient** of the curve at the point P.

   Hence, equation of tangent at P is $y-f(a)=f'(a)(x-a)$.

2. Slope of normal at $P(a, f(a))$ is $-\dfrac{1}{f'(a)}$, if $f'(a)\neq 0$ and equation of normal at P is $y-f(a)=-\dfrac{1}{f'(a)}(x-a)$.

Solved Examples	3 or 4 marks each

Ex. 1. Find the equations of tangent and normal to the curve $y=3x^2-x+1$ at P(1, 3).

Solution : $y=3x^2-x+1$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(3x^2-x+1)=3\times 2x-1+0=6x-1$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(1,\,3)}=6(1)-1=5$$

$$= \text{slope of the tangent at } (1, 3)$$

$\therefore$ the equation of the tangent at (1, 3) is $y-3=5(x-1)$

$\therefore$ $y-3=5x-5$

$\therefore$ $5x-y-2=0$.

The slope of the normal at $(1, 3)=-\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{\text{at }(1,\,3)}}=-\dfrac{1}{5}$

$\therefore$ the equation of the normal at (1, 3) is $y-3=-\dfrac{1}{5}(x-1)$

$\therefore$ $5y-15=-x+1$

$\therefore\ x+5y-16=0$

Hence, the equations of the tangent and normal are $5x-y-2=0$ and $x+5y-16=0$ respectively.

Ex. 2. Find the equations of the tangent and normal to the curve $y=6-x^2$ where the normal is parallel to the line $x-4y+3=0$.

Solution : Let $P(x_1, y_1)$ be the point on the curve $y=6-x^2$, where the normal is parallel to the line $x-4y+3=0$.

Differentiating $y=6-x^2$ w.r.t. x we get

$$\frac{dy}{dx}=\frac{d}{dx}(6-x^2)=0-2x=-2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)}=-2x_1=\text{slope of the tangent at }(x_1, y_1)$$

$$\therefore \text{slope of normal at }(x_1, y_1)=\frac{-1}{-2x_1}=\frac{1}{2x_1}$$

Let $m_1=\dfrac{1}{2x_1}$

The slope of the line $x-4y+3=0$ is $m_2=-\dfrac{1}{(-4)}=\dfrac{1}{4}$

Since the normal at $P(x_1, y_1)$ is parallel to the line $x-4y+3=0$, $m_1=m_2$

$\therefore \dfrac{1}{2x_1}=\dfrac{1}{4} \qquad \therefore 2x_1=4 \qquad \therefore x_1=2$

Since (x_1, y_1) lies on the curve $y=6-x^2$, $y_1=6-x_1{}^2$, where $x_1=2$

$\therefore y_1=6-(2)^2=6-4=2$

$\therefore$ the coordinates of the point are (2, 2) and the slope of the normal

$=m_1=m_2=\dfrac{1}{4}$.

Slope of the tangent $=-2x_1=-2(2)=-4$

$\therefore$ the equation of the tangent at (2, 2) is $y-2=-4(x-2)$

$\therefore y-2=-4x+8 \qquad \therefore 4x+y-10=0$

Also, the equation of normal at (2, 2) is $y-2=\dfrac{1}{4}(x-2)$

$\therefore 4y-8=x-2 \qquad \therefore x-4y+6=0$

Hence, the equations of tangent and normal are $4x+y-10=0$ and $x-4y+6=0$ respectively.

Examples for Practice	3 or 4 marks each

1. Find the equations of tangent and normal to the following curves at the given point on it :
   (1) $y = x^2 + 4x + 1$ at $(-1, -2)$. (2) $2x^2 + 3y^2 = 5$ at $(1, 1)$.
2. Find the equations of the tangent and normal to the curve $y = x^2 + 5$, where the tangent is parallel to the line $4x - y + 1 = 0$.
3. If the line $y = 4x - 5$ touches the curve $y^2 = ax^3 + b$ at the point $(2, 3)$, show that $7a + 2b = 0$.

Answers

1. (1) $2x - y = 0$, $x + 2y + 5 = 0$. (2) $2x + 3y - 5 = 0$, $3x - 2y - 1 = 0$.
2. $4x - y + 1 = 0$ and $x + 4y - 38 = 0$.

4.2 INCREASING AND DECREASING FUNCTIONS

Remember :

A function f is said to be

(i) increasing in (a, b), if $f'(x) > 0$ for all $x \in (a, b)$

(ii) decreasing in (a, b), if $f'(x) < 0$ for all $x \in (a, b)$.

Solved Examples	3 or 4 marks each

Ex. 3. Test whether the following functions are increasing or decreasing :

(1) $f(x) = x^3 - 6x^2 + 12x - 16$, $x \in R$.

(2) $f(x) = 2 - 3x + 3x^2 - x^3$, $x \in R$.

Solution :

(1) $f(x) = x^3 - 6x^2 + 12x - 16$

$$\therefore f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 12x - 16) = 3x^2 - 6 \times 2x + 12 \times 1 - 0$$

$$= 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$

$$= 3(x-2)^2 > 0 \text{ for all } x \in R,\ x \neq 2$$

$\therefore f'(x) > 0$ for all $x \in R - \{2\}$

$\therefore f$ is increasing for all $x \in R - \{2\}$.

(2) $f(x)=2-3x+3x^2-x^3$

$\therefore f'(x)=\dfrac{d}{dx}(2-3x+3x^2-x^3)=0-3\times 1+3\times 2x-3x^2$

$=-3+6x-3x^2=-3(x^2-2x+1)$

$=-3(x-1)^2<0$ for all $x\in R$, $x\neq 1$

$\therefore f'(x)<0$ for all $x\in R-\{1\}$

$\therefore f$ is decreasing for all $x\in R-\{1\}$.

Ex. 4. Find the values of x, such that $f(x)=2x^3-15x^2+36x+1$ is increasing function. *(Sept. '21)*

Solution :

$f(x)=2x^3-15x^2+36x+1$

$\therefore f'(x)=\dfrac{d}{dx}(2x^3-15x^2+36x+1)=2\times 3x^2-15\times 2x+36\times 1+0$

$=6x^2-30x+36=6(x^2-5x+6)$

f is increasing, if $f'(x)>0$

i.e. if $6(x^2-5x+6)>0$

i.e. if $x^2-5x+6>0$

i.e. if $x^2-5x>-6$

i.e. if $x^2-5x+\dfrac{25}{4}>-6+\dfrac{25}{4}$

i.e. if $\left(x-\dfrac{5}{2}\right)^2>\dfrac{1}{4}$

i.e. if $x-\dfrac{5}{2}>\dfrac{1}{2}$ or $x-\dfrac{5}{2}<-\dfrac{1}{2}$

i.e. if $x>3$ or $x<2$

i.e. if $x\in(-\infty, 2)\cup(3, \infty)$

$\therefore f$ is increasing, if $x\in(-\infty, 2)\cup(3, \infty)$.

Ex. 5. Find the values of x for which $f(x)=2x^3-15x^2-84x-7$ is decreasing function.

Solution : $f(x)=2x^3-15x^2-84x-7$

$\therefore f'(x)=\dfrac{d}{dx}(2x^3-15x^2-84x-7)=2\times 3x^2-15\times 2x-84\times 1-0$

$=6x^2-30x-84=6(x^2-5x-14)$

f is decreasing, if $f'(x) < 0$

i.e. if $6(x^2 - 5x - 14) < 0$

i.e. if $x^2 - 5x - 14 < 0$

i.e. if $x^2 - 5x < 14$

i.e. if $x^2 - 5x + \frac{25}{4} < 14 + \frac{25}{4}$

i.e. if $\left(x - \frac{5}{2}\right)^2 < \frac{81}{4}$

i.e. if $-\frac{9}{2} < x - \frac{5}{2} < \frac{9}{2}$

i.e. if $-\frac{9}{2} + \frac{5}{2} < x - \frac{5}{2} + \frac{5}{2} < \frac{9}{2} + \frac{5}{2}$

i.e. if $-2 < x < 7$

$\therefore$ f is decreasing, if $-2 < x < 7$, i.e. $x \in (-2, 7)$.

Examples for Practice	**3 or 4 marks each**

1. Test whether the following functions are increasing or decreasing :

   (1) $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$.

   (2) $f(x) = \frac{7}{x} - 3$, $x \in R$, $x \neq 0$.

2. Find the value of x, such that $f(x) = x^3 + 12x^2 + 36x + 6$ is increasing function.

3. Find the values of x, such that $f(x) = 2x^3 - 9x^2 + 12x + 2$ is decreasing function.

4. Show that the function $f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing.

Answers

1. (1) increasing for all $x \in R - \{1\}$

   (2) decreasing for all $x \in R$, $x \neq 0$.

2. $x < -6$ or $x > -2$.

3. $1 < x < 2$.

4.3 MAXIMA AND MINIMA

Remember :

Procedure for finding Maxima and Minima :

(1) Find $f'(x)$ and $f''(x)$.

(2) Find the roots of the equation $f'(x)=0$.

(3) If c is the root of this equation, find $f''(c)$.

(i) If $f''(c)<0$, then f has maximum at $x=c$ and $f(c)$ is the maximum value.

(ii) If $f''(c)>0$, then f has minimum at $x=c$ and $f(c)$ is the minimum value.

(4) Use the same procedure for the other roots of $f'(c)=0$, as in (3).

Solved Examples	3 or 4 marks each

Ex. 6. Examine the function f for maxima and minima, where $f(x)=2x^3-21x^2+36x-20$.

Solution : $f(x)=2x^3-21x^2+36x-20$

$$\therefore\ f'(x)=\frac{d}{dx}(2x^3-21x^2+36x-20)$$

$$=2\times 3x^2-21\times 2x+36\times 1-0$$

$$=6x^2-42x+36$$

$$\text{and } f''(x)=\frac{d}{dx}(6x^2-42x+36)=6\times 2x-42\times 1+0=12x-42$$

$f'(x)=0$ gives $6x^2-42x+36=0$.

$\therefore\ x^2-7x+6=0 \qquad \therefore\ (x-1)(x-6)=0$

$\therefore$ the roots of $f'(x)=0$ are $x_1=1$ and $x_2=6$.

For $x=1$, $f''(1)=12(1)-42=-30<0$

$\therefore$ by the second derivative test, f has maximum at $x=1$ and maximum value of f at $x=1$

$=f(1)=2(1)^3-21(1)^2+36(1)-20$

$=2-21+36-20=-3$

For $x=6$, $f''(6)=12(6)-42=30>0$

$\therefore$ by the second derivative test, f has minimum at $x=6$ and minimum value of f at $x=6$

$= f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20$

$= 432 - 756 + 216 - 20 = -128$

Hence, the function f has maximum value -3 at $x = 1$ and minimum value -128 at $x = 6$.

Ex. 7. Show that $f(x) = \dfrac{\log x}{x}$, $x \neq 0$ is maximum at $x = e$.

Solution : $f(x) = \dfrac{\log x}{x}$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{x\dfrac{d}{dx}(\log x) - (\log x)\cdot\dfrac{d}{dx}(x)}{x^2}$$

$$= \frac{x \times \dfrac{1}{x} - (\log x) \times 1}{x^2} = \frac{1 - \log x}{x^2}$$

and $f''(x) = \dfrac{d}{dx}\left(\dfrac{1 - \log x}{x^2}\right)$

$$= \frac{x^2 \cdot \dfrac{d}{dx}(1 - \log x) - (1 - \log x)\dfrac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left(0 - \dfrac{1}{x}\right) - (1 - \log x)\,(2x)}{x^4}$$

$$= \frac{-x - 2x + 2x\log x}{x^4} = \frac{-3x + 2x\log x}{x^4}$$

$$= \frac{2\log x - 3}{x^3} \qquad \ldots [\because x \neq 0]$$

Now, $f'(x) = 0$, if $\dfrac{1 - \log x}{x^2} = 0$

i.e. if $1 - \log x = 0$

i.e. if $\log x = 1 = \log e$, i.e. if $x = e$

When $x = e$, $f''(x) = f''(e) = \dfrac{2\log e - 3}{e^3} = \dfrac{2 - 3}{e^3} = -\dfrac{1}{e^3} < 0$

$\therefore$ by the second derivative test, f is maximum at $x = e$.

Maximum value of f at $x = e$

$= \frac{\log e}{e} = \frac{1}{e}$. ... [$\because \log e = 1$]

Ex. 8. A rod of 108 metres long is bent to form a rectangle. Find its dimensions, if the area of rectangle is maximum. ***(March '23)***

Solution : Let x metres and y metres be the length and breadth of the rectangle.

Then its perimeter is $2(x+y) = 108$

$\therefore x + y = 54 \quad \therefore y = 54 - x$

Area of the rectangle $= xy = x(54-x)$

Let $f(x) = x(54-x) = 54x - x^2$

Then $f'(x) = \frac{d}{dx}(54x - x^2) = 54 \times 1 - 2x = 54 - 2x$

and $f''(x) = \frac{d}{dx}(54 - 2x) = 0 - 2 \times 1 = -2$

Now, $f'(x) = 0$, if $54 - 2x = 0$

i.e. if $x = 27$ and $f''(27) = -2 < 0$

$\therefore$ by the second derivative test, f has maximum value at $x = 27$.

When $x = 27$, $y = 54 - 27 = 27$

Hence, the dimensions of rectangle are 27 m × 27 m.

Ex. 9. The total cost of producing x units is ₹ $(x^2 + 60x + 50)$ and the price is ₹ $(180 - x)$ per unit. For what units is the profit maximum? ***(July '23)***

Solution : Let the number of units sold be x.

Then profit = S.P. – C.P.

$\therefore \; P(x) = (180 - x)x - (x^2 + 60x + 50)$

$\qquad = 180x - x^2 - x^2 - 60x - 50$

$\therefore \; P(x) = 120x - 2x^2 - 50$

$\therefore \; P'(x) = \frac{d}{dx}(120x - 2x^2 - 50) = 120 \times 1 - 2 \times 2x - 0 = 120 - 4x$

and $P''(x) = \frac{d}{dx}(120 - 4x) = 0 - 4 \times 1 = -4$

$P'(x) = 0$, if $120 - 4x = 0$

i.e. if $x = 30$ and $P''(30) = -4 < 0$

$\therefore$ by the second derivative test, $P(x)$ is maximum when $x = 30$.

Hence, the number of units sold for maximum profit is 30.

Examples for Practice	3 or 4 marks each

1. Determine the maximum and minimum values of the following functions :
   (1) $f(x) = x^3 - 9x^2 + 24x$ (2) $f(x) = 3x^3 - 9x^2 - 27x + 15$.
2. Determine the minimum value of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$. *(March '24)*
3. Show that $f(x) = x \log x$ is minimum at $x = \frac{1}{e}$. Find the minimum value of $f(x)$.
4. A metal wire of 36 cm length is bent to form a rectangle. Find its dimensions, when its area is maximum. *(July '24)*
5. If $x + y = 3$, show that maximum value of x^2y is 4.
6. Divide the number 20 into two parts such that their product is maximum. *(March '22)*

Answers

1.

	Max. at	Max. value	Min. at	Min. value
(1)	2	20	4	16
(2)	-1	30	3	-66

2. -128 3. $-\frac{1}{e}$ 4. square of side 9 cm 6. 10 and 10.

4.4 APPLICATIONS OF DERIVATIVE IN ECONOMICS

Remember :

1. Demand function $D = f(P)$ Marginal Demand (MD) $= D_m = \frac{dD}{dP}$.
2. Supply function $S = g(P)$ Marginal Supply (MS) $= \frac{dS}{dP}$
3. Total cost function $C = f(x)$, where x is the number of articles produced.

   Marginal Cost (MC) $= C_m = \frac{dC}{dx}$ Average Cost (AC) $= C_A = \frac{C}{x}$

4. Total revenue $R = P \cdot D$, where P is the price and D is the demand.

Average Revenue $= R_A = \frac{R}{D} = \frac{P \cdot D}{D} = P$

Total profit $= \pi = R - C$.

Note : π is just a symbol for total profit. It should not be mistaken for the constant value $\frac{22}{7} \approx 3.141$.

5. **Elasticity of Demand :**

If Demand (D) of a commodity is a function of its Price (P), then Elasticity of Demand with respect to price is given by

$$\eta = -\frac{P}{D} \cdot \frac{dD}{dP}$$

Remarks :

(i) Demand is decreasing function of price

$$\therefore \frac{dD}{dP} < 0$$

Also, price P and the demand D are always positive.

$$\therefore \eta = -\frac{P}{D} \cdot \frac{dD}{dP} > 0$$

(ii) If $\eta = 0$, the demand D is constant function of price P.

$$\therefore \frac{dD}{dP} = 0$$

In this case, demand is perfectly inelastic.

(iii) If $0 < \eta < 1$, the demand is relatively inelastic.

(iv) If $\eta = 1$, the demand is exactly proportional to the price and demand is said to be unitary elastic.

(v) If $\eta > 1$, the demand is relatively elastic.

6. $R_m = P\left(1 - \frac{1}{\eta}\right) = R_A\left(1 - \frac{1}{\eta}\right)$, where R_m is the marginal revenue, R_A is the average revenue and η is the elasticity of demand.

7. (i) Marginal propensity to consume (MPC) $= \frac{dE_c}{dx}$, where consumption expenditure (E_c) depends upon income x, i.e. $E_c = f(x)$.

(ii) Average propensity to consume (APC) $= \frac{E_c}{x}$.

8. If S is the saving of a person with income x, then

(i) Average propensity to save (APS) $= \frac{S}{x}$.

(ii) Marginal propensity to save (MPS) $= \frac{dS}{dx}$.

9. For a person with income x, consumption expenditure E_c and saving S.

(i) $x = E_c + S$

(ii) $\text{MPC} + \text{MPS} = 1$

(iii) $\text{APC} + \text{APS} = 1$

Solved Examples	**3 or 4 marks each**

Ex. 10. For manufacturing x units, labour cost is $150 - 54x$ and processing cost is x^2. Price of each unit is $p = 10800 - 4x^2$. Find the values of x for which

(a) total cost is decreasing

(b) revenue is increasing. ***(March '23)***

Solution :

(a) Total cost C = labour cost + processing cost

$\therefore C = 150 - 54x + x^2$

$\therefore \frac{dC}{dx} = \frac{d}{dx}(150 - 54x + x^2) = 0 - 54 \times 1 + 2x = -54 + 2x$

The total cost is decreasing, if $\frac{dC}{dx} < 0$

i.e. if $-54 + 2x < 0$

i.e. if $2x < 54$

i.e. if $x < 27$

Hence, the total cost is decreasing, if $x < 27$.

(b) The total revenue R is given as $R = p \cdot x = (10800 - 4x^2)x$

$\therefore R = 10800x - 4x^3$

$\therefore \frac{dR}{dx} = \frac{d}{dx}(10800x - 4x^3) = 10800 \times 1 - 4 \times 3x^2 = 10800 - 12x^2$

The revenue is increasing, if $\frac{dR}{dx} > 0$

i.e. if $10800 - 12x^2 > 0$

i.e. if $10800 > 12x^2$

i.e. if $x^2 < 900$

i.e. if $x < 30$ ... $[\because x > 0]$

Hence, the revenue is increasing, if $x < 30$.

Ex. 11. Find the elasticity of demand, if the marginal revenue is 50 and price is ₹ 75.

Solution : Given $R_m = 50$ and $R_A = 75$

Now, $R_m = R_A\left(1 - \frac{1}{\eta}\right)$

$\therefore 50 = 75\left(1 - \frac{1}{\eta}\right)$ $\quad \therefore 1 - \frac{1}{\eta} = \frac{50}{75} = \frac{2}{3}$

$\therefore \frac{1}{\eta} = 1 - \frac{2}{3} = \frac{1}{3}$ $\quad \therefore \eta = 3$

Hence, the elasticity of demand = 3.

Ex. 12. If the demand function is $D = \frac{p+6}{p-3}$, find the elasticity of demand at $p = 4$. ***(July '23)***

Solution : The demand function is

$$D = \frac{p+6}{p-3}$$

$$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(\frac{p+6}{p-3}\right) = \frac{(p-3)\frac{d}{dp}(p+6) - (p+6)\frac{d}{dp}(p-3)}{(p-3)^2}$$

$$= \frac{(p-3)(1+0) - (p+6)(1-0)}{(p-3)^2}$$

$$= \frac{p-3-p-6}{(p-3)^2} = -\frac{9}{(p-3)^2}$$

Elasticity of demand is given by

$$\eta = -\frac{p}{D} \cdot \frac{dD}{dp} = -\frac{p}{\left(\frac{p+6}{p-3}\right)} \times \left[-\frac{9}{(p-3)^2}\right] = \frac{9p}{(p+6)(p-3)}$$

When $p = 4$, then

$$\eta = \frac{9(4)}{(4+6)(4-3)} = \frac{36}{10 \times 1} = 3.6$$

Hence, the elasticity of demand at $p = 4$ is 3.6.

Ex. 13. The total cost of manufacturing x articles is $C = 47x + 300x^2 - x^4$. Find x, for which average cost is

(1) increasing (2) decreasing.

Solution : The total cost is given as

$C = 47x + 300x^2 - x^4$

$\therefore$ average cost is given by

$$C_A = \frac{C}{x} = \frac{47x + 300x^2 - x^4}{x}$$

$\therefore$ $C_A = 47 + 300x - x^3$

$$\therefore \frac{dC_A}{dx} = \frac{d}{dx}(47 + 300x - x^3) = 0 + 300 \times 1 - 3x^2 = 300 - 3x^2$$

(1) C_A is increasing, if $\frac{dC_A}{dx} > 0$

i.e. if $300 - 3x^2 > 0$

i.e. if $300 > 3x^2$

i.e. if $x^2 < 100$

i.e. if $x < 10$ ... $[\because x > 0]$

Hence, the average cost is increasing, if $x < 10$.

(2) C_A is decreasing, if $\frac{dC_A}{dx} < 0$

i.e. if $300 - 3x^2 < 0$

i.e. if $300 < 3x^2$

i.e. if $x^2 > 100$

i.e. if $x > 10$ ... [$\because x > 0$]

Hence, the average cost is decreasing, if $x > 10$.

Ex. 14. Find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as $E_c = (0.0003)I^2 + (0.075)I$, when $I = 1000$.

(July '22; March '25)

Solution : $E_c = (0.0003)I^2 + (0.075)I$

$$\text{MPC} = \frac{dE_c}{dI} = \frac{d}{dI}[(0.0003)I^2 + (0.075)I]$$

$$= (0.0003)(2I) + (0.075)(1) = (0.0006)I + 0.075$$

When $I = 1000$, then

$\text{MPC} = (0.0006)(1000) + 0.075 = 0.6 + 0.075 = 0.675.$

$\because \text{MPC} + \text{MPS} = 1$

$\therefore 0.675 + \text{MPS} = 1$

$\therefore \text{MPS} = 1 - 0.675 = 0.325$

Now, $\text{APC} = \dfrac{E_c}{I} = \dfrac{(0.0003)I^2 + (0.075)I}{I} = (0.0003)I + (0.075)$

When $I = 1000$, then

$\text{APC} = (0.0003)(1000) + 0.075 = 0.3 + 0.075 = 0.375$

$\because \text{APC} + \text{APS} = 1$

$\therefore 0.375 + \text{APS} = 1$

$\therefore \text{APS} = 1 - 0.375 = 0.625$

Hence, MPC = 0.675, MPS = 0.325, APC = 0.375, APS = 0.625.

Examples for Practice	3 or 4 marks each

1. The daily production cost 'C' to manufacture 'x' items of a commodity is given by $C = 200 + 12x + \dfrac{x^2}{2}$. Find x for which the average cost is decreasing.
2. A manufacturing company produces x items at a total cost of ₹$(40 + 2x)$. Their price is given as $p = 120 - x$. Find the value of x for which
   (1) revenue is increasing, (2) profit is increasing.
   (3) Also, find elasticity of demand for price 80.

3. The cost C of producing x articles is given as $C = x^3 - 16x^2 + 47x$. For what values of x, the average cost is decreasing?

4. (1) If the average revenue R_A is 50 and elasticity of demand η is 5, find marginal revenue R_m.

   (2) Find the price, if the marginal revenue is 28 and elasticity of demand is 3.

5. Demand function x for a certain commodity is given as $x = 200 - 4p$, where p is the unit price. Find

   (1) elasticity of demand as a function of p.

   (2) elasticity of demand when $p = 10$. Interpret your result.

   (3) the price p for which elasticity of demand is equal to one.

6. In a factory, for production of Q articles, standing charges are ₹ 500, labour charges are ₹ 700 and processing charges are $50Q$. The price of an article is $1700 - 3Q$. For what values of Q, the profit is increasing?

7. Find the price for the demand function $D = \frac{2p+3}{3p-1}$, when elasticity of demand is $\frac{11}{14}$.

8. The consumption expenditure E_c of a person with income x is given by $E_c = (0.0006)x^2 + (0.003)x$.

   Find average propensity to consume, marginal propensity to consume when his income is ₹ 200. Also, find his marginal propensity to save. ***(March '24)***

Answers

1. $x < 20$ **2.** (1) $x < 60$ (2) $x < 59$ (3) 2 **3.** $x < 8$

4. (1) 40 (2) 42 **5.** (1) $\eta = \frac{p}{50-p}$, $(p < 50)$ (2) 0·25, inelastic (3) 25 units

6. $Q < 275$ **7.** $p = \frac{3}{2}$ **8.** APC = 0.123, MPC = 0.243, MPS = 0.757.

ACTIVITIES	4 marks each

1. Complete the following activity to find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as :
$E_c = (0.0003)I^2 + (0.075)I$, when $I = 1000$.

Given : $E_c = (0.0003)\,I^2 + (0.075)\,I$

We have APC $= \dfrac{E_c}{I}$

At $I = 1000$, APC $= \boxed{}$

Now, MPC $= \dfrac{d}{dI}(E_c)$

At $I = 1000$, MPC $= \boxed{}$

At $I = 1000$, MPS $= \boxed{}$

At $I = 1000$, APS $= \boxed{}$ *(March '22)*

Solution :

Given : $E_c = (0.0003)I^2 + (0.075)I$

We have APC $= \dfrac{E_c}{I} = \dfrac{(0.0003)I^2 + (0.075)I}{I}$

$\therefore$ APC $= (0.0003)I + 0.075$

At $I = 1000$, APC $= (0.0003)(1000) + 0.075 = 0.3 + 0.075 = \boxed{0.375}$

Now, MPC $= \dfrac{d}{dI}(E_c) = \dfrac{d}{dI}[(0.0003)I^2 + (0.075)I]$

$= (0.0003) \times 2I + (0.075) \times 1 = (0.0006)I + 0.075$

At $I = 1000$, MPC $= (0.0006)(1000) + 0.075 = 0.6 + 0.075 = \boxed{0.675}$

$\because$ MPC + MPS = 1 $\therefore$ 0.675 + MPS = 1

$\therefore$ At $I = 1000$, MPS $= 1 - 0.675 = \boxed{0.325}$

Also, APC + APS = 1 $\therefore$ 0.375 + APS = 1

At $I = 1000$, APS $= 1 - 0.375 = \boxed{0.625}$

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2. A metal wire of 36 cm long is bent to form a rectangle. Find its dimensions when its area is maximum by completing the following activity :

Let x and y be the length and breadth of the rectangle.

Then $x + y = \boxed{}$

Area of rectangle $= 18x - x^2$

Let $f(x)$ be the area of the rectangle.

Then $f(x) = 18x - x^2$

$\therefore f'(x) = \boxed{\quad\quad}$

$f''(x) = -2$

For extreme values, $f'(x) = 0$

$\therefore x = \boxed{\quad\quad}$

Now, $f''(9) < 0$

$\therefore f(x)$ [Area] is maximum when dimensions (length and breadth) are $\boxed{\quad\quad}$.

(July '22)

Solution : Let x and y be the length and breadth of the rectangle.

Then $2x + 2y = 36$ $\quad\quad \therefore x + y = \boxed{18}$

Area of rectangle $= xy = x(18 - x) = 18x - x^2$

Let $f(x)$ be the area of the rectangle.

Then $f(x) = 18x - x^2$

$$\therefore f'(x) = \frac{d}{dx}(18x - x^2) = 18 \times 1 - 2x$$

$$f'(x) = \boxed{18 - 2x}$$

$$\text{and } f''(x) = \frac{d}{dx}(18 - 2x) = 0 - 2 \times 1$$

$\therefore f''(x) = -2$

For extreme values, $f'(x) = 0$

$\therefore 18 - 2x = 0$

$\therefore x = \boxed{9}$

Now, $f''(9) = -2 < 0$

$\therefore f(x)$ is maximum when $x = 9$

When $x = 9, y = 18 - 9 = 9$

$\therefore f(x)$ [Area] is maximum when dimensions (length and breadth) are

$\boxed{9 \text{ cm and } 9 \text{ cm}}$.

3. The cost C for producing x articles is given as $C = x^3 - 16x^2 + 47x$. For what values of x, the average cost is decreasing?

Given : $C = x^3 - 16x^2 + 47x$

Average cost $C_A = \frac{C}{x}$

$\therefore C_A = \boxed{}$

Differentiating w.r.t. x, we get

$\frac{d}{dx}(C_A) = \boxed{}$

We know that C_A is decreasing, if

$\frac{d}{dx}(C_A) \boxed{} 0$

$\therefore 2x - 16 < 0 \qquad \therefore 2x < 16$

$\therefore x < \boxed{}$

$\therefore$ average cost is decreasing for $x \in (0, 8)$. *(July '24)*

Solution : Given : $C = x^3 - 16x^2 + 47x$

Average cost $C_A = \frac{C}{x} = \frac{x^3 - 16x^2 + 47x}{x}$

$\therefore C_A = \boxed{x^2 - 16x + 47}$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(C_A) = \frac{d}{dx}(x^2 - 16x + 47) = 2x - 16 \times 1 + 0 = \boxed{2x - 16}$$

We know that C_A is decreasing, if

$\frac{d}{dx}(C_A) \boxed{<} 0$

$\therefore 2x - 16 < 0 \qquad \therefore 2x < 16 \qquad \therefore x < \boxed{8}$

Also, $x > 0$

$\therefore$ average cost in decreasing for $x \in (0, 8)$.

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4. If the demand function is $D = 50 - 3P - P^2$, where P is the unit price, complete the following activity to find the elasticity of demand at

(a) $P = 5$ (b) $P = 2$.

The demand function is $D = 50 - 3P - P^2$.

Elasticity of demand is $\eta = -\frac{P}{D} \cdot \frac{dD}{dP}$

(a) When $P = 5$, $\eta = \boxed{}$

$\therefore$ demand is $\boxed{}$ for $P = 5$.

(b) When $P = 2$, $\eta = \boxed{}$

$\therefore$ demand is $\boxed{}$ for $P = 2$. ***(Sept. '21)***

Solution : The demand function is $D = 50 - 3P - P^2$.

$$\therefore \frac{dD}{dP} = \frac{d}{dP}(50 - 3P - P^2) = 0 - 3 \times 1 - 2P = -3 - 2P$$

Elasticity of demand is

$$\eta = -\frac{P}{D} \cdot \frac{dD}{dP} = -\frac{P}{50 - 3P - P^2} \times (-3 - 2P) = \frac{P(3 + 2P)}{50 - 3P - P^2}$$

(a) When $P = 5$, $\eta = \frac{5(3 + 10)}{50 - 15 - 25} = \frac{65}{10} = \boxed{6.5}$

$\therefore$ demand is $\boxed{\text{elastic}}$ for $P = 5$.

(b) When $P = 2$, $\eta = \frac{2(3 + 4)}{50 - 6 - 4} = \frac{14}{40} = \frac{7}{20} = \boxed{0.35}$

$\therefore$ demand is $\boxed{\text{inelastic}}$ for $P = 2$.

---

5. Divide the number 84 into two parts such that the product of one part and square of the other is maximum.

Let one part be x. Then the other part be $84 - x$.

$\therefore f(x) = \boxed{}$

$\therefore f'(x) = 168x - 3x^2$

For extreme values, $f'(x) = 0$

$\therefore 168x - 3x^2 = 0$

$\therefore 3x(56 - x) = 0$

$\therefore x = \boxed{}$ OR $\boxed{}$

$\therefore f''(x) = 168 - 6x$

If $x = 0$, $f''(0) = 168 - 6(0) = 168 > 0$

$\therefore$ function attains minimum at $x = 0$

If $x = 56$, $f''(56) = \boxed{} < 0$

$\therefore$ function attains maximum at $x = 56$

$\therefore$ two parts of 84 are $\boxed{}$ and $\boxed{}$ **(March '25)**

Solution : Let one part be x. Then the other part be $84 - x$.

$\therefore f(x) = x^2(84 - x) = \boxed{84x^2 - x^3}$

$\therefore f'(x) = \frac{d}{dx}(84x^2 - x^3) = 84 \times 2x - 3x^2 = 168x - 3x^2$

For extreme values, $f'(x) = 0$

$\therefore 168x - 3x^2 = 0 \qquad \therefore 3x(56 - x) = 0$

$\therefore x = \boxed{0}$ OR $\boxed{56}$

$f''(x) = \frac{d}{dx}(168x - 3x^2) = 168 \times 1 - 3 \times 2x = 168 - 6x$

If $x = 0$, $f''(0) = 168 - 6(0) = 168 > 0$

$\therefore$ function attains minimum at $x = 0$

If $x = 56$, $f''(56) = 168 - 6(56) = 168 - 336 = \boxed{-168} < 0$

$\therefore$ function attains maximum at $x = 56$

$\therefore$ two parts of 84 are $\boxed{56}$ and $\boxed{28}$.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The equation of tangent to the curve $y = x^2 + 4x + 1$ at $(-1, -2)$ is

 (a) $2x - y = 0$ (b) $x + 2y + 5 = 0$

 (c) $2x + 4 = 3y$ (d) $5x + y = 1$ **(July '24)**

2. The slope of a tangent to the curve $y = 3x^2 - x + 1$ at $(1, 3)$ is

 (a) 5 (b) -5 (c) $-\frac{1}{5}$ (d) $\frac{1}{5}$ **(March '22)**

3. A function f is said to be increasing at a point c, if

 (a) $f'(c) = 0$ (b) $f'(c) > 0$ (c) $f'(c) < 0$ (d) $f'(c) = 1$

 (March '23)

4. If elasticity of demand $\eta = 1$, then demand is

(a) constant (b) inelastic (c) unitary elastic (d) elastic

(July '22–'23)

5. The function $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$ is

(a) increasing for all $x \in R$, $x \neq 1$ (b) decreasing

(c) neither increasing nor decreasing (d) decreasing for all $x \in R$, $x \neq 1$

6. $f(x) = \dfrac{\log x}{x}$ is increasing in

(a) $(1, 2e)$ (b) $(0, e)$ (c) $(2, 2e)$ (d) $\left(\dfrac{1}{e}, 2e\right)$

7. If $f(x) = 3x^3 - 9x^2 - 27x + 15$, then

(a) f has maximum value 66 (b) f has minimum value 30

(c) f has maxima at $x = -1$ (d) f has minima at $x = -1$

8. The function $y = a \log x + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$, then

(a) $a = 2, b = \dfrac{1}{2}$ (b) $a = -2, b = \dfrac{1}{2}$

(c) $a = 2, b = -1$ (d) $a = 2, b = -\dfrac{1}{2}$.

Answers

1. (a) $2x - y = 0$
2. (a) 5
3. (b) $f'(c) > 0$
4. (c) unitary elastic
5. (a) increasing for all $x \in R$, $x \neq 1$
6. (b) $(0, e)$
7. (c) f has maxima at $x = -1$
8. (d) $a = 2, b = -\dfrac{1}{2}$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The equation of tangent to the curve $y = 4xe^x$ at $\left(-1, -\dfrac{4}{e}\right)$ is $y \cdot e + 4 = 0$.

2. $x + 10y + 21 = 0$ is the equation of the normal to the curve $y = 3x^2 + 4x - 5$ at $(1, 2)$.

3. If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing in the interval (a, b).

(March '22)

4. The function $f(x) = x^3 - 6x^2 + 9x - 2$ has maximum at $x = -1$.
5. A function f has minimum at $x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

Answers

1. True 2. False 3. False 4. False 5. False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. The slope of tangent at any point (a, b) is called as .... *(March '24)*
2. The function $f(x) = x^3 + 6x^2 - 36x + 7$ is decreasing, if ....
3. At $x = 3$, the function $f(x) = 2x^3 - 15x^2 + 36x + 10$ is ....
4. If the average revenue R_A is 50 and elasticity of demand η is 5, then marginal revenue $R_m =$ .... *(March '25)*
5. If $0 < \eta < 1$, then demand is .... *(March '23)*
6. If for the function $f(x)$, $f'(x) = 6x^2 - 42x + 36$, then the function $f(x)$ has minimum at $x =$ .... *(Sept. '21)*
7. If the marginal revenue $R_m = 40$ and elasticity of demand η is 5, then the average revenue R_A will be .... *(July '24)*

Answers

1. gradient 2. $-6 < x < 2$ 3. minimum 4. 40 5. relatively inelastic
6. 6 7. 50.

Question Set 5

INTEGRATION

(Marks with option : 07)

Remember :

1. **Definition :** If f and g are two functions such that $\frac{d}{dx}[g(x)] = f(x)$, then integral of $f(x)$ w.r.t. x is $g(x)+c$ and we write it as $\int f(x)\,dx = g(x)+c$, where c is the constant of integration.
2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant
4. If $\int f(x)\,dx = g(x)+c$, then $\int f(ax+b)\,dx = \frac{1}{a}\cdot g(ax+b)+c,\ a \neq 0$
5. **Integrals of standard functions :**

(1) $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c,\ (n \neq -1)$

(2) $\int 1\,dx = x+c$

(3) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c,\ (n \neq -1)$

(4) $\int a^x\,dx = \frac{a^x}{\log a}+c,\ (a \neq 1)$

(5) $\int e^x\,dx = e^x+c$

(6) $\int \frac{1}{x}\,dx = \log|x|+c$

(7) $\int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a}+c$

(8) $\int a^{px+q}\,dx = \frac{a^{px+q}}{p\cdot\log a}+c$

(9) $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a}+c$

(10) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(11) $\int \frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

(12) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx=\log\left|x+\sqrt{a^2+x^2}\right|+c$

(13) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(14) $\int \sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

(15) $\int \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

6. $\int \frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c$

7. $\int [f(x)]^n\cdot f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}+c,\ (n\neq-1)$

8. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx=2\sqrt{f(x)}+c$

9. $\int uv\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\int v\,dx\right)dx$

The order in which u and v are to be chosen, is according to the serial order of the letters of the word ***LAE***.

L : Logarithmic,

A : Algebraic and

E : Exponential functions.

10. $\int e^x[f(x)+f'(x)]\,dx=e^x\cdot f(x)+c.$

5.1 INTEGRAL OF STANDARD FUNCTIONS

Solved Examples | **3 marks each**

Ex. 1. Evaluate : $\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}}\,dx.$

Solution : $\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}}\,dx$

$$=\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}} \times \frac{\sqrt{5x-4}+\sqrt{5x-2}}{\sqrt{5x-4}+\sqrt{5x-2}}\, dx$$

$$=\int \frac{-2(\sqrt{5x-4}+\sqrt{5x-2})}{(5x-4)-(5x-2)}\, dx$$

$$=\int(\sqrt{5x-4}+\sqrt{5x-2})\, dx$$

$$=\int(5x-4)^{\frac{1}{2}}\, dx+\int(5x-2)^{\frac{1}{2}}\, dx$$

$$=\frac{(5x-4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \times \frac{1}{5}+\frac{(5x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \times \frac{1}{5}+c$$

$$=\frac{2}{15}[(5x-4)^{\frac{3}{2}}+(5x-2)^{\frac{3}{2}}]+c.$$

Ex. 2. Evaluate : $\int x^3\left(2-\frac{3}{x}\right)^2 dx.$

Solution : $\int x^3\left(2-\frac{3}{x}\right)^2 dx=\int x^3\left(4-\frac{12}{x}+\frac{9}{x^2}\right) dx$

$$=\int (4x^3-12x^2+9x)dx=4\int x^3dx-12\int x^2dx+9\int x dx$$

$$=4\times\frac{x^4}{4}-12\times\frac{x^3}{3}+9\times\frac{x^2}{2}+c$$

$$=x^4-4x^3+\frac{9}{2}x^2+c.$$

Ex. 3. Evaluate : $\int \frac{x-1}{\sqrt{x+4}}\, dx.$

Solution : $\int \frac{x-1}{\sqrt{x+4}}\, dx=\int \frac{(x+4)-5}{\sqrt{x+4}}\, dx$

$$=\int\left(\frac{x+4}{\sqrt{x+4}}-\frac{5}{\sqrt{x+4}}\right) dx=\int\left(\sqrt{x+4}-\frac{5}{\sqrt{x+4}}\right) dx$$

$$=\int (x+4)^{\frac{1}{2}}\, dx-5\int (x+4)^{-\frac{1}{2}}\, dx$$

$$=\frac{(x+4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}-5\cdot\frac{(x+4)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$= \frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c.$

Ex. 4. If $f'(x) = 4x^3 - 3x^2 + 2x + k$, $f(0) = 1$ and $f(1) = 4$, find $f(x)$.

(March '24)

Solution : By the definition of integral

$f(x) = \int f'(x)\, dx = \int (4x^3 - 3x^2 + 2x + k)\, dx$

$= 4\int x^3\, dx - 3\int x^2\, dx + 2\int x\, dx + k\int 1\, dx$

$= 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + kx + c$

$\therefore f(x) = x^4 - x^3 + x^2 + kx + c$ ... (1)

Now, $f(0) = 1$ gives

$f(0) = 0 - 0 + 0 + 0 + c = 1 \qquad \therefore c = 1$

$\therefore$ from (1), $f(x) = x^4 - x^3 + x^2 + kx + 1$ ... (2)

Further $f(1) = 4$ gives

$f(1) = 1 - 1 + 1 + k + 1 = 4 \qquad \therefore k = 2$

$\therefore$ from (2), $f(x) = x^4 - x^3 + x^2 + 2x + 1.$

Examples for Practice	3 marks each

1. Evaluate the following :

(1) $\int x^2\left(1 - \frac{2}{x}\right)^2 dx$

(2) $\int \left(x + \frac{1}{x}\right)^3 dx$

(3) $\int \frac{5(x^6 + 1)}{x^2 + 1}\, dx$

(4) $\int \frac{dx}{\sqrt{x} + \sqrt{x-2}}$

(5) $\int \frac{3x + 5}{\sqrt{2x - 1}}\, dx$

(6) $\int \frac{5x^2 - 6x + 3}{2x - 3}\, dx$

(7) $\int \frac{1}{x(x-1)}\, dx$

(8) $\int (e^{a \log x} + e^{x \log a})\, dx.$

2. If $f'(x) = 8x^3 + 3x^2 - 10x - k$, $f(0) = -3$ and $f(-1) = 0$, find $f(x)$.

Answers

1. (1) $\frac{(x-2)^3}{3} + c$

(2) $\frac{x^4}{4} + \frac{3x^2}{2} + 3 \log |x| - \frac{1}{2x^2} + c$

(3) $x^5-\frac{5}{3}x^3+5x+c$ (4) $\frac{1}{3}\left[x^{\frac{3}{2}}-(x-2)^{\frac{3}{2}}\right]+c$

(5) $\frac{1}{2}(2x-1)^{\frac{3}{2}}+\frac{13}{2}(2x-1)^{\frac{1}{2}}+c$ (6) $\frac{5}{4}x^2+\frac{3}{4}x+\frac{21}{8}\log|2x-3|+c$

(7) $\log\left|\frac{x-1}{x}\right|+c$ (8) $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c.$

2. $f(x)=2x^4+x^3-5x^2-7x-3.$

5.2 METHOD OF SUBSTITUTION

Result : If $x=\phi(t)$ is the differentiable function of t, then

$$\int f(x)\,dx=\int f[\phi(t)]\cdot\phi'(t)\,dt.$$

Solved Examples	3 marks each

Ex. 5. Evaluate : $\int\frac{x^3}{\sqrt{1+x^4}}\,dx.$

Solution : Let $I=\int\frac{x^3}{\sqrt{1+x^4}}\,dx$

Put $1+x^4=t$ $\therefore 4x^3dx=dt$ $\therefore x^3dx=\frac{dt}{4}$

$$\therefore I=\int\frac{1}{\sqrt{t}}\cdot\frac{dt}{4}=\frac{1}{4}\int t^{-\frac{1}{2}}\,dt$$

$$=\frac{1}{4}\cdot\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c=\frac{1}{2}\sqrt{1+x^4}+c.$$

Ex. 6. Evaluate : $\int\frac{e^{x-1}+x^{e-1}}{e^x+x^e}\,dx.$

Solution : Let $I=\int\frac{e^{x-1}+x^{e-1}}{e^x+x^e}\,dx$

Put $e^x+x^e=t$ $\therefore (e^x+ex^{e-1})\,dx=dt$

$\therefore e(e^{x-1}+x^{e-1})\,dx=dt$ $\therefore (e^{x-1}+x^{e-1})\,dx=\frac{dt}{e}$

$$\therefore I=\int \frac{1}{t}\cdot\frac{dt}{e}=\frac{1}{e}\int \frac{1}{t}\,dt$$

$$=\frac{1}{e}\log|t|+c=\frac{1}{e}\log|e^x+x^e|+c.$$

Ex. 7. Evaluate : $\int \frac{x^5}{x^2+1}\,dx.$

Solution : Let $I=\int \frac{x^5}{x^2+1}\,dx=\int \frac{x^4}{x^2+1}\cdot x\,dx$

$$=\int \frac{(x^2)^2}{x^2+1}\cdot x\,dx$$

Put $x^2+1=t \qquad \therefore 2x\,dx=dt$

$\therefore x\,dx=\frac{dt}{2}$ and $x^2=t-1$

$$\therefore I=\int \frac{(t-1)^2}{t}\cdot\frac{dt}{2}=\frac{1}{2}\int\left(\frac{t^2-2t+1}{t}\right)dt=\frac{1}{2}\int\left(t-2+\frac{1}{t}\right)dt$$

$$=\frac{1}{2}\int t\,dt-\int 1\,dt+\frac{1}{2}\int\frac{1}{t}\,dt=\frac{1}{2}\cdot\frac{t^2}{2}-t+\frac{1}{2}\log|t|+c$$

$$=\frac{1}{4}(x^2+1)^2-(x^2+1)+\frac{1}{2}\log|x^2+1|+c.$$

Ex. 8. Evaluate : $\int \frac{1}{x(x^6+1)}\,dx.$ ***(March '25)***

Solution : Let $I=\int \frac{1}{x(x^6+1)}\,dx=\int \frac{x^5}{x^6(x^6+1)}\,dx$

Put $x^6=t \qquad \therefore 6x^5\,dx=dt \qquad \therefore x^5\,dx=\frac{1}{6}\,dt$

$$\therefore I=\int \frac{1}{t(t+1)}\cdot\frac{dt}{6}=\frac{1}{6}\int\frac{(t+1)-t}{t(t+1)}\,dt=\frac{1}{6}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{6}\left[\int\frac{1}{t}\,dt-\int\frac{1}{t+1}\,dt\right]$$

$$= \frac{1}{6}[\log(t) - \log|t+1|] + c$$

$$= \frac{1}{6}\log\left|\frac{t}{t+1}\right| + c = \frac{1}{6}\log\left|\frac{x^6}{x^6+1}\right| + c.$$

Ex. 9. Evaluate : $\int (2x+1)\sqrt{x-4}\, dx$.

Solution : Let $I = \int (2x+1)\sqrt{x-4}\, dx$

Put $x - 4 = t$ $\quad\quad \therefore dx = dt$

Also, $x = t + 4$

$$\therefore I = \int [2(t+4)+1]\sqrt{t}\, dt$$

$$= \int (2t+9)\sqrt{t}\, dt = \int \left(2t^{\frac{3}{2}} + 9t^{\frac{1}{2}}\right) dt$$

$$= 2\int t^{\frac{3}{2}}\, dt + 9\int t^{\frac{1}{2}}\, dt = 2\cdot\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + 9\cdot\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$$

$$= \frac{4}{5}(x-4)^{\frac{5}{2}} + 6(x-4)^{\frac{3}{2}} + c.$$

Examples for Practice	3 marks each

Evaluate the following :

1. $\int \frac{3x^2}{\sqrt{1+x^3}}\, dx$

2. $\int \frac{4x-6}{(x^2-3x+5)^{\frac{3}{2}}}\, dx$

3. $\int \frac{1+x}{x+e^{-x}}\, dx$

4. $\int \frac{(x+1)(x+\log x)^4}{-3x}\, dx$

5. $\int \frac{1}{\sqrt{x}+x}\, dx$

6. $\int \frac{e^{2x}-1}{e^{2x}+1}\, dx$

7. $\int \frac{1}{2x+x^{-n}}\, dx$

8. $\int \frac{e^{3x}}{e^{3x}+1}\, dx$

9. $\int (5-3x)(2-3x)^{-\frac{1}{2}}\, dx$

10. $\int \frac{x^7}{(1+x^4)^2}\, dx.$

Answers

1. $2\sqrt{1+x^3}+c$

2. $-\dfrac{4}{\sqrt{x^2-3x+5}}+c$

3. $\log|xe^x+1|+c$

4. $-\dfrac{1}{15}(x+\log x)^5+c$

5. $2\log|1+\sqrt{x}|+c$

6. $\log|e^x+e^{-x}|+c$

7. $\dfrac{1}{2(n+1)}\log|2x^{n+1}+1|+c$

8. $\dfrac{1}{3}\log|e^{3x}+1|+c$

9. $-2\sqrt{2-3x}-\dfrac{2}{9}(2-3x)^{\frac{3}{2}}+c$

10. $\dfrac{1}{4}\log|1+x^4|+\dfrac{1}{4(1+x^4)}+c.$

Solved Examples	3 or 4 marks each

Ex. 10. Evaluate : $\displaystyle\int \frac{4e^x-25}{2e^x-5}\,dx.$

Solution : Let $I=\displaystyle\int \frac{4e^x-25}{2e^x-5}\,dx$

Put, Numerator $=A(\text{Denominator})+B\left[\dfrac{d}{dx}(\text{Denominator})\right]$

$\therefore\ 4e^x-25=A(2e^x-5)+B\left[\dfrac{d}{dx}(2e^x-5)\right]$

$=A(2e^x-5)+B(2e^x-0)$

$\therefore\ 4e^x-25=(2A+2B)\,e^x-5A$

Equating the coefficient of e^x and constant on both sides, we get

$2A+2B=4$ ... (1)

and $-5A=25 \qquad \therefore\ A=5$

$\therefore$ from (1), $2(5)+2B=4 \qquad \therefore\ 2B=-6 \qquad \therefore\ B=-3$

$\therefore\ 4e^x-25=5(2e^x-5)-3(2e^x)$

$\therefore\ I=\displaystyle\int\left[\frac{5(2e^x-5)-3(2e^x)}{2e^x-5}\right]dx$

$$=\int\left[5-\frac{3(2e^x)}{2e^x-5}\right]dx=5\int 1\,dx-3\int\frac{2e^x}{2e^x-5}\,dx$$

$$=5x-3\log|2e^x-5|+c \qquad \ldots\left[\because\int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c\right]$$

Ex. 11. Evaluate the following :

(1) $\displaystyle\int\frac{1}{4x^2-20x+17}\,dx$ (2) $\displaystyle\int\frac{1}{7+6x-x^2}\,dx$ *(March '23)*

(3) $\displaystyle\int\frac{e^x}{e^{2x}+6e^x+5}\,dx.$

Solution :

(1) $$\int\frac{1}{4x^2-20x+17}\,dx=\frac{1}{4}\int\frac{1}{x^2-5x+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int\frac{1}{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int\frac{1}{\left(x-\frac{5}{2}\right)^2-(\sqrt{2})^2}\,dx$$

$$=\frac{1}{4}\times\frac{1}{2\sqrt{2}}\log\left|\frac{x-\frac{5}{2}-\sqrt{2}}{x-\frac{5}{2}+\sqrt{2}}\right|+c$$

$$=\frac{1}{8\sqrt{2}}\log\left|\frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}}\right|+c.$$

---

(2) $$\int\frac{1}{7+6x-x^2}\,dx=\int\frac{1}{7-(x^2-6x+9)+9}\,dx$$

$$=\int\frac{1}{(4)^2-(x-3)^2}\,dx$$

$$=\frac{1}{2\times 4}\log\left|\frac{4+x-3}{4-x+3}\right|+c$$

$$=\frac{1}{8}\log\left|\frac{1+x}{7-x}\right|+c.$$

---

(3) Let $I=\int \frac{e^x}{e^{2x}+6e^x+5}dx$

Put $e^x=t \quad \therefore e^x dx=dt$

$$\therefore I=\int \frac{1}{t^2+6t+5}dt=\int \frac{1}{(t^2+6t+9)-4}dt$$

$$=\int \frac{1}{(t+3)^2-(2)^2}dt$$

$$=\frac{1}{2\times 2}\log\left|\frac{t+3-2}{t+3+2}\right|+c=\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c.$$

Ex. 12. Evaluate the following :

(1) $\int \frac{1}{\sqrt{(x-2)(x-3)}}dx$ **(2)** $\int \frac{e^x}{\sqrt{e^{2x}+4e^x+13}}dx.$ *(March '22)*

Solution :

(1) $\int \frac{1}{\sqrt{(x-2)(x-3)}}dx=\int \frac{1}{\sqrt{x^2-5x+6}}dx$

$$=\int \frac{1}{\sqrt{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+6}}dx=\int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$=\log\left|\left(x-\frac{5}{2}\right)+\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$=\log\left|\left(x-\frac{5}{2}\right)+\sqrt{x^2-5x+6}\right|+c.$$

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(2) Let $I=\int \frac{e^x}{\sqrt{e^{2x}+4e^x+13}}dx$

Put $e^x dx \quad \therefore e^x dx=dt$

$$\therefore I=\int \frac{1}{\sqrt{t^2+4t+13}}dt=\int \frac{1}{\sqrt{(t^2+4t+4)+9}}dt$$

$$=\int \frac{1}{\sqrt{(t+2)^2+(3)^2}}dt=\log|(t+2)+\sqrt{(t+2)^2+(3)^2}|+c$$

$$= \log |(t+2) + \sqrt{t^2+4t+13}| + c$$

$$= \log |(e^x+2) + \sqrt{e^{2x}+4e^x+13}| + c.$$

Examples for Practice	**3 or 4 marks each**

Evaluate the following :

1. (1) $\int \frac{20-12e^x}{3e^x-4}\,dx$ (2) $\int \frac{3e^x+4}{2e^x-8}\,dx$

2. (1) $\int \frac{1}{x^2+4x-5}\,dx$ ***(July '22)*** (2) $\int \frac{1}{2x^2+x-1}\,dx$

(3) $\int \frac{1}{1+x-x^2}\,dx$ (4) $\int \frac{1}{3-2x-x^2}\,dx$

3. (1) $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$ (2) $\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$

(3) $\int \frac{1}{\sqrt{2x^2+3x+5}}\,dx$ (4) $\int \frac{1}{\sqrt{3x^2+5x+7}}\,dx$

4. (1) $\int \frac{dx}{x\sqrt{(\log x)^2-5}}$ (2) $\int \frac{x}{4x^4-20x^2-3}\,dx$ ***(July '24)***

(3) $\int \frac{dx}{x[(\log x)^2+4\log x-1]}$ (4) $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}\,dx.$

Answers

1. (1) $-5x+\log|3e^x-4|+c$ (2) $-\frac{1}{2}x+2\log|2e^x-8|+c$

2. (1) $\frac{1}{6}\log\left|\frac{x-1}{x+5}\right|+c$ (2) $\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$

(3) $\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right|+c$ (4) $\frac{1}{4}\log\left|\frac{3+x}{1-x}\right|+c$

3. (1) $\log|(x+2)+\sqrt{x^2+4x+29}|+c$

(2) $\log|(x-4)+\sqrt{x^2-8x-20}|+c$

(3) $\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+\frac{5}{2}}\right|+c$

(4) $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right|+c$

4. (1) $\log\left|\log x+\sqrt{(\log x)^2-5}\right|+c$ (2) $\frac{1}{16\sqrt{7}}\log\left|\frac{2x^2-5-2\sqrt{7}}{2x^2-5+2\sqrt{7}}\right|+c$

(3) $\frac{1}{2\sqrt{5}}\log\left|\frac{\log x+2-\sqrt{5}}{\log x+2+\sqrt{5}}\right|+c$ (4) $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+c.$

5.3 INTEGRATION BY PARTS

Result : If u and v are functions of x, then

$$\int uv\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\int v\,dx\right)dx.$$

Solved Examples	3 or 4 marks each

Ex. 13. Evaluate : $\int x^3\log x\,dx.$ *(March '23)*

Solution : $\int x^3\log x\,dx=\int(\log x)\cdot x^3\,dx$

$$=(\log x)\int x^3\,dx-\int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx$$

$$=(\log x)\cdot\frac{x^4}{4}-\int\frac{1}{x}\cdot\frac{x^4}{4}\,dx$$

$$=\frac{1}{4}x^4\log x-\frac{1}{4}\int x^3\,dx$$

$$=\frac{1}{4}x^4\log x-\frac{1}{4}\times\frac{x^4}{4}+c$$

$$=\frac{1}{4}x^4\log x-\frac{1}{16}x^4+c.$$

Ex. 14. Evaluate : $\int x^2e^{4x}dx.$ *(Sept. '21)*

Solution : $\int x^2e^{4x}dx=x^2\int e^{4x}dx-\int\left[\frac{d}{dx}(x^2)\int e^{4x}dx\right]dx$

$$=x^2\cdot\frac{e^{4x}}{4}-\int 2x\cdot\frac{e^{4x}}{4}dx=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\int xe^{4x}dx$$

$$=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\left[x\int e^{4x}dx-\int\left\{\frac{d}{dx}(x)\int e^{4x}dx\right\}dx\right]$$

$$=\frac{1}{4}x^2e^{4x}-\frac{1}{2}\left[x\cdot\frac{e^{4x}}{4}-\int 1\cdot\frac{e^{4x}}{4}dx\right]$$

$$=\frac{1}{4}x^2e^{4x}-\frac{1}{8}xe^{4x}+\frac{1}{8}\int e^{4x}dx$$

$$=\frac{1}{4}x^2e^{4x}-\frac{1}{8}xe^{4x}+\frac{1}{8}\cdot\frac{e^{4x}}{4}+c$$

$$=\frac{1}{4}e^{4x}\left[x^2-\frac{x}{2}+\frac{1}{8}\right]+c.$$

Ex. 15. Evaluate : $\int\frac{\log(\log x)}{x}\,dx.$

Solution : Let $I=\int\frac{\log(\log x)}{x}\,dx=\int\log(\log x)\cdot\frac{1}{x}dx$

Put $\log x=t$ $\quad\therefore\ \frac{1}{x}dx=dt$

$\therefore\ I=\int\log t\,dt=\int(\log t)\cdot 1\,dt$

$$=(\log t)\int 1\,dt-\int\left[\frac{d}{dt}(\log t)\int 1\,dt\right]dt$$

$$=(\log t)\,t-\int\frac{1}{t}\times t\,dt=t\log t-\int 1\,dt$$

$$=t\log t-t+c=t(\log t-1)+c$$

$$=(\log x)\,[\log(\log x)-1]+c.$$

Ex. 16. Evaluate : $\int x^3e^{x^2}dx.$ ***(July '22)***

Solution : Let $I=\int x^3e^{x^2}dx=\int x^2e^{x^2}x\,dx$

Put $x^2=t$ $\quad\therefore\ 2x\,dx=dt$ $\quad\therefore\ x\,dx=\frac{dt}{2}$

$\therefore\ I=\int te^t\cdot\frac{dt}{2}=\frac{1}{2}\int te^tdt$

$$=\frac{1}{2}\left[t\int e^tdt-\int\left\{\frac{d}{dt}(t)\int e^tdt\right\}dt\right]$$

$$=\frac{1}{2}[te^t-\int 1\cdot e^t dt]$$

$$=\frac{1}{2}[te^t-e^t]+c=\frac{1}{2}(t-1)e^t+c$$

$$=\frac{1}{2}(x^2-1)\,e^{x^2}+c.$$

Ex. 17. Evaluate : $\int\sqrt{x^2-8x+7}\,dx.$

Solution : $\int\sqrt{x^2-8x+7}\,dx=\int\sqrt{(x^2-8x+16)-9}\,dx$

$$=\int\sqrt{(x-4)^2-(3)^2}\,dx$$

$$=\frac{(x-4)}{2}\sqrt{(x-4)^2-(3)^2}-\frac{(3)^2}{2}\log\left|(x-4)+\sqrt{(x-4)^2-(3)^2}\right|+c$$

$$=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+c.$$

Ex. 18. Evaluate : $\int e^x\cdot\frac{x+3}{(x+4)^2}\,dx.$

Solution : Let $I=\int e^x\cdot\frac{x+3}{(x+4)^2}\,dx=\int e^x\left[\frac{(x+4)-1}{(x+4)^2}\right]dx$

$$=\int e^x\left[\frac{1}{x+4}-\frac{1}{(x+4)^2}\right]dx$$

Put $f(x)=\frac{1}{x+4}$

Then $f'(x)=\frac{d}{dx}(x+4)^{-1}=-1(x+4)^{-2}\cdot\frac{d}{dx}(x+4)$

$$=-\frac{1}{(x+4)^2}\times(1+0)=-\frac{1}{(x+4)^2}$$

$\therefore\ I=\int e^x\,[f(x)+f'(x)]\,dx$

$$=e^x\cdot f(x)+c=e^x\cdot\frac{1}{x+4}+c=\frac{e^x}{x+4}+c.$$

Ex. 19. Evaluate : $\int \frac{\log x}{(1+\log x)^2}\,dx.$

Solution : Let $I=\int \frac{\log x}{(1+\log x)^2}\,dx$

Put $\log x = t \quad \therefore x = e^t \quad \therefore dx = e^t\,dt$

$$\therefore I=\int \frac{t}{(1+t)^2}\cdot e^t\,dt=\int e^t\left[\frac{(1+t)-1}{(1+t)^2}\right]dt$$

$$=\int e^t\left[\frac{1}{1+t}-\frac{1}{(1+t)^2}\right]dt$$

Let $f(t)=\frac{1}{1+t}$.

$$\therefore f'(t)=\frac{d}{dt}(1+t)^{-1}=-1(1+t)^{-2}(0+1)=-\frac{1}{(1+t)^2}$$

$$\therefore I=\int e^t[f(t)+f'(t)]\,dt$$

$$=e^t\cdot f(t)+c=e^t\times\frac{1}{1+t}+c=\frac{x}{1+\log x}+c.$$

Examples for Practice	3 marks each

Evaluate the following :

1. $\int x\log x\,dx$
2. $\int e^{\sqrt{x}}dx$
3. $\int x^2e^{3x}dx$
4. $\int \log x\,dx$
5. $\int (\log x)^2\,dx$
6. $\int e^x\cdot\frac{(x-1)}{(x+1)^3}\,dx$
7. $\int e^x\cdot\frac{1+x}{(2+x)^2}\,dx$
8. $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$
9. $\int\sqrt{x^2+2x+5}\,dx$
10. $\int e^x\sqrt{e^{2x}+1}\,dx.$

Answers

1. $\frac{x^2}{2}\log x-\frac{x^2}{4}+c$
2. $2(\sqrt{x}-1)\,e^{\sqrt{x}}+c$

3. $\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+c$

4. $x\log x-x+c$

5. $x(\log x)^2-2x\log x+2x+c$

6. $\frac{e^x}{(x+1)^2}+c$

7. $\frac{e^x}{(2+x)}+c$

8. $\frac{x}{\log x}+c$

9. $\frac{(x+1)}{2}\sqrt{x^2+2x+5}+2\log|(x+1))+\sqrt{x^2+2x+5}|+c$

10. $\frac{e^x}{2}\cdot\sqrt{e^{2x}+1}+\frac{1}{2}\log|e^x+\sqrt{e^{2x}+1}|+c.$

5.4 INTEGRATION BY METHOD OF PARTIAL FRACTIONS

Remember :

Integral of the type $\int\frac{P(x)}{Q(x)}dx$, where

(i) $P(x)$ and $Q(x)$ are polynomials in x

(ii) degree $P(x)<$ degree $Q(x)$

(iii) no common polynomial factors in $P(x)$ and $Q(x)$.

Case (i) : If the denominator $Q(x)$ consists of distinct linear factors,

i.e. $\frac{P(x)}{Q(x)}=\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}$, then we need to find the constants $A_1, A_2, A_3, \ldots, A_n$ such that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\ldots+\frac{A_n}{a_nx+b_n}$$

Case (ii) : If the denominator has repeated linear factor,

i.e. $Q(x)=(x-a)^k(x-a_1)(x-a_2)\ldots(x-a_r)$, then we assume

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a_1}+\frac{B_2}{x-a_2}+\ldots+\frac{B_r}{x-a_r}$$

where $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_r$ are constants.

Case (iii) : If the denominator $Q(x)$ has non-repeated quadratic factors, then corresponding to each quadratic factor ax^2+bx+c, we assume the partial fraction $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants.

> **Remark :** If degree $P(x) >$ degree $Q(x)$, then divide $P(x)$ by $Q(x)$ till the degree of remainder $f(x)$ is less than $Q(x)$.
>
> $\therefore \dfrac{P(x)}{Q(x)} = r(x) + \dfrac{f(x)}{Q(x)}$, where $r(x)$ is the quotient.

Solved Examples	**3 or 4 marks each**

Ex. 20. Evaluate : $\displaystyle\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\,dx$

Solution : Let $I = \displaystyle\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\,dx$

Let $\dfrac{x^2+2}{(x-1)(x+2)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

$\therefore x^2+2 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$ ... (1)

Put $x-1=0$, i.e. $x=1$ in (1), we get

$1+2 = A(3)(4) + B(0)(4) + C(0)(3)$

$\therefore 3 = 12A \qquad \therefore A = \dfrac{1}{4}$

Put $x+2=0$, i.e. $x=-2$ in (1), we get

$4+2 = A(0)(1) + B(-3)(1) + C(-3)(0)$

$\therefore 6 = -3B \qquad \therefore B = -2$

Put $x+3=0$, i.e. $x=-3$ in (1), we get

$9+2 = A(-1)(0) + B(-4)(0) + C(-4)(-1)$

$\therefore 11 = 4C \qquad \therefore C = \dfrac{11}{4}$

$$\therefore \frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{\left(\frac{1}{4}\right)}{x-1} + \frac{(-2)}{x+2} + \frac{\left(\frac{11}{4}\right)}{x+3}$$

$$\therefore I = \int \left[\frac{\left(\frac{1}{4}\right)}{x-1} + \frac{(-2)}{x+2} + \frac{\left(\frac{11}{4}\right)}{x+3}\right] dx$$

$$=\frac{1}{4}\int\frac{1}{x-1}\,dx-2\int\frac{1}{x+2}\,dx+\frac{11}{4}\int\frac{1}{x+3}\,dx$$

$$=\frac{1}{4}\log|x-1|-2\log|x+2|+\frac{11}{4}\log|x+3|+c.$$

Ex. 21. Evaluate : $\int\frac{x}{(x-1)^2(x+2)}\,dx.$ ***(July '23)***

Solution : Let $I=\int\frac{x}{(x-1)^2(x+2)}\,dx$

Let $\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$

$\therefore\ x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$ ... (1)

Put $x-1=0$, i.e. $x=1$ in (1), we get

$1=A(0)(3)+B(3)+C(0)$ $\therefore\ B=\frac{1}{3}$

Put $x+2=0$, i.e. $x=-2$ in (1), we get

$-2=A(-3)(0)+B(0)+C(9)$ $\therefore\ C=-\frac{2}{9}$

Put $x=-1$ in (1), we get

$-1=A(-2)(1)+B(1)+C(4)$

But $B=\frac{1}{3}$ and $C=-\frac{2}{9}$

$\therefore\ -1=-2A+\frac{1}{3}-\frac{8}{9}$ $\therefore\ 2A=-\frac{5}{9}+1=\frac{4}{9}$ $\therefore\ A=\frac{2}{9}$

$$\therefore\ \frac{x}{(x-1)^2(x+2)}=\frac{(2/9)}{x-1}+\frac{(1/3)}{(x-1)^2}+\frac{(-2/9)}{x+2}$$

$$\therefore\ I=\int\left[\frac{(2/9)}{x-1}+\frac{(1/3)}{(x-1)^2}+\frac{(-2/9)}{x+2}\right]dx$$

$$=\frac{2}{9}\int\frac{1}{x-1}\,dx+\frac{1}{3}\int(x-1)^{-2}dx-\frac{2}{9}\int\frac{1}{x+2}\,dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\cdot\frac{(x-1)^{-1}}{-1}-\frac{2}{9}\log|x+2|+c$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + c.$$

Ex. 22. Evaluate : $\displaystyle\int \frac{1+\log x}{x(3+\log x)(2+3\log x)}\, dx.$

Solution : Let $I = \displaystyle\int \frac{1+\log x}{x(3+\log x)(2+3\log x)}\, dx$

$$= \int \frac{1+\log x}{(3+\log x)(2+3\log x)} \cdot \frac{1}{x}\, dx$$

Put $\log x = t$ $\quad \therefore \frac{1}{x} dx = dt$

$$\therefore I = \int \frac{1+t}{(3+t)(2+3t)}\, dt$$

Let $\dfrac{1+t}{(3+t)(2+3t)} = \dfrac{A}{3+t} + \dfrac{B}{2+3t}$

$\therefore 1+t = A(2+3t) + B(3+t)$ ... (1)

Put $3+t=0$, i.e. $t=-3$ in (1), we get

$1-3 = A(-7) + B(0)$

$\therefore -2 = -7A \quad \therefore A = \dfrac{2}{7}$

Put $2+3t=0$, i.e. $t = -\dfrac{2}{3}$ in (1), we get

$1 - \dfrac{2}{3} = A(0) + B\left(\dfrac{7}{3}\right)$

$\therefore \dfrac{1}{3} = \dfrac{7}{3} B \quad \therefore B = \dfrac{1}{7}$

$$\therefore \frac{1+t}{(3+t)(2+3t)} = \frac{\left(\frac{2}{7}\right)}{3+t} + \frac{\left(\frac{1}{7}\right)}{2+3t}$$

$$\therefore I = \int \left[\frac{\left(\frac{2}{7}\right)}{3+t} + \frac{\left(\frac{1}{7}\right)}{2+3t} \right] dt$$

$$=\frac{2}{7}\int\frac{1}{3+t}\,dt+\frac{1}{7}\int\frac{1}{2+3t}\,dt=\frac{2}{7}\log|3+t|+\frac{1}{7}\cdot\frac{\log|2+3t|}{3}+c$$

$$=\frac{2}{7}\log|3+\log x|+\frac{1}{21}\log|2+3\log x|+c.$$

Examples for Practice	3 or 4 marks each

Evaluate the following :

1. $\int\frac{2x+1}{(x+1)(x-2)}\,dx$
2. $\int\frac{2x+1}{x(x-1)(x-4)}\,dx$ *(July '24)*
3. $\int\frac{3x-1}{2x^2-x-1}\,dx$
4. $\int\frac{3x+1}{(x-2)^2(x+2)}\,dx$
5. $\int\frac{3x-2}{(x+1)^2(x+3)}\,dx$
6. $\int\frac{x^2+x+1}{(x-1)^3}\,dx$
7. $\int\frac{1}{x(x^5+1)}\,dx$
8. $\int\frac{\log x}{x(1+\log x)(2+\log x)}\,dx.$

Answers

1. $\frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+c$
2. $\frac{1}{4}\log|x|-\log|x-1|+\frac{3}{4}\log|x-4|+c$
3. $\frac{2}{3}\log|x-1|+\frac{5}{6}\log|2x+1|+c$
4. $\frac{5}{16}\log\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$
5. $\frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+c$
6. $\log|x-1|-\frac{3}{x-1}-\frac{3}{2(x-1)^2}+c$
7. $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c$
8. $2\log|2+\log x|-\log|1+\log x|+c.$

ACTIVITIES	4 marks each

1. Complete the following activity :

Let $I=\int x^5\cdot\sqrt{a^3+x^3}\,dx$

$=\int x^3\cdot\sqrt{a^3+x^3}\cdot\boxed{}\,dx$

Put $a^3+x^3=t \quad \therefore x^2dx=\frac{dt}{3}$

$$\therefore I=\int (t-a^3)\sqrt{t}\cdot\frac{dt}{3}=\frac{1}{3}\int\left(t^{\frac{3}{2}}-a^3\boxed{}\right)dt$$

$$=\frac{1}{3}\cdot\left[\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}-a^3\boxed{}\right]+c=\frac{2}{15}(a^3+x^3)^{\frac{5}{2}}-\boxed{}+c$$

Solution : Let $I=\int x^5\cdot\sqrt{a^3+x^3}\,dx=\int x^3\cdot\sqrt{a^3+x^3}\,\boxed{x^2}\,dx$

Put $a^3+x^3=t \quad \therefore 3x^2dx=dt \quad \therefore x^2dx=\frac{dt}{3}$

$$\therefore I=\int (t-a^3)\sqrt{t}.\frac{dt}{3}=\frac{1}{3}\int\left(t^{\frac{3}{2}}-a^3\boxed{t^{\frac{1}{2}}}\right)dt$$

$$=\frac{1}{3}\int t^{\frac{3}{2}}dt-a^3\int t^{\frac{1}{2}}dt$$

$$=\frac{1}{3}.\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}-a^3.\boxed{\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}}+c$$

$$=\frac{2}{15}(a^3+x^3)^{\frac{5}{2}}-\boxed{\frac{2a^3}{3}(a^3+x^3)^{\frac{3}{2}}}+c.$$

---

2. Complete the following activity :

$$\int x^2\log x\,dx=\int(\log x)\cdot x^2\,dx$$

$$=(\log x)\int x^2\,dx-\int\left[\frac{d}{dx}(\log x)\int x^2\,dx\right]dx$$

$$=(\log x)\cdot\frac{x^3}{3}-\int\boxed{}\,dx$$

$$=(\log x)\cdot\frac{x^3}{3}-\frac{1}{3}\int\boxed{}\,dx$$

$$=\frac{1}{3}x^3\log x-\boxed{}+c=\frac{x^3}{9}\left(\boxed{}\right)+c.$$

Solution : $\int x^2 \log x\,dx = \int (\log x)\cdot x^2\,dx$

$$= (\log x)\int x^2\,dx - \int\left[\frac{d}{dx}(\log x)\int x^2\,dx\right]dx$$

$$= (\log x)\cdot\frac{x^3}{3} - \int \boxed{\frac{1}{x}\cdot\frac{x^3}{3}}\,dx$$

$$= (\log x)\cdot\frac{x^3}{3} - \frac{1}{3}\int \boxed{x^2}\,dx$$

$$= \frac{1}{3}x^3\log x - \boxed{\frac{1}{3}\cdot\frac{x^3}{3}} + c$$

$$= \frac{x^3}{9}\left(\boxed{3\log x - 1}\right) + c.$$

---

3. Complete the following activity :

Let $I = \int \dfrac{2x+8}{\sqrt{x^2+6x+13}}\,dx$

Let $2x+8 = A\dfrac{d}{dx}(x^2+6x+13) + B$

$\therefore\ 2x+8 = A(2x+6) + B$

$\therefore\ A = \boxed{},\ B = \boxed{}$

$$\therefore\ I = \int \frac{2x+6}{\sqrt{x^2+6x+13}}\,dx + \int \frac{\boxed{}}{\sqrt{x^2+6x+13}}\,dx$$

$$= 2\sqrt{x^2+6x+13} + 2\,\boxed{} + c \qquad \ldots\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \boxed{} + c\right]$$

Solution : Let $I = \int \dfrac{2x+8}{\sqrt{x^2+6x+13}}\,dx$

Let $2x+8 = A\dfrac{d}{dx}(x^2+6x+13) + B$

$\therefore\ 2x+8 = A(2x+6) + B$

$\therefore\ 2x+8 = 2Ax + (6A+B)$

$\therefore\ 2 = 2A$ and $8 = 6A+B$

$\therefore\ A = \boxed{1}$ and $8 = 6+B$ $\quad\therefore\ B = \boxed{2}$

$\therefore\ 2x+8 = (2x+6) + 2$

$$\therefore I=\int \frac{(2x+6)+2}{\sqrt{x^2+6x+13}}\,dx$$

$$=\int \frac{2x+6}{\sqrt{x^2+6x+13}}\,dx+\int \frac{\boxed{2}}{\sqrt{x^2+6x+13}}\,dx$$

$$=\int \frac{2x+6}{\sqrt{x^2+6x+13}}\,dx+2\int \frac{1}{\sqrt{(x+3)^2+(2)^2}}\,dx$$

$$=2\sqrt{x^2+6x+13}+2\,\boxed{\log|(x+3)+\sqrt{x^2+6x+13}|}+c$$

$$\ldots\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx=\boxed{2\sqrt{f(x)}}+c\right]$$

---

4. Complete the following activity :

Let $I=\int \frac{2x}{x^2-5x+4}\,dx$

Let $\frac{2x}{x^2-5x+4}=\frac{2x}{(x-1)(x-4)}=\frac{A}{x-1}+\frac{B}{x-4}$

$\therefore 2x=A(x-4)+B(x-1)$

$\therefore A=\boxed{}, B=\boxed{}$

$\therefore I=\int\left[\frac{\boxed{}}{(x-1)}+\frac{\boxed{}}{(x-4)}\right]dx=\boxed{}+\boxed{}+c.$

Solution : Let $I=\int \frac{2x}{x^2-5x+4}\,dx$

Let $\frac{2x}{x^2-5x+4}=\frac{2x}{(x-1)(x-4)}=\frac{A}{x-1}+\frac{B}{x-4}$

$\therefore 2x=A(x-4)+B(x-1)$ ... (1)

Put $x-1=0$, i.e. $x=1$ in (1), we get

$2=A(-3)+B(0)$ $\quad\therefore A=\boxed{-\frac{2}{3}}$

Put $x-4=0$, i.e. $x=4$ in (1), we get

$8=A(0)+B(3)$ $\quad\therefore B=\boxed{\frac{8}{3}}$

$$\therefore \frac{2x}{x^2-5x+4}=\frac{\left(-\frac{2}{3}\right)}{x-1}+\frac{\left(\frac{8}{3}\right)}{x-4}$$

$$\therefore I=\int\left[\frac{\boxed{-\frac{2}{3}}}{(x-1)}+\frac{\boxed{\frac{8}{3}}}{(x-4)}\right]dx=-\frac{2}{3}\int\frac{1}{x-1}\,dx+\frac{8}{3}\int\frac{1}{x-4}\,dx$$

$$=\boxed{-\frac{2}{3}\log|x-1|}+\boxed{\frac{8}{3}\log|x-4|}+c.$$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The value of $\int\frac{dx}{\sqrt{1-x}}$ is

(a) $2\sqrt{1-x}+c$ (b) $-2\sqrt{1-x}+c$

(c) $\sqrt{x}+c$ (d) $x+c$ *(March '24)*

2. $\int(1-x)^{-2}dx=$ ....

(a) $(1+x)^{-1}+c$ (b) $(1-x)^{-1}-x+c$

(c) $(1-x)^{-1}+c$ (d) $(1-x)^{-1}+x+c$ *(July '23)*

3. $\int\frac{1}{x^2-9}\,dx=$ ....

(a) $\log\left|\frac{x-3}{x+3}\right|+c$ (b) $\frac{1}{3}\log\left|\frac{x-3}{x+3}\right|+c$

(c) $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+c$ (d) $\frac{1}{9}\log\left|\frac{x-3}{x+3}\right|+c$ *(Sept. '21)*

4. $\int\frac{1}{\sqrt{x^2-9}}\,dx=$ ....

(a) $\frac{1}{3}\log|x+\sqrt{x^2-9}|+c$ (b) $\log|x+\sqrt{x^2-9}|+c$

(c) $3\log|x+\sqrt{x^2-9}|+c$ (d) $\log|x-\sqrt{x^2-9}|+c$ *(March '22)*

5. $\int \frac{dx}{(x-8)(x+7)} =$ ....

(a) $\frac{1}{15}\log\left|\frac{x+2}{x+1}\right|+c$

(b) $\frac{1}{15}\log\left|\frac{x+8}{x+7}\right|+c$

(c) $\frac{1}{15}\log\left|\frac{x-8}{x+7}\right|+c$

(d) $(x-8)(x+7)+c$ ***(March '24)***

6. $\int \left(\frac{e^{2x}+e^{-2x}}{e^{x}}\right)dx =$ ....

(a) $e^{x}-\frac{1}{3e^{3x}}+c$

(b) $e^{x}+\frac{1}{3e^{3x}}+c$

(c) $e^{-x}+\frac{1}{3e^{3x}}+c$

(d) $e^{-x}-\frac{1}{3e^{3x}}+c$

7. $\int \left(x+\frac{1}{x}\right)^{3} dx =$ ....

(a) $\frac{1}{4}\left(x+\frac{1}{x}\right)^{4}+c$

(b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

(c) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x+\frac{1}{x^2}+c$

(d) $(x+x^{-1})^{3}+c$ ***(March '25)***

8. $\int \log x \, dx =$ ....

(a) $\frac{1}{2}(\log x)^2+c$

(b) $x\log x-x+c$

(c) $x\log x+x+c$

(d) $2(\log x)^2+c$

Answers

1. (b) $-2\sqrt{1-x}+c$

2. (c) $(1-x)^{-1}+c$

3. (c) $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+c$

4. (b) $\log|x+\sqrt{x^2-9}|+c$

5. (c) $\frac{1}{15}\log\left|\frac{x-8}{x+7}\right|+c$

6. (a) $e^{x}-\frac{1}{3e^{3x}}+c$

7. (b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

8. (b) $x\log x-x+c$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. $\int e^x [x^2+2x]\, dx = x^2 e^x + c.$ **(Sept. '21)**

2. If $\int x f(x)\, dx = \frac{f(x)}{2}$, then $f(x) = e^{x^2}$.

3. $\int (1-x)^{-2} dx = (1-x)^{-1} + c.$ **(March '23)**

4. For $\int \frac{x-1}{(x+1)^3} e^x\, dx = e^x \cdot f(x) + c,\ f(x) = (x+1)^2.$ **(March '24)**

5. If $\int \frac{4e^x - 25}{2e^x - 5}\, dx = Ax - 3 \log |2e^x - 5| + c$, where c is the constant of integration, then $A = 5$. **(March '22)**

6. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = \sqrt{f(x)} + c.$ **(July '23)**

7. $\int \log x\, dx = x \log x + x + c.$ **(March '25)**

Answers

1. True **2.** True **3.** True **4.** False **5.** True **6.** False **7.** False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. $\int \frac{5(x^6+1)}{x^2+1} dx = x^5 + \ldots\ldots + 5x + c.$

2. If $f'(x) = \frac{1}{x} + x$ and $f(1) = \frac{5}{2}$, then $f(x) = \log x + \frac{x^2}{2} + \ldots\ldots\ldots$ **(March '24)**

3. $\int \frac{1+x}{x+e^{-x}} dx = \log |\ldots\ldots\ldots + 1| + c$, where c is the constant of integration. **(Sept. '21)**

4. $\int \frac{1}{x^3} [\log x^x]^2\, dx = p(\log x)^3 + c$, then $p = \ldots\ldots\ldots$

5. $\int \frac{x}{(x+2)(x+3)} dx = \ldots\ldots\ldots + \int \frac{3}{x+3} dx$ **(March '22)**

6. To find the value of $\int \frac{(1+\log x)}{x}\,dx$, the proper substitution is ....

(July '23)

7. $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=$ .... $+\,c$. (March '25)

8. If $f'(x)=x^2+5$ and $f(0)=-1$, then $f(x)=$ .... (March '25)

Answers

1. $-\frac{5}{3}x^3$ 2. 2   3. xe^x 4. $\frac{1}{3}$ 5. $\int \frac{-2}{x+2}\,dx$

6. $1+\log x=t$ or $\log x=t$ 7. $\frac{e^x}{x}$ 8. $\frac{x^3}{3}+5x-1$.

Question Set 6

DEFINITE INTEGRATION

(Marks with option : 05)

6.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and $\int f(x)\,dx = g(x)$, then

$$\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a).$$

This result is known as the *fundamental theorem of integral calculus.*

The number $g(b) - g(a)$ is called the definite integral of f over the interval $[a, b]$.

Solved Examples	3 or 4 marks each

Ex. 1. Evaluate : $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx.$

Solution : $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}} \times \frac{\sqrt{1+x}-\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\,dx$$

$$= \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{1+x-x}\,dx$$

$$= \int_0^1 (\sqrt{1+x}-\sqrt{x})\,dx = \int_0^1 (1+x)^{\frac{1}{2}}\,dx - \int_0^1 x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(1+x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 - \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$$

$$=\frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^1-\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1=\frac{2}{3}(2^{\frac{3}{2}}-1)-\frac{2}{3}(1-0)$$

$$=\frac{2}{3}(2^{\frac{3}{2}}-1-1)=\frac{2}{3}(2\sqrt{2}-2)=\frac{4}{3}(\sqrt{2}-1).$$

Ex. 2. Evaluate : $\displaystyle\int_1^2 \frac{x+3}{x(x+2)}\,dx.$

Solution : Let $I=\displaystyle\int_1^2 \frac{x+3}{x(x+2)}\,dx$

Let $\dfrac{x+3}{x(x+2)}=\dfrac{A}{x}+\dfrac{B}{x+2}$

$\therefore\ x+3=A(x+2)+Bx$ ... (1)

Put $x=0$ in (1), we get

$3=A(2)+B(0) \qquad \therefore\ A=\dfrac{3}{2}$

Put $x+2=0$, i.e. $x=-2$ in (1), we get

$-2+3=A(0)+B(-2)$

$\therefore\ 1=-2B \qquad \therefore\ B=-\dfrac{1}{2}$

$$\therefore\ \frac{x+3}{x(x+2)}=\frac{\left(\frac{3}{2}\right)}{x}+\frac{\left(-\frac{1}{2}\right)}{x+2}$$

$$\therefore\ I=\int_1^2\left[\frac{\left(\frac{3}{2}\right)}{x}+\frac{\left(-\frac{1}{2}\right)}{x+2}\right]dx=\frac{3}{2}\int_1^2\frac{1}{x}\,dx-\frac{1}{2}\int_1^2\frac{1}{x+2}\,dx$$

$$=\frac{3}{2}\left[\log|x|\right]_1^2-\frac{1}{2}\left[\log|x+2|\right]_1^2$$

$$=\frac{3}{2}(\log 2-\log 1)-\frac{1}{2}(\log 4-\log 3)$$

$$=\frac{3}{2}\log 2-\frac{1}{2}\log 4+\frac{1}{2}\log 3$$ ... [$\because\ \log 1=0$]

$$= \frac{1}{2}(3 \log 2 - \log 4 + \log 3) = \frac{1}{2}(\log 8 - \log 4 + \log 3)$$

$$= \frac{1}{2} \log\left(\frac{8 \times 3}{4}\right) = \frac{1}{2} \log 6.$$

Ex. 3. Evaluate : $\int_1^3 \log x dx.$

Solution : $\int_1^3 x^2 \log x \, dx = \int_1^3 (\log x) \cdot x^2 \, dx$

$$= [(\log x) \int x^2 dx]_1^3 - \int_1^3 \left[\frac{d}{dx}(\log x) \int x^2 dx\right] dx$$

$$= \left[(\log x)\left(\frac{x^3}{3}\right)\right]_1^3 - \int_1^3 \frac{1}{x} \times \frac{x^3}{3} dx$$

$$= \frac{1}{3}[x^3 \log x]_1^3 - \frac{1}{3} \int_1^3 x^2 dx$$

$$= \frac{1}{3}[27 \log 3 - 0] - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^3 \qquad \ldots [\because \log 1 = 0]$$

$$= 9 \log 3 - \frac{1}{9}(27 - 1) = 9 \log 3 - \frac{26}{9}.$$

Ex. 4. Evaluate : $\int_0^1 e^{x^2} \cdot x^3 \, dx.$

Solution : Let $I = \int_0^1 e^{x^2} \cdot x^3 \, dx = \int_0^1 e^{x^2} \cdot x^2 \cdot x dx$

Put $x^2 = t \qquad \therefore 2x dx = dt \qquad \therefore x dx = \frac{dt}{2}$

When $x = 0, t = 0$

When $x = 1, t = 1$

$$\therefore I = \int_0^1 e^t \cdot t \cdot \frac{dt}{2} = \frac{1}{2} \int_0^1 t e^t \, dt$$

$$= \frac{1}{2}\left\{[t \int e^t \, dt]_0^1 - \int_0^1 \left[\frac{d}{dt}(t) \int e^t \, dt\right] dt\right\}$$

$$= \frac{1}{2}[te^t]_0^1 - \frac{1}{2} \int_0^1 1 \cdot e^t \, dt = \frac{1}{2}(e - 0) - \frac{1}{2}[e^t]_0^1$$

$$= \frac{e}{2} - \frac{1}{2}(e-1) = \frac{e}{2} - \frac{e}{2} + \frac{1}{2} = \frac{1}{2}.$$

Ex. 5. Evaluate : $\displaystyle\int_1^3 \frac{1}{x(1+x^2)}\,dx.$

Solution :

$$\text{Let } I = \int_1^3 \frac{1}{x(1+x^2)}\,dx = \int_1^3 \frac{(1+x^2)-x^2}{x(1+x^2)}\,dx = \int_1^3 \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

$$= \int_1^3 \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= [\log|x|]_1^3 - \frac{1}{2}[\log|1+x^2|]_1^3$$

$$\ldots\left[\because \frac{d}{dx}(1+x^2) = 2x \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= (\log 3 - \log 1) - \frac{1}{2}(\log 10 - \log 2) = \log 3 - \frac{1}{2}\log\left(\frac{10}{2}\right)$$

$$= \log 3 - \frac{1}{2}\log 5 = \log 3 - \log\sqrt{5} = \log\left(\frac{3}{\sqrt{5}}\right).$$

Ex. 6. Evaluate : $\displaystyle\int_1^2 e^{2x}\left(\frac{1}{x} - \frac{1}{2x^2}\right)dx.$

Solution : Let $I = \displaystyle\int_1^2 e^{2x}\left(\frac{1}{x} - \frac{1}{2x^2}\right)dx$

Put $2x = t$ $\quad \therefore 2dx = dt \quad \therefore dx = \frac{dt}{2}$ and $x = \frac{t}{2}$

When $x = 1, t = 2$ $\quad$ When $x = 2, t = 4$

$$\therefore I = \int_2^4 e^t\left(\frac{2}{t} - \frac{2}{t^2}\right)\frac{dt}{2} = \frac{1}{2}\int_2^4 e^t\left(\frac{2}{t} - \frac{2}{t^2}\right)dt$$

Let $f(t) = \frac{2}{t}$, then $f'(t) = 2\left(-\frac{1}{t^2}\right) = -\frac{2}{t^2}$

$$\therefore I=\frac{1}{2}\int_{2}^{4} e^{t}[f(t)+f'(t)]\,dt$$

$$=\frac{1}{2}[e^{t}\cdot f(t)]_{2}^{4}=\frac{1}{2}\left[e^{t}\cdot\frac{2}{t}\right]_{2}^{4}$$

$$=\frac{1}{2}\left[e^{4}\times\frac{2}{4}-e^{2}\times\frac{2}{2}\right]=\frac{e^{4}}{4}-\frac{e^{2}}{2}.$$

Ex. 7. Evaluate the following :

(1) $\displaystyle\int_{1}^{2}\frac{dx}{x^{2}+6x-5}$ *(March '25)* (2) $\displaystyle\int_{0}^{1}\frac{dx}{\sqrt{x^{2}-x+1}}$.

Solution :

(1) $$\int_{1}^{2}\frac{dx}{x^{2}+6x+5}=\int_{1}^{2}\frac{dx}{(x^{2}+6x+9)-4}=\int_{1}^{2}\frac{1}{(x+3)^{2}-(2)^{2}}\,dx$$

$$=\frac{1}{2(2)}\left[\log\left|\frac{x+3-2}{x+3+2}\right|\right]_{1}^{2}=\frac{1}{4}\left[\log\left|\frac{x+1}{x+5}\right|\right]_{1}^{2}$$

$$=\frac{1}{4}\left[\log\frac{3}{7}-\log\frac{2}{6}\right]$$

$$=\frac{1}{4}\log\left(\frac{3}{7}\times\frac{6}{2}\right)=\frac{1}{4}\log\left(\frac{9}{7}\right).$$

---

(2) $$\int_{0}^{1}\frac{dx}{\sqrt{x^{2}-x+1}}=\int_{0}^{1}\frac{dx}{\sqrt{\left(x^{2}-x+\frac{1}{4}\right)+\frac{3}{4}}}=\int_{0}^{1}\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^{2}+\left(\frac{\sqrt{3}}{2}\right)^{2}}}$$

$$=\left[\log\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^{2}+\left(\frac{\sqrt{3}}{2}\right)^{2}}\right|\right]_{0}^{1}$$

$$=\left[\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^{2}-x+1}\right|\right]_{0}^{1}$$

$$=\log\left|\frac{1}{2}+\sqrt{1-1+1}\right|-\log\left|-\frac{1}{2}+\sqrt{1}\right|$$

$$=\log\frac{3}{2}-\log\frac{1}{2}=\log 3.$$

Ex. 8. If $\int_0^1 (3x^2+2x+a)\,dx = 0$, find a.

Solution : $\int_0^1 (3x^2+2x+a)\,dx = 0$

$\therefore 3\int_0^1 x^2 dx + 2\int_0^1 x\,dx + a\int_0^1 1\,dx = 0$

$\therefore 3\left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^2}{2}\right]_0^1 + a\,[x]_0^1 = 0$

$\therefore [x^3]_0^1 + [x^2]_0^1 + a\,[x]_0^1 = 0$

$\therefore (1-0)+(1-0)+a\,(1-0) = 0$

$\therefore 1+1+a = 0 \qquad \therefore a = -2.$

Examples for Practice	**3 or 4 marks each**

1. Evaluate the following :

(1) $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}\,dx$ (2) $\int_3^5 \frac{dx}{\sqrt{x+4}+\sqrt{x-2}}$ (3) $\int_1^2 \frac{3x}{9x^2-1}\,dx$

(4) $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$ (5) $\int_2^3 \frac{x}{(x+2)(x+3)}\,dx$ ***(Sept. '21)*** (6) $\int_0^2 \frac{1}{4+x+x^2}\,dx$

(7) $\int_1^2 \frac{\log x}{x^2}\,dx$ (8) $\int_1^3 \log x\,dx$ ***(March '22; July '24)*** (9) $\int_2^3 \frac{dx}{x(x^3-1)}\,dx$

(10) $\int_2^3 \frac{x}{x^2-1}\,dx$ (11) $\int_1^2 \frac{1}{(x+1)(x+3)}\,dx$ (12) $\int_1^2 \frac{dx}{x(1+\log x)^2}$

2. If $f(x) = a+bx+cx^2$, show that $\int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$.

Answers

1. (1) $\frac{32}{5}$ (2) $\frac{1}{9}(28-3\sqrt{3}-7\sqrt{7})$ (3) $\frac{1}{6}\log\left(\frac{35}{8}\right)$ (4) $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$

(5) $\log\left(\frac{3456}{3125}\right)$ (6) $\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$ (7) $\frac{1}{2}\log\left(\frac{e}{2}\right)$

(8) $9\log 3-\frac{26}{9}$ (9) $\frac{1}{3}\log\left(\frac{208}{189}\right)$ (10) $\frac{1}{2}\log\left(\frac{8}{3}\right)$ (11) $\frac{1}{2}\log\left(\frac{6}{5}\right)$

(12) $\frac{\log 2}{1+\log 2}$.

6.2 PROPERTIES OF DEFINITE INTEGRALS

Property 1 : $\int_a^a f(x)\,dx=0$

Property 2 : $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

Property 3 : $\int_a^b f(x)\,dx=\int_a^b f(t)\,dt$

Property 4 : $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$, where $a<c<b$

Property 5 : $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

Property 6 : $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

Property 7 : $\int_0^{2a} f(x)\,dx=\int_0^a [f(x)+f(2a-x)]\,dx$

Property 8 : $\int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx$, if f is an even function

$=0$, if f is an odd function

Note : f is an even function, if $f(-x)=f(x)$ and f is an odd function, if $f(-x)=-f(x)$.

Solved Examples	3 or 4 marks each

Ex. 9. Evaluate : $\int_0^a x^2(a-x)^{\frac{3}{2}}dx$.

Solution : We use the property, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$\therefore \int_0^a x^2(a-x)^{\frac{3}{2}}dx = \int_0^a (a-x)^2(a-a+x)^{\frac{3}{2}}dx$$

$$= \int_0^a (a^2-2ax+x^2)x^{\frac{3}{2}}dx = \int_0^a (a^2x^{\frac{3}{2}} - 2ax^{\frac{5}{2}} + x^{\frac{7}{2}})dx$$

$$= a^2\int_0^a x^{\frac{3}{2}}dx - 2a\int_0^a x^{\frac{5}{2}}dx + \int_0^a x^{\frac{7}{2}}dx$$

$$= a^2\left[\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right]_0^a - 2a\left[\frac{x^{\frac{7}{2}}}{\left(\frac{7}{2}\right)}\right]_0^a + \left[\frac{x^{\frac{9}{2}}}{\left(\frac{9}{2}\right)}\right]_0^a$$

$$= \frac{2a^2}{5}[a^{\frac{5}{2}}-0] - \frac{4a}{7}[a^{\frac{7}{2}}-0] + \frac{2}{9}[a^{\frac{9}{2}}-0]$$

$$= \frac{2}{5}a^{\frac{9}{2}} - \frac{4}{7}a^{\frac{9}{2}} + \frac{2}{9}a^{\frac{9}{2}} = \left(\frac{2}{5}-\frac{4}{7}+\frac{2}{9}\right)a^{\frac{9}{2}}$$

$$= \left(\frac{126-180+70}{315}\right)a^{\frac{9}{2}} = \frac{16}{315}a^{\frac{9}{2}}.$$

Ex. 10. Evaluate : $\displaystyle\int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx.$

Solution : Let $\displaystyle I = \int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx$ ... (1)

We use the property, $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

Hence, in I, we change x by $3-x$.

$$\therefore I = \int_0^3 \frac{\sqrt[3]{3-x+4}}{\sqrt[3]{3-x+4}+\sqrt[3]{7-3+x}}\,dx$$

$$= \int_0^3 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x}+\sqrt[3]{x+4}}\,dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I=\int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx+\int_0^3 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x}+\sqrt[3]{x+4}}\,dx$$

$$=\int_0^3 \frac{\sqrt[3]{x+4}+\sqrt[3]{7-x}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx=\int_0^3 1\,dx=[x]_0^3=3-0=3$$

$$\therefore I=\frac{3}{2}$$

Hence, $\int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx=\frac{3}{2}$.

Ex. 11. Evaluate : $\int_0^1 \log\left(\frac{1}{x}-1\right)dx.$

Solution : Let $I=\int_0^1 \log\left(\frac{1}{x}-1\right)dx=\int_0^1 \log\left(\frac{1-x}{x}\right)dx$

We use the property, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$\therefore I=\int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right]dx=\int_0^1 \log\left(\frac{x}{1-x}\right)dx$$

$$=\int_0^1 -\log\left(\frac{1-x}{x}\right)dx=-\int_0^1 \log\left(\frac{1-x}{x}\right)dx$$

$\therefore I=-I \qquad \therefore 2I=0 \qquad \therefore I=0$

Hence, $\int_0^1 \log\left(\frac{1}{x}-1\right)dx=0.$

Ex. 12. Evaluate : $\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx.$ ***(March '23)***

Solution : Let $I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$ ... (1)

We use the property, $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

Hence in I, we replace x by $1+3-x$.

$$\therefore\ I=\int_1^3 \frac{\sqrt[3]{1+3-x+5}}{\sqrt[3]{1+3-x+5}+\sqrt[3]{9-1-3+x}}\,dx$$

$$=\int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx \qquad \text{... (2)}$$

Adding (1) and (2), we get

$$2I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx+\int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx$$

$$=\int_1^3 \frac{\sqrt[3]{x+5}+\sqrt[3]{9-x}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$$

$$=\int_1^3 1\,dx=[x]_1^3=3-1=2 \qquad \therefore\ I=1$$

Hence, $\displaystyle\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx=1.$

Ex. 13. Evaluate : $\displaystyle\int_{-2}^{2} \frac{x^2}{x^2-1}\,dx.$

Solution : Let $I=\displaystyle\int_{-2}^{2} \frac{x^2}{x^2-1}\,dx$

Let $f(x)=\dfrac{x^2}{x^2-1}$

Then $f(-x)=\dfrac{(-x)^2}{(-x)^2-1}=\dfrac{x^2}{x^2-1}=f(x)$

$\therefore$ f is an even function.

$$\therefore\ \int_{-2}^{2} f(x)\,dx=2\int_0^2 f(x)\,dx$$

$$\therefore\ I=2\int_0^2 \frac{x^2}{x^2-1}\,dx$$

$$=2\int_0^2 \frac{(x^2-1)+1}{x^2-1}\,dx=2\int_0^2 \left(1+\frac{1}{x^2-1}\right)dx$$

$$=2\left[x+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_0^2=2\left[2+\frac{1}{2}\log\frac{1}{3}-0\right]$$

$$= 4 + \log \frac{1}{3}$$

Hence, $\int_{-2}^{2} \frac{x^2}{x^2 - 1} dx = 4 + \log \frac{1}{3}$.

Ex. 14. Evaluate : $\int_{-9}^{9} \frac{x^3}{4 - x^2} dx$.

Solution : Let $I = \int_{-9}^{9} \frac{x^3}{4 - x^2} dx$

Let $f(x) = \frac{x^3}{4 - x^2}$

$\therefore f(-x) = \frac{(-x)^3}{4 - (-x)^2} = \frac{-x^3}{4 + x^2} = -f(x)$

$\therefore f$ is an odd function.

$\therefore \int_{-9}^{9} f(x)\, dx = 0$, i.e. $\int_{-9}^{9} \frac{x^3}{4 - x^2} dx = 0$.

Examples for Practice	3 or 4 marks each

Evaluate the following :

1. $\int_{0}^{1} x(1 - x)^n dx$

2. $\int_{0}^{3} x^2 \sqrt{3 - x}\, dx$

3. $\int_{0}^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$

4. $\int_{2}^{7} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9 - x}} dx$ ***(July '22)***

5. $\int_{1}^{2} \frac{\sqrt{x} - \sqrt{3 - x}}{1 + \sqrt{x(3 - x)}} dx$

6. $\int_{4}^{7} \frac{(11 - x)^2}{x^2 + (11 - x)^2} dx$

7. $\int_{-2}^{2} \frac{x^3}{4 + x^6} dx$

8. $\int_{-4}^{4} \frac{1 + x^2}{1 - x^2} dx$.

Answers

1. $\dfrac{1}{(n+1)(n+2)}$ 2. $\dfrac{144\sqrt{3}}{35}$ 3. $\dfrac{a}{2}$ 4. $\dfrac{5}{2}$

5. 0 6. $\dfrac{3}{2}$ 7. 0 8. $2\log\left(\dfrac{5}{3}\right)-8.$

ACTIVITIES	4 marks each

1. Complete the following activity :

$$\text{Let } I=\int_1^3 \frac{2}{x(1+x^2)}\,dx=\int_1^3 \frac{\boxed{}}{x^2(1+x^2)}\,dx$$

Put $1+x^2=t$ $\therefore 2x\,dx=dt$ and $x^2=t-1$

when $x=1,\ t=1+1=2$

when $x=3,\ t=1+9=10$

$$\therefore I=\int_2^{10} \frac{1}{(t-1)\,t}\,dt=\int_2^{10} \frac{t-\boxed{}}{(t-1)\,t}\,dt$$

$$=\int_2^{10}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt=\Big[\boxed{}-\log t\Big]_2^{10}=\boxed{}.$$

Solution :

$$\text{Let } I=\int_1^3 \frac{2}{x(1+x^2)}\,dx=\int_1^3 \frac{\boxed{2x}}{x^2(1+x^2)}\,dx$$

Put $1+x^2=t$ $\therefore 2xdx=dt$ and $x^2=t-1$

When $x=1,\ t=1+1=2$

When $x=3,\ t=1+9=10$

$$\therefore I=\int_2^{10}\left(\frac{1}{(t-1)\,t}\right)dt$$

$$=\int_2^{10} \frac{t-\boxed{(t-1)}}{(t-1)t}\,dt=\int_2^{10}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$

$$=\left[\boxed{\log|t-1|}-\log t\right]_2^{10}=\left[\log\left|\frac{t-1}{t}\right|\right]_2^{10}$$

$$=\log\left(\frac{9}{10}\right)-\log\left(\frac{1}{2}\right)=\log\left(\frac{9}{10}\times 2\right)=\boxed{\log\left(\frac{9}{5}\right)}.$$

---

2. Complete the following activity :

Let $I=\int_0^1 \log\left(\frac{1}{x}-1\right)dx=\int_0^1 \log\left(\frac{1-x}{\boxed{}}\right)dx$ ... (1)

We use the property, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$\therefore I=\int_0^1 \log\left(\boxed{}\right)dx$ ... (2)

Adding (1) and (2), we get

$2I=\int_0^1 \boxed{}\,dx \qquad \therefore I=\boxed{}$

Solution :

Let $I=\int_0^1 \log\left(\frac{1}{x}-1\right)dx=\int_0^1 \log\left(\frac{1-x}{\boxed{x}}\right)dx$ ... (1)

We use the property, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$\therefore I=\int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right]dx=\int_0^1 \log\left(\boxed{\frac{x}{1-x}}\right)dx$ ... (2)

Adding (1) and (2), we get

$$2I=\int_0^1 \log\left(\frac{1-x}{x}\right)dx+\int_0^1 \log\left(\frac{x}{1-x}\right)dx$$

$$=\int_0^1\left[\log\left(\frac{1-x}{x}\right)+\log\left(\frac{x}{1-x}\right)\right]dx$$

$$=\int_0^1 \log\left(\frac{1-x}{x}\times\frac{x}{1-x}\right)dx=\int_0^1 \log 1\,dx$$

$2I=\int_0^1 \boxed{0}\,dx=0 \qquad \therefore I=\boxed{0}.$

---

3. Evaluate : $\displaystyle\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx.$

$$\text{Let } I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx \qquad \ldots (1)$$

By property $\displaystyle\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

$$I=\int_1^3 \frac{\boxed{}}{\sqrt[3]{9-x}+\boxed{}}\,dx \qquad \ldots (2)$$

Adding (1) and (2), we get

$$I+I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx+\int_1^3 \frac{\boxed{}}{\sqrt[3]{9-x}+\boxed{}}\,dx=\int_1^3 \frac{\sqrt[3]{x+5}+\sqrt[3]{9-x}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$$

$$\therefore\ 2I=\int_1^3 \boxed{}\,dx=\left[\boxed{}\right]_1^3$$

$\therefore\ 2I=\boxed{}$ $\qquad \therefore\ I=1.$ **(July '23)**

Solution : $\displaystyle \text{Let } I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx \qquad \ldots (1)$

By property $\displaystyle\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

$$I=\int_1^3 \frac{\sqrt[3]{1+3-x+5}}{\sqrt[3]{1+3-x+5}+\sqrt[3]{9-1-3+x}}\,dx$$

$$=\int_1^3 \frac{\boxed{\sqrt[3]{9-x}}}{\sqrt[3]{9-x}+\boxed{\sqrt[3]{x+5}}}\,dx \qquad \ldots (2)$$

Adding (1) and (2), we get

$$I+I=\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx+\int_1^3 \frac{\boxed{\sqrt[3]{9-x}}}{\sqrt[3]{9-x}+\boxed{\sqrt[3]{x+5}}}\,dx$$

$$=\int_1^3 \frac{\sqrt[3]{x+5}+\sqrt[3]{9-x}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$$

$$\therefore\ 2I=\int_1^3 \boxed{1}\,dx=\left[\boxed{x}\right]_1^3$$

$$\therefore\ 2I=3-1=\boxed{2} \qquad \therefore\ I=1.$$

---

4. Complete the following activity :

$$\int_0^2 \frac{dx}{4+x-x^2}=\int_0^2 \frac{dx}{-x^2+\boxed{}+\boxed{}}$$

$$=\int_0^2 \frac{dx}{-x^2+x+\frac{1}{4}-\boxed{}+4}$$

$$=-\int_0^2 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\boxed{}\right)^2}$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right).$$

(March '24)

Solution : $$\int_0^2 \frac{dx}{4+x-x^2}=\int_0^2 \frac{dx}{-x^2+\boxed{x}+\boxed{4}}$$

$$=\int_0^2 \frac{dx}{-x^2+x+\frac{1}{4}-\boxed{\frac{1}{4}}+4}$$

$$=-\int_0^2 \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}-4}$$

$$=-\int_0^2 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\boxed{\frac{\sqrt{17}}{2}}\right)^2}$$

$$=-\frac{1}{2\times\frac{\sqrt{17}}{2}}\left[\log\left|\frac{x-\frac{1}{2}-\frac{\sqrt{17}}{2}}{x-\frac{1}{2}+\frac{\sqrt{17}}{2}}\right|\right]_0^2$$

$$= -\frac{1}{\sqrt{17}}\left[\log\left|\frac{2x-1-\sqrt{17}}{2x-1+\sqrt{17}}\right|\right]_0^2$$

$$= -\frac{1}{\sqrt{17}}\left[\log\left(\frac{3-\sqrt{17}}{3+\sqrt{17}}\right) - \log\left(\frac{-1-\sqrt{17}}{-1+\sqrt{17}}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[-\log\left(\frac{3-\sqrt{17}}{3+\sqrt{17}}\right) + \log\left(\frac{-1-\sqrt{17}}{-1+\sqrt{17}}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{-1-\sqrt{17}}{-1+\sqrt{17}} \times \frac{3+\sqrt{17}}{3-\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{-3-\sqrt{17}-3\sqrt{17}-17}{-3+\sqrt{17}+3\sqrt{17}-17}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{-20-4\sqrt{17}}{-20+4\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right).$$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. $\displaystyle\int_2^3 \frac{x}{x^2-1}\,dx =$ ....

(a) $\log\left(\frac{8}{3}\right)$ (b) $-\log\left(\frac{8}{3}\right)$ (c) $\frac{1}{2}\log\left(\frac{8}{3}\right)$ (d) $-\frac{1}{2}\log\left(\frac{8}{3}\right)$

2. If $\displaystyle\int_0^a 5x^4\,dx = 32$, $a \in R$, then $a =$ ....

(a) 2 (b) 1 (c) $\frac{2}{5}$ (d) 10 ***(Sept. '21)***

3. $\displaystyle\int_{-5}^{5} \frac{x^7}{x^4+10}\,dx =$ ....

(a) 10 (b) 5 (c) 0 (d) $\frac{1}{5}$ ***(March '22)***

4. $\int_{-7}^{7} \frac{x^3}{x^2+7}dx = \ldots\ldots\ldots$

(a) 7 (b) 49 (c) $\frac{7}{2}$ (d) 0 *(July '23)*

5. $\int_{-2}^{3} \frac{dx}{x+5} = \ldots\ldots\ldots$

(a) $-\log\left(\frac{8}{3}\right)$ (b) $\log\left(\frac{8}{3}\right)$ (c) $\log\left(\frac{3}{8}\right)$ (d) $-\log\left(\frac{3}{8}\right)$ *(July '24)*

6. $\int_{1}^{2} x\log x\,dx = \ldots\ldots\ldots$

(a) $\log 2-\frac{3}{4}$ (b) $2\log 2+\frac{3}{4}$ (c) $2\log 2-\frac{1}{4}$ (d) $2\log 2-\frac{3}{4}$

7. $\int_{0}^{2} e^x dx = \ldots\ldots\ldots$

(a) $e-1$ (b) $1-e$ (c) e^2-1 (d) $1-e^2$

(March '23)

8. $\int_{2}^{7} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}dx = \ldots\ldots\ldots$

(a) $\frac{7}{2}$ (b) $\frac{5}{2}$ (c) 7 (d) 2 *(March '25)*

Answers

1. (c) $\frac{1}{2}\log\left(\frac{8}{3}\right)$ **2.** (a) 2 **3.** (c) 0 **4.** (d) 0

5. (b) $\log\left(\frac{8}{3}\right)$ **6.** (d) $2\log 2-\frac{3}{4}$ **7.** (c) e^2-1 **8.** (b) $\frac{5}{2}$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$ *(March '24)*

2. $\int_4^7 \frac{(11-x)^2}{x^2+(11-x)^2}\,dx = \frac{3}{2}$.

3. If $f(a-x) = -f(x)$, then $\int_0^a f(x)\,dx = 0$.

4. If $\int_0^{\alpha} (3x^2+2x+1)\,dx = 14$, then $\alpha = -2$.

5. $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \log(\sqrt{2}+1)$.

Answers

1. True **2.** True **3.** True **4.** False **5.** True.

FILL IN THE BLANKS	**1 mark each**

Fill in the following blanks :

1. If $\int_0^a 3x^2 dx = 8$, then $a =$ .... ***(July '22)***

2. If $\int_0^1 (3x^2+2x+a)\,dx = 0$, then $a =$ ....

3. $\int_{-7}^{7} \frac{x^7}{x^4-9}\,dx =$ .... ***(Sept. '21)***

4. $\int_1^e \frac{\log x}{x}dx =$ ....

5. $\int_2^3 \frac{x}{x^2-1}\,dx =$ ....

Answers

1. 2 **2.** -2 **3.** 0 **4.** $\frac{1}{2}$ **5.** $\frac{1}{2}\log\left(\frac{8}{3}\right)$.

Question Set 7

APPLICATIONS OF DEFINITE INTEGRATION

(Marks with option : 04)

Remember :

1. The area bounded by the curve $y=f(x)$, the X-axis and the ordinates $x=a$ and $x=b$ is given by

$$A=\left|\int_a^b y\,dx\right|=\left|\int_a^b f(x)\,dx\right|.$$

2. If the two curves $y=f(x)$ and $y=g(x)$ intersect each other at $x=a$ and $x=b$, then the area between the curves is

$$\left|\int_a^b f(x)\,dx-\int_a^b g(x)\,dx\right|.$$

3. The area bounded by the curve $x=g(y)$, the Y-axis and the abscissas $y=c$ and $y=d$ is given by

$$A=\left|\int_c^d x\,dy\right|=\left|\int_c^d g(y)\,dy\right|.$$

Solved Examples	3 or 4 marks each

Ex. 1. Find the area of the region bounded by the curve $y=x^2+1$, lines $x=0$, $x=3$ and X-axis. *(July '22)*

Solution : Required area $=\int_0^3 y\,dx$, where $y=x^2+1$

$$=\int_0^3 (x^2+1)\,dx=\left[\frac{x^3}{3}+x\right]_0^3$$

$=9+3-0=12$ sq units.

Ex. 2. Find the area of the region bounded by the parabola $y^2=4x$ and the line $x=3$. *(Sept. '21; March '23)*

Solution :

Required area = area of the region OABO

$=2$ (area of the region OACO)

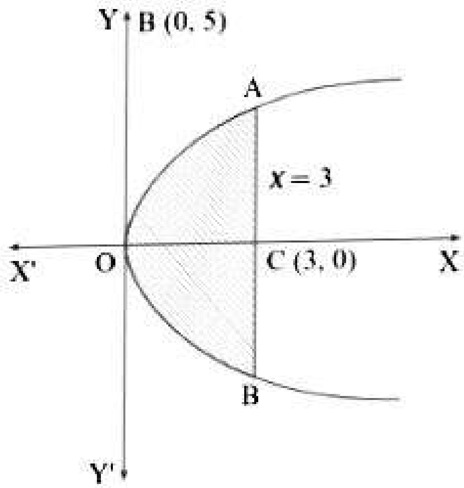
$$= 2\int_0^3 y\,dx \text{ where } y^2 = 4x, \text{ i.e. } y = 2\sqrt{x}$$

$$= 2\int_0^3 2\sqrt{x}\,dx$$

$$= 4\int_0^3 x^{\frac{1}{2}}\,dx$$

$$= 4\cdot\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^3 = \frac{8}{3}\left[x^{\frac{3}{2}}\right]_0^3$$

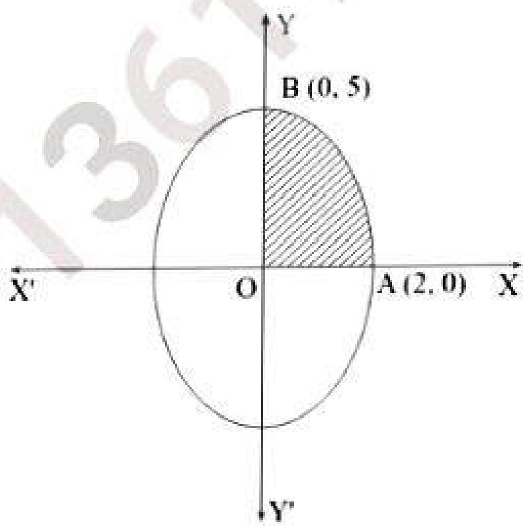
$$= \frac{8}{3}(3\sqrt{3} - 0) = 8\sqrt{3} \text{ sq units.}$$



Ex. 3. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$ by using integration.

$\left[\text{Given : } \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \text{ and } \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$

Solution :



By the symmetry of the ellipse, its area is equal to 4 times the area of the region OABO. Clearly for this region, the limits of integration are 0 and 2.

From the equation of the ellipse,

$$\frac{y^2}{25} = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4} \qquad \therefore\ y^2 = \frac{25}{4}(4 - x^2)$$

In the first quadrant, $y>0$

$\therefore y=\frac{5}{2}\sqrt{4-x^2}$

$\therefore$ area of ellipse $=4$ (area of the region OABO)

$$=4\int_0^2 y\,dx$$

$$=4\int_0^2 \frac{5}{2}\sqrt{4-x^2}\,dx$$

$$=10\int_0^2 \sqrt{4-x^2}\,dx$$

$$=10\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

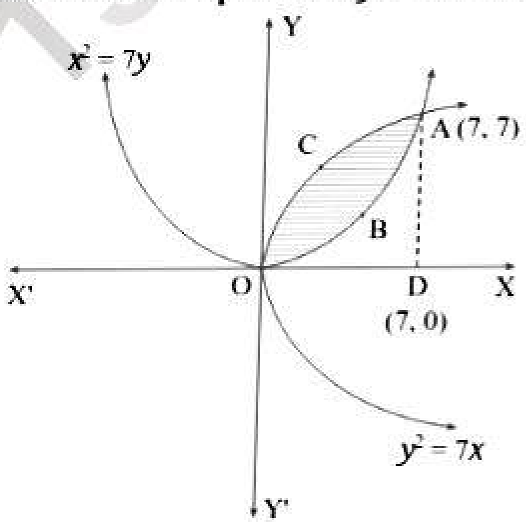
$$\ldots\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=10\left[\left\{\frac{2}{2}\sqrt{4-4}+2\sin^{-1}(1)\right\}-\left\{\frac{0}{2}\sqrt{4-0}+2\sin^{-1}(0)\right\}\right]$$

$$=10\times 2\times\frac{\pi}{2}=10\pi \text{ sq units} \qquad \ldots\left[\because \sin^{-1}(1)=\frac{\pi}{2}, \sin^{-1}(0)=0.\right]$$

Ex. 4. Find the area between the parabolas $y^2=7x$ and $x^2=7y$. *(March '22)*

Solution :



For finding the points of intersection of the two parabolas, we equate the values of y^2 from their equations.

From the equation $x^2 = 7y$, $y^2 = \frac{x^4}{49}$

$\therefore \frac{x^4}{49} = 7x \qquad \therefore x^4 = 343x$

$\therefore x^4 - 343x = 0 \qquad \therefore x(x^3 - 343) = 0$

$\therefore x = 0$ or $x^3 = 343$, i.e. $x = 7$

When $x = 0$, $y = 0$.

When $x = 7$, $7y = 49 \qquad \therefore y = 7$

$\therefore$ the points of intersection are O(0, 0) and A(7, 7).

Required area = area of the region OBACO

= (area of the region ODACO) – (area of the region ODABO)

Now, area of the region ODACO

= area under the parabola $y^2 = 7x$, i.e. $y = \sqrt{7}\sqrt{x}$ between $x = 0$ and $x = 7$

$$= \int_0^7 \sqrt{7}\sqrt{x}\,dx = \sqrt{7}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^7 = \sqrt{7} \times \frac{2}{3}\left[7^{\frac{3}{2}} - 0\right] = \frac{2\sqrt{7}}{3}[7\sqrt{7} - 0] = \frac{98}{3}$$

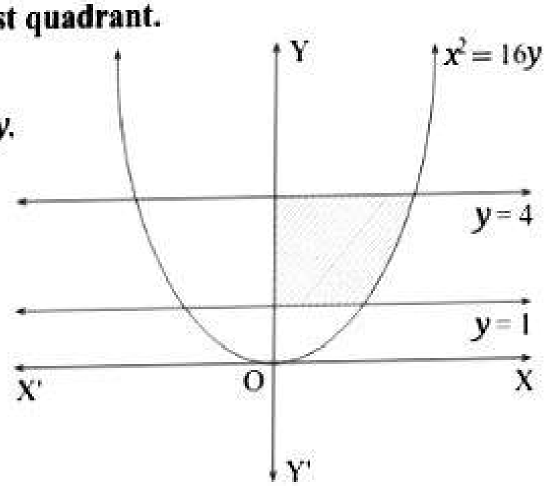
Area of the region ODABO

= area under the parabola $x^2 = 7y$, i.e. $y = \frac{x^2}{7}$ between $x = 0$ and $x = 7$

$$= \int_0^7 \frac{x^2}{7}\,dx = \frac{1}{7}\left[\frac{x^3}{3}\right]_0^7 = \frac{1}{7}\left[\frac{7^3}{3} - 0\right] = \frac{7^2}{3} = \frac{49}{3}$$

$\therefore$ required area $= \frac{98}{3} - \frac{49}{3} = \frac{49}{3}$ sq units.

Ex. 5. Find the area of the region bounded by the curve $x^2 = 16y$, $y = 1$, $y = 4$ and the Y-axis lying in the first quadrant.



Solution :

Required area $= \int_1^4 x\,dy$, where $x^2 = 16y$,

i.e. $x = 4\sqrt{y}$

$$= \int_1^4 4\sqrt{y}\,dy$$

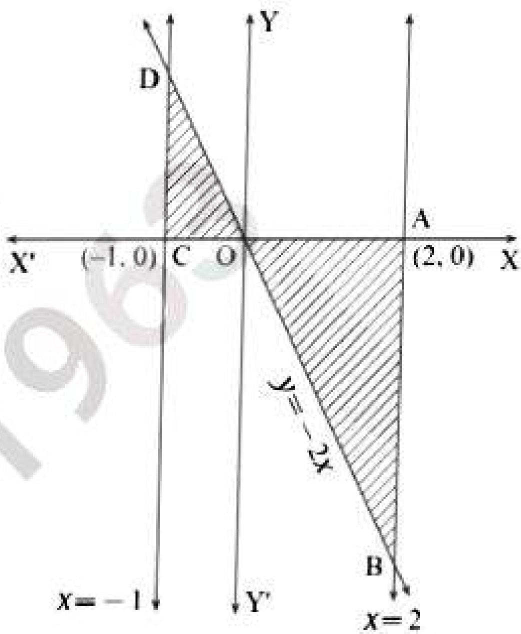
$$= 4\int_1^4 y^{\frac{1}{2}}\,dy$$

$$= 4\left[\frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_1^4 = \frac{8}{3}\left[y^{\frac{3}{2}}\right]_1^4$$

$$= \frac{8}{3}(8-1) = \frac{56}{3} \text{ sq units.}$$

Ex. 6. Find the area of the region bounded by the line $y = -2x$, the X-axis and the lines $x = -1$ and $x = 2$. ***(March '24)***

Solution :



Required area

= area of shaded region OCD

+ area of shaded region OAB

= (area under the line $y = -2x$ between $x = -1$ and $x = 0$) + (area under the line $y = -2x$ between $x = 0$ and $x = 2$)

$$= \int_{-1}^{0} -2x\,dx + \left|\int_{0}^{2} -2x\,dx\right|$$

$$= -2\int_{-1}^{0} x\,dx + \left|-2\int_{0}^{2} x\,dx\right|$$

$$= -2\left[\frac{x^2}{2}\right]_{-1}^{0} + \left|-2\left[\frac{x^2}{2}\right]_0^2\right|$$

$$= -2\left[0 - \frac{1}{2}\right] + |-2[2-0]|$$

$$= 1 + |-4| = 1 + 4 = 5 \text{ sq units.}$$

Examples for Practice	**3 or 4 marks each**

1. Find the area of the region bounded by the following curves, the X-axis and the given lines :

   (1) $2y = 5x + 7$, $x = 2$, $x = 8$

   (2) $y = 2 - x^2$, $x = -1$, $x = 1$

   (3) $y = \sqrt{6x + 4}$, $x = 0$, $x = 2$.

2. (1) Find the area of the region bounded by the parabola $y^2 = 25x$ and the line $x = 4$. ***(July '24)***

(2) Find the area of the region bounded by the parabola $y^2 = 25x$ and the line $x = 5$. ***(March '25)***

3. Find the area of the region bounded by the curve $y = 2x + 3x^2$, X-axis and the lines $x = 0$, $x = 2$.

4. Find the area of the circle $x^2 + y^2 = 25$ using integration.

[Given : $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and

$\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(0) = 0$.]

5. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using integration.

[Given : $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and

$\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(0) = 0$.]

6. Find the area of the region lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

7. Find the area of the region bounded by the curve $x^2 = 25y$, $y = 1$, $y = 4$ and the Y-axis lying in the first quadrant.

8. (1) Find the area of the region bounded by the curve $y = x^2$ and the line $y = 10$.

(2) Find the area of the region bounded by $y = x^2$ with X-axis and $x = 1$, $x = 4$. ***(July '23)***

Answers

1. (1) 96 sq units (2) $\frac{10}{3}$ sq units (3) $\frac{128}{3}$ sq units

2. (1) $\frac{160}{3}$ sq units (2) $\frac{100\sqrt{5}}{3}$ sq units **3.** 12 sq units **4.** 10π sq units

5. πab sq units **6.** $\frac{16a^2}{3}$ sq units **7.** $\frac{70}{3}$ sq units

8. (1) $\frac{40\sqrt{10}}{3}$ sq units (2) 21 sq units.

MULTIPLE CHOICE QUESTIONS | 1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The area of the region bounded by the curve $y=x^2$, $x=0$, $x=3$ and X-axis is

(a) 9 sq units (b) $\frac{26}{3}$ sq units (c) $\frac{52}{3}$ sq units (d) 18 sq units

(March '22)

2. The area of the region bounded by the curve $y=x^2$ and the line $y=4$ is

(a) $\frac{32}{3}$ sq units (b) $\frac{64}{3}$ sq units (c) $\frac{16}{3}$ sq units (d) 64 sq units

(March '25)

3. Area of the region bounded by $y=x^4$, $x=1$, $x=5$ and the X-axis is

(a) $\frac{3142}{5}$ sq units (b) $\frac{3124}{5}$ sq units (c) $\frac{3142}{3}$ sq units (d) $\frac{3124}{3}$ sq units

4. The area of the region bounded by the parabola $y^2=16x$ and its latus rectum is

(a) $\frac{32}{3}$ sq units (b) $\frac{64}{3}$ sq units (c) $\frac{128}{3}$ sq units (d) $\frac{256}{3}$ sq units

5. The area of the region bounded by the curve $y=2x+3$, the X-axis and lines $x=0$ and $x=3$ is

(a) 18 sq units (b) 9 sq units (c) 27 sq units (d) $\frac{81}{4}$ sq units

(Sept. '21)

Answers

1. (a) 9 sq units **2.** (a) $\frac{32}{3}$ sq units **3.** (b) $\frac{3124}{5}$ sq units

4. (c) $\frac{128}{3}$ sq units **5.** (a) 18 sq units.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The area bounded by two curves $y=f(x)$, $y=g(x)$ and X-axis is $\left|\int_a^b f(x)\,dx-\int_b^a g(x)\,dx\right|$.
2. The area of the portion lying above the X-axis is positive. ***(July '22)***
3. If the curve, under consideration, is below the X-axis, then the area bounded by curve, X-axis and lines $x=a$, $x=b$ is positive.
4. The area of the region bounded by the line $y=2x$, the X-axis and the lines $x=-2$ and $x=4$ is 12 sq units.

Answers

1. False **2.** True **3.** False **4.** False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. Area of the region bounded by $y=x^4$, $x=1$, $x=5$ and the X-axis will be .... ***(July '24)***
2. Area of the region bounded by $x^2=16y$, $y=1$, $y=4$ and the Y-axis lying in the first quadrant is ....
3. The area of the region bounded by $y^2=4x$, the X-axis and the lines $x=1$, $x=4$ is ....
4. The area of the region bounded by the curve $2y+x=8$, the X-axis and the lines $x=2$, $x=4$ is ....
5. Using definite integration, area of circle $x^2+y^2=25$ is .... ***(July '23)***

Answers

1. $\frac{3124}{5}$ sq units **2.** $\frac{56}{3}$ sq units **3.** $\frac{28}{3}$ sq units **4.** 5 sq units

5. 25π sq units.

Question Set 8

DIFFERENTIAL EQUATIONS AND APPLICATIONS

(Marks with option : 10)

8.1 FORMATION OF DIFFERENTIAL EQUATION

Solved Examples	3 marks each

Ex. 1. Obtain the differential equation by eliminating arbitrary constants from the following equations :

(1) $y = Ae^{3x} + Be^{-3x}$ *(March '25)* (2) $x^3 + y^3 = 35ax$ *(March '24)*

(3) $y = c_1 e^{3x} + c_2 e^{2x}$ (4) $Ax^2 + By^2 = 1$.

Solution :

(1) $y = Ae^{3x} + Be^{-3x}$ ... (1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{3x} \times 3 + Be^{-3x} \times (-3)$$

$$\therefore \frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$$

and $\dfrac{d^2y}{dx^2} = 3Ae^{3x} \times 3 - 3Be^{-3x} \times (-3) = 9Ae^{3x} + 9Be^{-3x}$

$= 9(Ae^{3x} + Be^{-3x}) = 9y$ ... [By (1)]

$$\therefore \frac{d^2y}{dx^2} = 9y$$

This is the required D.E.

---

(2) $x^3 + y^3 = 35ax$ ... (1)

Differentiating w.r.t. x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 35a$$

Substituting the value of $35a$ in (1), we get

$$x^3 + y^3 = \left(3x^2 + 3y^2 \frac{dy}{dx}\right)x$$

$$\therefore x^3 + y^3 = 3x^3 + 3xy^2 \frac{dy}{dx}$$

$$\therefore 2x^3 - y^3 + 3xy^2\frac{dy}{dx} = 0$$

This is the required D.E.

---

(3) $y = c_1e^{3x} + c_2e^{2x}$

Dividing both sides by e^{2x}, we get

$$e^{-2x}y = c_1e^x + c_2$$

Differentiating w.r.t. x, we get

$$e^{-2x}\frac{dy}{dx} + y \cdot e^{-2x}(-2) = c_1e^x + 0$$

$$\therefore e^{-2x}\left(\frac{dy}{dx} - 2y\right) = c_1e^x$$

Dividing both sides by e^x, we get

$$e^{-3x}\left(\frac{dy}{dx} - 2y\right) = c_1$$

Differentiating w.r.t. x, we get

$$e^{-3x}\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\right) + \left(\frac{dy}{dx} - 2y\right) \cdot e^{-3x}(-3) = 0$$

$$\therefore e^{-3x}\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3\frac{dy}{dx} + 6y\right) = 0$$

$$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

This is the required D.E.

---

(4) $Ax^2 + By^2 = 1$

Differentiating w.r.t. x, we get

$$A \times 2x + B \times 2y\frac{dy}{dx} = 0$$

$$\therefore Ax + By\frac{dy}{dx} = 0 \qquad \dots (1)$$

Differentiating again w.r.t. x, we get

$$A \times 1 + B\left[y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx}\right] = 0$$

$$\therefore A = -B\left[y \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

Substituting the value of A in (1), we get

$$-Bx\left[y\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]+By\frac{dy}{dx}=0$$

$$\therefore -x\left[y\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]+y\frac{dy}{dx}=0 \qquad \therefore -xy\frac{d^2y}{dx^2}-x\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=0$$

$$\therefore xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

This is the required D.E.

Ex. 2. Verify that :

(1) $y=e^{-x}+ax+b$, where $a, b\in R$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right)=1$.

(2) $y=ae^x+be^{-2x}$ is the solution of the D.E. $\frac{d^2y}{dx^2}+\frac{dy}{dx}=2y$.

Solution :

(1) $y=e^{-x}+ax+b$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{-x}(-1)+a\times1+0=-e^{-x}+a$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-e^{x}(-1)+0=e^{-x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{1}{e^x} \qquad \therefore e^x\left(\frac{d^2y}{dx^2}\right)=1$$

This shows that $y=e^{-x}+ax+b$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right)=1$.

---

(2) $y=ae^x+be^{-2x}$ ... (1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=ae^x+be^{-2x}\times(-2)=ae^x-2be^{-2x}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=ae^x-2be^{-2x}\times(-2)=ae^x+4be^{-2x}$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}=ae^x+4be^{-2x}+ae^x-2be^{-2x}$$
$$=2ae^x+2be^{-2x}$$
$$=2\,(ae^x+be^{-2x})=2y \qquad \text{... [By (1)]}$$

Hence, $y=ae^x+be^{-2x}$ is solution of the D.E. $\frac{d^2y}{dx^2}+\frac{dy}{dx}=2y$.

Examples for Practice	3 marks each

1. Obtain the differential equations by eliminating arbitrary constants from the following equations :

(1) $y=(c_1+c_2x)e^x$

(2) $y=c_2+\frac{c_1}{x}$

(3) $x^3+y^3=4ax$

(4) $y=Ae^{5x}+Be^{-5x}$

(5) $xy=ae^{3x}+be^{-3x}$

(6) $y=Ae^{3x}+Be^{-2x}$

(7) $y=ax+\frac{b}{x}$

(8) $Ax^3+By^2=5$.

2. Verify that :

(1) $y=\log x+c$ is the solution of the D.E. $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$.

(2) $y=a+\frac{b}{x}$ is a solution of the D.E. $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$.

(3) $x^2+y^2=r^2$ is a solution of the D.E. $y=\frac{dy}{dx}+r.\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Answers

1. (1) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

(2) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

(3) $3xy^2\frac{dy}{dx}=y^3-2x^3$

(4) $\frac{d^2y}{dx^2}=25y$

(5) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=9xy$

(6) $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=6y=0$

(7) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$

(8) $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}=0$.

8.2 SOLUTION OF DIFFERENTIAL EQUATION

1. Variable Separable Method

Solved Examples	3 marks each

Ex. 3. Solve the following differential equations :

(1) $(x^2-yx^2)\,dy+(y^2+xy^2)\,dx=0$

(2) $\log\left(\frac{dy}{dx}\right)=2x+3y.$

Solution :

(1) $(x^2-yx^2)\,dy+(y^2+xy^2)\,dx=0$

$\therefore x^2(1-y)\,dy+y^2(1+x)\,dx=0$

$\therefore \frac{1-y}{y^2}dy+\frac{1+x}{x^2}dx=0$

Integrating, we get

$\int\frac{1-y}{y^2}dy+\int\frac{1+x}{x^2}dx=c$

$\therefore \int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy+\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx=c$

$\therefore \int y^{-2}dy-\int\frac{1}{y}dy+\int x^{-2}dx+\int\frac{1}{x}dx=c$

$\therefore \frac{y^{-1}}{-1}-\log|y|+\frac{x^{-1}}{-1}+\log|x|=c$

$\therefore -\frac{1}{y}-\log|y|-\frac{1}{x}+\log|x|=c$

$\therefore \log|x|-\log|y|=\frac{1}{x}+\frac{1}{y}+c$

This is the general solution.

---

(2) $\log\left(\frac{dy}{dx}\right)=2x+3y$ $\quad\therefore \frac{dy}{dx}=e^{2x+3y}$

$\therefore \frac{dy}{dx}=e^{2x}\cdot e^{3y}$ $\qquad\therefore e^{-3y}dy=e^{2x}dx$

$$\therefore \int e^{2x}\,dx - \int e^{-3y}\,dy = c_1$$

$$\therefore \frac{e^{2x}}{2} + \frac{e^{-3y}}{3} = c_1 \qquad \therefore 3e^{2x} + 2e^{-3y} = 6c_1$$

$$\therefore 3e^{2x} + 2e^{-3y} = c, \text{ where } c = 6c_1$$

This is the general solution.

Examples for Practice	3 marks each

Solve the following differential equations :

1. $\dfrac{dy}{dx} = x^2y + y$

2. $\dfrac{dy}{dx} = 1 + x + y + xy$

3. $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$

4. $y - x\dfrac{dy}{dx} = 3\left(1 + x^2\dfrac{dy}{dx}\right)$

5. $\dfrac{dy}{dx} = e^{x+y} + x^2e^y$

6. $3e^x\,dx + (1+e^x)\,dy = 0$

7. $\dfrac{dy}{dx} = \log x$

8. $y - x\dfrac{dy}{dx} = 0.$

Answers

1. $\log|y| = \dfrac{x^3}{3} + x + c$

2. $\log|1+y| = x + \dfrac{x^2}{2} + c$

3. $(1+x^2)(1+y^2) = c$

4. $(y-3)(3x+1) = cx$

5. $e^x + e^{-y} + \dfrac{x^3}{3} + c = 0$

6. $3\log|1+e^x| + y = c$

7. $y = x\log x - x + c$

8. $x = cy.$

2. General Solution by Substitution

Solved Examples	3 or 4 marks each

Ex. 4. Solve the following differential equations with the help of the substitution shown against them :

(1) $(x-y)^2\dfrac{dy}{dx} = a^2,\ x - y = u$ **(2)** $\dfrac{dy}{dx} = 4x + y + 1,\ 4x + y + 1 = v$

(3) $(x+2y+1)\,dx - (2x+4y+3)\,dy = 0,\ x + 2y = u.$

Solution :

(1) $(x-y)^2\dfrac{dy}{dx}=a^2$ ... (1)

Put $x-y=u$ $\therefore$ $x-u=y$ $\therefore$ $1-\dfrac{du}{dx}=\dfrac{dy}{dx}$

$\therefore$ (1) becomes, $u^2\left(1-\dfrac{du}{dx}\right)=a^2$

$\therefore$ $u^2-u^2\dfrac{du}{dx}=a^2$

$\therefore$ $u^2-a^2=u^2\dfrac{du}{dx}$ $\therefore$ $dx=\dfrac{u^2}{u^2-a^2}\,du$

Integrating, we get

$$\int dx=\int\frac{(u^2-a^2)+a^2}{u^2-a^2}\,du$$

$\therefore$ $x=\int 1\,du+a^2\displaystyle\int\frac{du}{u^2-a^2}$

$\therefore$ $x=u+a^2\cdot\dfrac{1}{2a}\log\left|\dfrac{u-a}{u+a}\right|+c_1$

$\therefore$ $x=x-y+\dfrac{a}{2}\log\left|\dfrac{x-y-a}{x-y+a}\right|+c_1$

$\therefore$ $-c_1+y=\dfrac{a}{2}\log\left|\dfrac{x-y-a}{x-y+a}\right|$

$\therefore$ $-2c_1+2y=a\log\left|\dfrac{x-y-a}{x-y+a}\right|$

$\therefore$ $c+2y=a\log\left|\dfrac{x-y-a}{x-y+a}\right|$, where $c=-2c_1$

This is the general solution.

---

(2) $\dfrac{dy}{dx}=4x+y+1$

Put $4x+y+1=v$

$\therefore$ $4+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{dv}{dx} - 4$

$\therefore$ the given D.E. becomes

$\frac{dv}{dx} - 4 = v \qquad \therefore \frac{dv}{dx} = 4 + v \qquad \therefore \frac{dv}{v+4} = dx$

Integrating, we get

$\int \frac{dv}{v+4} = \int dx$

$\therefore \log|v+4| = x + c$

$\therefore \log|4x+y+1+4| = x + c$

i.e. $\log|4x+y+5| = x + c$

This is the general solution.

---

(3) $(x+2y+1)\,dx - (2x+4y+3)\,dy = 0$

$\therefore (2x+4y+3)\,dy = (x+2y+1)\,dx$

$\therefore \frac{dy}{dx} = \frac{(x+2y)+1}{2(x+2y)+3}$ ... (1)

Put $x + 2y = u \qquad \therefore 1 + 2\frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 1\right)$

$\therefore$ (1) becomes, $\frac{1}{2}\left(\frac{du}{dx} - 1\right) = \frac{u+1}{2u+3}$

$\therefore \frac{du}{dx} - 1 = \frac{2u+2}{2u+3}$

$\therefore \frac{du}{dx} = \frac{2u+2}{2u+3} + 1 = \frac{2u+2+2u+3}{2u+3}$

$\therefore \frac{du}{dx} = \frac{4u+5}{2u+3} \qquad \therefore \frac{2u+3}{4u+5}\,du = dx$

Integrating, we get

$\int \frac{2u+3}{4u+5}\,du = \int dx$

$\therefore \int \frac{\frac{1}{2}(4u+5)+\frac{1}{2}}{4u+5}\,du = \int dx$ $\quad \therefore \int \left[\frac{1}{2}+\frac{\left(\frac{1}{2}\right)}{4u+5}\right] du = \int dx$

$\therefore \frac{1}{2}\int 1\,du + \frac{1}{2}\int \frac{1}{4u+5}\,du = \int dx$

$\therefore \frac{1}{2}u + \frac{1}{2}\cdot\frac{\log|4u+5|}{4} = x + c_1$

$\therefore 4u + \log|4u+5| = 8x + 8c_1$

$\therefore 4(x+2y) + \log\ |4\,(x+2y)+5| = 8x + c$, where $c = 8c_1$

$\therefore \log|4x+8y+5| = 4x - 8y + c$

This is the general solution.

Examples for Practice	**3 or 4 marks each**

Solve the following differential equations using the substitution shown against them :

1. $(x+y)\frac{dy}{dx}+y=0,\ x+y=u$

2. $(x-y)\left(1-\frac{dy}{dx}\right)=e^x,\ x-y=u$

3. $x\left(\frac{dy}{dx}-y\right)=x(x-1),\ x^2+y^2=u$

4. $\frac{dy}{dx}=\frac{6x-4y+3}{3x-2y+1},\ 3x-2y=u.$

Answers

1. $y(2x+y)=c$

2. $(x-y)^2=2e^x+c$

3. $\log(x^2+y^2)=2x+c$

4. $4x-2y-2\log|3x-2y+3|=c.$

3. Homogeneous Differential Equations

A homogeneous differential equation of the first order is of the type $\frac{dy}{dx}=-\frac{f_1(x,y)}{f_2(x,y)}$ or $f_1(x,\ y)\,dx+f_2(x,\ y)\,dy=0$, where $f_1(x,\ y)$ and $f_2(x,\ y)$ are homogeneous functions of the same degree in x and y. Such an equation can be reduced to the variable separable form by the substitution $y=vx$.

Solved Examples	3 or 4 marks each

Ex. 5. Solve the following differential equations :

(1) $x^2y\,dx-(x^3+y^3)\,dy=0$ *(March '22; July '23)*

(2) $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0.$

Solution :

(1) $x^2y\,dx-(x^3+y^3)\,dy=0$

$\therefore (x^3+y^3)\,dy=x^2y\,dx$

$$\therefore \frac{dy}{dx}=\frac{x^2y}{x^3+y^3} \qquad \text{... (1)}$$

Put $y=vx$ $\qquad \therefore \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$$\therefore \text{(1) becomes, } v+x\frac{dv}{dx}=\frac{x^2\cdot vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{1+v^3}-v=\frac{v-v-v^4}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$$

$$\therefore \frac{1+v^3}{v^4}dv=-\frac{1}{x}dx$$

Integrating, we get

$$\int\frac{1+v^3}{v^4}dv=-\int\frac{1}{x}dx$$

$$\therefore \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \int v^{-4}dv+\int\frac{1}{v}dv=-\int\frac{1}{x}dx$$

$$\therefore \frac{v^{-3}}{-3}+\log|v|=-\log|x|+c_1$$

$$\therefore -\frac{1}{3v^3}+\log|v|=-\log|x|+c_1$$

$$\therefore -\frac{1}{3}\cdot\frac{1}{\left(\frac{y}{x}\right)^3}+\log\left|\frac{y}{x}\right|=-\log|x|+c_1$$

$$\therefore -\frac{x^3}{3y^3}+\log|y|-\log|x|=-\log|x|-\log c,\text{ where } c_1=-\log c$$

$$\therefore \frac{x^3}{3y^3}=\log c+\log y$$

$$\therefore \frac{x^3}{3y^3}=\log|cy|$$

This is the general solution.

---

(2) $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

$$\therefore 2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=-(1+2e^{\frac{x}{y}})\,dx$$

$$\therefore \frac{dy}{dx}=-\frac{(1+2e^{\frac{x}{y}})}{2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)} \quad \text{... (1)}$$

Put $y=vx$ i.e. $\frac{x}{y}=\frac{1}{v}$ $\quad\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v+x\frac{dv}{dx}=-\frac{(1+2e^{\frac{1}{v}})}{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}$

$$\therefore x\frac{dv}{dx}=-\frac{(1+2e^{\frac{1}{v}})}{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}-v=\frac{-1-2e^{\frac{1}{v}}-2ve^{\frac{1}{v}}+2e^{\frac{1}{v}}}{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}$$

$$\therefore x\frac{dv}{dx}=\frac{-1-2ve^{\frac{1}{v}}}{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}=\frac{-(1+2ve^{\frac{1}{v}})}{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}$$

$$\therefore \frac{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}{1+2ve^{\frac{1}{v}}}\,dv=-\frac{dx}{x}$$

Integrating, we get

$$\int \frac{2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)}{1+2ve^{\frac{1}{v}}}\,dv = -\int \frac{1}{x}\,dx \qquad \text{... (2)}$$

Put $1+2ve^{\frac{1}{v}} = t \qquad \therefore 2\left[ve^{\frac{1}{v}}\left(-\frac{1}{v^2}\right)+e^{\frac{1}{v}}\right]dv = dt$

$$\therefore 2e^{\frac{1}{v}}\left(1-\frac{1}{v}\right)dv = dt$$

$\therefore$ (2) becomes, $\int \frac{1}{t}\,dt = -\int \frac{1}{x}\,dx$

$$\therefore \log|t| = -\log|x| + \log c$$

$$\therefore \log\left|1+2ve^{\frac{1}{v}}\right| + \log|x| = \log c$$

$$\therefore \log\left|1+2\frac{y}{x}\cdot e^{\frac{x}{y}}\right| + \log|x| = \log c$$

$$\therefore \log\left|\frac{x+2ye^{\frac{x}{y}}}{x}\times x\right| = \log c \qquad \therefore x+2ye^{\frac{x}{y}} = c$$

This is the general solution.

Examples for Practice	3 or 4 marks each

Solve the following differential equations :

1. $y^2dx+(xy+x^2)\,dy=0$

2. $(x^2+y^2)\,dx-2xy\,dy=0$

3. $y^2\,dx+(xy+x^2)\,dy=0$

4. $x^2\frac{dy}{dx}=x^2+xy-y^2$

5. $\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$.

Answers

1. $xy^2=c^2(x+2y)$

2. $x^2-y^2=cx$

3. $xy^2=c^2\,(x+2y)$

4. $\frac{x+y}{x-y}=cx^2$

5. $y+\sqrt{x^2+y^2}=cx^2$.

4. Linear Differential Equation

1. The general form of a linear differential equation of the first order is $\frac{dy}{dx}+P\cdot y=Q$, where P and Q are the functions of x only or constants.

   The solution of the linear differential equation is given by $y(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$, where $\text{I.F.}=e^{\int P\,dx}$.

2. If the linear differential equation is of the form $\frac{dx}{dy}+P\cdot x=Q$, where P and Q are functions of y or constants, then its solution is $x\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dy+c$, where $\text{I.F.}=e^{\int P\,dy}$.

Solved Examples	3 or 4 marks each

Ex. 6. Solve the following differential equations :

(1) $x\frac{dy}{dx}+2y=x^2\cdot\log x$ ***(Sept. '21)***

(2) $(x+2y^3)\frac{dy}{dx}=y$.

Solution :

(1) $x\frac{dy}{dx}+2y=x^2\cdot\log x$

$\therefore \frac{dy}{dx}+\left(\frac{2}{x}\right)\cdot y=x\cdot\log x$ ... (1)

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{2}{x}$ and $Q=x\cdot\log x$

$\therefore \text{I.F.}=e^{\int P\,dx}=e^{\int\frac{2}{x}dx}=e^{2\int\frac{1}{x}dx}$

$=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore y\cdot x^2=\int(x\log x)\cdot x^2\,dx+c$

$\therefore x^2\cdot y=\int x^3\cdot\log x\,dx+c$

$=(\log x)\int x^3\,dx-\int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx+c$

$$= (\log x) \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4} dx + c$$

$$= \frac{1}{4} x^4 \log x - \frac{1}{4} \int x^3 dx + c$$

$$\therefore x^2 \cdot y = \frac{1}{4} x^4 \log x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$

$$\therefore y \cdot x^2 = \frac{x^4 \log x}{4} - \frac{x^4}{16} + c$$

This is the general solution.

---

(2) $(x + 2y^3)\dfrac{dy}{dx} = y$

$$\therefore x + 2y^3 = y\frac{dx}{dy}$$

$$\therefore \frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\therefore \frac{dx}{dy} - \frac{x}{y} = 2y^2 \qquad \ldots (1)$$

This is a linear differential equation of the form

$\dfrac{dx}{dy} + Px = Q$, where $P = -\dfrac{1}{y}$, $Q = 2y^2$

$$\therefore \text{I.F.} = e^{\int P\,dy} = e^{\int -\frac{1}{y} dy} = e^{-\int \frac{1}{y} dy}$$

$$= e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dy + c$$

$$\therefore \frac{x}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c = 2 \int y\,dy + c$$

$$= 2\left(\frac{y^2}{2}\right) + c$$

$$\therefore x = y(y^2 + c)$$

This is the general solution.

Examples for Practice	3 or 4 marks each

Solve the following differential equations :

1. $\frac{dy}{dx}+y=e^{-x}$

2. $\frac{dy}{dx}+\frac{y}{x}=x^3-3$

3. $y\,dx-x\,dy+\log x\,dx=0$

4. $\frac{dy}{dx}+2xy=x$

5. $y\,dx+(x-y^2)\,dy=0$

6. $\frac{dy}{dx}+\frac{2}{x}\cdot y=x^2.$

Answers

1. $ye^x=x+c$

2. $xy=\frac{x^5}{5}-\frac{3x^2}{2}+c$

3. $y=cx-(1+\log x)$

4. $ye^{x^2}=\frac{1}{2}e^{x^2}+c$

5. $3xy=y^2+c$

6. $3x^2y=x^5+c.$

8.3 PARTICULAR SOLUTION

Solved Examples	3 or 4 marks each

Ex. 7. Find the particular solutions of the following differential equations :

(1) $(x-y^2x)\,dx-(y+x^2y)\,dy=0$, **when** $x=2, y=0.$ ***(March '23)***

(2) $(x+1)\frac{dy}{dx}-1=2e^{-y}$ **when** $y=0, x=1.$

(3) $\frac{dy}{dx}=\frac{x+y+1}{x+y-1}$, **when** $x=\frac{2}{3}$ **and** $y=\frac{1}{3}.$

Solution :

(1) $(x-y^2x)\,dx-(y+x^2y)\,dy=0$

$\therefore x(1-y^2)\,dx-y(1+x^2)\,dy=0$

$\therefore \frac{x}{1+x^2}dx-\frac{y}{1-y^2}dy=0$

$\therefore \frac{2x}{1+x^2}-\frac{2y}{1-y^2}dy=0$

Integrating, we get

$$\int \frac{2x}{1+x^2}dx+\int \frac{-2y}{1-y^2}dy=c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx=\log|f(x)|+c$$

$\therefore$ the general solution is

$$\log|1+x^2|+\log|1-y^2|=\log c, \text{ where } c_1=\log c$$

$\therefore \log|(1+x^2)(1-y^2)|=\log c$

$\therefore (1+x^2)(1-y^2)=c$

When $x=2, y=0$, we have

$(1+4)(1-0)=c \qquad \therefore c=5$

$\therefore$ the particular solution is $(1+x^2)(1-y^2)=5.$

---

(2) $(x+1)\dfrac{dy}{dx}-1=2e^{-y}$

$\therefore (x+1)\dfrac{dy}{dx}=\dfrac{2}{e^y}+1=\dfrac{2+e^y}{e^y}$

$\therefore \dfrac{e^y}{2+e^y}dy=\dfrac{1}{x+1}dx$

Integrating, we get

$$\int \frac{e^y}{2+e^y}dy=\int \frac{1}{x+1}dx$$

$\therefore \log|2+e^y|=\log|x+1|+\log c$

$$\ldots\left[\because \frac{d}{dy}(2+e^y)=e^y \text{ and } \int \frac{f'(y)}{f(y)}dy=\log|f(y)|+c\right]$$

$\therefore \log|2+e^y|=\log|c(x+1)|$

$\therefore 2+e^y=c(x+1)$

This is the general solution.

Now, $y=0$, when $x=1$

$\therefore 2+e^0=c(1+1)$

$\therefore 3=2c \qquad \therefore c=3/2$

$\therefore$ the particular solution is

$$2+e^{y}=\frac{3}{2}(x+1)$$

$\therefore \ 4+2e^{y}=3x+3$

$\therefore \ 3x-2e^{y}-1=0.$

---

(3) $\dfrac{dy}{dx}=\dfrac{x+y+1}{x+y-1}$ ... (1)

Put $x+y=v$ $\quad \therefore \ 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$ $\quad \therefore \ \dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

$\therefore$ (1) becomes, $\dfrac{dv}{dx}-1=\dfrac{v+1}{v-1}$

$$\therefore \ \frac{dv}{dx}=\frac{v+1}{v-1}+1=\frac{v+1+v-1}{v-1}$$

$$\therefore \ \frac{dv}{dx}=\frac{2v}{v-1}$$

$$\therefore \ \frac{v-1}{v}dv=2dx$$

Integrating, we get

$$\int\frac{v-1}{v}dv=2\int dx$$

$$\therefore \int\left(1-\frac{1}{v}\right)dv=2\int dx+c$$

$\therefore \ v-\log|v|=2x+c$

$\therefore \ x+y-\log|x+y|=2x+c$

$\therefore \ \log|x+y|=y-x-c$

This is the general solution.

When $x=\dfrac{2}{3}$ and $y=\dfrac{1}{3}$, we get

$\log\left|\dfrac{2}{3}+\dfrac{1}{3}\right|=\dfrac{1}{3}-\dfrac{2}{3}-c$ $\quad \therefore \ \log 1=-\dfrac{1}{3}-c$

$\therefore\ 0=-\frac{1}{3}-c \qquad \therefore\ c=-\frac{1}{3}$

$\therefore$ the particular solution is

$\log|x+y|=y-x+\frac{1}{3}$.

Examples for Practice	3 or 4 marks each

Find the particular solutions of the following differential equations :

1. $y(1+\log x)\frac{dx}{dy}-x\log x=0$, when $x=e$, $y=e^2$.

2. $\frac{dy}{dx}=3^{x+y}$, when $x=y=0$.

3. $xdy=y(1-y)\,dx$, if $y=2$, when $x=-4$.

4. $\log\frac{dy}{dx}=3x+4y$, given that $y=0$, when $x=0$.

5. $(2x-2y+3)\,dx-(x-y+1)\,dy=0$, when $x=0$, $y=1$.

6. $\frac{dy}{dx}-y=e^x$, when $x=0$, $y=1$.

Answers

1. $y=ex\log x$

2. $3^x+3^{-y}=2$

3. $2y=x(1-y)$

4. $4e^{3x}+3e^{-4y}=7$

5. $(2x-y)-\log|x-y+2|+1=0$

6. $y=(x+1)e^x$

8.4 APPLICATIONS OF DIFFERENTIAL EQUATIONS

If x denotes the amount or quantity (which grows or decay) at time t, then rate of change of x w.r.t. time t is $\frac{dx}{dt}$.

In case of growth, x increases as t increases.

$\therefore\ \frac{dx}{dt}$ is positive. Hence, $\frac{dx}{dt}=kx$, where $k>0$.

In case of decay, x decreases as t increases.

$\therefore\ \frac{dx}{dt}$ is negative. Hence, $\frac{dx}{dt}=-kx$, where $k>0$.

Solved Examples	4 marks each

Ex. 8. Bacteria increase at the rate proportional to the number of bacteria present. If the original number N doubles in 3 hours, find in how many hours, the number of bacteria will be $4N$?

Solution : Let x be the number of bacteria in the culture at time t.

Then the rate of increase is $\dfrac{dx}{dt}$ which is proportional to x.

$\therefore \dfrac{dx}{dt} \propto x \qquad \therefore \dfrac{dx}{dt} = kx$, where k is a constant

$\therefore \dfrac{dx}{x} = k\,dt$

On integrating, we get

$\displaystyle\int \dfrac{dx}{x} = k\int dt$

$\therefore \log x = kt + c$

Initially, i.e. when $t = 0$, let $x = N$

$\therefore \log N = k \times 0 + c \qquad \therefore c = \log N$

$\therefore \log x = kt + \log N \qquad \therefore \log x - \log N = kt$

$\therefore \log\left(\dfrac{x}{N}\right) = kt$ ... (1)

Since the number doubles in 3 hours, i.e. when $t = 3$, $x = 2N$

$\therefore \log\left(\dfrac{2N}{N}\right) = 3k \qquad \therefore k = \dfrac{1}{3}\log 2$

$\therefore$ (1) becomes, $\log\left(\dfrac{x}{N}\right) = \dfrac{t}{3}\log 2$

When $x = 4N$, we get

$\log\left(\dfrac{4N}{N}\right) = \dfrac{t}{3}\log 2$

$\therefore \log 4 = \log 2^{\frac{t}{3}}$

$\therefore \log 2^2 = \log 2^{\frac{t}{3}}$

$\therefore \dfrac{t}{3} = 2 \qquad \therefore t = 6$

Hence, the number of bacteria will be $4N$ in 6 hours.

Ex. 9. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000. ***(July '22)***

Solution : Let P be the population of the city at time t.

Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to P.

$\therefore \frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt} = kP$, where k is a constant.

$\therefore \frac{dP}{P} = k dt$

On integrating, we get

$$\int \frac{1}{P} dP = k \int dt$$

$\therefore \log P = kt + c$

Initially, i.e. when $t = 0$, $P = 30000$

$\therefore \log 30000 = k \times 0 + c \qquad \therefore c = \log 30000$

$\therefore \log P = kt + \log 30000$

$\therefore \log P - \log 30000 = kt$

$\therefore \log\left(\frac{P}{30000}\right) = kt \qquad \dots (1)$

Now, when $t = 40$, $P = 40000$

$\therefore \log\left(\frac{40000}{30000}\right) = k \times 40$

$\therefore k = \frac{1}{40} \log\left(\frac{4}{3}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{P}{30000}\right) = \frac{t}{40} \log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}} \qquad \therefore P = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$

Hence, the population of the city at time $t = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$.

Ex. 10. The rate of disintegration of a radioactive element at any time t is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.

Solution : Let m be the mass of the radioactive element at time t.

Then the rate of disintegration is $\dfrac{dm}{dt}$ which is proportional to m

$$\therefore \frac{dm}{dt} \propto m$$

$$\therefore \frac{dm}{dt} = -km, \text{ where } k > 0$$

$$\therefore \frac{dm}{m} = -k\,dt$$

On integrating, we get

$$\int \frac{1}{m}\,dm = -k \int dt$$

$$\therefore \log m = -kt + c$$

Initially, i.e. when $t = 0$, $m = 1.5$

$$\therefore \log(1.5) = -k \times 0 + c \qquad \therefore c = \log\left(\frac{3}{2}\right)$$

$$\therefore \log m = -kt + \log\left(\frac{3}{2}\right)$$

$$\therefore \log m - \log\left(\frac{3}{2}\right) = -kt \qquad \therefore \log\left(\frac{2m}{3}\right) = -kt$$

When $m = 0.5 = \dfrac{1}{2}$, then

$$\log\left(\frac{2 \times \frac{1}{2}}{3}\right) = -kt$$

$$\therefore \log\left(\frac{1}{3}\right) = -kt \qquad \therefore \log(3)^{-1} = -kt$$

$$\therefore -\log 3 = -kt \qquad \therefore t = \frac{1}{k}\log 3$$

Hence, the original mass will disintegrate to 0.5 gm, when $t = \dfrac{1}{k}\log 3$.

Examples for Practice	4 marks each

1. The rate of growth of population is proportional to the number present. If the population doubled in the last 25 years and the present population is 1 lakh, when will the city have population of 4,00,000 ?
2. The population of a town increases at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years? $\left(\text{Given : } \sqrt{\frac{3}{2}} = 1.2247\right)$
3. In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours. ***(July '24)***
4. The rate of disintegration of a radioactive element at time t is proportional to its mass at that time. The original mass of 800 gm will disintegrate into its mass of 400 gm after 5 days. Find the mass remaining after 30 days.
5. The rate of reduction of a person's assets is proportional to the square root of the existing assets. If the assets dwindle from 25 lakh to 6.25 lakh in 2 years, in how many years the person be bankrupt?

Answers

1. 50 years **2.** 73,482 **3.** 8 times **4.** 12.5 gm **5.** 4 years.

ACTIVITIES	4 marks each

1. Solve the differential equation $\frac{dy}{dx} - y = 2x$.

The differential equation $\frac{dy}{dx} - y = 2x$ is in the form of $\frac{dy}{dx} + Py = Q$,

where $P = -1$ and $Q = 2x$

$\therefore$ I.F. $= e^{\int P dx} = \boxed{}$

The solution of linear differential equation is

$$y\boxed{} = \int 2x\boxed{}\,dx + c$$

$$= 2\left\{x\int e^{-x}dx - \left[\int e^{-x}dx \cdot \frac{d}{dx}(x)\right]dx\right\} + c$$

$$= 2\left\{x\boxed{\quad\quad} - \int \boxed{\quad\quad} dx\right\} + c$$

$\therefore ye^{-x} = -2xe^{-x} + 2\int e^{-x}dx + c$

$\therefore ye^{-x} = -2xe^{-x} + 2\left(\frac{e^{-x}}{-1}\right) + c$

$\therefore y + \boxed{\quad\quad} = ce^{x}$ is the required solution of the given differential equation.

(July '23)

Solution :

The differential equation $\frac{dy}{dx} - y = 2x$ is in the form of $\frac{dy}{dx} + Py = Q$,

where $P = -1$ and $Q = 2x$

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int -1dx} = \boxed{e^{-x}}$

The solution of linear differential equation is

$y\boxed{e^{-x}} = \int 2x\boxed{e^{-x}}dx + c$

$$= 2\left\{x\int e^{-x}dx - \int\left[\int e^{-x}dx \cdot \frac{d}{dx}(x)\right]dx\right\} + c$$

$$= 2\left\{x\boxed{\frac{e^{-x}}{-1}} - \int \boxed{\frac{e^{-x}}{-1} \times 1}\, dx\right\} + c$$

$\therefore ye^{-x} = -2xe^{-x} + 2\int e^{-x}dx + c$

$\therefore ye^{-x} = -2xe^{-x} + 2\left(\frac{e^{-x}}{-1}\right) + c$

$\therefore ye^{-x} = -2xe^{-x} - 2e^{-x} + c$

$\therefore ye^{-x} + 2xe^{-x} + 2e^{-x} = c$

$\therefore y + \boxed{2x+2} = ce^{x}$ is the required solution of the given differential equation.

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2. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour; complete the following activity to find the number of bacteria after $\frac{5}{2}$ hours. **(Given $\sqrt{2} = 1.414$)**

Let N be the number of bacteria present at time t. Since growth of bacteria is proportional to the number present, the differential equation can be written as :

$\frac{dN}{dt} = k\boxed{\quad\quad}$, where k is the constant of proportionality.

$\therefore \log N = \boxed{} + c$ ... (1)

(i) When $t = 0$, $N = 1000$

$\therefore$ from equation (1)

$\log 1000 = 0 + c$

$\therefore c = \log 1000$

(ii) When $t = 1$, $N = 2000$

$\therefore$ from equation (1)

$k = \boxed{}$

(iii) When $t = \frac{5}{2}$ hours

$N = \boxed{}$ ***(Sept. '21)***

Solution : Let N be the number of bacteria present at time t. Since growth of bacteria is proportional to the number present,

$$\frac{dN}{dt} \propto N$$

$\therefore$ the differential equation is $\frac{dN}{dt} = k\boxed{N}$, where k is the constant of proportionality.

$$\therefore \frac{1}{N} dN = k\,dt$$

On integrating, we get

$$\int \frac{1}{N} dN = k \int dt$$

$\therefore \log N = \boxed{kt} + c$ ... (1)

(i) When $t = 0$, $N = 1000$

$\therefore$ from equation (1)

$\log 1000 = 0 + c$ $\quad \therefore c = \log 1000$

$\therefore \log N = kt + \log 1000$

(ii) When $t = 1$, $N = 2000$

$\therefore$ from equation (1)

$\log 2000 = k + \log 1000$

$\therefore k = \log 2000 - \log 1000 = \log\left(\frac{2000}{1000}\right)$

$\therefore k = \boxed{\log 2}$

$\therefore \log N = t \log 2 + \log 1000$

(iii) When $t=\frac{5}{2}$ hours

$\log N=\frac{5}{2}\log 2+\log 1000=\log(2^{\frac{5}{2}}\times 1000)$

$\therefore N=1000\times 4\sqrt{2}=4000\times 1.414$

$\therefore N=\boxed{5656}$.

---

3. In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, complete the following activity to find the number of times the bacteria are increased in 12 hours.

Let N be the number of bacteria present at time t.

Since the rate of increase is proportional to the number present,

$\therefore \frac{dN}{dt}=k\boxed{}$, where k is the constant of proportionality.

Integrating on both sides, we get

$\log N=k\boxed{}+c$ ... (1)

(i) If $t=0$, then $N=N_0$.

from equation (1)

$\log N_0=0+c$

$\therefore c=\log N_0$

(ii) If $t=4$ hours, then $N=2N_0$,

from equation (1)

$k=\boxed{}$

(iii) When $t=12$ hours

$N=\boxed{}N_0$. *(March '22–'23)*

Solution : Let N be the number of bacteria present at time t.

Since the rate of increase is proportional to the number present,

$\therefore \frac{dN}{dt}\propto N$

$\therefore \frac{dN}{dt}=k\boxed{N}$, where k is the constant of proportionality.

$\therefore \frac{1}{N} dN = k\,dt$

Integrating on both sides, we get

$\int \frac{1}{N} dN = k \int dt$ ... (1)

$\therefore \log N = k \boxed{t} + c$

(i) If $t = 0$, then $N = N_0$,

from equation (1)

$\log N_0 = 0 + c$

$\therefore c = \log N_0$

$\therefore \log N = kt + \log N_0$

(ii) If $t = 4$ hours, then $N = 2N_0$,

$\therefore$ from equation (1)

$\log 2N_0 = 4k + \log N_0$

$\therefore 4k = \log 2N_0 - \log N_0 = \log\left(\frac{2N_0}{N_0}\right)$

$\therefore k = \boxed{\frac{1}{4} \log 2}$

$\therefore \log N = \frac{t}{4} \log 2 + \log N_0$.

(iii) When $t = 12$ hours,

$\log N = \frac{12}{4} \log 2 + \log N_0 = 3 \log 2 + \log N_0 = \log 8\, N_0$

$N = \boxed{8}\, N_0$.

---

4. Solve the following differential equation $x^2y\,dx - (x^3 + y^3)\,dy = 0$ by filling the boxes in the following activity :

$x^2y\,dx - (x^3 + y^3)\,dy = 0$

This is homogeneous equation.

$\therefore$ we put $y = vx$

$\therefore \frac{dy}{dx} = \boxed{}$

From given equation

$$\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$$

$$\therefore v+x\frac{dv}{dx}=\frac{x^2\cdot vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{v^3+1}-v=\frac{v-v^4-v}{v^3+1}$$

$$\therefore x\frac{dv}{dx}=-\frac{v^4}{v^3+1}$$

$$\therefore \boxed{}\,dv=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{1}{v}+\frac{1}{v^4}\right)dv=-\int\frac{dx}{x}$$

$$\boxed{}+\log c=-\log x$$

Substituting the value of v, we get, $\boxed{}$ is the general solution. ***(July '22)***

Solution : $x^2ydx-(x^3+y^3)dy=0$

This is homogeneous equation.

$\therefore$ we put $y=vx$

$$\therefore \frac{dy}{dx}=\boxed{v+x\frac{dv}{dx}}$$

From given equation

$$\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$$

$$\therefore v+x\frac{dv}{dx}=\frac{x^2\cdot vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{v^3+1}-v=\frac{v-v^4-v}{v^3+1}$$

$$\therefore x\frac{dv}{dx}=-\frac{v^4}{v^3+1}$$

$$\therefore \boxed{\frac{v^3+1}{v^4}}\,dv=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{1}{v}+\frac{1}{v^4}\right)dv=-\int\frac{dx}{x}$$

$$\boxed{\log v+\frac{v^{-3}}{-3}}+\log c=-\log x$$

Substituting the value of v, we get

$$\log\left(\frac{y}{x}\right)-\frac{1}{3}\cdot\frac{1}{\left(\frac{y^3}{x^3}\right)}+\log c=-\log x$$

$$\therefore\ \log y-\log x-\frac{x^3}{3y^3}+\log c=-\log x$$

$$\therefore\ \log y-\frac{x^3}{3y^3}+\log c=0$$

$\therefore$ $\boxed{\log cy=\frac{x^3}{3y^3}}$ is the general solution.

---

5. The rate of growth of population is proportional to the number of inhabitants. If the population doubles in 25 years and the present population is 1,00,000, when will the city have population 4,00,000?

Let P be the population of time t.

Since the rate of growth of population is proportional to the number of inhabitants :

$$\frac{dP}{dt}\propto P$$

$\therefore$ differential equation can be written as : $\frac{dP}{dt}=kP$, where k is the constant of proportionality.

$$\therefore\ \frac{dP}{P}=k\,dt$$

On integrating, we get

$$\boxed{\quad\quad}=kt+c \qquad \dots(1)$$

(i) When $t=0$, $P=1{,}00{,}000$

$\therefore$ from (1), $\log 100000=k(0)+c$

$\therefore$ $c=\boxed{\quad\quad}$

$$\therefore \log\left(\frac{P}{100000}\right) = kt \quad \dots (2)$$

(ii) When $t = 25$, $P = 2,00,000$ as population doubles in 25 years

$\therefore$ from (2), $\log 2 = 25k$

$\therefore k = \boxed{}$

$$\therefore \log\left(\frac{P}{100000}\right) = \left(\frac{1}{25}\log 2\right)t$$

(iii) When $P = 4,00,000$

$$\log\left(\frac{400000}{100000}\right) = \left(\frac{1}{25}\log 2\right)t$$

$$\therefore \log 4 = \left(\frac{1}{25}\log 2\right)t$$

$\therefore t = \boxed{}$ years. ***(March '24)***

Solution : Let P be the population of time t.

Since the rate of growth of population is proportional to the number of inhabitants :

$$\frac{dP}{dt} \propto P$$

$\therefore$ differential equation can be written as : $\frac{dP}{dt} = kP$, where k is the constant of proportionality.

$$\therefore \frac{dP}{P} = k\,dt$$

On integrating, we get

$$\int \frac{1}{P}dP = k\int dt$$

$$\therefore \boxed{\log P} = kt + c \quad \dots (1)$$

(i) When $t = 0$, $P = 1,00,000$

$\therefore$ from (1), $\log 100000 = k(0) + c$

$\therefore c = \boxed{\log 100000}$

$\therefore \log P = kt + \log 100000$

$\therefore \log P - \log 100000 = kt$

$$\therefore \log\left(\frac{P}{100000}\right) = kt \quad \dots (2)$$

(ii) When $t = 25$, $P = 2,00,000$ as population doubles in 25 years

$\therefore$ from (2), $\log\left(\frac{200000}{100000}\right) = k(25)$

$\therefore \log 2 = 25k$

$\therefore k = \boxed{\frac{1}{25}\log 2}$

$\therefore \log\left(\frac{P}{100000}\right) = \left(\frac{1}{25}\log 2\right)t$

(iii) When $P = 4,00,000$

$\log\left(\frac{400000}{100000}\right) = \left(\frac{1}{25}\log 2\right)t$

$\therefore \log 4 = \left(\frac{1}{25}\log 2\right)t$

$\therefore \log 2^2 = \log 2^{t/25}$

$\therefore 2 = \frac{t}{25}$ $\qquad \therefore t = \boxed{50}$ years.

---

6. Solve the differential equation : $y - x\frac{dy}{dx} = 0$.

Given equation is $y - x\frac{dy}{dx} = 0$.

Separating the variables, we get

$$\frac{dx}{\square} = \frac{dy}{\square}$$

Integrating, we get

$$\int \frac{dx}{\square} = \int \frac{dy}{\square} + c$$

$\therefore \log x = \boxed{} + c$

$\therefore \log x - \log y = \log c_1$, where $c = \log c_1$

$\therefore \log\left(\frac{x}{y}\right) = \log c_1$

$\therefore \frac{x}{\square} = c_1$

Hence, the required solution is $x = c_1 y$. ***(July '24)***

Solution : Given equation is $y - x\dfrac{dy}{dx} = 0.$

Separating the variable, we get

$$\frac{dx}{\boxed{x}} = \frac{dy}{\boxed{y}}$$

Integrating, we get

$$\int \frac{dx}{\boxed{x}} = \int \frac{dy}{\boxed{y}} + c$$

$$\therefore \log x = \boxed{\log y} + c$$

$\therefore \log x - \log y = \log c_1$, where $c = \log c_1$

$$\therefore \log\left(\frac{x}{y}\right) = \log c_1$$

$$\therefore \frac{x}{\boxed{y}} = c_1$$

Hence, the required solution is $x = c_1 y$.

---

7. Solve the following differential equation :

$(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0.$

Separating the variables, the given equation can be written as :

$$\boxed{\quad}\, dy + \boxed{\quad}\, dx = 0$$

$$\therefore \left(y^{-2} - \frac{1}{y}\right) dy + \left(x^{-2} + \frac{1}{x}\right) dx = 0$$

$$\therefore \boxed{\quad}\, dy - \frac{1}{y}\, dy + x^{-2} dx + \boxed{\quad}\, dx = 0$$

Integrating, we get

$$\int y^{-2}\, dy - \int \frac{1}{y}\, dy + \int x^{-2}\, dx + \int \frac{1}{x}\, dx = 0$$

$$\therefore \frac{y^{-1}}{-1} - \boxed{\quad} + \frac{x^{-1}}{-1} + \boxed{\quad} = c$$

$$\therefore -\frac{1}{y} - \frac{1}{x} + \log x - \log y = c$$

$\therefore \log x - \log y = \boxed{\quad} + c$ is the required solution. ***(March '25)***

Solution : Separating the variables, the given equation can be written as :

$$\boxed{\frac{1-y}{y^2}}dy+\boxed{\frac{1+x}{x^2}}dx=0$$

$$\therefore \left(y^{-2}-\frac{1}{y}\right)dy+\left(x^{-2}+\frac{1}{x}\right)dx=0$$

$$\therefore \boxed{y^{-2}}dy-\frac{1}{y}dy+x^{-2}dx+\boxed{\frac{1}{x}}dx=0$$

Integrating, we get

$$\int y^{-2}dy-\int\frac{1}{y}dy+\int x^{-2}dx+\int\frac{1}{x}dx=0$$

$$\therefore \frac{y^{-1}}{-1}-\boxed{\log y}+\frac{x^{-1}}{-1}+\boxed{\log x}=c$$

$$\therefore -\frac{1}{y}-\frac{1}{x}+\log x-\log y=c$$

$\therefore \log x-\log y=\boxed{\frac{1}{x}+\frac{1}{y}}+c$ is the required solution.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The order and degree of $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}}=8\frac{d^3y}{dx^3}$ are respectively

   (a) 3, 1 (b) 1, 3 (c) 3, 3 (d) 1, 1 ***(March '22)***

2. The order and degree of the differential equation $\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+8=0$ are

   (a) order = 2, degree = 2 (b) order = 1, degree = 2
   (c) order = 1, degree = 1 (d) order = 2, degree = 1 ***(July '24)***

3. The order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=a^x$ are .... respectively.

   (a) 1, 1 (b) 1, 2 (c) 2, 2 (d) 2, 1 ***(March '25)***

4. The differential equation of $y=k_1+\frac{k_2}{x}$ is

(a) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ (b) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

(c) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=0$ (d) $x\frac{d^2y}{dx^2}-2\frac{dy}{dx}=0$

5. The differential equation of $y=k_1e^x+k_2e^{-x}$ is

(a) $\frac{d^2y}{dx^2}-y=0$ (b) $\frac{d^2y}{dx^2}+\frac{dy}{dx}=c$

(c) $\frac{d^2y}{dx^2}+y\frac{dy}{dx}=0$ (d) $\frac{d^2y}{dx^2}+y=0$ ***(March '24)***

6. The solution of the differential equation $\frac{dy}{dx}=1$ is

(a) $y+x=c$ (b) $xy=c$

(c) $x^2+y^2=c$ (d) $y=x+c$ ***(Sept. '21)***

7. If $y=Ae^{3x}+Be^{-3x}$ is the general solution of the differential equation, then the differential equation is

(a) $\frac{d^2y}{dx^2}=9y$ (b) $\frac{d^2y}{dx^2}=3y$ (c) $\frac{d^2y}{dx^2}=-3y$ (d) $\frac{d^2y}{dx^2}=-9y$ ***(Sept. '21)***

8. The solution of $\frac{dy}{dx}+\frac{x^2}{y^2}=0$ is

(a) $x^3+y^2=7$ (b) $x^2+y^2=c$

(c) $x^3+y^3=c$ (d) $x+y=c$

9. The integrating factor of $\frac{dy}{dx}-y=e^x$ is e^{-x}, then its solution is

(a) $ye^{-x}=x+c$ (b) $ye^x=x+c$

(c) $ye^x=2x+c$ (d) $ye^{-x}=2x+c$

10. The integrating factor of $\frac{dy}{dx}+y=e^{-x}$ is

(a) x (b) $-x$ (c) e^x (d) e^{-x} ***(March '23)***

11. The integrating factor of the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^3-3$ is

(a) $\log x$ (b) e^x (c) $\frac{1}{x}$ (d) x ***(March '25)***

12. Bacteria increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be $4M$ in

(a) 4 hours (b) 6 hours (c) 8 hours (d) 10 hours

Answers

1. (c) 3, 3
2. (d) order = 2, degree = 1
3. (c) 2, 2
4. (b) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$
5. (a) $\frac{d^2y}{dx^2}-y=0$
6. (d) $y=x+c$
7. (a) $\frac{d^2y}{dx^2}=9y$
8. (c) $x^3+y^3=c$
9. (a) $ye^{-x}=x+c$
10. (c) e^x
11. (d) x
12. (b) 6 hours.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The degree of differential equation $e^{\frac{dy}{dx}}=\frac{dy}{dx}+x^2$ is not defined. ***(Sept. '21)***

2. The degree of differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3=a^x$ is 3. ***(March '23)***

3. $y=e^x$ is a general solution of $\frac{dy}{dx}=y$.

4. The integrating factor of $\frac{dy}{dx}-y=x$ is e^x. ***(July '24)***

5. The integrating factor of the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^3$ is $-x$. ***(March '22)***

6. The differential equation of $x^2+4y^2=4b^2$ is $x+y\frac{dy}{dx}=0$.

7. The order of highest order derivative occurring in the differential equation is called degree of the differential equation. ***(July '22)***

8. The integrating factor of the differential equation $\frac{dy}{dx}-y=x$ is e^{-x}. ***(July '23)***

9. Order and degree of differential equation are always positive integers. **(March '24)**

10. The differential equation obtained by eliminating arbitrary constants from $bx+ay=ab$ is $\dfrac{d^2y}{dx^2}=0$. **(March '25)**

Answers

1. True **2.** False **3.** False **4.** False **5.** False **6.** False **7.** False
8. True **9.** True **10.** True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. $y^2=(x+c)^3$ is the general solution of the differential equation .... **(March '22)**

2. Order and degree of a differential equation are always .... integers. **(July '22)**

3. The differential equation by eliminating arbitrary constants from $bx+ay=ab$ is ....

4. The differential equation of $y=c^2+\dfrac{c}{x}$ is ....

5. The order and degree of $\left(\dfrac{dy}{dx}\right)^3-\dfrac{d^3y}{dx^3}+ye^x=0$ are .... **(July '23)**

6. A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called .... solution. **(March '24)**

Answers

1. $\dfrac{dy}{dx}=\dfrac{3}{2}y^{\frac{1}{3}}$ **2.** positive **3.** $\dfrac{d^2y}{dx^2}=0$ **4.** $x^4\left(\dfrac{dy}{dx}\right)^2-x\dfrac{dy}{dx}=y$

5. order $=3$, degree $=1$ **6.** particular.

SECTION – II

Question Set 9
COMMISSION, BROKERAGE AND DISCOUNT

(Marks with option : 06)

9.1 COMMISSION AND BROKERAGE

Remember :

1. **Commission :** The charges paid to an agent for doing the work on behalf of some other person.
2. **Brokerage :** The commission, the broker gets. It is charged to both the parties.
3. **Del credere commission :** Additional commission received by an agent other than the usual commission.

Solved Examples	3 or 4 marks each

Ex. 1. A shop is sold at 30 % profit. The amount of brokerage at the rate of $\frac{3}{4}$% amounts to ₹ 73,125. Find the cost of the shop.

Solution :

Let the sale price of the shop be ₹ x. Brokerage is ₹ 73,125.

Brokerage at $\frac{3}{4}$% on ₹ $x = x \times \frac{3}{4} \times \frac{1}{100}$

$\therefore\ 73125 = \frac{3x}{400}$ $\qquad \therefore\ x = \frac{73125 \times 400}{3}$ $\qquad \therefore\ x = ₹\ 97{,}50{,}000$

The selling price of the shop is ₹ 97,50,000. ... (1)

Let the cost price of the shop be ₹ y. Sold at 30% profit.

$\therefore$ profit $= y \times \frac{30}{100} = \frac{3y}{100}$

$\therefore$ selling price $= ₹ \left(y + \frac{3y}{10}\right) = ₹\ \frac{13y}{10}$ ... (2)

From (2) and (1). $\frac{13y}{10} = 9750000$

$\therefore y = \frac{9750000 \times 10}{13} = 7500000$

The cost of the shop is ₹ 75,00,000.

Ex. 2. Ms Saraswati was paid ₹ 88,000 as commission on the sale of computers at the rate of 12.5%. If the price of each computer was ₹ 32,000, how many computers did she sell? ***(March '23)***

Solution :

Commission at 12.5% on a computer costing ₹ 32,000

$= ₹\ 32000 \times \frac{12.5}{100} = 4000$

Ms Saraswati received ₹ 88,000 as commission on selling computers.

The number of computers sold

$= \frac{\text{total commission}}{\text{commission on 1 computer}} = \frac{88000}{4000} = 22$

Hence, Ms Saraswati sold 22 computers.

Ex. 3. Deepak's salary was increased from ₹ 4000 to ₹ 5000. The sales being the same, due to reduction in the rate of commission from 3 % to 2 %, his income remained unchanged. Find his sales. ***(March '24)***

Solution :

Let Deepak's sale be ₹ x.

Commission at 3 % $= ₹\ x \times \frac{3}{100} = ₹\ \frac{3x}{100}$

$\therefore$ his income is ₹ $\left(4000 + \frac{3x}{100}\right)$ ... (1)

Now, the commission is 2 %

$\therefore$ commission $= ₹\ x \times \frac{2}{100} = ₹\ \frac{2x}{100}$.

But now the salary is ₹ 5000.

$\therefore$ his income is ₹ $\left(5000 + \frac{2x}{100}\right)$ ... (2)

There is no change in his income.

$$\therefore \left(4000+\frac{3x}{100}\right)=\left(5000+\frac{2x}{100}\right) \qquad \text{... [By (1) and (2)]}$$

$$\therefore \frac{3x}{100}-\frac{2x}{100}=5000-4000 \qquad \therefore \frac{x}{100}=1000$$

$\therefore x=100000$

Deepak's sales is ₹ 1,00,000.

Ex. 4. Mr Pavan is paid a fixed weekly salary plus commission based on percentage of sales made by him. If on the sale of ₹ 68,000 and ₹ 73,000 in two successive weeks, he received in all ₹9880 and ₹10,180, find his weekly salary and the rate of commission paid to him. ***(Sept. '21)***

Solution :

Income of Mr Pavan = weekly salary + commission on sales

Salary + commission on ₹ 68,000 = ₹ 9880 ... (1)

Salary + commission on ₹ 73,000 = ₹ 10,180 ... (2)

Subtracting (1) from (2), we get

Commission on ₹ 5000 [₹ 73000 – ₹ 68000]

= ₹ 300 [₹ 10180 – ₹ 9880]

$\therefore$ the rate of commission $=\frac{300}{5000}\times 100=6\,\%$.

Commission on ₹ 68,000 at 6%

= ₹ $68000\times\frac{6}{100}$ = ₹ 4080 ... (3)

From (1) and (3),

Salary = ₹(9880 – 4080) = ₹ 5800

Hence, fixed weekly salary is ₹ 5800 and the rate of commission is 6 %.

Ex. 5. A salesman receives 8% commission on the total sales. If his sales exceeds ₹ 20,000, he receive an additional commission at 2% on the sales over ₹ 20,000. If he receives ₹ 7600 as commission, find his total sales.

Solution :

Let the total sales be ₹ x.

$\therefore$ commission on total sales at the rate of 8% $= x\times\frac{8}{100}=\frac{8x}{100}$

Sales exceeding ₹ 20,000 = ₹ $(x-20000)$

$\therefore$ commission on excess sales at the rate of 2%

$$= (x - 20000) \times \frac{2}{100} = \frac{2(x-20000)}{100}$$

$$\therefore \text{ total commission} = \frac{8x}{100} + \frac{2(x-20000)}{100}$$

This is given to be ₹ 7600

$$\therefore \frac{8x}{100} + \frac{2(x-20000)}{100} = 7600$$

$$\therefore 8x + 2x - 40000 = 760000$$

$$\therefore 10x = 760000 + 40000 = 800000$$

$$\therefore x = 80000$$

Hence, total sales is ₹ 80,000.

Examples for Practice	**3 or 4 marks each**

1. A salesman receives 3% commission on the sales upto ₹ 50,000 and 4% commission on the sales over ₹ 50,000. Find his total income on the sale of ₹ 2,00,000.
2. Anita is allowed 6.5 % commission on the total sales made by her, plus a bonus of $\frac{1}{2}$% on the sale over ₹ 20,000. If her total commission amount to ₹ 3400. Find the sales made by her.
3. A merchant gives his agent 5% ordinary commission plus 2% del credere commission on sale of goods, worth ₹ 55,000. How much does the agent receive? How much does the merchant receive?
4. The income of an agent remains unchanged though the rate of commission is increased from 5% to 6.25%. Find the percentage reduction in the value of business.
5. A salesman is paid a fixed monthly salary plus a commission on the sales. If on the sales of ₹ 96,000 and ₹ 1,08,000 in two successive months he receives in all ₹ 17,600 and ₹ 18,800 respectively, find his monthly salary and rate of commission paid to him.

Answers

1. ₹ 7500 **2.** ₹ 50,000 **3.** ₹ 3850, ₹ 51,150 **4.** 20% **5.** ₹ 8000, 10%.

9.2 DISCOUNT

Remember :

1. **Discount :** It is reduction in the price of an article.
2. **Trade Discount :** Discount allowed by one trader to another is called Trade discount. Trade discount is allowed on list price or marked price of the goods.
3. **Cash Discount :** Discount allowed on cash payment. It is allowed after deducting the trade discount from list price. Cash discount is calculated on invoice price.
4. (1) Invoice Price = List Price – Trade Discount [IP = LP – TD]

   (2) The selling price / Net selling price = Invoice Price – Cash Discount [NP = IP – CD]

   (3) Profit = Net Selling Price – Cost Price [P = NP – CP]

   (4) Loss = Cost Price – Net Selling Price [L = CP – NP]
5. **Present Worth (PW) :** Payment of goods made on the spot.
6. **Sum Due (SD) :** It is sum of the present worth and Trade discount.

   Sum Due (SD) = Present Worth (PW) + True Discount (TD).

   $$\text{TD} = \frac{\text{PW} \times n \times r}{100} \qquad \therefore \text{SD} = \text{PW} + \frac{\text{PW} \times n \times r}{100}$$

   $$\therefore \text{SD} = \text{PW}\left[1 + \frac{n \times r}{100}\right]$$
7. **Face Value (FV) :** Amount of the bill drawn. It is sum due on present worth.
8. **Banker's Discount (BD) :** Amount deducted from the face value of the bill. It is calculated on face value (sum due).
9. **Cash Value (CV) :** Amount paid of the bill after deducting banker's discount.
10. **Banker's Gain (BG) :** It is the difference between the banker's discount and the true discount. It is interest on true discount.
11. (1) $\text{BD} = \dfrac{\text{SD} \times n \times r}{100}$ (2) BG = BD – TD

(3) $BG = \dfrac{TD \times n \times r}{100}$ (4) $BD = TD\left(1 + \dfrac{nr}{100}\right)$

(5) Cash value $=$ SD $-$ BD

Solved Examples	**3 or 4 marks each**

Ex. 6. What is the true discount on a sum of ₹ 12,720 due 9 months hence at 8% p.a. simple interest?

Solution : Given : SD $=$ ₹ 12,720, $n = \dfrac{9}{12}$ years, $r = 8\%$

Now, SD $=$ PW $+$ TD

$$\therefore\ 12720 = PW + \frac{PW \times n \times r}{100} = PW\left[1 + \frac{n \times r}{100}\right]$$

$$= PW\left[\frac{1 + \left(\frac{9}{12}\right) \times 8}{100}\right]$$

$$= PW\left[1 + \frac{6}{100}\right] = PW\left(\frac{106}{100}\right)$$

$$\therefore\ PW = \frac{12720 \times 100}{106} = 12000$$

$$\therefore\ TD = SD - PW = 12720 - 12000 = 720$$

Hence, true discount (TD) $=$ ₹ 720.

Ex. 7. The difference between true discount and banker's discount on a bill due 6 months hence at 4% is ₹ 160. Calculate true discount, banker's discount and amount of bill.

Solution : Let TD be ₹ x.

Given : $n = 6$ months $= \dfrac{1}{2}$ years, $r = 4\%$

BG $=$ BD $-$ TD $=$ ₹ 160

But BG $=$ Interest on TD for 6 months at 4% p.a.

$$\therefore\ 160 = x \times \frac{1}{2} \times \frac{4}{100} = \frac{x}{50}$$

$$\therefore\ x = 8000$$

$\therefore$ true discount (TD) = ₹ 8000

BD = BG + TD = 160 + 8000 = 8160

$\therefore$ banker's discount (BD) = ₹ 8160

Let the amount of bill (FV) be ₹ y.

Now, BD = Interest on FV for 6 months at 4% p.a.

$\therefore 8160 = y \times \frac{1}{2} \times \frac{4}{100} = \frac{y}{50}$

$\therefore y = 408000$

$\therefore$ the amount of bill is ₹ 4,08,000

Hence, the true discount is ₹ 8000, banker's discount is ₹ 8160 and the amount of bill is ₹ 4,08,000.

Ex. 8. A bill drawn on 5th June for 6 months was discounted at the rate of 5% p.a. on 19th October. If the cash value of the bill is ₹ 43,500, find the face value of the bill.

Solution : Cash value = ₹ 43,500, $r = 5\%$

Date of drawing = 5th June

Period of the bill = 6 months

Nominal due date = 5th December

Legal due date = 8th December

Date of discounting = 19th October

$\therefore$ number of days from the date of discounting to the legal due date is as follows :

Oct.	Nov.	Dec.	Total
12	30	8	50

$\therefore$ period $n = \frac{50}{365}$ years

Let the face value (SD) of the bill be ₹ x

$$BD = \frac{FV \times n \times r}{100} = x \times \frac{50}{365} \times \frac{5}{100} = \frac{x}{146}$$

Also, BD = FV − CV = $x - 43500$

$$\therefore \frac{x}{146} = x - 43500$$

$\therefore\ 43500 = x - \frac{x}{146}$

$\therefore\ 43500 = \frac{146x - x}{146}$ $\qquad \therefore\ 43500 \times 146 = 145x$

$\therefore\ x = \frac{43500 \times 146}{145}$

$\therefore\ x = 43800$

Hence, the face value of the bill is ₹ 43,800.

Ex. 9. A certain sum due 3 months hence is $\frac{21}{20}$ of the present worth. What is the rate of interest?

Solution :

$SD = \frac{21}{20}(PW),\ n = 3 \text{ months} = \frac{1}{4} \text{ years}$

$SD = PW + TD$

$\therefore\ PW + TD = \frac{21}{20} PW$

$\therefore\ TD = \frac{21}{20} PW - PW$

$\therefore\ TD = PW\left(\frac{21}{20} - 1\right)$

$\therefore\ TD = PW\left(\frac{21 - 20}{20}\right) = \frac{1}{20} \times PW$

Also, $TD = \frac{PW \times n \times r}{100} = PW \times \frac{1}{4} \times \frac{r}{100}$

$\therefore\ \frac{1}{20} \times PW = PW \times \frac{1}{4} \times \frac{r}{100}$

$\therefore\ \frac{1}{20} = \frac{r}{400}$

$\therefore\ \frac{400}{20} = r$ $\qquad \therefore\ r = 20$

Hence, the rate of interest is 20% p.a.

Ex. 10. A bill of ₹ 15,000 drawn on 15^{th} February 2015 for 10 months was discounted on 13th May 2015 at $3\frac{3}{4}$ % p.a. Calculate banker's discount.

Solution :

Face value of the bill = ₹ 15,000, $r = 3\frac{3}{4}\% = \frac{15}{4}\%$

Date of drawing = 15^{th} February 2015

Period of the bill = 10 months

Nominal due date = 15^{th} December 2015

Legal due date = 18^{th} December 2015

Date of discounting = 13^{th} May 2015

Number of days from the date of discounting to the legal due date is as follows :

May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Total
18	30	31	31	30	31	30	18	219

$\therefore$ period $= \frac{219}{365} = \frac{3}{5}$ years

BD = Interest on FV for 219 days at the rate of $\frac{15}{4}$ % p.a.

$$= \frac{15000 \times \frac{3}{5} \times \frac{15}{4}}{100} = 337.5$$

Hence, the banker's discount = **₹ 337.5**.

Examples for Practice	**3 or 4 marks each**

1. The true discount on the sum due 8 months hence at 12% p.a. is ₹ 560. Find the sum due and present worth of the bill.
2. A trader offers 25% discount on the catalogue price of a radio and yet makes 20% profit. If he gains ₹ 160 per radio, what must be the catalogue price of the radio?
3. A bill of a certain sum drawn on 28^{th} February 2007 for 8 months was encashed on 26^{th} March 2007 for ₹ 10,992 at 14% p.a. Find the face value of the bill. ***(July '23)***
4. A bill of ₹ 10,100 drawn on 14^{th} January for 5 months was discounted on 26^{th} March. The customer was paid ₹ 9939.25. Calculate the rate of interest.
5. (1) A bill was drawn on 14^{th} April 2005 for ₹ 3500 and was discounted on 6^{th} July 2005 at 5% per annum. The banker paid ₹ 3465 for the bill. Find the period of the bill. ***(March '22)***

(2) A bill was drawn on 14th April for ₹ 7000 and was discounted on 6th July at 5% p.a. The banker paid ₹ 6930 for the bills. Find the period of the bill. ***(July '24)***

6. A bill of ₹ 51,000 was drawn on 18th February 2010 for 9 months. It was encashed on 28th June 2010 at 5% p.a. Calculate the banker's gain and true discount.

Answers

1. Sum due = ₹ 7560, present worth = ₹ 7000

2. ₹ 1280 **3.** ₹ 12,000 **4.** 7%

5. (1) 14th April 2005 to 14th September 2005, i.e. 5 months (2) 5 months

6. BG = ₹ 20, TD = ₹ 1000.

ACTIVITIES | 4 marks each

1. A bill of ₹ 18,000 was discounted for ₹ 17,568 at a bank on 25th October 2017. If the rate of interest was 12% p.a., what is the legal due date?

Given : SD = ₹ 18,000, CV = ₹ 17,568, r = 12% p.a.

Now, BD = $\boxed{}$ = 18000 – 17568 = ₹ 432

Also, BD = $\boxed{}$

$\therefore 432 = \dfrac{18000 \times n \times 12}{100}$

$\therefore n = \dfrac{432 \times 100}{18000 \times 12}$

$\therefore n = \dfrac{1}{5}$ years = $\boxed{}$ days

The period for which the discount is deducted is 73 days which is counted from the date of discounting, i.e. 25th October 2017 :

October	November	December	January	Total
6	30	31	6	73

Hence, legal due date is $\boxed{}$. ***(March '25)***

Solution : Given : SD = ₹ 18,000, CV = ₹ 17,568, r = 12% p.a.

Now, BD = $\boxed{\text{SD} - \text{CV}}$ = 18000 – 17568 = ₹ 432

Also, $BD = \boxed{\dfrac{SD \times n \times r}{100}}$

$\therefore 432 = \dfrac{18000 \times n \times 12}{100}$

$\therefore n = \dfrac{432 \times 100}{18000 \times 12}$

$\therefore n = \dfrac{1}{5}$ years $= \dfrac{1}{5} \times 365 = \boxed{73}$ days

The period for which the discount is deducted is 73 days which is counted from the date of discounting, i.e. 25th October 2017 :

October	November	December	January	Total
6	30	31	6	73

Hence, legal due date is $\boxed{\text{6th January 2018}}$

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2. Prakash gets a commission at 10% on cash sales and 8% on credit sales. If he receives ₹ 4400 as commission on the total sales of ₹ 50,000. Find the sales made by him in cash and on credit.

Let the cash sales be ₹ x.

$\therefore$ credit sales $=$ ₹ $\boxed{}$

Total commission $= 10\%$ on $x + 8\%$ on $(50000 - x)$

$\therefore 4400 = x \times \dfrac{10}{100} + \boxed{} = \dfrac{400000 + \boxed{}}{100}$

$\therefore x = \boxed{}$

Hence, Prakash's cash sales is ₹ 20,000 and his credit sales is ₹ (50,000 – 20,000) = ₹ 30,000.

Solution : Let the cash sales be ₹ x.

$\therefore$ credit sales $=$ ₹ $\boxed{(50000 - x)}$

Total commission $= 10\%$ on $x + 8\%$ on $(50000 - x)$

$\therefore 4400 = x \times \dfrac{10}{100} + \boxed{(50000 - x) \times \dfrac{8}{100}}$

$$=\frac{10x}{100}+\frac{400000-8x}{100}$$

$$=\frac{400000+\boxed{2x}}{100}$$

$\therefore\ 440000=400000+2x \qquad \therefore\ 2x=40000$

$\therefore\ x=\boxed{20000}$

Hence, Prakash's cash is ₹ 20,000 and his credit sales is

₹ (50,000 − 20,000) = ₹ 30,000.

---

3. Find the true discount on the sum of ₹ 12,720 due 9 months hence at 8% p.a. simple interest by completing the following activity :

SD = ₹ 12,720, $n=\frac{9}{12}=\frac{3}{4}$ years, $r=8\%$

Now, $\text{SD}=\text{PW}+\text{TD}=\text{PW}+\frac{\text{PW}\times n\times r}{100}=\text{PW}\left[\boxed{}\right]$

$\therefore\ 12720=\text{PW}\left[\boxed{}\right]$

$\therefore\ \text{PW}=\boxed{}$

$\therefore\ \text{TD}=\text{SD}-\text{PW}=\boxed{}$.

Solution : SD = ₹ 12,720, $n=\frac{9}{12}=\frac{3}{4}$ years, $r=8\%$

Now, $\text{SD}=\text{PW}+\text{TD}=\text{PW}+\frac{\text{PW}\times n\times r}{100}$

$$=\text{PW}\left[\boxed{1+\frac{n\times r}{100}}\right]$$

$$\therefore\ 12720=\text{PW}\left[\boxed{1+\frac{\frac{3}{4}\times 8}{100}}\right]$$

$$=\text{PW}\left[1+\frac{6}{100}\right]=\text{PW}\left(\frac{106}{100}\right)$$

$\therefore\ \text{PW}=\frac{12720\times 100}{106}=\boxed{₹\ 12{,}000}$

$\therefore\ \text{TD}=\text{SD}-\text{PW}=12720-12000=\boxed{₹\ 720}$.

---

4. A bill was drawn on 14th April for ₹ 7000 and was discounted on 6th July at 5% p.a. The banker paid ₹ 6930 for the bill. Complete the following activity to find, the period of the bill by filling boxes in the following activity.

SD = ₹ 7000, $r = 5\%$, CV = ₹ 6930

BD = $\boxed{}$

$$BD = \frac{SD \times n \times r}{100}$$

$n = \boxed{}$ days

Number of days to be counted from date of discounting, i.e. 6th July.

July	Aug.	Sept.	Total

∴ period of bill is $\boxed{}$ months. *(July '22)*

Solution :

SD = ₹ 7000, $r = 5\%$, CV = ₹ 6930

BD = SD − CV = 7000 − 6930

BD = $\boxed{70}$

$$BD = \frac{SD \times n \times r}{100}$$

$$70 = \frac{7000 \times n \times 5}{100}$$

$$n = \frac{70 \times 100}{7000 \times 5} = \frac{1}{5} \text{ years}$$

$$n = \frac{1}{5} \times 365 \text{ days}$$

$n = \boxed{73}$ days

Number of days to be counted from date of discounting, i.e. 6th July.

July	Aug.	Sept.	Total
25	31	17	73

Hence, the legal due date is 17th September.

∴ nominal due date is 14th September.

Date of drawing is 14^{th} April.

$\therefore$ period of bill is 14^{th} April to 14^{th} September.

$\therefore$ period of bill is $\boxed{5}$ months.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The marked price is also called as

(a) cost price (b) selling price

(c) list price (d) invoice price *(July '22)*

2. An agent who gives guarantee to his principal that the party will pay the sale price of goods is called

(a) auctioneer (b) del credere agent

(c) factor (d) broker *(July '22; March '25)*

3. The payment date after adding 3 days of grace period is known as

(a) the legal due date (b) the nominal due date

(c) days of grace (d) date of drawing *(July '23)*

4. The sum due is also called as

(a) face value (b) present value

(c) cash value (d) true discount *(March '23)*

5. Banker's gain is the simple interest on

(a) banker's discount (b) face value

(c) cash value (d) true discount

6. The date on which the period of the bill expires is called

(a) legal due date (b) grace date

(c) nominal due date (d) date of drawing *(Sept. '21; July '24)*

7. If TD is ₹ 720 and SD is ₹ 12,720, then PW is

(a) ₹ 13,440 (b) ₹ 12,000

(c) ₹ 13,400 (d) ₹ 12,700

8. The difference between face value and present worth is called

(a) Banker's Discount (b) True Discount

(c) Banker's Gain (d) Cash Value *(March '22–'24)*

9. If the present worth of a bill due six months hence is ₹ 23000 at 8% p.a., then TD is

(a) ₹ 920 (b) ₹ 1840

(c) ₹ 11,040 (d) ₹ 230

Answers

1. (c) list price 2. (b) del credere agent 3. (a) the legal due date
4. (a) face value 5. (d) true discount 6. (c) nominal due date
7. (b) ₹ 12,000 8. (b) True Discount 9. (a) ₹ 920.

TRUE OR FALSE | 1 mark each

State whether the following statements are True or False :

1. The date on which the period of the bill expires is called the nominal due date.
2. Broker is an agent who gives a guarantee to the seller that the buyer will pay the sale price of goods. ***(March '24)***
3. The difference between the banker's discount and true discount is called sum due.
4. Trade discount is allowed on catalogue price.
5. The banker's discount is also called as true discount. ***(March '23)***

Answers

1. True 2. False 3. False 4. True 5. True.

FILL IN THE BLANKS | 1 mark each

Fill in the following blanks :

1. The banker's discount is also called ....
2. If legal due date is 18^{th} December, then nominal due date is ....
3. .... = List price (Catalogue Price) – Trade Discount.
4. The difference between the banker's discount and the true discount is called .... ***(July '23)***
5. The date by which the buyer is legally allowed to pay the amount is known as .... ***(March '23)***

6. A wholeseller allows 25% trade discount and 5% cash discount. The net price of an article marked at ₹ 1600 is .... ***(March '22)***

7. The payment worth of a sum of ₹ 10,920 due six months hence at 8% p.a. simple interest is .... ***(Sept. '21)***

8. The banker's discount is always .... than the true discount. ***(March '24)***

9. If an agent charges 12% commission on the sales of ₹ 48,000, then the total commission is ₹ .... ***(July '24)***

10. The amount paid to the holder of the bill after deducting banker's discount is known as .... ***(March '25)***

Answers

1. Commercial discount **2.** 15^{th} December **3.** Invoice price
4. banker's gain **5.** legal due date **6.** ₹ 1140 **7.** ₹ 10,500
8. higher **9.** 5760 **10.** cash value.

Question Set 10

INSURANCE AND ANNUITY

(Marks with option : 04)

10.1 INSURANCE

Remember :

1. Premium = Rate of premium × Policy value
2. General Insurance : Claim $= \dfrac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$

Solved Examples | **3 or 4 marks each**

Ex. 1. An agent places insurance for ₹ 4,00,000 on life of a person. The premium is to be paid annually at the rate of ₹ 35 per thousand per annum. find the agent's commission at 15% on the premium.

(March '23)

Solution : Here, the policy value = ₹ 4,00,000 and the rate of premium is ₹ 35 per thousand per annum.

∴ amount of premium $= \dfrac{35}{1000} \times 400000 =$ ₹ 14,400.

Now, rate of commission = 15%

∴ amount of commission $= \dfrac{15}{100} \times 14000 =$ ₹ 2100

Hence, agent's commission @ 15% is ₹ 2100.

Ex. 2. A building is insured for 75% of its value. The annual premium at 0.70% amounts to ₹ 2625. If the building is damaged to the extent of 60% due to fire, how much can be claimed under the policy? ***(July '22)***

Solution :

Let the value of the building be ₹ x

Then the policy value of the building $= x \times \dfrac{75}{100} =$ ₹ $\dfrac{3x}{4}$

Rate of premium is 0.70%

Premium amount = ₹ 2625

Amount of the premium = Policy value × Rate of premium

$\therefore 2625 = \frac{3x}{4} \times \frac{0.70}{100}$

$\therefore x = \frac{2625 \times 4 \times 100}{3 \times 0.70}$

$\therefore x = 500000$

$\therefore$ the value of the building = ₹ 5,00,000

Damage = 60% of the value of the building

$= 500000 \times \frac{60}{100} =$ ₹ 3,00,000

Policy value of the building $= \frac{3x}{4} = \frac{3}{4} \times 500000 =$ ₹ 3,75,000

$\text{Claim} = \frac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$

$= \frac{375000}{500000} \times 300000 =$ ₹ 2,25,000

Hence, the claim under the policy = ₹ 2,25,000.

Ex. 3. A house valued at ₹ 8,00,000 is insured at 75% of its value. If the rate of premium is 0.80%. Find the premium paid by the owner of the house. If agent's commission is 9% of the premium, find agent's commission.

(March '25)

Solution :

The value of the house = ₹ 8,00,000.

The policy value of the house = ₹ $\left(800000 \times \frac{75}{100}\right) =$ ₹ 600000

The rate of premium = 0.80%

Amount of premium = ₹ $600000 \times \frac{0.80}{100} =$ ₹ 4800

Agent's commission at 9% of the premium = ₹ $4800 \times \frac{9}{100} =$ ₹ 432

The premium paid by the owner = ₹ 4800;

Agent's commission = ₹ 432.

Ex. 4. A shopkeeper insures his shop valued ₹ 20,00,000 for 80% of its value. He pays a premium of ₹ 80,000. Find the rate of premium. If the agent gets commission at 12%, find the agent's commission.

Solution :

Property value = ₹ 20,00,000

Insured value = 80% of property value $= 2000000 \times \frac{80}{100} =$ ₹ 16,00,000.

Premium = ₹ 80,000

$\therefore$ rate of premium $= \frac{80000 \times 100}{1600000} = 5\%$

Commission paid at the rate of 12% on the premium

$\therefore$ agent's commission $= 80000 \times \frac{12}{100} =$ ₹ 9600

Hence, rate of premium = 5% and agent's commission = ₹ 9600.

Ex. 5. A warehouse valued at ₹ 40,000 contains goods worth ₹ 2,40,000. The warehouse is insured against fire for ₹ 16,000 and the goods to the extent of 90 % of their value. Goods worth ₹ 80,000 are completely destroyed, while the remaining goods are destroyed to 80 % of their value due to a fire. The damage to the warehouse is to the extent of ₹ 8000. Find the total amount that can be claimed. ***(March '22)***

Solution :

The value of the warehouse = ₹ 40,000.

Goods in warehouse worth = ₹ 2,40,000.

Goods is insured for 90 % of its value.

$\therefore$ policy value of the goods = ₹ $240000 \times \frac{90}{100} =$ ₹ 2,16,000

Goods worth ₹ 80,000 are completely destroyed.

$\therefore$ loss is ₹ 80,000.

$$\text{Claim} = \frac{\text{Policy value}}{\text{Value of the goods}} \times \text{Loss} = \frac{216000}{240000} \times 80000 = ₹\ 72{,}000 \qquad \ldots (1)$$

The remaining goods, i.e. goods worth

₹ (240000 – 80000) = ₹ 1,60,000 was destroyed to 80 %.

$\therefore$ the loss of the remaining goods = ₹ $160000 \times \frac{80}{100} =$ ₹ 1,28,000.

Loss incurred = ₹ 1,28,000

$$\text{Claim} = \frac{\text{Policy value}}{\text{Value of the goods}} \times \text{Loss} = \frac{216000}{240000} \times 128000 = ₹\ 1{,}15{,}200 \qquad \ldots (2)$$

The damage to the warehouse = ₹ 8000

The value of the warehouse = ₹ 40,000

Policy value of the warehouse = ₹ 16,000

Claim for the warehouse $= \frac{16000}{40000} \times 8000 =$ ₹ 3200 ... (3)

Total claim = ₹ (72000 + 115200 + 3200) ... [By (1), (2) and (3)]

= ₹ 1,90,400

Hence, the total amount that can be claimed is ₹ 1,90,400.

Examples for Practice	3 or 4 marks each

1. A building worth ₹ 50,00,000 is insured for $\left(\frac{4}{5}\right)^{\text{th}}$ of its value at a payment of 5%. Find the amount of premium. Also, find commission of the agent, if the rate of commission is 3%.
2. A person insures his office valued at ₹ 5,00,000 for 80% of its value. Find the rate of premium, if he pays ₹ 13,000 as premium. Also, find agent's commission at 11%.
3. A merchant takes out fire insurance policy to cover 80% of the value of his stock. Stock worth ₹ 80,000 was completely destroyed in a fire, while the rest of stock was reduced to 20% of its value. If the proportional compensation under the policy was ₹ 67,200, find the value of the stock.
4. A person takes a life policy of ₹ 2,00,000 for 15 years. The rate of premium is ₹ 55 per thousand per annum. If the bonus is paid at the average rate of ₹ 6 per thousand, what is the benefit to the insured?
5. The rate of premium is 2% and other expenses are 0.75%. A cargo worth ₹ 3,50,100 is to be insured so that all its value and the cost of insurance will be recovered in the event of total loss. Find the policy value of the cargo.
6. A car valued at ₹ 8,00,000 is insured for ₹ 5,00,000. The rate of premium is 5% less 20%. How much will the owner bear including the premium, if value of the car is reduced to 60% of its original value.

Answers

1. Premium = ₹ 2,00,000, Commission = ₹ 6000 **2.** 3.25%, ₹ 1430
3. ₹ 1,00,000 **4.** ₹ 53,000 **5.** ₹ 3,60,000 **6.** 1,40,000.

10.2 ANNUITY

Remember :

1. **Annuity :** It is a sequence of payments of equal amounts with a fixed frequency.
2. **Annuity certain :** It is an investment that provides a series of payments for a set of period of time to the person's beneficiary.
3. **Annuity Immediate (Ordinary Annuity) :** If the payment is made at the end of each period, it is called Annuity Immediate.
4. **Annuity Due :** The periodic payment which is made at the commencement of each period.
5. **Annuity Immediate :**

   (1) $A=\frac{C}{i}\left[(1+i)^n-1\right]$, where $i=\frac{r}{100}$

   $A=$ Accumulated value $\quad C=$ Annual payments

   $n=$ period, $r=$ rate of interest.

   (2) $P=\frac{C}{i}\left[1-(1+i)^{-n}\right]$ $\quad P=$ Present value

   Relations : (i) $A=P(1+i)^n$; $\quad$ (ii) $\frac{1}{P}-\frac{1}{A}=\frac{i}{C}$.

6. **Annuity Due :**

   The accumulated value A' is given by

   $A'=\frac{C(1+i)}{i}\left[(1+i)^n-1\right]$, $i=\frac{r}{100}$

   The present value P' is given by

   $P'=\frac{C(1+i)}{i}\left[1-(1+i)^{-n}\right]$

   A' and P' have the following relations : $A'=P'(1+i)^n$

   $\frac{1}{P'}-\frac{1}{A'}=\frac{i}{C(1+i)}$

Solved Examples | **3 or 4 marks each**

Ex. 6. Find the amount of an ordinary annuity, if payment of ₹ 500 is made at the end of every quarter for 5 years at the rate of 12 % p.a. compounded quarterly.

Solution : $C=₹\ 500$, $n=5$, $r=12\%$.

The period of payment is every quarter.

$\therefore\ n=5\times4=20,\ r=\dfrac{12}{4}=3\%,\ A=?$

$$i=\frac{r}{100}=\frac{3}{100}=0.03$$

$$A=\frac{C}{i}[(1+i)^n-1]=\frac{500}{0.03}[(1+0.03)^{20}-1]$$

$$=\frac{500}{0.03}[1.8061-1]=\frac{500\times0.8061}{0.03}=13435$$

Hence, the amount of an ordinary annuity is ₹ 13,435.

Ex. 7. Find the accumulated value after 3 years of an immediate annuity of ₹ 5000 p.a. with interest compounded at 4% p.a.

[Given : $(1.04)^3=1.1249$]

Solution : Given : $C=₹\ 5000$, $r=4\%$ p.a., $n=3$ years.

Accumulated value is given by

$$A=\frac{C}{i}\left[(1+i)^n-1\right],\text{ where } i=\frac{r}{100}=\frac{4}{100}=0.04$$

$$\therefore\ A=\frac{5000}{0.04}[(1+0.04)^3-1]=\frac{5000\times100}{4}[(1.04)^3-1]$$

$$=12500\,[1.1249-1] \qquad \ldots[\because (1.04)^3=1.1249]$$

$$=125000\times0.1249=15612.50$$

Hence, the accumulated value $=$ ₹ 15,612.50.

Ex. 8. Find the rate of interest compounded annually, if an annuity immediate at ₹ 20,000 per year amounts to ₹ 2,60,000 in 3 years. ***(Sept. '21)***

Solution : Here, $C=₹\ 20{,}000$, $A=₹\ 2{,}60{,}000$, $n=3$, $r=?$

$$A=\frac{C}{i}\left[(1+i)^n-1\right]$$

$$\therefore\ 260000=\frac{20000}{i}\left[(1+i)^3-1\right]$$

$$\therefore\ \frac{260000}{20000}=\frac{1}{i}\left[(1+i)^3-1\right]$$

$\therefore\ 13=\dfrac{1+3i+3i^2+i^3-1}{i}$ $\qquad\qquad$ $\therefore\ 13=\dfrac{i^3+3i^2+3i}{i}$

$\therefore\ 13=i^2+3i+3$

$\therefore\ i^2+3i+3-13=0$ $\qquad\qquad$ $\therefore\ i^2+3i-10=0$

$\therefore\ i^2+5i-2i-10=0$

$\therefore\ i(i+5)-2(i+5)=0$ $\qquad\qquad$ $\therefore\ (i+5)(i-2)=0$

$\therefore\ (i+5)=0$ or $i-2=0$ $\qquad\qquad$ $\therefore\ i=-5$ or $i=2$

$i=-5$ not acceptable, $\qquad$ $\because\ i>0$

$\therefore\ i=2$

$i=\dfrac{r}{100}$

$\therefore\ 2=\dfrac{r}{100}$ $\qquad\qquad$ $\therefore\ r=200\ \%$

Hence, the rate of interest is 200 %.

Ex. 9. After how many years would an annuity due of ₹ 3000 p.a. accumulated ₹ 19,324.80 at 20% p.a. compounded yearly?

[Given : $(1.2)^4=2.0736$]

Solution :

Here, $A'=$ ₹ 19324.80, $C=$ ₹ 3000, $r=20\ \%$, $n=?$

$i=\dfrac{r}{100}=\dfrac{20}{100}=0.2$

Now, $A'=\dfrac{C(1+i)}{i}[(1+i)^n-1]$

$\therefore\ 19324.80=\dfrac{3000(1+0.2)}{0.2}[(1+0.2)^n-1]$

$\therefore\ 19324.80=\dfrac{3000\times 1.2}{0.2}[(1.2)^n-1]$

$\therefore\ 19324.80=18000[(1.2)^n-1]$

$\therefore\ \dfrac{19324.80}{18000}=(1.2)^n-1$

$\therefore\ 1.0736=(1.2)^n-1$

$\therefore\ 1.0736+1=(1.2)^n$

$\therefore\ 2.0736 = (1.2)^n$

$\therefore\ (1.2)^4 = (1.2)^n$

$\therefore\ n = 4$

An annuity due is to be accumulated for 4 years.

Examples for Practice	**3 or 4 marks each**

1. Find the rate of interest compounded annually, if an ordinary annuity of ₹ 20,000 per year amounts to ₹ 41,000 in 2 years.
2. A person deposited ₹ 15,000 every six months for 2 years. The rate of interest is 10% p.a. compounded half yearly. Find the amount accumulated at the end of 2 years. [Given : $(1.05)^4 = 1.2155$]
3. A lady plans to save for her daughter's marriage. She wishes to accumulate a sum of ₹ 4,64,100 at the end of 4 years. What amount should she invest every year, if she can get interest of 10% p.a. compounded annually?
   [Given : $(1.1)^4 = 1.4641$] ***(July '23)***
4. Find the number of years for which an annuity of ₹ 500 is paid at the end of every year, if the accumulated amount works out to be ₹ 1655 when interest compounded at 10% p.a.?
5. A person sets up a sinking fund in order to have ₹ 1,00,000 after 10 years. What amount should be deposited biannually in the account that pays him 5% p.a. compounded semiannually? [Given : $(1.025)^{20} = 1.675$]
6. Find accumulated value after 1 year of annuity immediate in which ₹ 10,000 is invested every quarter at 16% p.a. compounded quarterly.
   [Given : $(1.04)^4 = 1.1699$]

Answers

1. 5% **2.** ₹ 64,650 **3.** ₹ 1,00,000 **4.** 3 years **5.** ₹ 3703.70
6. ₹ 42,475.

ACTIVITIES	**4 marks each**

1. A property worth ₹ 4,00,000 is insured with three companies X, Y and Z for amounts ₹ 1,20,000, ₹ 80,000 and ₹ 1,00,000 respectively. A fire caused a loss for ₹ 2,40,000. Calculate the amounts that can be claimed from the three companies.

Loss = ₹ 2,40,000

$$\text{Claim} = \boxed{} \times \frac{\text{Policy value}}{\text{Property value}}$$

$$\text{Claim from company X} = 240000 \times \frac{120000}{400000} = ₹\ \boxed{}$$

Claim from company Y = ₹ $\boxed{}$

Claim from company Z = ₹ $\boxed{}$.

Solution : Loss = ₹ 2,40,000

$$\text{Claim} = \boxed{\text{Loss}} \times \frac{\text{Policy value}}{\text{Property value}}$$

$$\text{Claim from company X} = 240000 \times \frac{120000}{400000} = ₹\ \boxed{72{,}000}$$

$$\text{Claim from company Y} = 240000 \times \frac{80000}{400000} = ₹\ \boxed{48{,}000}$$

$$\text{Claim from company Z} = 240000 \times \frac{100000}{400000} = ₹\ \boxed{60{,}000}.$$

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2. Complete the following activity to find the present value of an immediate annuity of ₹ 50,000 per annum for 4 years with interest compounded at 10% p.a. **[Given : $(1.1)^{-4} = 0.6830$]**

Given : $C = 50{,}000$, $n = 4$ years, $r = 10\%$ p.a.

$\therefore i = \boxed{}$

The present value P is given by

$$P = \frac{C}{i}\left[\boxed{}\right]$$

$$\therefore P = \frac{50000}{0.1}\left[\boxed{}\right] = \boxed{}.$$

Solution : Given : $C = 50{,}000$, $n = 4$ years, $r = 10\%$ p.a.

$$\therefore i = \frac{r}{100} = \frac{10}{100} = \boxed{0.1}$$

The present value P is given by

$$P = \frac{C}{i}\left[\boxed{1-(1+i)^{-n}}\right]$$

$$\therefore P = \frac{50000}{0.1}\left[\boxed{1-(1+0.1)^{-4}}\right]$$

$$= \frac{50000}{0.1}[1-(1.1)^{-4}]$$

$$= 500000\,[1-0.6830] = 500000\,(0.3170)$$

$$= \boxed{1{,}58{,}500}.$$

---

3. Complete the following activity :

Policy value = ₹ 70,000

Period of policy = 15 years

Rate of premium = ₹ 56.50 per thousand p.a.

$\therefore$ amount of premium per year $= \frac{56.50}{1000} \times \boxed{} =$ ₹ 3955

$\therefore$ total premium paid = ₹ $\boxed{}$

Rate of bonus = ₹ 6 per thousand p.a.

$\therefore$ bonus for 15 years = ₹ $\boxed{}$

$\therefore$ total amount the person received = ₹ (70000 + 6300) = ₹ 76,300

$\therefore$ benefit = ₹ $\boxed{}$.

Solution : Policy value = ₹ 70,000

Period of policy = 15 years

Rate of premium = ₹ 56.50 per thousand p.a.

$\therefore$ amount of premium per year $= \frac{56.50}{1000} \times \boxed{70000} =$ ₹ 3955

$\therefore$ total premium paid = ₹ (3955×15) = ₹ $\boxed{59{,}325}$

Rate of bonus = ₹ 6 per thousand p.a.

$\therefore$ bonus for 1 year $= \frac{6}{1000} \times 70000 =$ ₹ 420

$\therefore$ bonus for 15 years = ₹ (420×15) = ₹ $\boxed{6300}$

$\therefore$ total amount the person received = ₹ (70000 + 6300) = 76,300

$\therefore$ benefit = ₹ (76300 − 59325) = ₹ $\boxed{16{,}975}$.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. In an ordinary annuity, payments or receipts occur at
   (a) beginning of each period (b) end of each period
   (c) mid of each period (d) quarterly basis *(March '24 – '25)*
2. You get payments of ₹ 8000 at the beginning of each year for five years at 6%, what is the value of this annuity?
   (a) ₹ 34,720 (b) ₹ 39,320 (c) ₹ 35,720 (d) ₹ 40,000
3. .... is a series of cashflows over a limited period of time.
   (a) Policy value (b) Annuity
   (c) Present value (d) Future value *(March '23)*
4. Insurance companies collect a fixed amount from their customers at a fixed interval of time. This amount is called
   (a) EMI (b) instalment (c) contribution (d) premium
5. A person insured a property of ₹ 4,00,000. The rate of premium is ₹ 35 per thousand p.a. The amount of annual premium is
   (a) ₹ 14,000 (b) ₹ 24,000 (c) ₹ 34,000 (d) ₹ 15,000 *(July '24)*
6. The car worth ₹ 10,00,000 insured for ₹ 7,00,000 is damaged to the extent of ₹ 5,00,000 in an accident, then the amount of compensation that can be claimed under the policy is ₹ ....
   (a) 3,50,000 (b) 7,00,000 (c) 2,50,000 (d) 5,00,000
   (Sept. '21)

Answers

1. (b) end of each period 2. (c) ₹ 35,720 3. (b) Annuity
4. (d) premium 5. (a) ₹ 14,000 6. (a) 3,50,000.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The amount of claim cannot exceed the amount of loss. *(July '23)*
2. In the immediate annuity, the instalment is to be paid at the end of each period.
3. Payment of every annuity is called an instalment.

4. The future value of annuity is the accumulated values of all instalments.
5. Premium is the amount paid to the insurance company every month.

Answers

1. True 2. True 3. False 4. False 5. True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. In an ordinary annuity, payments or receipts occur at ....
2. If payments of an annuity fall due at the end of every period, the series is called ....
3. General insurance covers all risks except ....
4. The value of insured property is called ....
5. The payment of each single annuity is called ....
6. An annuity where payments continue forever is called ....
7. The present worth of a sum of ₹ 10,920 due six months hence at 8% p.a. simple interest is .... *(Sept. '21)*

Answers

1. end of each period 2. ordinary annuity or immediate annuity 3. life
4. policy value 5. instalment 6. perpetuity 7. ₹ 10,500.

Question Set 11

LINEAR REGRESSION

(Marks with option : 08)

11.1 LINEAR REGRESSION

Remember :

1. **Linear regression :** Linear regression is the statistical method which helps to formulate a functional relationship between two or more variables in the form of linear equations and predicts the value of one variable given the value of the other variable.

2. **The linear regression model of Y on X :**

$y-\bar{y}=b_{yx}(x-\bar{x})$ ***OR*** $y=a+bx$, where $b=b_{yx}$, $a=\bar{y}-b\bar{x}$

3. **The linear regression model of X on Y :**

$x-\bar{x}=b_{xy}(y-\bar{y})$ ***OR*** $x=a'+b'y$, where $b'=b_{xy}$, $a'=\bar{x}-b'\bar{y}$

4. **Regression Coefficient of Y on X :**

$$b_{yx}=\frac{\text{Cov}(x,y)}{\sigma_x^2}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

$$=\frac{\frac{\Sigma xy}{n}-\bar{x}\bar{y}}{\frac{\Sigma x^2}{n}-(\bar{x})^2}=\frac{\Sigma xy-n\bar{x}\bar{y}}{\Sigma x^2-n(\bar{x})^2}=r\cdot\frac{\sigma_y}{\sigma_x}$$

[**Note :** $b_{yx}=b$]

5. **Regression Coefficient of X on Y :**

$$b_{xy}=\frac{\text{Cov}(x,y)}{\sigma_y^2}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$=\frac{\frac{\Sigma xy}{n}-\bar{x}\bar{y}}{\frac{\Sigma y^2}{n}-(\bar{y})^2}=\frac{\Sigma xy-n\bar{x}\bar{y}}{\Sigma y^2-n(\bar{y})^2}=r\cdot\frac{\sigma_x}{\sigma_y}$$

[**Note :** $b_{xy}=b'$]

Solved Examples	3 or 4 marks each

Ex. 1. For a bivariate data : $\Sigma(x-\bar{x})^2 = 1200$, $\Sigma(y-\bar{y})^2 = 300$, $\Sigma(x-\bar{x})(y-\bar{y}) = -250$.
Find (a) b_{yx} (b) b_{xy} (c) correlation coefficient between x and y.

(March '23)

Solution : Given : $\Sigma(x-\bar{x})^2 = 1200$, $\Sigma(y-\bar{y})^2 = 300$, $\Sigma(x-\bar{x})(y-\bar{y}) = -250$.

(a) $b_{yx} = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \dfrac{-250}{1200} = -\dfrac{5}{24}$

(b) $b_{xy} = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \dfrac{-250}{300} = -\dfrac{5}{6}$

(c) Correlation coefficient between x and y

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}} = \pm\sqrt{\left(-\frac{5}{24}\right)\left(-\frac{5}{6}\right)} = \pm\frac{5}{12}$$

$\therefore r = -\dfrac{5}{12}$ ... [$\because$ b_{yx} and b_{xy} both are negative]

Ex. 2. Compute the appropriate regression equation for the following data :

X	1	2	3	4	5
Y	5	7	9	11	13

X is the independent variable and Y is the dependent variable.

(March '25)

Solution :
We prepare the following table for calculation :

x	y	xy	x^2	y^2
1	5	5	1	25
2	7	14	4	49
3	9	27	9	81
4	11	44	16	121
5	13	65	25	169
$\Sigma x = 15$	$\Sigma y = 45$	$\Sigma xy = 155$	$\Sigma x^2 = 55$	$\Sigma y^2 = 445$

Here, $n = 5$

$$\bar{x}=\frac{\Sigma x}{n}=\frac{15}{5}=3;\ \bar{y}=\frac{\Sigma y}{n}=\frac{45}{5}=9$$

Regression equation of Y on X :

$y=a+b_{yx}\cdot x$

$$b_{yx}=\frac{\Sigma xy-n\bar{x}\bar{y}}{\Sigma x^2-n(\bar{x})^2}=\frac{155-5(3\times 9)}{55-5(3)^2}=\frac{155-135}{55-45}=\frac{20}{10}=2$$

$a=\bar{y}-b_{yx}\cdot\bar{x}$

Putting $\bar{y}=9$, $b_{yx}=2$, $\bar{x}=3$, we get

$a=9-2(3)=9-6=3$

Putting $a=3$ and $b_{yx}=2$ in $y=a+b_{yx}\cdot x$, we get, the regression equation Y on X as follows :

$y=3+2x$, i.e. $y=2x+3$.

Ex. 3. For the following data, find the regression line of Y on X :

X	**1**	**2**	**3**
Y	**2**	**1**	**6**

Hence, find the most likely value of Y when $X=4$. ***(March '23)***

Solution : We prepare the following table for calculation :

x	y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
1	2	-1	-1	1	1
2	1	0	-2	0	0
3	6	1	3	3	1
$\Sigma x=6$	$\Sigma y=9$	$\Sigma(x-\bar{x})=0$	$\Sigma(y-\bar{y})=0$	$\Sigma(x-\bar{x})(y-\bar{y})=4$	$\Sigma(x-\bar{x})^2=2$

Here, $n=3$

$$\bar{x}=\frac{\Sigma x}{n}=\frac{6}{3}=2;\ \bar{y}=\frac{\Sigma y}{n}=\frac{9}{3}=3$$

Regression equation of Y on X :

$y=a+b_{yx}x$

$$b_{yx}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}=\frac{4}{2}=2$$

$a=\bar{y}-b_{yx}\bar{x}$

Putting $\bar{y}=3$, $b_{yx}=2$, $\bar{x}=2$, we get

$a=3-2(2)=-1$

Hence, regression line of Y on X is $y=-1+2x$

$\therefore y=2x-1$

By putting $x=4$ in $y=2x-1$, we get

$y=2(4)-1 \qquad \therefore y=8-1=7$

Hence, most likely value of Y is 7 when $X=4$.

Examples for Practice	**3 or 4 marks each**

1. From the following data, find the regression equation of Y on X and estimate Y when $X=10$:

X	1	2	3	4	5	6
Y	2	4	7	6	5	6

(July '22)

2. For a certain bivariate data on 5 pairs of observations given

$\Sigma x=20$, $\Sigma y=20$, $\Sigma x^2=90$, $\Sigma y^2=90$, $\Sigma xy=76$.

Calculate (i) $\text{cov}(x, y)$ (ii) b_{yx} and b_{xy} (iii) r.

3. A departmental store gives in service training to the salesmen followed by a test. It is experienced that the performance regarding sales of any salesmen is linearly related to the scores secured by him. The following data give test scores and sales made by nine salesmen during fixed period :

Test scores (X)	16	22	28	24	29	25	16	23	24
Sales ('00 ₹) (Y)	35	42	57	40	54	51	34	47	45

(i) Obtain the line of regression of Y on X.

(ii) Estimate Y when $X=17$.

4. Find the line of regression of X on Y for the following data :

$n=8$, $\Sigma(x_i-\bar{x})^2=36$, $\Sigma(y_i-\bar{y})^2=44$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})=24$,

$\Sigma x_i=32$, $\Sigma y_i=40$.

5. From the data of 20 pairs of observations on X and Y, following results are obtained :

$\bar{x}=199$, $\bar{y}=94$, $\Sigma(x_i-\bar{x})^2=1200$, $\Sigma(y_i-\bar{y})^2=300$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})=-250$

Find : (a) the line of regression Y on X.

(b) Estimate Y when $X = 211$. *(Sept. '21)*

Answers

1. $y = 0.63x + 2.8$, $y = 9.1$
2. (i) -0.8 (ii) $b_{yx} = -0.4$, $b_{xy} = -0.4$ (iii) -0.4
3. (i) $y = 1.6325x + 7.4525$ (ii) 35.205
4. $x = \frac{6}{11}y + \frac{14}{11}$
5. (a) $5x + 4y = 3251$ (b) 91.5.

11.2 PROPERTIES OF REGRESSION COEFFICIENTS

Remember :

1. $r^2 = b_{yx} \cdot b_{xy}$; $0 \leqslant b_{yx} \cdot b_{xy} \leqslant 1$

   $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

   sign of r is same as the signs of b_{yx} and b_{xy}.
2. b_{yx} and b_{xy} are independent of change of origin but not of scale.
3. If $b_{yx} > 1$, then $b_{xy} < 1$
4. $\left|\frac{b_{yx} + b_{xy}}{2}\right| \geqslant |r|$
5. If $u = \frac{x-a}{h}$ and $v = \frac{y-b}{k}$, then $b_{yx} = \frac{k}{h} b_{vu}$ and $b_{xy} = \frac{h}{k} b_{uv}$.

Solved Examples | **3 or 4 marks each**

Ex. 4. The equations of two regression lines are $10x - 4y = 80$ and $10y - 9x = -40$. Find

(a) $\bar{x}$ and $\bar{y}$ (b) b_{yx} and b_{xy} (c) r

(d) If Var $(Y) = 36$, obtain Var (X). *(March '25)*

Solution : Given : $10x - 4y = 80$, $10y - 9x = -40$

(a) $\bar{x}$ and $\bar{y}$:

$$10x - 4y = 80 \quad \text{... (1)}$$

$$-9x + 10y = -40 \quad \text{... (2)}$$

Multiplying equation (1) by 9 and equation (2) by 10 and then adding them, we get

$$\begin{array}{r}90x - 36y = 720 \\ -90x + 100y = -400 \\ \hline \therefore 64y = 320\end{array}$$

$$\therefore y = \frac{320}{64} = 5$$

Put $y = 5$ in equation (1), we get

$$10x - 4(5) = 80$$

$$\therefore 10x = 80 + 20 \qquad \therefore x = \frac{100}{10} = 10$$

Hence, $\bar{x} = 10$, $\bar{y} = 5$.

(b) b_{yx} and b_{xy} :

Let regression equation of X on Y be $10x - 4y = 80$

$$\therefore 10x = 4y + 80$$

$$\therefore x = \frac{4}{10}y + 8 \qquad \therefore b_{xy} = 0.4.$$

And another equation $10y - 9x = -40$ be the regression equation of Y on X.

$$\therefore 10y = 9x - 40$$

$$\therefore y = \frac{9}{10}x - 4 \qquad \therefore b_{yx} = \frac{9}{10} = 0.9$$

Hence, $b_{yx} = 0.9$ and $b_{xy} = 0.4$.

(c) Coefficient of correlation r :

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}} = \pm\sqrt{0.9 \times 0.4} = \pm\sqrt{0.36}$$

$\therefore r = 0.6$ ... [$\because$ b_{yx} and b_{xy} are positive]

(d) Var (X), if Var (Y) = 36, i.e. $\sigma_y = 6$:

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$$\therefore 0.9 = 0.6 \times \frac{6}{\sigma_x}$$

$$\therefore \frac{0.9}{0.6 \times 6} = \frac{1}{\sigma_x} \qquad \therefore \frac{1}{4} = \frac{1}{\sigma_x} \qquad \therefore \sigma_x = 4$$

$\therefore$ Var $(X) = \sigma_x^2 = (4)^2 = 16.$

Ex. 5. Find the equation of line of regression of Y on X for the following data :

$n=8$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})=120$, $\bar{x}=20$, $\bar{y}=36$, $\sigma_x=2$, $\sigma_y=3$. ***(March '22)***

Solution : Given : $n=8$, $\bar{x}=20$, $\bar{y}=36$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})=120$, $\sigma_x=2$, $\sigma_y=3$.

$$\text{Cov}(x, y)=\frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n}=\frac{120}{8}=15$$

$$\therefore b_{yx}=\frac{\text{Cov}(x, y)}{\sigma_x^2}=\frac{15}{(2)^2}=\frac{15}{4}=3.75$$

Line of regression of Y on X is

$y=a+b_{yx}\cdot x$

$a=\bar{y}-b_{yx}\cdot\bar{x}=36-3.75(20)=36-75=-39$

$\therefore a=-39$

$\therefore$ line of regression of Y on X is $y=-39+3.75x$, i.e. $y=3.75x-39$.

Ex. 6. For a bivariate data, the regression coefficient of Y on X is 0.4 and the regression coefficient of X on Y is 0.9. Find the value of variance of Y, if variance of X is 9. ***(March '24)***

Solution : Given : $b_{yx}=0.4$, $b_{xy}=0.9$, $\sigma_x^2=9$

$r=\pm\sqrt{b_{yx}\cdot b_{xy}}=\pm\sqrt{0.4\times0.9}=\pm\sqrt{0.36}$

$=0.6$ ... [$\because b_{yx}$ and b_{xy} are positive]

Variance of Y :

$b_{yx}=0.4$

$$\therefore r\cdot\frac{\sigma_y}{\sigma_x}=0.4$$

$$\therefore 0.6\times\frac{\sigma_y}{3}=0.4$$ ... [$\because \sigma_x^2=9, \therefore \sigma_x=3$]

$\therefore 0.2\,\sigma_y=0.4$

$$\therefore \sigma_y=\frac{0.4}{0.2}=2$$

$\therefore \sigma_y^2=(2)^2=4$

Hence, the variance of Y is 4.

Examples for Practice	3 or 4 marks each

1. The following results were obtained from records of age (X) and systolic blood pressure (Y) of a group of 10 men :

	X	Y
Mean	50	140
Variance	150	165

and $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 1120$.

Find the prediction of blood pressure of a man of age 40 years. ***(July '23)***

2. Given the following data, obtain linear regression estimate of X for $Y = 10$:
$\bar{x} = 7.6$, $\bar{y} = 14.8$, $\sigma_x = 3.2$, $\sigma_y = 16$ and $r = 0.7$.

3. The two regression equations are $5x - 6y + 90 = 0$ and $15x - 8y - 130 = 0$. Find $\bar{x}, \bar{y}, r$. ***(July '23)***

4. If for a bivariate data $\bar{x} = 10$, $\bar{y} = 12$, $V(X) = 9$, $\sigma_y = 4$ and $r = 0.6$. Estimate y when $x = 5$.

5. In a partially destroyed laboratory record of an analysis of regression data, the following data are legible : Variance of $X = 9$.

Regression equations : $8x - 10y + 66 = 0$ and $40x - 18y = 214$.

Find on the basis of above information :

(i) The mean values of X and Y.

(ii) Correlation coefficient between X and Y. ***(Sept. '21)***

6. For a certain bivariate data the following information are available :

	X	Y
A.M.	13	17
S.D.	3	2

Correlation coefficient between x and y is 0.6. Estimate x when $y = 15$ and estimate y when $x = 10$.

7. For 50 students of a class, the regression equation of marks in statistics (X) on the marks in accountancy (Y) is $3y - 5x + 180 = 0$. The variance of marks in statistics is $\left(\frac{9}{16}\right)^{\text{th}}$ of the variance of marks in accountancy. Find the correlation coefficient between marks in two subjects. ***(March '22)***

8. For a bivariate data $\bar{x} = 53$, $\bar{y} = 28$, $b_{yx} = -1.2$, $b_{xy} = -0.3$, find
   (a) correlation coefficient between X and Y.
   (b) Estimate Y for $X = 50$. ***(July '22–'24)***

Answers

1. $y = 0.75x + 102.5$, $y = 132.5$
2. $x = 0.14y + 5.528$, $x = 6.928$
3. $\bar{x} = 30$, $\bar{y} = 40$, $r = \dfrac{2}{3}$
4. 8
5. (i) $\bar{x} = 13$, $\bar{y} = 17$ (ii) 0.6
6. $x = 11.2$ when $y = 15$, $y = 15.8$ when $x = 10$
7. $r = 0.8$ 8. $r = -0.6$, $y = -1.2x + 91.6$, $y = 31.6$.

ACTIVITIES	4 marks each

1. Complete the following activity :

Given : $n = 8$, $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 120$, $\bar{y} = 36$, $\sigma_x = 2$, $\sigma_y = 3$

$$\therefore \text{Cov}(x, y) = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n} = \boxed{}$$

$$b_{yx} = \frac{\boxed{}}{\sigma_x^2} = \boxed{}$$

$$b_{xy} = \frac{\text{Cov}(x, y)}{\boxed{}} = \boxed{}$$

Regression equation of Y on X :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\therefore y = \boxed{}.$$

Solution : Given : $n = 8$, $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 120$, $\bar{y} = 36$, $\sigma_x = 2$, $\sigma_y = 3$

$$\therefore \text{Cov}(x, y) = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{120}{8} = \boxed{15}$$

$$b_{yx} = \frac{\boxed{\text{Cov}(x, y)}}{\sigma_x^2} = \boxed{\frac{15}{4}}$$

$$b_{xy} = \frac{\text{Cov}(x, y)}{\boxed{\sigma_y^2}} = \boxed{\frac{15}{9}}$$

Regression equation of Y on X :

$y - \bar{y} = b_{yx}(x - \bar{x})$

$\therefore y - 36 = \frac{15}{4}(x - \bar{x})$

$\therefore y = \boxed{\frac{15}{4}(x - \bar{x}) + 36}$.

---

2. The equation of the two lines of regression are $3x + 2y - 26 = 0$ and $6x + y - 31 = 0$. Obtain the correlation coefficient between x and y :

To find correlation coefficient, we have to find the regression coefficients b_{yx} and b_{xy}.

Let $3x + 2y = 26$ be the equation of the line of regression of y on x.

This gives $y = \boxed{}\, x + 13$

$\therefore b_{yx} = -\frac{3}{2}$

Now, consider $6x + y = 31$ as equation of the line of regression of x on y.

This can be written as $x = \boxed{}\, y + \frac{31}{6}$

$\therefore b_{xy} = -\frac{1}{6}$

Now, $r^2 = \boxed{} = 0.25$

As both b_{yx} and b_{xy} are negative

$\therefore r = \boxed{}$. *(July '24)*

Solution : To find correlation coefficient, we have to find the regression coefficients b_{yx} and b_{xy}.

Let $3x + 2y = 26$ be the equation of the line of regression of y on x.

This gives $y = \boxed{-\frac{3}{2}}\, x + 13$

$\therefore b_{yx} = -\frac{3}{2}$

Now, consider $6x + y = 31$ as equation of the line of regression of x on y.

This can be written as $x = \boxed{-\frac{1}{6}}\, y + \frac{31}{6}$

$\therefore b_{xy} = -\frac{1}{6}$

Now, $r^2 = \boxed{b_{yx} \times b_{xy}} = \left(-\frac{3}{2}\right) \times \left(-\frac{1}{6}\right) = 0.25$

As both b_{yx} and b_{xy} are negative

$\therefore r = \boxed{-0.25}$.

---

3. For a bivariate data : $\bar{x} = 10, \bar{y} = 12, V(X) = 9, \sigma_y = 4$ and $r = 0.6$. Estimate y when $x = 5$.

Line of regression of Y on X is

$Y - \bar{y} = \boxed{} (X - \bar{x})$

$\therefore Y - 12 = r \cdot \frac{\sigma_y}{\sigma_x} (X - 10)$

$\therefore Y - 12 = 0.6 \times \frac{4}{\boxed{}} (X - 10)$

When $x = 5$

$Y - 12 = \boxed{} (5 - 10)$

$\therefore Y - 12 = -4$

$\therefore Y = \boxed{}$ *(March '24)*

Solution : Line of regression of Y on X is

$Y - \bar{y} = \boxed{b_{yx}} (X - \bar{x})$

$\therefore Y - 12 = r \cdot \frac{\sigma_y}{\sigma_x} (X - 10)$

$\therefore Y - 12 = 0.6 \times \frac{4}{\boxed{3}} (X - 10)$

When $x = 5$

$Y - 12 = \boxed{0.8} (5 - 10)$

$\therefore Y - 12 = -4$

$\therefore Y = \boxed{8}$.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. 'r' is ....

(a) regression coefficient of Y on X

(b) regression coefficient of X on Y

(c) correlation coefficient between X and Y

(d) covariance between X and Y.

2. $|b_{xy}+b_{yx}| \geqslant$ ....

(a) $|r|$ (b) $2|r|$ (c) r (d) $2r$

3. b_{yx} is ....

(a) regression coefficient of Y on X.

(b) regression coefficient of X on Y.

(c) correlation coefficient between X and Y.

(d) covariance between X and Y. ***(March '23)***

4. b_{xy} and b_{yx} are

(a) independent of change of origin and scale

(b) independent of change of origin but not of scale

(c) independent of change of scale but not of origin

(d) affected by change of origin and scale ***(March '24)***

5. $b_{yx}=$ ....

(a) $r\frac{\sigma_x}{\sigma_y}$ (b) $r\frac{\sigma_y}{\sigma_x}$ (c) $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$ (d) $\frac{1}{r}\frac{\sigma_x}{\sigma_y}$

6. If $u=\frac{x-a}{c}$ and $v=\frac{y-b}{d}$, then $b_{xy}=$ ....

(a) $\frac{d}{c}b_{uv}$ (b) $\frac{c}{d}b_{uv}$ (c) $\frac{a}{b}b_{uv}$ (d) $\frac{b}{a}b_{uv}$

7. $\text{Cov}(x, y)=$ ....

(a) $\Sigma(x-\bar{x})(y-\bar{y})$ (b) $\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$

(c) $\frac{\Sigma xy}{n}-(\bar{x})(\bar{y})$ (d) (b) and (c) both

8. If $b_{xy} < 0$ and $b_{yx} < 0$, then 'r' is

(a) >0 (b) <0 (c) >1 (d) not found

9. If $b_{yx} > 1$, then b_{xy} is ....

(a) >1 (b) <1 (c) >0 (d) <0

10. $b_{xy} \cdot b_{yx} =$ ....

(a) $V(X)$ (b) σ_x (c) r^2 (d) σ_y^2

(March '22; July '23)

Answers

1. (c) correlation coefficient between X and Y
2. (b) $2|r|$
3. (a) regression coefficient of Y on X
4. (b) independent of change of origin but not of scale

5. (b) $r\frac{\sigma_y}{\sigma_x}$ 6. (b) $\frac{c}{d} b_{uv}$ 7. (d) (b) and (c) both

8. (b) <0 9. (b) <1 10. (c) r^2.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. Regression equation of Y on X is $y - \bar{y} = b_{yx}(x - \bar{x})$.
2. If $b_{xy} < 0$ and $b_{yx} < 0$, then r is positive.
3. b_{xy} and b_{yx} are independent of change of origin and scale.
4. If $u = x - a$ and $v = y - b$, then $b_{xy} = b_{uv}$.
5. If $b_{yx} + b_{xy} = 1.30$ and $r = 0.75$, then the given data is inconsistent.

(March '25)

6. In the regression equation of Y on X, b_{xy} represents slope of the line.

(Sept. '21)

7. Regression equation of X on Y is $y - \bar{y} = b_{yx}(x - \bar{x})$. ***(July '22)***
8. In the regression of Y on X, X is the independent variable and Y is the dependent variable. ***(July '24)***

Answers

1. True **2.** False **3.** False **4.** True **5.** True **6.** False **7.** False **8.** True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. If $r = 0.6$, $b_{yx} = 0.9$, $\text{Var}(Y) = 36$, then $\text{Var}(X) =$ ....
2. $|b_{xy} + b_{yx}| \geqslant$ ....
3. If $r = 0.5$, $\sigma_x = 8$, $\sigma_y = 10$, then $b_{yx} =$ ....
4. If $b_{yx} > 1$, then b_{xy} is ....
5. $\text{Cov}(X, -X) =$ ....
6. For a certain bivariate data on 5 pairs of observations given :
   $\Sigma x = 20$, $\Sigma y = 20$, $\Sigma x^2 = 90$, $\Sigma y^2 = 90$, $\Sigma xy = 76$, then $b_{xy} =$ ....

(March '22)

Answers

1. 16 **2.** $2|r|$ **3.** 0.625 **4.** <1 **5.** -1 **6.** -0.4.

Question Set 12

TIME SERIES

(Marks with option : 07)

Remember :

1. Time series is a sequence of observations made on a variable at regular time intervals over a specified period of time.

2. **Components of Time Series :**
   - Trend (T)
   - Seasonal variations (S)
   - Cyclical variations (C)
   - Irregular variations (I)

3. **Linear trend :** $x_t = a + bt$

4. **Normal equations to fit a linear trend :**

   $\Sigma x_t = na + b\Sigma t$

   $\Sigma tx_t = a\Sigma t + b\Sigma t^2$

5. **Short-cut method :**

   Two normal equations are :

   $\Sigma x_t = na' + b'\Sigma u$

   $\Sigma ux_t = a'\Sigma u + b'\Sigma u^2$

   The equation of linear trend : $x_t = a' + b'u$,

   where $u = \dfrac{t - \text{middle } t \text{ value}}{h}$

   h = distance between two consecutive t values

6. **Methods of measuring trend :**
   - Graphical method
   - Least square method
   - Moving averages method.

Solved Examples	3 or 4 marks each

Ex. 1. Obtain the trend values for the following data using 5-yearly moving averages :

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Production	10	15	20	25	30	35	40	45	50	55

(March '22)

Solution :

We construct the following table to obtain 5-yearly moving averages :

Year	Production	5-yearly moving total	5-yearly moving averages (Trend Value)
2000	10	–	–
2001	15	–	–
2002	20	100	20
2003	25	125	25
2004	30	150	30
2005	35	175	35
2006	40	200	40
2007	45	225	45
2008	50	–	–
2009	55	–	–

Ex. 2. Following table shows the amount of sugar production (in lakh tonnes) for the years 1971 to 1982 :

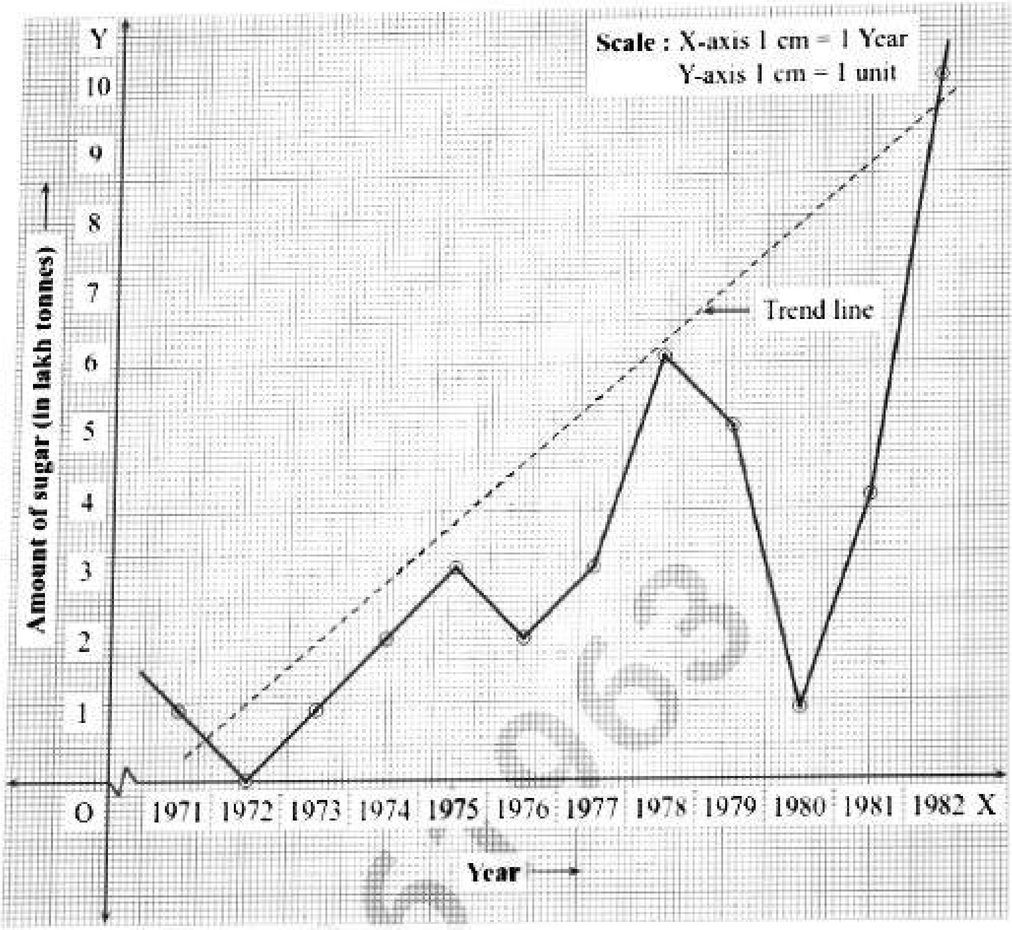
Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Production	1	0	1	2	3	2	3	6	5	1	4	10

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and production on Y-axis, we plot the points for production corresponding to years. Joining these points, we get, the graph of time series.

We fit a trend line as shown in the figure :



Ex. 3. Following table shows the number of traffic fatalities (in a state) resulting from drunken driving from years 1975 to 1983 :

Year	1975	1976	1977	1978	1979	1980	1981	1982	1983
Number of deaths	0	6	3	8	2	9	4	5	10

Fit a trend line to the above data by the method of least squares.

(July '23)

Solution :

Here, $n = 9$. We transform year t to u by taking $u = t - 1979$.

We construct the following table for calculation :

Year t	Number of deaths x_t	$u = t - 1979$	u^2	ux_t
1975	0	-4	16	0
1976	6	-3	9	-18
1977	3	-2	4	-6
1978	8	-1	1	-8
1979	2	0	0	0
1980	9	1	1	9
1981	4	2	4	8
1982	5	3	9	15
1983	10	4	16	40
Total	$\Sigma x_t = 47$	$\Sigma u = 0$	$\Sigma u^2 = 60$	72 -32 $\Sigma ux_t = 40$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$\Sigma x_t = na' + b'\Sigma u$ ... (1)

$\Sigma ux_t = a'\Sigma u + b'\Sigma u^2$ ... (2)

Here, $n = 9$, $\Sigma x_t = 47$, $\Sigma u = 0$, $\Sigma u^2 = 60$, $\Sigma ux_t = 40$.

Putting these values in normal equations, we get

$47 = 9a' + b'(0)$ ... (3)

$40 = a'(0) + b'(60)$ ... (4)

From equation (3), we get

$$a' = \frac{47}{9} = 5.2222$$

From equation (4), we get

$$b' = \frac{40}{60} = 0.6667$$

Putting $a' = 5.2222$ and $b' = 0.6667$ in $x_t = a' + b'u$, we get

the equation of trend line as $x_t = 5.2222 + 0.6667u$, where $u = (t - 1979)$.

Ex. 4. Following table shows the all India infant mortality rates (per '000) for years 1980 to 2010 :

Year	1980	1985	1990	1995	2000	2005	2010
IMR	10	7	5	4	3	1	0

Fit the trend line to the above data by the method of least squares.

(March '24)

Solution :

Here, $n = 7$ and interval between the year t is 5.

We transform year t to u by taking $u = \dfrac{t - 1995}{5}$.

We construct the following table for calculation :

Year t	**Infant Mortality Rate** x_t	$u = \dfrac{t-1995}{5}$	u^2	ux_t
1980	10	-3	9	-30
1985	7	-2	4	-14
1990	5	-1	1	-5
1995	4	0	0	0
2000	3	1	1	3
2005	1	2	4	2
2010	0	3	9	0
Total	$\Sigma x_t = 30$	$\Sigma u = 0$	$\Sigma u^2 = 28$	05 -49 $\Sigma ux_t = -44$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$$\Sigma x_t = na' + b'\Sigma u \qquad \text{... (1)}$$

$$\Sigma ux_t = a'\Sigma u + b'\Sigma u^2 \qquad \text{... (2)}$$

Here, $n = 7$, $\Sigma x_t = 30$, $\Sigma u = 0$, $\Sigma u^2 = 28$, $\Sigma ux_t = -44$

Putting these values in normal equations, we get

$$30 = 7a' + b'(0) \qquad \text{... (3)}$$

$$-44 = a'(0) + b'(28) \qquad \text{... (4)}$$

From equation (3), we get

$$a' = \frac{30}{7} = 4.286$$

From equation (4), we get

$$b' = \frac{-44}{28} = -1.571$$

Putting $a' = 4.286$ and $b' = -1.571$ in $x_t = a' + b'u$, we get, the equation of trend line as

$x_t = 4.286 - 1.571\,u$, where $u = \frac{t - 1995}{5}$.

Ex. 5. The following table shows the index of industrial production for the period from 1976 to 1985, using the year 1976 as the base year. Obtain the trend values for the following data using 4-yearly centred moving averages :

Year	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985
Index	0	2	3	3	2	4	5	6	7	10

(March '24–'25)

Solution :

We construct the following table to obtain 4-yearly centred moving averages :

Year t	**Index** x_t	4-yearly moving **total**	**4-yearly moving** averages	**2-unit moving total**	**4-yearly centred moving averages (Trend value)**
1976	0	–	–	–	–
1977	2	8	2.0	–	–
1978	3	10	2.5	4.5	2.25
1979	3	12	3.0	5.5	2.75
1980	2	14	3.5	6.5	3.25
1981	4	17	4.25	7.75	3.875
1982	5	22	5.5	9.75	4.875
1983	6	28	7.0	12.5	6.25
1984	7	–	–	–	–
1985	10			–	–

Examples for Practice	3 or 4 marks each

1. The publisher of a magazine wants to determine the rate of increase in the number of subscribers. The following table shows the subscription information for eight consecutive years :

Year	1976	1977	1978	1979	1980	1981	1982	1983
Number of subscribers	12	11	19	17	19	18	20	23

Fit a trend line by graphical method. ***(March '23)***

2. The following data gives the production of bleaching powder (in '000 tonnes) for the years 1962 to 1972 :

Year	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972
Production	0	0	1	1	4	2	4	9	7	10	8

Fit a trend line by graphical method to the above data.

3. Following table shows the amount of sugar production (in lakh tonnes) for the years 1971 to 1982 :

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Production	1	0	1	2	3	2	3	6	5	1	4	10

Fit a trend line to the above data by the method of least squares.

4. Following table shows the all India infant mortality rates (per '000) for years 1980 to 2010 :

Year	1980	1985	1990	1995	2000	2005	2010
IMR	10	7	5	4	3	1	2

Obtain trend values for the above data using 3-yearly moving averages.

5. The following table gives the production of steel (in millions of tonnes) for years 1976 to 1986 :

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Fit a trend line by the method of least squares. ***(July '24)***

6. Following table shows the amount of sugar production (in lakh tonnes) for years 1931 to 1942 :

Year	1931	1932	1933	1934	1935	1936
Production	0	1	2	3	2	3
Year	1937	1938	1939	1940	1941	1942
Production	4	5	6	7	4	8

Obtain the trend values using 4-yearly centred moving averages. ***(Sept. '21)***

7. The following tables shows the production of gasoline in U.S.A. for the years 1962 to 1976. Obtain trend values using 5-yearly moving averages :

Year	1962	1963	1964	1965	1966	1967	1968	1969
Production	0	0	1	1	2	3	4	5
Year	1970	1971	1972	1973	1974	1975	1976	
Production	6	7	8	9	8	9	10	

(July '22–'23)

8. The following table shows gross capital information (in crore ₹) for years 1966 to 1975 :

Year	1966	1967	1968	1969	1970
Gross capital information	20	25	25	30	35
Year	1971	1972	1973	1974	1975
Gross capital information	30	45	40	55	65

Obtain the trend values using 5-yearly moving averages. ***(March '23)***

9. The following data gives the production of bleaching powder (in '000 tonnes) for the years 1962 to 1972 :

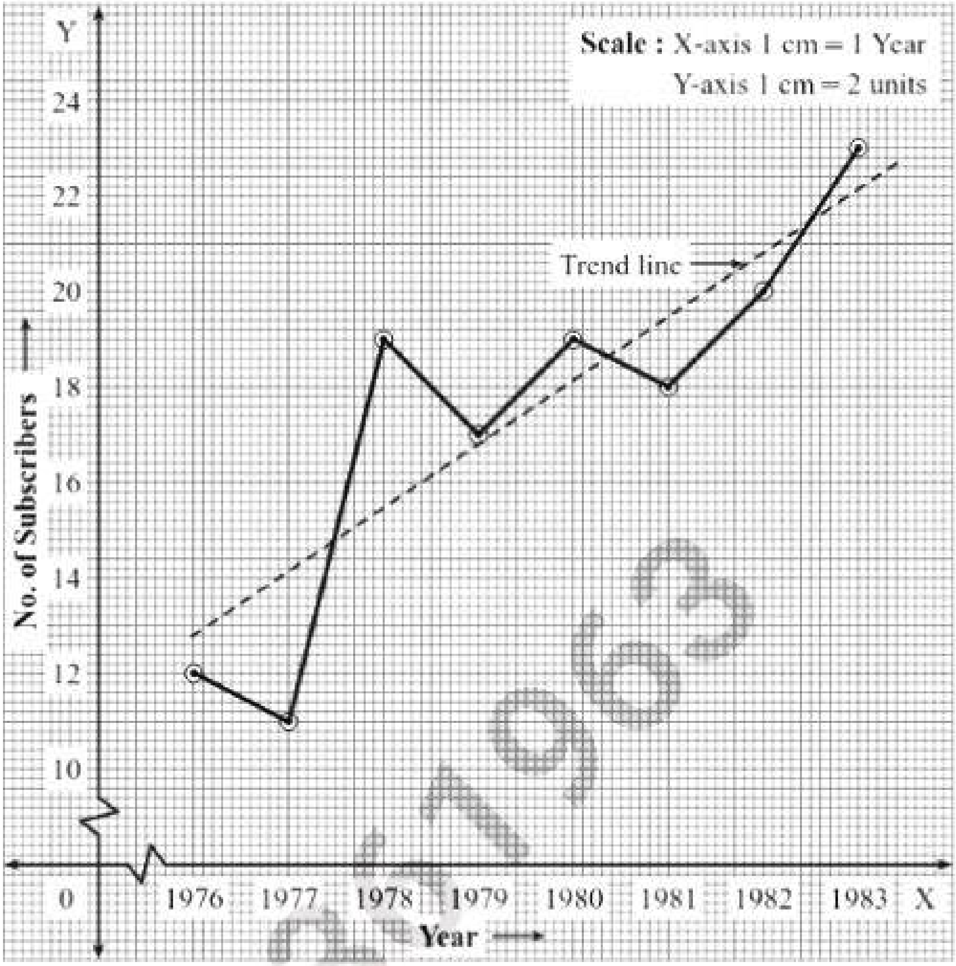
Year	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972
Production	0	0	1	1	4	2	4	9	7	10	8

Obtain the trend line for the above data using 5-yearly moving averages.

(July '24)

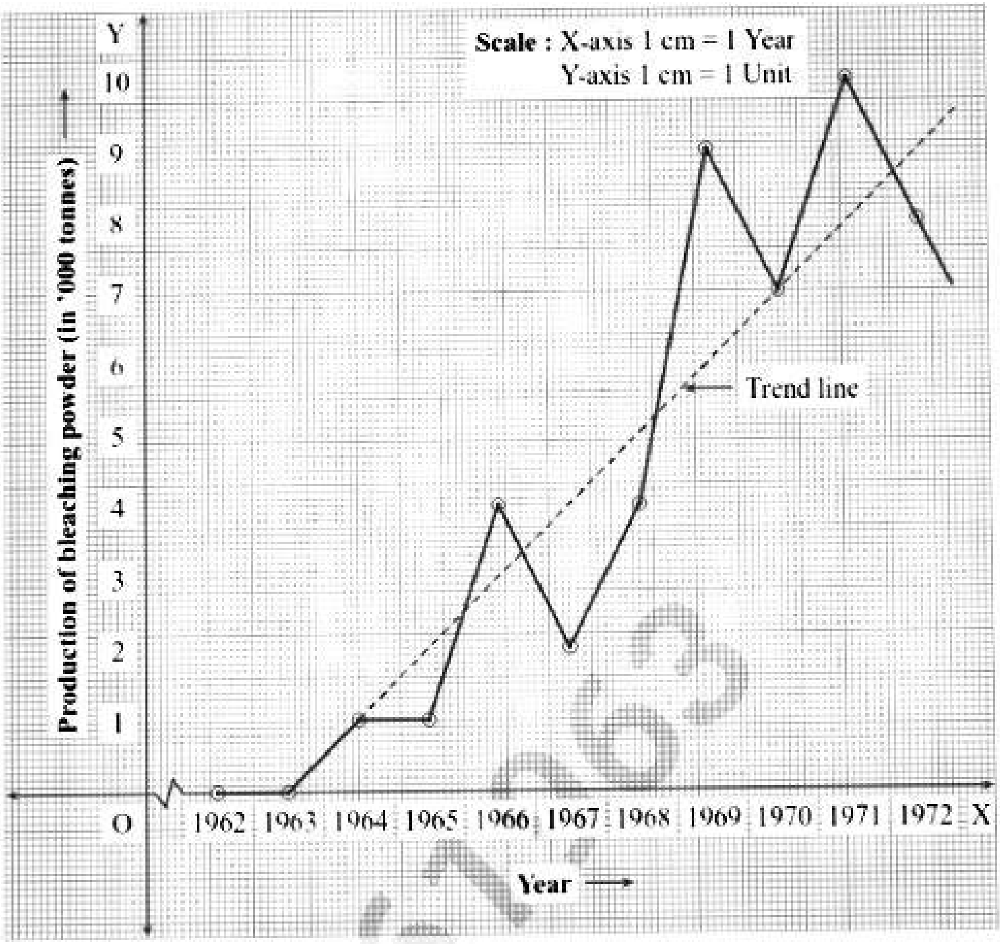
Answers

1.



Scale : X-axis 1 cm = 1 Year
Y-axis 1 cm = 2 units
Y
24
22
20
18
16
14
12
10
0
Trend line
No. of Subscribers
1976
1977
1978
1979
1980
1981
1982
1983
X
Year

2.



3. $x_t = 3.1667 + 0.2797u$, where $u = 2(t - 1976.5)$

4. —, 7.3333, 5.3333, 4.0000, 2.6667, 1.3333, —

5. $x_t = 5.6364 + 0.7909u$, where $u = t - 1981$.

6. —, —, 1.75, 2.25, 2.75, 3.25, 4, 5, 5.5, 5.875, —, —

7. —, —, 0.8, 1.4, 2.2, 3.0, 4.0, 5.0, 6.0, 7.0, 7.6, 8.2, 8.8, —, —.

8. —, —, 27, 29, 33, 36, 41, 47, —, —

9. —, —, 1.2, 1.6, 2.4, 4.0, 5.2, 6.4, 7.6, —, —

ACTIVITIES	**4 marks each**

1. Complete the following activity to obtain 4-yearly centred moving averages for the following time series :

Year	1987	1988	1989	1990	1991	1992	1993	1994	1995
Annual sales (in lakh ₹)	3.6	4.3	4.3	3.4	4.4	5.2	3.8	4.9	5.4

We construct the following table to find 4-yearly centred moving averages for the given data :

Year	Annual sales (in lakh ₹)	4-yearly moving total	4-yearly moving averages	2-unit moving total	4-yearly centred moving averages (trend value)
1987	3.6	–	–	–	–
1988	4.3	–	–	–	–
1989	4.3	15.6	$\boxed{}$	–	–
1990	3.4	16.4	4.1	8	4
1991	4.4	$\boxed{}$	4.325	8.425 8.525	4.2125 4.2625
1992	5.2	16.8	4.2	$\boxed{}$	$\boxed{}$
1993	3.8	18.3	4.575	9.4	4.7
1994	4.9	19.3	4.825	–	–
1995	5.4	–	–	–	–

Solution : We construct the following table to find 4-yearly centred moving averages for the given data :

Year	Annual sales (in lakh ₹)	4-yearly moving total	4-yearly moving averages	2-unit moving total	4-yearly centred moving averages (trend value)
1987	3.6	–	–	–	–
1988	4.3	–	–	–	–
1989	4.3	15.6	$\boxed{3.9}$	–	–
1990	3.4	16.4	4.1	8	4
1991	4.4	$\boxed{17.3}$	4.325	8.425 8.525	4.2125 4.2625
1992	5.2	16.8	4.2	$\boxed{8.775}$	$\boxed{4.3875}$
1993	3.8	18.3	4.575	9.4	4.7
1994	4.9	19.3	4.825	–	–
1995	5.4	–	–	–	–

2. Complete the following activity to find the equation of trend line for the following data showing production of steel (in millions of tonnes) for years 1976 to 1986 by filling boxes in the following activity :

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Here, $n = 11$. We transform year t to u by taking $u = t - 1981$.

We construct the following table for calculation :

Year t	**Production** y_t	$u = t - 1981$	u^2	uy_t
1976	0	−5	25	0
1977	4	−4	16	−16
1978	4	−3	9	−12
1979	2	−2	4	−4
1980	6	−1	1	−6
1981	8	0	0	0
1982	5	1	1	5
1983	9	2	4	18
1984	4	3	9	12
1985	10	4	16	40
1986	10	5	25	50
Total	$\Sigma y_t = 62$	$\Sigma u = 0$	$\Sigma u^2 = 110$	$\Sigma uy_t = 87$

Let equation of trend line be

$y_t = \boxed{}$ ... (1)

$\Sigma y_t = na' + b' \Sigma u$ ... (2)

$\Sigma uy_t = a' \Sigma u + b' \Sigma u^2$ ... (3)

$a' = \boxed{}$

$b' = \boxed{}$

$\therefore$ equation of trend line is $y_t = \boxed{}$ *(July '22)*

Solution : Here, $n = 11$. We transform year t to u by taking $u = t - 1981$.

We construct the following table for calculation :

Year t	Production y_t	$u = t - 1981$	u^2	uy_t
1976	0	-5	25	0
1977	4	-4	16	-16
1978	4	-3	9	-12
1979	2	-2	4	-4
1980	6	-1	1	-6
1981	8	0	0	0
1982	5	1	1	5
1983	9	2	4	18
1984	4	3	9	12
1985	10	4	16	40
1986	10	5	25	50
Total	$\Sigma y_t = 62$	$\Sigma u = 0$	$\Sigma u^2 = 110$	$\Sigma u y_t = 87$

Let equation of trend line be

$y_t = \boxed{a' + b'u}$ ... (1)

$\Sigma y_t = na' + b' \Sigma u$ ... (2)

$\Sigma u y_t = a' \Sigma u + b' \Sigma u^2$ ... (3)

Here, $n = 11$, $\Sigma y_t = 62$, $\Sigma u = 0$, $\Sigma u^2 = 110$, $\Sigma u y_t = 87$

Putting these values in normal equation, we get

$62 = 11a' + b'(0)$

$87 = a'(0) + b'(110)$

$\therefore\ a' = \boxed{\frac{62}{11}}$, $\quad b' = \boxed{\frac{87}{110}}$

$\therefore$ equation of trend line is $y_t = \boxed{\frac{62}{11} + \frac{87}{110}u}$

3. Following table shows the amount of sugar production (in lakh tonnes) for the years 1931 to 1941 :

Year	1931	1932	1933	1934	1935	1936	1937	1938	1939	1940	1941
Production	1	0	1	2	3	2	8	6	5	1	4

Complete the following activity to fit a trend line by method of least squares :

Let y_t be the trend line represented by the equation $y_t = a + bt$

Let $u = \dfrac{t - \text{mid value}}{h}$, mid value = 1936 and $h = 1$

Year (t)	y_t	u	u^2	uy_t
1931	1	-5	25	-5
1932	0	-4	16	0
1933	1	-3	9	-3
1934	2	-2	4	-4
1935	3	-1	1	-3
1936	2	0	0	0
1937	8	1	1	8
1938	6	2	4	12
1939	5	3	9	15
1940	1	4	16	4
1941	4	5	25	20
	$\Sigma y_t = 33$	$\Sigma u = 0$	$\Sigma u^2 = 110$	$\boxed{\quad}$

The equation of the trend line becomes.

$$y_t = a' + b'u \qquad \text{... (1)}$$

Two normal equations are

$$\Sigma y_t = na' + b' \Sigma u \qquad \text{... (2)}$$

$$\Sigma u \cdot y_t = a' \Sigma u + b' \Sigma u^2 \qquad \text{... (3)}$$

From equation (2), we get, $a' = \boxed{\quad}$

From equation (3), we get, $b' = \boxed{\quad}$

The equation of trend line is given by $y_t = \boxed{\quad}$ ***(March '22)***

Solution :

Year (t)	y_t	$u=\frac{t-1936}{1}$	u^2	uy_t
1931	1	-5	25	-5
1932	0	-4	16	0
1933	1	-3	9	-3
1934	2	-2	4	-4
1935	3	-1	1	-3
1936	2	0	0	0
1937	8	1	1	8
1938	6	2	4	12
1939	5	3	9	15
1940	1	4	16	4
1941	4	5	25	20
	$\Sigma y_t=33$	$\Sigma u=0$	$\Sigma u^2=110$	$\boxed{\Sigma uy_t=44}$

The equation of the trend line becomes, $y_t=a'+b'u$ ... (1)

Two normal equations are

$$\Sigma y_t=na'+b'\Sigma u$$

$$\Sigma u\cdot y_t=a'\Sigma u+b'\Sigma u^2$$

From the table, $\Sigma y_t=33$, $\Sigma u=0$, $\Sigma u^2=110, \Sigma u\cdot y_t=44, n=11$

$\therefore\ 33=11a'+b'(0)$ ... (2)

and $44=a'(0)+b'(110)$ ... (3)

From equation (2), we get, $a'=\frac{33}{11}=\boxed{3}$

From equation (3), we get, $b'=\frac{44}{110}=\boxed{0.4}$

The equation of trend line is given by $y_t=\boxed{3+0.4u}$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. What is a disadvantage of the graphical method of determining a trend line?

(a) Provides quick approximations (b) Is subject to human error

(c) Provides accurate forecasts (d) Is too difficult to calculate

2. Moving averages are useful in identifying

(a) seasonal component (b) irregular component

(c) trend component (d) cyclical component *(March '25)*

3. An overall upward or downward pattern in an annual time series would be contained in which component of the time series.

(a) trend (b) cyclical (c) irregular (d) seasonal

4. The complicated but efficient method of measuring trend of time series is ......

(a) graphical method (b) method of moving averages

(c) method of least squares (d) method of addition *(March '23)*

5. Which component of time series refers to erratic time series movements that follow no recognizable or regular pattern?

(a) Trend (b) Seasonal (c) Cyclical (d) Irregular

6. The following trend line equation was developed for annual sales from 1984 to 1990 with 1984 as base year, $y = 500 + 60x$, (in 1000 ₹). The estimated sales for 1984 (in 1000 ₹) is ....

(a) 500 (b) 560 (c) 1040 (d) 1100 *(Sept. '21)*

Answers

1. (b) Is subject to human error **2.** (c) trend component **3.** (a) trend

4. (c) method of least squares **5.** (d) Irregular **6.** (a) 500.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. Cyclical variation can occure several times in a year. *(March '25)*

2. Additive method of time series does not require the assumption of independence of its components.

3. All the three methods of measuring trend will always give the same results.
4. Seasonal variation can be observed over several years.
5. Least squares method of finding trend is very simple and does not involve any calculations.
6. Irregular variation is not a random component of time series. *(Sept. '21)*

Answers

1. False 2. False 3. False 4. True 5. False 6. False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. The complicated but efficient method of measuring trend of time series is ....
2. .... component of time series is indicated by periodic variation year after year.
3. The method of measuring trend of time series using only averages is ....
4. Multiplicative models of time series .... independence of its components.
5. The simplest method of measuring trend of time series is .... *(March '25)*
6.

Year	1991	1992	1993
Production	2	3	4

From the above data, the trend value using 3-yearly moving average is .... *(Sept. '21)*

7. If $\Sigma u y_t = 61$, $\Sigma u = 0$, $\Sigma u^2 = 110$, then $b' =$ ....

Answers

1. least square 2. seasonal 3. moving average 4. does not assume
5. Graphical 6. 3 7. 0.5545.

Question Set 13

INDEX NUMBERS

(Marks with option : 07)

13.1 SIMPLE AGGREGATE METHOD

Remember :

1. Price Index Number :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

2. Quantity Index Number :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

3. Value Index Number :

Value = Price × Quantity

Base year's value $= p_0 q_0$

Current year's value $= p_1 q_1$

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

Solved Examples | **3 marks each**

Ex. 1. Calculate the Price Index Number using Simple Aggregate Method for the following data. Use 2000 as base year :

Commodity	A	B	C	D	E
Price (in ₹) in 2000	30	35	45	55	25
Price (in ₹) in 2003	40	50	70	75	40

Solution :

Here, Base year = 2000

$\therefore p_0$ = Price in the year 2000 and

p_1 = Price in the year 2003.

Commodity	Price (in ₹)	
	p_0	p_1
A	30	40
B	35	50
C	45	70
D	55	75
E	25	40
Total	$\Sigma p_0 = 190$	$\Sigma p_1 = 275$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100 = \frac{275}{190} \times 100$$

$$= 1.4474 \times 100 = 144.74$$

Hence, price index number is 144.74.

Ex. 2. Find the Quantity Index Number using Simple Aggregate Method for the following data :

Commodity	I	II	III	IV	V
Base Year quantities	140	120	100	200	220
Current Year quantities	100	80	70	150	185

Solution :

Let q_0 = Quantity of base year and q_1 = Quantity of current year.

Commodity	Quantity	
	q_0	q_1
I	140	100
II	120	80
III	100	70
IV	200	150
V	220	185
Total	$\Sigma q_0 = 780$	$\Sigma q_1 = 585$

Quantity Index number by Simple Aggregate Method :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100 = \frac{585}{780} \times 100 = 0.75 \times 100 = 75$$

Hence, quantity index number is 75.

Ex. 3. Find the Value Index Number using Simple Aggregate Method for the following data :

Commodity	Base Year		Current Year	
	Price (in ₹)	Quantity	Price (in ₹)	Quantity
A	30	22	40	18
B	40	15	60	12
C	10	38	15	24
D	50	12	60	16
E	20	28	25	36

Solution : Here, p_0=Price in base year, p_1=Price in current year,

q_0 = Quantity of base year and q_1 = Quantity of current year.

Commodity	Base Year		Current Year		p_0q_0	p_1q_1
	p_0	q_0	p_1	q_1		
A	30	22	40	18	660	720
B	40	15	60	12	600	720
C	10	38	15	24	380	360
D	50	12	60	16	600	960
E	20	28	25	36	560	900
Total					Σp_0q_0 $=2800$	Σp_1q_1 $=3660$

Value Index Number by Simple Aggregate Method :

$$V_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \times 100 = \frac{3660}{2800} \times 100 = 1.3071 \times 100 = 130.71$$

Hence, value index number is 130.71.

Ex. 4. Find x, if Price Index Number by Simple Aggregate Method is 125 :

Commodity	P	Q	R	S	T
Base Year Price (in ₹)	8	12	16	22	18
Current Year Price (in ₹)	12	18	x	28	22

(July '23)

Solution : Given : $P_{01} = 125$, $x = ?$

Commodity	Price (in ₹)	
	Base year	Current year
	p_0	p_1
P	8	12
Q	12	18
R	16	x
S	22	28
T	18	22
Total	$\Sigma p_0 = 76$	$\Sigma p_1 = 80 + x$

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$\therefore\ 125 = \frac{80 + x}{76} \times 100 \qquad \therefore\ \frac{125 \times 76}{100} = 80 + x$$

$$\therefore\ 95 = 80 + x \qquad \therefore\ 95 - 80 = x \qquad \therefore\ x = 15$$

Hence, the value of x is ₹ 15.

Examples for Practice	**3 marks each**

1. Find Price Index Number using Simple Aggregate Method. Consider 1995 as base year :

Commodity	A	B	C	D	E
Price (in ₹) in 1995	42	30	58	70	120
Price (in ₹) in 2005	60	55	75	110	140

2. Find the Quantity Index Number using Simple Aggregate Method :

Commodity	Base Year Quantity	Current Year Quantity
A	100	130
B	170	200
C	210	250
D	90	110
E	50	150

3. Find y, if the Price Index Number by Simple Aggregate Method is 120, taking 1995 as base year :

Commodity	A	B	C	D
Price (in ₹) in 1995	95	y	80	35
Price (in ₹) in 2003	116	74	92	42

4. Find the Value Index Number using Simple Aggregate Method for the following data :

Commodity	Base Year		Current Year	
	Price (in ₹)	Quantity	Price (in ₹)	Quantity
A	30	22	40	18
B	40	16	60	12
C	10	38	15	24
D	50	12	60	16
E	20	28	25	36

(July '24)

5. Using 1995 as base year in the following problem, find the Price Index Number by Simple Aggregate Method :

Commodity	A	B	C	D	E
Price (in ₹) in 1995	42	30	54	70	120
Price (in ₹) in 2005	60	55	74	110	140

(July '22)

Answers

1. 137.5 2. 135.48 3. $y = 60$ 4. 128. 87 5. 138.92.

13.2 WEIGHTED AGGREGATE METHOD

Remember :

1. **Laspeyre's Price Index Number :**

$$P_{01}(L) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

2. **Paasche's Price Index Number :**

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

3. **Dorbish-Bowley's Price Index Number :**

$$P_{01}(D\text{-}B) = \frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100 = \frac{P_{01}(L) + P_{01}(P)}{2}$$

4. **Fisher's Price Index Number :**

$$P_{01}(F) = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100 = \sqrt{P_{01}(L) \times P_{01}(P)}$$

5. **Marshall-Edgeworth's Price Index Number :**

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1 (q_0 + q_1)}{\Sigma p_0 (q_0 + q_1)} \times 100 = \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$

6. **Walsch's Price Index Number :**

$$P_{01}(W) = \frac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100$$

Solved Examples	**3 or 4 marks each**

Ex. 5. Calculate Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers for the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	8	20	11	15
B	7	10	12	10
C	3	30	5	25
D	2	50	4	35

Solution :

	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
Commodity	p_0	q_0	p_1	q_1				
A	8	20	11	15	220	160	165	120
B	7	10	12	10	120	70	120	70
C	3	30	5	25	150	90	125	75
D	2	50	4	35	200	100	140	70
Total					$\Sigma p_1q_0 = 690$	$\Sigma p_0q_0 = 420$	$\Sigma p_1q_1 = 550$	$\Sigma p_0q_1 = 335$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{690}{420} \times 100 = 1.6429 \times 100 = 164.29$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100 = \frac{550}{335} \times 100 = 1.6418 \times 100 = 164.18$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} + \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2} \times 100 = \frac{1.6429 + 1.6418}{2} \times 100 = \frac{3.2847}{2} \times 100$$

$$= 1.6424 \times 100 = 164.24$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100 = \frac{690 + 550}{420 + 335} \times 100 = \frac{1240}{755} \times 100$$

$$= 1.6424 \times 100 = 164.24$$

Hence, $P_{01}(L) = 164.29$, $P_{01}(P) = 164.18$, $P_{01}(D\text{-}B) = 164.24$ and $P_{01}(M\text{-}E) = 164.24$.

Ex. 6. Find x, if Walsch's Price Index Number is 150 for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	5	3	10	3
B	x	4	16	9
C	15	5	23	5
D	10	2	26	8

(March '24)

Solution :

Commodity	Base Year		Current Year		q_0q_1	$\sqrt{q_0q_1}$	$p_1\sqrt{q_0q_1}$	$p_0\sqrt{q_0q_1}$
	p_0	q_0	p_1	q_1				
A	5	3	10	3	9	3	30	15
B	x	4	16	9	36	6	96	$6x$
C	15	5	23	5	25	5	115	75
D	10	2	26	8	16	4	104	40
Total							$\Sigma p_1\sqrt{q_0q_1} = 345$	$\Sigma p_0\sqrt{q_0q_1} = 130+6x$

Walsch's Price Index Number :

$$P_{01}(W) = \frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}} \times 100 = \frac{345}{130+6x} \times 100$$

It is given that $P_{01}(W) = 150$

$$\therefore\ 150 = \frac{345}{130+6x} \times 100$$

$$\therefore\ 150(130+6x) = 345 \times 100$$

$$\therefore\ 19500 + 900x = 34500$$

$$\therefore\ 900x = 34500 - 19500$$

$$\therefore\ 900x = 15000$$

$$\therefore\ x = \frac{15000}{900} \qquad \therefore\ x = 16.66$$

Hence, x is 16.66.

Ex. 7. Find x, if Laspeyre's Price Index Number is same as Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	3	x	2	5
B	4	6	3	5

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
A	3	x	2	5	$2x$	$3x$	10	15
B	4	6	3	5	18	24	15	20
Total					$\Sigma p_1q_0 = 18+2x$	$\Sigma p_0q_0 = 24+3x$	$\Sigma p_1q_1 = 25$	$\Sigma p_0q_1 = 35$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{18+2x}{24+3x} \times 100$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100 = \frac{25}{35} \times 100$$

Given : $P_{01}(L) = P_{01}(P)$

$$\therefore \frac{18+2x}{24+3x} \times 100 = \frac{25}{35} \times 100$$

$$\therefore \frac{18+2x}{24+3x} = \frac{25}{35}$$

$\therefore 35(18+2x) = 25(24+3x)$

$\therefore 630 + 70x = 600 + 75x$

$\therefore 630 - 600 = 75x - 70x$

$\therefore 30 = 5x \quad \therefore x = \frac{30}{5} \quad \therefore x = 6$

Hence, x is 6.

Ex. 8. If $\Sigma p_0q_0 = 140$, $\Sigma p_0q_1 = 200$, $\Sigma p_1q_0 = 350$ and $\Sigma p_1q_1 = 460$. Find Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers. ***(July '22–'23)***

Solution : Given : $\Sigma p_0q_0 = 140$, $\Sigma p_0q_1 = 200$, $\Sigma p_1q_0 = 350$, $\Sigma p_1q_1 = 460$.

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{350}{140} \times 100 = 2.5 \times 100 = 250$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100 = \frac{460}{200} \times 100 = 2.3 \times 100 = 230$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} + \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2} \times 100 = \frac{2.5 + 2.3}{2} \times 100$$

$$= \frac{4.8}{2} \times 100 = 2.4 \times 100 = 240$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1q_0\Sigma + p_1q_1}{\Sigma P_0q_0 + \Sigma p_0q_1} \times 100 = \frac{350 + 460}{140 + 200} \times 100$$

$$= \frac{810}{340} \times 100 = 2.3824 \times 100 = 238.24$$

Hence, $P_{01}(L) = 250$, $P_{01}(P) = 230$, $P_{01}(D\text{-}B) = 240$ and $P_{01}(M\text{-}E) = 238.24$.

Examples for Practice	**3 or 4 marks each**

1. Calculate Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers for the following data :

Commodity	**Base Year**		**Current Year**	
	Price	**Quantity**	**Price**	**Quantity**
P	12	20	18	24
Q	14	12	21	16
R	8	10	12	18
S	16	15	20	25

2. Calculate Laspeyre's and Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	20	18	30	5
B	25	8	28	4
C	32	5	40	5
D	12	10	18	20

3. Calculate Marshall-Edgeworth's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
P	12	20	18	24
Q	14	12	21	16
R	8	10	12	18
S	16	15	20	25

(March '23)

4. Given that, $\Sigma p_0q_0 = 220$, $\Sigma p_0q_1 = 380$, $\Sigma p_0q_1 = 350$ and Marshall-Edgeworth's Price Index Number is 150. Find Laspeyre's Price Index Number. ***(July '24)***

5. Given : $\Sigma p_0q_0 = 130$, $\Sigma p_1q_1 = 140$, $\Sigma p_0q_1 = 160$ and $\Sigma p_1q_0 = 200$.
Find Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers

6. If $\Sigma p_0q_0 = 180$, $\Sigma p_0q_1 = 200$, $\Sigma p_1q_1 = 280$ and $p_{01}(\text{M-E}) = 150$, find $P_{01}(\text{P})$.

Answers

1. $P_{01}(L) = 141.76$, $P_{01}(P) = 140.53$, $P_{01}(D\text{-}B) = 141.15$, $P_{01}(M\text{-}E) = 141.03$

2. $P_{01}(L) = 136.19$, $P_{01}(P) = 137$ **3.** $P_{01}(M\text{-}E) = 141.03$ **4.** 250

5. $P_{01}(L) = 153.85$, $P_{01}(P) = 87.5$, $P_{01}(D\text{-}B) = 120.68$, $P_{01}(M\text{-}E) = 117.24$

6. $P_{01}(P) = 200$.

13.3 COST OF LIVING INDEX NUMBER

Remember :

1. Cost of Living Index Number (CLI) :

The percentage change occurring in the cost of living of a particular section of people at a given place during a certain period in relation to base period is called Cost of Living Index Number of that particular section of people.

2. (1) Aggregate Expenditure Method :

$$\text{CLI} = \frac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$$

[**Note :** The above formula is equivalent to formula of

Laspeyre's Index Number.]

(2) Family Budget Method (Weighted Relative Method) :

In this method, Price relatives (I) for all items are obtained.

i.e. $\text{I} = \frac{p_1}{p_0} \times 100$

The base year's expenditure of items ($p_0 q_0$) are taken as weights of Price relatives I.

Hence, Cost of Living Index Number

$\text{CLI} = \frac{\Sigma \text{IW}}{\Sigma \text{W}}$, where $\text{I} = \frac{p_1}{p_0} \times 100$ and $\text{W} = p_0 q_0$.

[**Note :** This formula is equivalent to formula of

Aggregate Expenditure Method.]

(3) It is used to calculate purchasing power of money.

$$\text{Purchasing power of money} = \frac{1}{\text{Cost of Living Index Number}}$$

(4) It is used to determine the real wages

$$\text{Real wages} = \frac{\text{Actual Wage}}{\text{Cost of Living Index Number}} \times 100$$

Solved Examples	3 or 4 marks each

Ex. 9. Find Cost of Living Index Number using Aggregate Expenditure Method for the following data :

Group	Base Year		Current Year
	Price	Quantity	Price
Food	120	15	170
Clothing	150	20	190
Fuel and Lighting	130	30	220
House Rent	160	10	180
Miscellaneous	200	12	200

(March '22)

Solution :

Group	Base Year		Current Year	p_1q_0	p_0q_0
	p_0	q_0	p_1		
Food	120	15	170	2550	1800
Clothing	150	20	190	3800	3000
Fuel and Lighting	130	30	220	6600	3900
House Rent	160	10	180	1800	1600
Miscellaneous	200	12	200	2400	2400
Total				$\Sigma p_1q_0 = 17150$	$\Sigma p_0q_0 = 12700$

By Aggregate Expenditure Method,

$$\text{CLI} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{17150}{12700} \times 100 = 1.3504 \times 100 = 135.04$$

Hence, Cost of Living Index Number is 135.04.

Ex. 10. Calculate the Cost of Living Index Number for the year 1999 by Family Budget Method from the following data. Also, find the expenditure of the person in 1999, if his expenditure in year 1995 was 800 :

Group	Price in Year 1995	Price in Year 1999	W
Food	8	24	6
Clothing	18	36	12
Fuel and Lighting	20	40	8
House Rent	15	30	4
Miscellaneous	10	22	10

Solution :

Group	Price in year 1995	Price in year 1999	$I=\frac{p_1}{p_0}\times 100$	W	IW
	p_0	p_1			
Food	8	24	$\frac{24}{8}\times 100=300$	6	1800
Clothing	18	36	$\frac{36}{18}\times 100=200$	12	2400
Fuel and Lighting	20	40	$\frac{40}{20}\times 100=200$	8	1600
House Rent	15	30	$\frac{30}{15}\times 100=200$	4	800
Miscellaneous	10	22	$\frac{22}{10}\times 100=220$	10	2200
Total	–	–	–	$\Sigma W = 40$	$\Sigma IW = 8800$

Cost of Living Index Number : $CLI=\frac{\Sigma IW}{\Sigma W}=\frac{8800}{40}=220$

Given : Expenditure in 1995 = ₹ 800

Now, expenditure in 1999 $=\frac{\text{CLI for 1999}\times\text{Expenditure in 1995}}{100}$

$$=\frac{220\times 800}{100}=₹\ 1760$$

Hence, CLI is 220 and expenditure in year 1999 is ₹ 1760.

Ex. 11. Find x, if Cost of Living Index is 150 : *(March '25)*

Group	Food	Clothing	Fuel and Electricity	House Rent	Miscellaneous
I	180	120	300	100	160
W	4	5	6	x	3

Solution :

Group	I	W	IW
Food	180	4	720
Clothing	120	5	600
Fuel and Electricity	300	6	1800
House Rent	100	x	$100x$
Miscellaneous	160	3	480
Total		$\Sigma W = 18 + x$	$\Sigma IW = 3600 + 100x$

By Family budget method.

$$CLI = \frac{\Sigma IW}{\Sigma W}$$

Given : $CLI = 150$

$$\therefore \ 150 = \frac{3600 + 100x}{18 + x}$$

$\therefore \ 150(18 + x) = 3600 + 100x$

$\therefore \ 2700 + 150x = 3600 + 100x$

$\therefore \ 150x - 100x = 3600 - 2700$

$\therefore \ 50x = 900 \qquad \therefore \ x = 18$

Hence, x is 18.

Ex. 12. The Cost of Living Index Number for the years 2000 and 2003 are 150 and 210 respectively. A person earns ₹ 13,500 per month in the year 2000. What should be his monthly earning in the year 2003 in order to maintain the same standard of living ?

Solution :

Given : CLI for 2000 = 150

CLI for 2003 = 210

The earning p.m. of a person in the year 2000 = ₹ 13500.

Now, the earning of a person in the year 2003

$$= \frac{\text{Earning in the year 2000} \times \text{CLI for 2003}}{\text{CLI for 2000}}$$

$$= \frac{13500 \times 210}{150} = ₹\ 18{,}900$$

Hence, the earning of a person should be ₹ 18,900 p.m. so as to maintain his former standard of living.

Examples for Practice	**3 or 4 marks each**

1. Find the Cost of Living Index Number by the Weighted Aggregate Method :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	70	90	100	60	80
W	5	3	2	4	6

2. Find Cost of Living Index Number using an appropriate method for the following data :

Group	Base Year		Current Year
	Price	Quantity	Price
Food and Clothing	40	3	70
Fuel and Lighting	30	5	60
House Rent	50	2	50
Miscellaneous	60	3	90

3. Find x, if the cost of Living Index number is 193 for the following data :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	221	198	171	183	161
W	35	14	x	8	20

4. Calculate the Cost of Living Index Number for the following data :

Group	Base Year		Current Year
	Price	Quantity	Price
Food	130	10	170
Clothing	150	12	160
Fuel and Lighting	162	20	180
House Rent	170	18	195
Miscellaneous	120	5	120

Answers

1. 77 **2.** 160 **3.** $x = 15$ **4.** 113.3.

ACTIVITIES	4 marks each

1. Complete the following activity to find x, if Paasche's Price Index Number is 140 for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	20	8	40	7
B	50	10	60	10
C	40	15	60	x
D	12	15	15	15

Commodity	Base Year		Current Year		$p_1 q_1$	$p_0 q_1$
	p_0	q_0	p_1	q_1		
A	20	8	40	7	280	140
B	50	10	60	10	600	500
C	40	15	60	x	$60x$	$40x$
D	12	15	15	15	225	180
Total					$\Sigma p_1 q_1 = \boxed{}$	$\Sigma p_0 q_1 = \boxed{}$

$$P_{01}(P) = \boxed{} \times 100$$

$$\therefore\ 140 = \frac{1105 + 60x}{820 + 40x} \times 100$$

$$\therefore\ x = \boxed{}.$$

Solution :

Commodity	Base Year		Current Year		$p_1 q_1$	$p_0 q_1$
	p_0	q_0	p_1	q_1		
A	20	8	40	7	280	140
B	50	10	60	10	600	500
C	40	15	60	x	$60x$	$40x$
D	12	15	15	15	225	180
Total					$\Sigma p_1 q_1 = \boxed{1105 + 60x}$	$\Sigma p_0 q_1 = \boxed{820 + 40x}$

$$P_{01}(P) = \boxed{\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$\therefore\ 140 = \frac{1105 + 60x}{820 + 40x} \times 100 \qquad \therefore\ \frac{140}{100} = \frac{1105 + 60x}{820 + 40x}$$

$$\therefore\ \frac{14}{10} = \frac{1105 + 60x}{820 + 40x} \qquad \therefore\ 11480 + 560x = 11050 + 600x$$

$$\therefore\ 40x = 430 \qquad \therefore\ x = \frac{430}{40} = \boxed{10.75}.$$

---

2. Complete the following activity to find x, if the Cost of Living Index Number is 193 for the following data :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	221	198	171	183	161
W	35	14	x	8	20

We construct the following table to find x :

Group	I	W	IW
Food	221	35	7735
Clothing	198	14	2772
Fuel and Lighting	171	x	$171x$
House Rent	183	8	1464
Miscellaneous	161	20	3220
		$\Sigma W = \boxed{\quad}$	$\Sigma IW = \boxed{\quad}$

Cost of Living Index Number is given by

CLI $= \boxed{\quad}$

$\therefore x = \boxed{\quad}$

Solution : We construct the following table to find x :

Group	I	W	IW
Food	221	35	7735
Clothing	198	14	2772
Fuel and Lighting	171	x	$171x$
House Rent	183	8	1464
Miscellaneous	161	20	3220
		$\Sigma W = \boxed{77 + x}$	$\Sigma IW = \boxed{15191 + 171x}$

Cost of Living Index Number is given by $\text{CLI} = \boxed{\dfrac{\Sigma IW}{\Sigma W}} = \dfrac{15191 + 171x}{77 + x}$

But, it is given, CLI = 193

$$\therefore 193 = \frac{15191 + 171x}{77 + x}$$

$$\therefore 193(77 + x) = 15191 + 171x$$

$$\therefore 14861 + 193x = 15191 + 171x$$

$$\therefore 193x - 171x = 15191 - 14861$$

$$\therefore 22x = 330 \qquad \therefore x = \frac{330}{22} \qquad \therefore x = \boxed{15}$$

3. In the following table, Laspeyre's and Paasche's Price Index Numbers are equal. Complete the following activity to find x :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	2	10	2	5
B	2	5	x	2

$P_{01}(L) = P_{01}(P)$

$$\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{\boxed{}}{\Sigma p_0 q_1} \times 100$$

$$\therefore \frac{20+5x}{\boxed{}} \times 100 = \frac{\boxed{}}{14} \times 100$$

$\therefore x = \boxed{}$ **(Sept. '21)**

Solution :

Commodity	Base Year		Current Year		p_0q_0	p_1q_0	p_1q_1	p_0q_1
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)				
A	2	10	2	5	20	20	10	10
B	2	5	x	2	10	$5x$	$2x$	4
					$\Sigma p_0q_0 = 30$	$\Sigma p_1q_0 = 20+5x$	$\Sigma p_1q_1 = 10+2x$	$\Sigma p_0q_1 = 14$

$P_{01}(L) = P_{01}(P)$

$$\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{\boxed{\Sigma p_1 q_1}}{\Sigma p_0 q_1} \times 100$$

$$\therefore \frac{20+5x}{\boxed{30}} \times 100 = \frac{\boxed{10+2x}}{14} \times 100$$

$$\therefore \frac{20+5x}{30} = \frac{10+2x}{14}$$

$\therefore 280 + 70x = 300 + 60x$

$\therefore 10x = 20 \qquad \therefore x = \boxed{2}$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. Marshall-Edgeworth's Price Index Number is given by

(a) $\dfrac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)}\times 100$ (b) $\dfrac{\Sigma p_0(q_0+q_1)}{\Sigma p_1(q_0+q_1)}\times 100$

(c) $\dfrac{\Sigma q_1(p_0+p_1)}{\Sigma q_0(p_0+p_1)}\times 100$ (d) $\dfrac{\Sigma q_0(p_0+p_1)}{\Sigma q_1(p_0+p_1)}\times 10$

2. Paasche's Price Index Number is given by

(a) $\dfrac{\Sigma p_0q_0}{\Sigma p_1q_0}\times 100$ (b) $\dfrac{\Sigma p_0q_1}{\Sigma p_1q_1}\times 100$ (c) $\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0}\times 100$ (d) $\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}\times 100$

(July '24)

3. Walsch's Price Index Number is given by

(a) $\dfrac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}}\times 100$ (b) $\dfrac{\Sigma p_0\sqrt{q_0q_1}}{\Sigma p_1\sqrt{q_0q_1}}\times 100$

(c) $\dfrac{\Sigma q_1\sqrt{p_0p_1}}{\Sigma q_0\sqrt{p_0p_1}}\times 100$ (d) $\dfrac{\Sigma q_0\sqrt{p_0p_1}}{\Sigma q_1\sqrt{p_0p_1}}\times 100$

4. Quantity Index Number by Weighted Aggregate Method is given by

(a) $\Sigma\dfrac{q_1w}{q_0w}\times 100$ (b) $\Sigma\dfrac{q_0w}{q_1w}\times 100$ (c) $\dfrac{\Sigma q_0w}{\Sigma q_1w}\times 100$ (d) $\dfrac{\Sigma q_1w}{\Sigma q_0w}\times 100$

(March '23)

5. The Cost of Living Index Number using Weighted Relative Method is given by

(a) $\dfrac{\Sigma \text{IW}}{\Sigma \text{W}}$ (b) $\Sigma\dfrac{\text{W}}{\text{IW}}$ (c) $\dfrac{\Sigma \text{W}}{\Sigma \text{IW}}$ (d) $\Sigma\dfrac{\text{IW}}{\text{W}}$

6. Dorbish-Bowley's Price Index Number is given by

(a) $\dfrac{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_1}+\dfrac{\Sigma p_0q_1}{\Sigma p_1q_0}}{2}\times 100$ (b) $\dfrac{\dfrac{\Sigma p_1q_1}{\Sigma p_0q_0}+\dfrac{\Sigma p_0q_0}{\Sigma p_1q_1}}{2}\times 100$

(c) $\dfrac{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0}+\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2}\times 100$ (d) $\dfrac{\dfrac{\Sigma p_0q_0}{\Sigma p_1q_0}+\dfrac{\Sigma p_0q_1}{\Sigma p_1q_1}}{2}\times 100$

(March '24)

7. If $P_{01}(L) = 90$, $P_{01}(P) = 40$, then $P_{01}(F) =$ ....

(a) 3600 (b) 60 (c) 65 (d) 90 ***(Sept. '21)***

8. Price Index Number by Weighted Aggregate Method is given by ....

(a) $\Sigma \dfrac{p_1 w}{p_0 w} \times 100$ (b) $\Sigma \dfrac{p_0 w}{p_1 w} \times 100$

(c) $\dfrac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$ (d) $\dfrac{\Sigma p_0 w}{\Sigma p_1 w} \times 100$ ***(March '22)***

9. The Cost of Living Index Number using Aggregate Expenditure Method is given by

(a) $\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (b) $\Sigma \dfrac{p_1 q_1}{p_0 q_1} \times 100$ (c) $\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ (b) $\Sigma \dfrac{p_1 q_0}{p_0 q_0} \times 100$

(July '22)

10. Laspeyre's Price Index Number is given by

(a) $\dfrac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \times 100$ (b) $\dfrac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times 100$ (c) $\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (b) $\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$

11. If $P_{01}(L) = 90$ and $P_{01}(P) = 40$, then $P_{01}(D\text{-}B)$ is.........

(a) 65 (b) 50 (c) 25 (d) 130

(March '25)

Answers

1. (a) $\dfrac{\Sigma p_1(q_0 + q_1)}{\Sigma p_0(q_0 + q_1)} \times 100$ 2. (d) $\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ 3. (a) $\dfrac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100$

4. (d) $\dfrac{\Sigma q_1 w}{\Sigma q_0 w} \times 100$ 5. (a) $\dfrac{\Sigma IW}{\Sigma W}$ 6. (c) $\dfrac{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$

7. (b) 60 8. (c) $\dfrac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$ 9. (a) $\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ 10. (c) $\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$

11. (a) 65.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. $\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \dfrac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$ is Dorbish-Bowley's Price Index Number.

2. $\dfrac{\Sigma p_0(q_0+q_1)}{\Sigma p_1(q_0+q_1)} \times 100$ is Marshall-Edgeworth's Price Index Number. ***(July '23)***

3. $\Sigma \dfrac{p_0q_0}{p_1q_1} \times 100$ is Value Index Number by Simple Aggregate Method. ***(March '24)***

4. The Quantity Index Number according to Weighted Aggregate Method is given by $\dfrac{\Sigma q_1w}{\Sigma q_0w} \times 100$. ***(Sept. '21)***

5. Dorbish-Bowley's Price Index Number is square root of product of Laspeyre's and Paasche's Index Numbers. ***(March '22)***

6. Laspeyre's Price Index Number uses current year's quantities as weights. ***(March '23)***

7. Cost of Living Index Number is used in calculating purchasing power of money. ***(March '25)***

Answers

1. False 2. False 3. False 4. True 5. False 6. False 7. True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. Walsch's Price Index Number is given by .... ***(March '23)***

2. Fisher's Price Index Number is given by ....

3. Quantity Index Number by Weighted Aggregate Method is given by .... ***(March '25)***

4. Price Index Number by Weighted Aggregate Method is given by ....

5. Paasche's Price Index Number is given by ....

6. The Marshall-Edgeworth's Price Index Number is given by .... ***(Sept. '21)***

7. If $P_{01}(L) = 121$, $P_{01}(P) = 100$, then $P_{01}(F) = \ldots\ldots\ldots$. *(March '22)*

8. If $\Sigma p_0 = 25 + x$, $\Sigma p_1 = 60$, $P_{01} = 120$, then $x = \ldots\ldots\ldots$.

9. The Cost of Living Index Number using Weighted Relative Method is given by .... *(March '24)*

Answers

1. $\dfrac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}} \times 100$

2. $\sqrt{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100$

3. $\dfrac{\Sigma q_1w}{\Sigma q_0w} \times 100$

4. $\dfrac{\Sigma p_1w}{\Sigma p_0w} \times 100$

5. $\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$

6. $\dfrac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100$

7. 110

8. 25

9. $\dfrac{\Sigma IW}{\Sigma W}$.

Question Set 14

LINEAR PROGRAMMING

(Marks with option : 06)

Solved Examples	3 or 4 marks each

Ex. 1. The company makes concrete bricks made up of cement and sand. The weight of a concrete brick has to be at least 5 kg. Cement costs ₹ 20 per kg and sand costs ₹ 6 per kg. Strength consideration dictate that a concrete brick should contain minimum 4 kg of cement and not more than 2 kg of sand. Formulate the LPP for the cost to be minimum.

(March '25)

Solution : Let the company use x_1 kg of cement and x_2 kg of sand to make concrete bricks.

Cement costs ₹ 20 per kg and sand costs ₹ 6 per kg.

$\therefore$ the total cost $c=$ ₹ $(20x_1+6x_2)$

This is a linear function which is to be minimized.

Hence, it is the objective function.

Total weight of brick $=(x_1+x_2)$ kg

Since the weight of concrete brick has to be at least 5 kg, $x_1+x_2 \geqslant 5$.

Since concrete brick should contain minimum 4 kg of cement and not more than 2 kg of sand, $x_1 \geqslant 4$ and $0 \leqslant x_2 \leqslant 2$

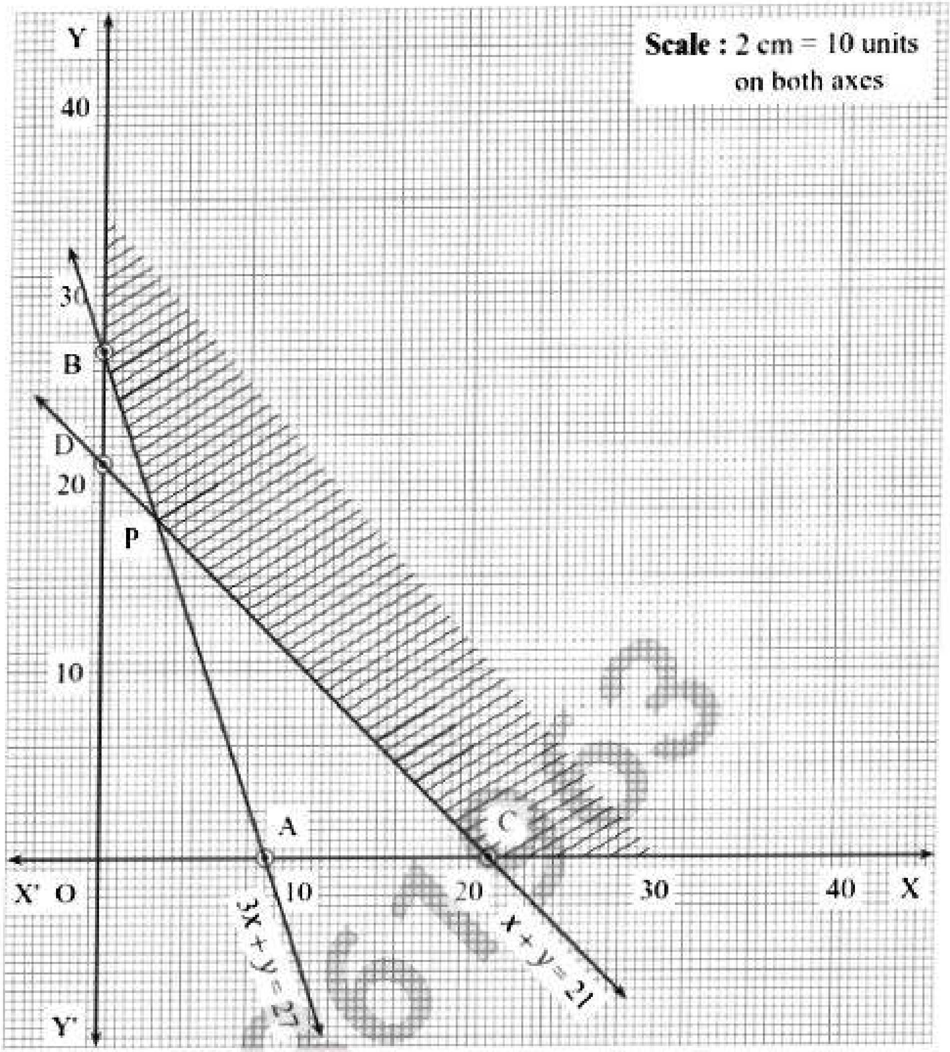
Hence, the given LPP can be formulated as :

Minimize $c=20x_1+6x_2$, subject to $x_1+x_2 \geqslant 5$, $x_1 \geqslant 4$, $0 \leqslant x_2 \leqslant 2$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

Ex. 2. Minimize $z=4x+2y$, subject to $3x+y \geqslant 27$, $x+y \geqslant 21$, $x \geqslant 0$, $y \geqslant 0$. ***(July '22)***

Solution : First we draw the lines AB and CD whose equations are $3x+y=27$ and $x+y=21$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x+y=27$	A(9, 0)	B(0, 27)	$\geqslant$	non-origin side of the line AB
CD	$x+y=21$	C(21, 0)	D(0, 21)	$\geqslant$	non-origin side of the line CD



XCPBY is unbounded feasible region which is shaded in the graph.

The vertices of the feasible region are C (21, 0), P and B (0, 27).

P is the point of intersection of lines $3x+y=27$ ... (1)

and $x+y=21$ ... (2)

Subtracting (2) from (1), we get, $2x=6$ $\therefore x=3$

By putting $x=3$ in (2), we get, $3+y=21$ $\therefore y=18$ $\therefore$ P is (3, 18)

The value of the objective function $z=4x+2y$ at these vertices are

$z(\text{C})=4(21)+2(0)=84$

$z(\text{P})=4(3)+2(18)=48$

$z(\text{B})=4(0)+2(27)=54$

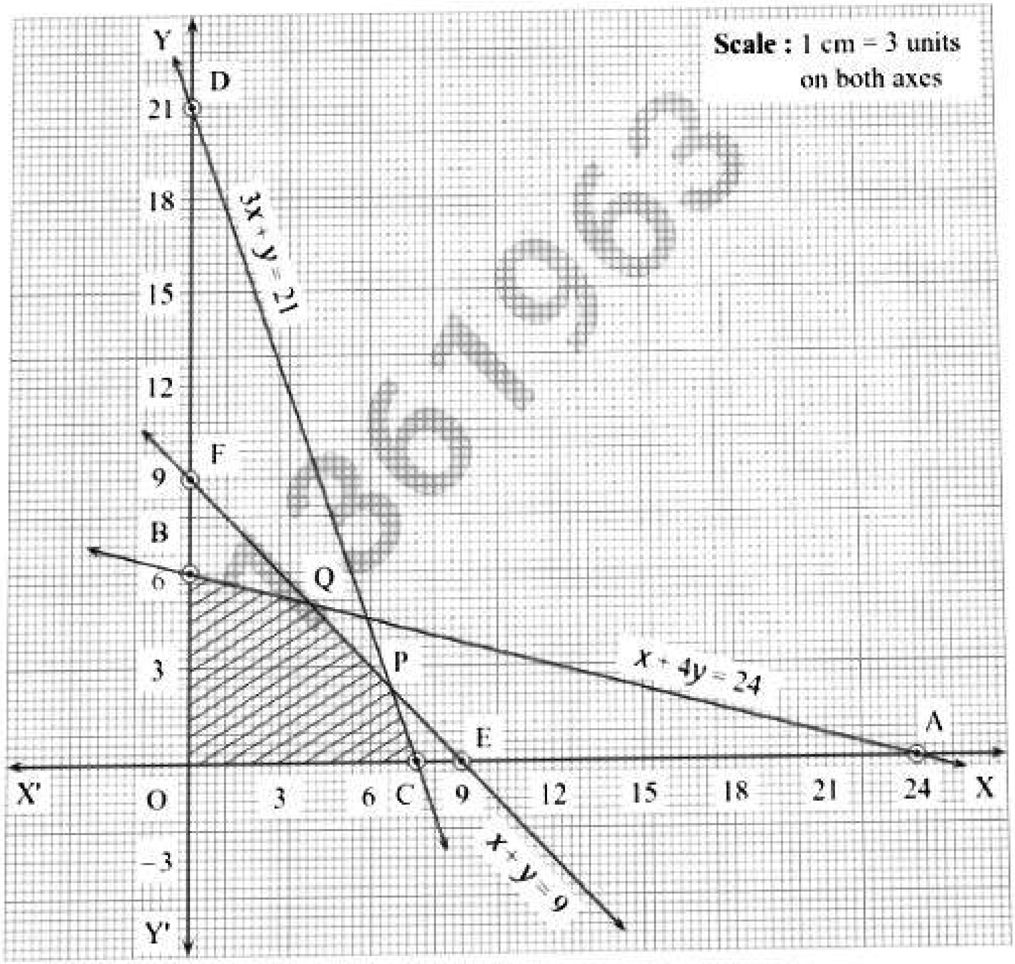
$\therefore$ at P (3, 18) the value of z is minimum.

Hence, z has minimum value 48, when $x=3$ and $y=18$.

Ex. 3. Maximize $z=3x+5y$, subject to $x+4y\leqslant 24$, $3x+y\leqslant 21$, $x+y\leqslant 9$, $x\geqslant 0, y\geqslant 0$.

Solution : First we draw the lines AB, CD and EF whose equations are $x+4y=24$, $3x+y=21$ and $x+y=9$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x+4y=24$	A(24, 0)	B(0, 6)	$\leqslant$	origin side of the line AB
CD	$3x+y=21$	C(7, 0)	D(0, 21)	$\leqslant$	origin side of the line CD
EF	$x+y=9$	E(9, 0)	F(0, 9)	$\leqslant$	origin side of the line EF



The feasible region is OCPQBO which is shaded in the figure.

The vertices of the feasible region are O(0, 0), C(7, 0), P, Q and B(0, 6).

P is the point of intersection of the lines

$3x+y=21$ ... (1)

and $x+y=9$ ... (2)

On subtracting, we get

$2x = 12 \quad \therefore \ x = 6$

Substituting $x = 6$ in equation (2), we get

$6 + y = 9 \quad \therefore \ y = 3 \quad \therefore \ P \equiv (6, 3)$

Q is the point of intersection of the lines

$x + 4y = 24$ ... (3)

and $x + y = 9$ ... (2)

On subtracting, we get

$3y = 15 \quad \therefore \ y = 5$

Substituting $y = 5$ in equation (2), we get

$x + 5 = 9 \quad \therefore \ x = 4 \quad \therefore \ Q \equiv (4, 5)$

$\therefore$ the corner points of the feasible region are

O(0, 0), C(7, 0), P(6, 3), Q(4, 5) and B(0, 6).

The values of the objective function $z = 3x + 5y$ at these corner points are

$z(O) = 3(0) + 5(0) = 0 + 0 = 0$

$z(C) = 3(7) + 5(0) = 21 + 0 = 21$

$z(P) = 3(6) + 5(3) = 18 + 15 = 33$

$z(Q) = 3(4) + 5(5) = 12 + 25 = 37$

$z(B) = 3(0) + 5(6) = 0 + 30 = 30$

$\therefore$ z has maximum value 37, when $x = 4$ and $y = 5$.

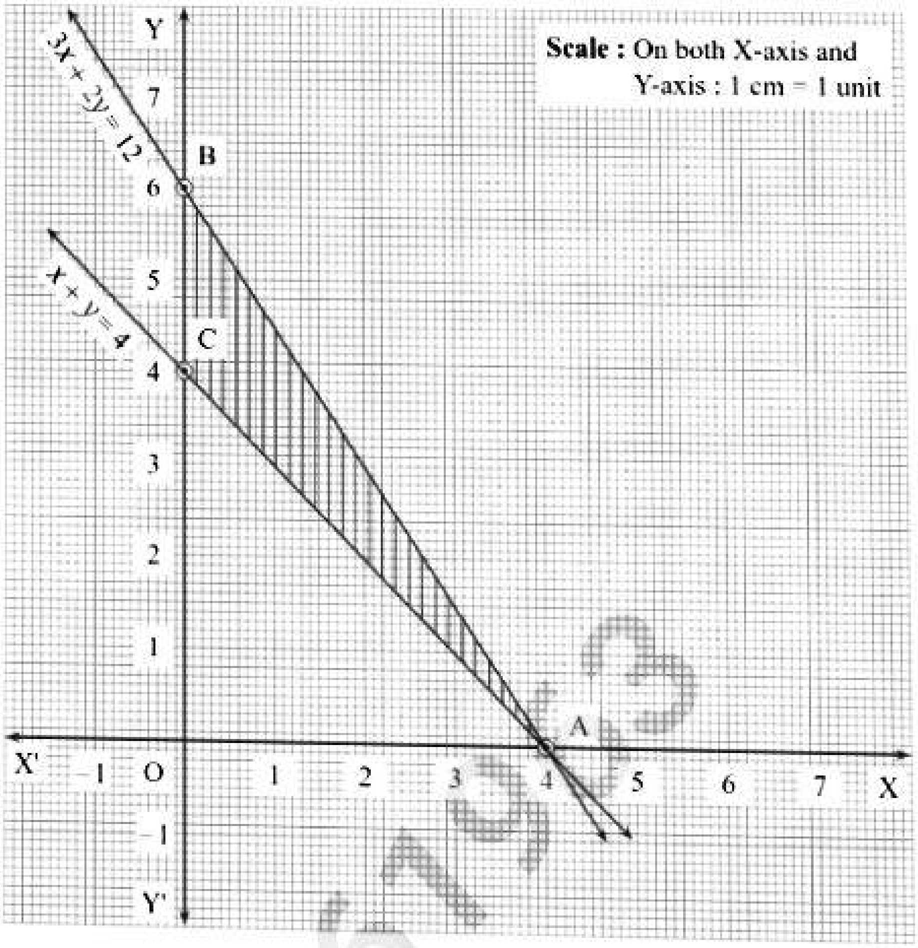
Ex. 4. Solve the following LPP by graphical method :

Maximize $z = 4x + 6y$, subject to $3x + 2y \leqslant 12$, $x + y \geqslant 4$, $x, y \geqslant 0$.

(March '23–'25)

Solution : First we draw the lines AB and AC whose equations are $3x + 2y = 12$ and $x + y = 4$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x+2y=12$	A(4, 0)	B(0, 6)	$\leqslant$	origin side of the line AB
AC	$x+y=4$	A(4, 0)	C(0, 4)	$\geqslant$	non-origin side of the line AC



The feasible region is the $\triangle ABC$, which is shaded in the graph.

The vertices of the feasible region (i.e. corner points) are A(4, 0), B(0, 6) and C(0, 4).

The values of the objective function $z = 4x + 6y$ at these vertices are

$z(A) = 4(4) + 6(0) = 16 + 0 = 16$

$z(B) = 4(0) + 6(6) = 0 + 36 = 36$

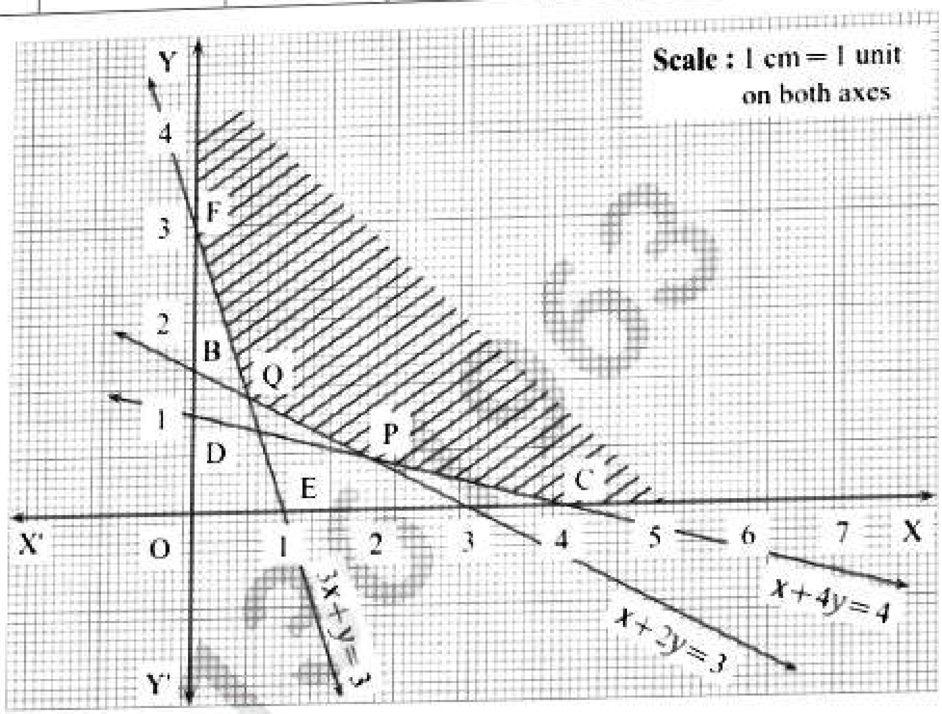
$z(C) = 4(0) + 6(4) = 0 + 24 = 24$

$\therefore$ z has maximum value 36, when $x = 0$, $y = 6$.

Ex. 5. Minimize $z = 6x + 2y$, subject to $x + 2y \geq 3$, $x + 4y \geq 4$, $3x + y \geq 3$, $x \geq 0, y \geq 0$. ***(March '24)***

Solution : First we draw the lines AB, CD and EF whose equations are $x + 2y = 3$, $x + 4y = 4$ and $3x + y = 3$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x+2y=3$	A (3, 0)	$B\left(0, \frac{3}{2}\right)$	$\geqslant$	non-origin side of the line AB
CD	$x+4y=4$	C (4, 0)	D (0, 1)	$\geqslant$	non-origin side of the line CD
EF	$3x+y=3$	E (1, 0)	F (0, 3)	$\geqslant$	non-origin side of the line EF



The feasible region is XCPQFY which is shaded in the figure.

The vertices of the feasible region are C (4, 0), P, Q and F (0, 3).

P is the point of intersection of the lines

$$x+4y=4$$

and $x+2y=3$

On subtracting, we get

$$2y=1 \qquad \therefore y=\frac{1}{2}$$

Substituting $y=\frac{1}{2}$ in $x+2y=3$, we get

$$x+2\left(\frac{1}{2}\right)=3 \qquad \therefore x=2$$

$\therefore P \equiv \left(2, \frac{1}{2}\right)$

Q is the point of intersection of the lines

$x + 2y = 3$ ... (1)

and $3x + y = 3$ ... (2)

Multiplying equation (1) by 3, we get

$3x + 6y = 9$

Subtracting equation (2) from this equation, we get

$5y = 6 \qquad \therefore y = \frac{6}{5}$

$\therefore$ from (1), $x + 2\left(\frac{6}{5}\right) = 3 \qquad \therefore x = 3 - \frac{12}{5} = \frac{3}{5}$

$\therefore Q \equiv \left(\frac{3}{5}, \frac{6}{5}\right)$

The values of the objective function $z = 6x + 2y$ at these vertices are

$z(C) = 6(4) + 2(0) = 24$

$z(P) = 6(2) + 2\left(\frac{1}{2}\right) = 12 + 1 = 13$

$z(Q) = 6\left(\frac{3}{5}\right) + 2\left(\frac{6}{5}\right) = \frac{18}{5} + \frac{12}{5} = 6$

$z(F) = 6(0) + 2(3) = 6$

Since z has minimum value 6 at two consecutive vertices Q and F of the feasible region, z has minimum value 6 at every point of segment QF.

Hence, there are infinite number of optimal solutions.

Examples for Practice	**3 or 4 marks each**

1. Diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1500 calories. Two foods F_1 and F_2 cost ₹ 50 and ₹ 75 per unit respectively. Each unit of food F_1 contains 200 units of vitamins, 1 unit of minerals and 40 calories, whereas each unit of food F_2 contain 100 units of vitamins, 2 units of minerals and 30 calories. Formulate the above problem as LPP to satisfy sick person's requirements at minimum cost.
2. Maximize $z = 7x + 11y$, subject to $3x + 4y \leqslant 24$, $5x + 3y \leqslant 30$, $x \geqslant 0, y \geqslant 0$.

(July '24)

3. Minimize $z = 8x + 10y$, subject to $2x + y \geqslant 7$, $2x + 3y \geqslant 15$, $y \geqslant 2$, $x \geqslant 0, y \geqslant 0$.

4. Maximize $z = 10x + 25y$, subject to $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$, $x + y \leqslant 5$.

5. Minimize $z = 2x + 3y$, subject to $x - y \leqslant 1$, $x + y \geqslant 3$, $x \geqslant 0$, $y \geqslant 0$.

6. A firm manufacturing two types of electrical items A and B, can make a profit of ₹ 20 per unit of A and ₹ 30 per unit of B. Both A and B make use of two essential components a motor and a transformer. Each unit of A requires 3 motors and 2 transformers and each units of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 210 motors and 300 transformers. How many units of A and B should be manufactured per month to maximize profit ? How much is the maximum profit ?

7. A company manufactures two types of chemicals A and B. Each chemical requires two types of raw materials P and Q. The table below shows number of units of P and Q required to manufacture one unit of A and one unit of B.

Chemicals → / **Raw materials ↓**	A	B	Availability
P	3	2	120
Q	2	5	160

The company gets profits of ₹ 350 and ₹ 400 by selling one unit of A and one unit of B respectively. Formulate the problem as LPP to maximize the profit. ***(Sept. '21)***

8. Solve the following LPP : Minimize $z = 7x + y$, subject to $5x + y \geqslant 5$, $x + y \geqslant 3$, $x \geqslant 0$, $y \geqslant 0$. ***(Sept. '21, July '23)***

9. In a cattle breeding firm, it is prescribed that the food ration for one animal must contain 14, 22 and 1 unit of nutrients A, B and C respectively. Two different kinds of fodder are available. Each unit weight of these two contains the following amounts of these three nutrients :

Fodder→ / **Nutrients ↓**	**Fodder 1**	**Fodder 2**
Nutrients A	2	1
Nutrients B	2	3
Nutrients C	1	1

The cost of fodder 1 is ₹ 3 per unit and that of fodder 2 is ₹ 2 per unit. Formulate the LPP to minimize the cost. *(March '22)*

10. Maximize $z = 13x + 9y$, subject to $3x + 2y \leqslant 12$, $x + y \geqslant 4$, $x \geqslant 0$, $y \geqslant 0$.

(March '22)

Answers

1. Minimize $z = 50x + 75y$, subject to $200x + 100y \geqslant 4000$, $x + 2y \geqslant 50$, $40x + 30y \geqslant 1500$, $x \geqslant 0$, $y \geqslant 0$.
2. $z_{max} = 66$, when $x = 0$, $y = 6$.
3. $z_{min} = 52$, when $x = 1.5$, $y = 4$.
4. $z_{max} = 95$, when $x = 2$ and $y = 3$.
5. $z_{min} = 7$, when $x = 2$ and $y = 1$.
6. The firm should manufactured 30 units of item A and 60 units of item B to get the maximum profit of ₹ 2400.
7. Maximize $z = 350x + 400y$, subject to $3x + 2y \leqslant 120$, $2x + 5y \leqslant 160$, $x \geqslant 0$, $y \geqslant 0$.
8. $z_{min} = 5$, when $x = 0$, $y = 5$.
9. Minimize $z = 3x + 2y$, subject to $2x + y \geqslant 14$, $2x + 3y \geqslant 22$, $x + y \geqslant 1$, $x \geqslant 0$, $y \geqslant 0$.
10. $z_{max} = 54$, when $x = 0$, $y = 6$.

MULTIPLE CHOICE QUESTIONS | 1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. Of all the points of the feasible region, the optimal value of z is obtained at a point

(a) inside the feasible region (b) at the boundary of the feasible region

(c) at vertex of feasible region (d) on X-axis

2. The corner points of the feasible region given by the inequations $x + y \leqslant 4$, $2x + y \leqslant 7$, $x \geqslant 0$, $y \geqslant 0$ are

(a) (0, 0), (4, 0), (3, 1), (0, 4) (b) $(0, 0), \left(\frac{7}{2}, 0\right), (3, 1), (0, 4)$

(c) $(0, 0), \left(\frac{7}{2}, 0\right), (3, 1), (5, 7)$ (d) (6, 0), (4, 0), (3, 1), (0, 7)

3. The point at which the maximum value of $z = x + y$ subject to the constraints $x + 2y \leqslant 70$, $2x + y \leqslant 95$, $x \geqslant 0$, $y \geqslant 0$ is
(a) (36, 25) (b) (20, 35) (c) (35, 20) (d) (40, 15)

4. Objective function of LPP is
(a) a constraint
(b) a function to be maximized or minimized
(c) a relation between the decision variables
(d) a feasible region *(March '24)*

5. If the corner points of the feasible region are (0, 10), (2, 2) and (4, 0), then the point of minimum $z = 3x + 2y$ is
(a) (2, 2) (b) (0, 10) (c) (4, 0) (d) (2, 4) *(July '23)*

6. If the corner points of the feasible region are (0, 0), (3, 0) (2, 1) and $\left(0, \frac{7}{3}\right)$, the maximum value $z = 4x + 5y$ is
(a) 12 (b) 13 (c) $\frac{35}{2}$ (d) 0 *(March '23)*

7. Solution of LPP to minimize $z = 2x + 3y$ subject to $x \geqslant 0$, $y \geqslant 0$, $1 \leqslant x + 2y \leqslant 10$ is
(a) $x = 0, y = \frac{1}{2}$ (b) $x = \frac{1}{2}, y = 0$ (c) $x = 1, y = -2$ (d) $x = y = \frac{1}{2}$

Answers

1. (c) at vertex of feasible region 2. (b) (0, 0), $\left(\frac{7}{2}, 0\right)$, (3, 1), (0, 4)

3. (d) (40, 15) 4. (b) a function to be maximized or minimized

5. (a) (2, 2) 6. (b) 13 7. (a) $x = 0, y = \frac{1}{2}$.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The optimum value of the objective function of LPP occurs at the centre of the feasible region. *(July '22; March '24)*

2. If LPP has two optimal solutions, then the LPP has infinitely many solutions.

3. Saina wants to invest at most ₹ 24,000 in bonds and fixed deposits. Mathematically this constraint is written as $x+y \leqslant 24000$, where x is investment in bond and y is in fixed deposits.
4. The point (1, 2) is not a vertex of the feasible region bounded by $2x+3y \leqslant 6$, $5x+3y \leqslant 15$, $x \geqslant 0$, $y \geqslant 0$.
5. The region represented by the inequalities $x \leqslant 0$, $y \leqslant 0$ lies in the first quadrant. ***(July '24)***
6. The common region represented by the inequalities $x+y \leqslant 400$, $4x \leqslant y$, $x \geqslant 40$ is a quadrilateral.

Answers

1. False 2. False 3. True 4. True 5. False 6. False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. The optimal value of the objective function is attained at the .... points of feasible region. ***(July '24)***
2. The conditions under which the objective function is to be optimized are called ....
3. The constraint that a factory has to employ more women (y) than men (x) is given by .... ***(July '22)***
4. A train carries at least twice as many first class passengers (y) as second class passengers (x). The constraint is given by .... ***(July '23)***
5. Graphical solution set of the inequations $x \geqslant 0$ and $y \leqslant 0$ lies in .... quadrant. ***(March '23)***
6. A set of values of variables satisfying all the constraints of LPP is known as ....

Answers

1. corner 2. constraints 3. $y > x$ 4. $x \geqslant 2y$ 5. IV
6. solution of LPP.

Question Set 15

ASSIGNMENT PROBLEM AND SEQUENCING

(Marks with option : 09)

15.1 ASSIGNMENT PROBLEM

Remember :

1. The assignment problem is a special case of transportation problem in which a number of origins (resources) are assigned to the equal number of destinations (activities) on one-to-one basis, so that the total cost is minimized (or profit is maximized).

2. **Special cases of Assignment Problem :** The assignment problem is generally defined as a problem of minimization. But some special assignment problems are as follows :

   (i) Unbalanced Assignment Problem :

   Cost matrix is not a square matrix. Adding 'dummy tasks/facilities with zero costs.'

   (ii) Maximization Assignment Problem :

   Convert into minimization problem (a) by subtracting all the elements from the largest element of cost matrix or (b) multiplying all the elements of cost matrix by -1.

   (iii) Restricted (Prohibited) Assignment Problem :

   Certain combinations of tasks/facilities are restricted. Assign very high cost (say infinity) at the restricted (prohibited) combinations.

   (iv) Alternate/Multiple Optimal Solution :

   Assignment matrix contains more than required number of zero elements.

3. A Hungarian Mathematician D. Konia developed the most effective method for solving an assignment problem.

Solved Examples	**3 or 4 marks each**

Ex. 1. A marketing manager has list of salesmen and territories. Considering the travelling cost of the salesmen and the nature of the territory, the marketing manager estimates the total cost per month (in thousand rupees) for each salesman in each territory. Suppose these amounts are as follows :

Salesmen	Territories				
	I	II	III	IV	V
A	11	16	18	15	15
B	7	19	11	13	17
C	9	6	14	14	7
D	13	12	17	11	13

Find the assignment of salesmen to territories that will result in minimum cost. ***(March '23)***

Solution :

Here, the number of salesmen are less than the number of territories, the problem is unbalanced. It can be balanced by introducing a dummy salesman E with zero cost as shown below :

Salesmen	Territories				
	I	II	III	IV	V
A	11	16	18	15	15
B	7	19	11	13	17
C	9	6	14	14	7
D	13	12	17	11	13
E	0	0	0	0	0

Step 1 : Subtract the smallest element in each row from every element of it. Therefore, new assignment matrix is obtained as follows :

Salesmen	Territories				
	I	II	III	IV	V
A	0	5	7	4	4
B	0	12	4	6	10
C	3	0	8	8	1
D	2	1	6	0	2
E	0	0	0	0	0

Step 2 : Since each column contains one zero, there is no need to subtract minimum element in each column. The new assignment matrix is obtained as above. Draw minimum number of vertical and horizontal lines as follows :

Salesmen	Territories				
	I	II	III	IV	V
A	0	5	7	4	4
B	0	12	4	6	10
C	3	0	8	8	1
D	2	1	6	0	2
E	0	0	0	0	0

Here, minimum number of lines covering all zeros = 4, which is not equal to order of matrix (i.e. 5).

$\therefore$ minimum number of lines < order of matrix.

$\therefore$ optimal solution has not reached.

Step 3 : Subtract the smallest uncovered element (i.e. 4) from all uncovered element and add it to element at the intersection of horizontal and vertical lines. All other elements on the line remain unchanged. Hence, we get, assignment matrix as follows :

Salesmen	Territories				
	I	II	III	IV	V
A	0	1	3	0	0
B	0	8	0	2	6
C	7	0	8	8	1
D	6	1	6	0	2
E	4	0	0	0	0

Step 4 : From step 3, as the minimum number of straight lines required to cover all zeros in the assignment matrix equal to the number of rows/columns. Hence, optimal solution has reached and optimal assignment scheduled as follows :

Salesmen	Territories				
	I	II	III	IV	V
A	$\boxed{0}$	1	3	$\cancel{0}$	$\cancel{0}$
B	$\cancel{0}$	8	$\boxed{0}$	2	6
C	7	$\boxed{0}$	8	8	1
D	6	1	6	$\boxed{0}$	2
E	4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

OR

Alternate optimum schedule :

Salesmen	Territories				
	I	II	III	IV	V
A	$\cancel{0}$	1	3	$\cancel{0}$	$\boxed{0}$
B	$\boxed{0}$	8	$\cancel{0}$	2	6
C	7	$\boxed{0}$	8	8	1
D	6	1	6	$\boxed{0}$	2
E	4	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Hence, optimum solution is :

Salesmen	Territories	Cost
A	I	11
B	III	11
C	II	6
D	IV	11
	Total	39

OR

Salesmen	Territories	Cost
A	V	15
B	I	7
C	II	6
D	IV	11
	Total	39

Hence, optimum cost = ₹ 39,000.

Ex. 2. A chartered accountants' firm has accepted five new cases. The estimated number of days required by each of their five employees for each case are given below, where – means that the particular employee cannot be assigned the particular case. Determine the optimal assignment of cases of the employees so that the total number of days required to complete these five cases will be minimum. Also, find the minimum number of days.

Employees	Cases				
	I	II	III	IV	V
E_1	6	4	5	7	8
E_2	7	–	8	6	9
E_3	8	6	7	9	10
E_4	5	7	–	4	6
E_5	9	5	3	10	–

Solution :

This is prohibited assignment problem. Therefore, we assign very high days say infinity ∞ to employees E_2 for case II, E_4 for case III and E_5 for case V.

Employees	Cases				
	I	II	III	IV	V
E_1	6	4	5	7	8
E_2	7	∞	8	6	9
E_3	8	6	7	9	10
E_4	5	7	∞	4	6
E_5	9	5	3	10	∞

Subtracting the lowest element of each row from the element of that row :

Employees	Cases				
	I	II	III	IV	V
E_1	2	0	1	3	4
E_2	1	∞	2	0	3
E_3	2	0	1	3	4
E_4	1	3	∞	0	2
E_5	6	2	0	7	∞

Subtracting the lowest element of each column from the elements of that column :

Employees	Cases				
	I	II	III	IV	V
E_1	1	0	1	3	2
E_2	0	∞	2	0	1
E_3	1	0	1	3	2
E_4	0	3	∞	0	0
E_5	5	2	0	7	∞

Minimum number of lines covering all zeros are 4, which is less than order of matrix (i.e. 5).

Therefore, select smallest element, i.e. 1 from all the uncovered elements and subtract it from all uncovered elements and add it to the intersection of lines.

We get

Employees	Cases				
	I	II	III	IV	V
E_1	0	0	0	2	1
E_2	0	∞	2	0	1
E_3	0	0	0	2	1
E_4	0	4	∞	0	0
E_5	5	3	0	7	∞

Draw minimum number of vertical and horizontal lines to cover all zeros. We get

Employees	Cases				
	I	II	III	IV	V
E_1	0	0	0	2	1
E_2	0	∞	2	0	1
E_3	0	0	0	2	1
E_4	0	4	∞	0	0
E_5	5	3	0	7	∞

We observe that minimum number of lines drawn are 5 which is equal to order of matrix (i.e. 5).

Hence, optimal solution has reached.

We allocate the cases to employees as follows :

Employees	Cases				
	I	II	III	IV	V
E_1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	2	1
E_2	$\cancel{0}$	∞	2	$\boxed{0}$	1
E_3	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{2}$	1
E_4	$\cancel{0}$	4	∞	0	$\boxed{0}$
E_5	5	2	$\boxed{0}$	7	∞

Hence, optimum assignment schedule is as follows :

Employees	Cases	Number of days
E_1	I	6
E_2	IV	6
E_3	II	6
E_4	V	6
E_5	III	3
	Total	27

$\therefore$ total number of minimum days = 27.

Ex. 3. The estimated sales (tonnes) per month in four different cities by five different managers are given below :

Managers	Sales of Cities (in tonnes)			
	P	Q	R	S
I	34	36	33	35
II	33	35	31	33
III	37	39	35	35
IV	36	36	34	34
V	35	36	35	33

Find out the assignment of managers to cities in order to maximize sales.

Solution :

Since this is an unbalanced assignment, add dummy city T with zero sales :

Managers	Sales of Cities (in tonnes)				
	P	Q	R	S	T
I	34	36	33	35	0
II	33	35	31	33	0
III	37	39	35	35	0
IV	36	36	34	34	0
V	35	36	35	33	0

To convert the maximization problem into minimization problem, subtract all the elements of given matrix from the maximum element 39 of the matrix.

Managers	Sales of Cities (in tonnes)				
	P	Q	R	S	T
I	5	3	6	4	39
II	6	4	8	6	39
III	2	0	4	4	39
IV	3	3	5	5	39
V	4	3	4	6	39

Subtract the lowest element of each row from the elements of that row :

Managers	Sales of Cities (in tonnes)				
	P	Q	R	S	T
I	2	0	3	1	36
II	2	0	4	2	35
III	2	0	4	4	39
IV	0	0	2	2	36
V	1	0	1	3	36

Subtract the lowest element of each column from the elements of that column :

Managers	Sales of Cities (in tonnes)				
	P	Q	R	S	T
I	2	0	2	0	1
II	2	0	3	1	0
III	2	0	3	3	4
IV	0	0	1	1	1
V	1	0	0	2	1

The number of lines covering all zeros is the same as the order of matrix.

Therefore, assignment is made as follows :

Managers	Sales of Cities (in tonnes)				
	P	Q	R	S	T
I	2	$\cancel{0}$	2	$\boxed{0}$	1
II	2	$\cancel{0}$	3	1	$\boxed{0}$
III	2	$\boxed{0}$	3	3	4
IV	$\boxed{0}$	$\cancel{0}$	1	1	1
V	1	$\cancel{0}$	$\boxed{0}$	2	1

The optimal assignment is shown as follows :

Managers	Cities	Sales (in tonnes)
I	S	35
II	T	0
III	Q	39
IV	P	36
V	R	35

Total sales = 145 tonnes.

Ex. 4. Three new machines M_1, M_2, M_3 are to be installed in a machine shop. There are four vacant places A, B, C, D. Due to limited space, machine M_2 cannot be placed at B. The cost matrix (in hundred ₹) is a follows :

Machines	Places			
	A	B	C	D
M_1	13	10	12	11
M_2	15	–	13	20
M_3	5	7	10	6

Determine the optimum assignment schedule and find the minimum cost. ***(March '24)***

Solution :

As the number of machines is less than the number of vacant places, the problem is unbalanced. It can be balanced by introducing dummy machine M_4 with zero cost.

Also, machine M_2 cannot be placed at B, a very high cost say ∞ is assigned to the corresponding cell. We get

Machines	Places			
	A	B	C	D
M_1	13	10	12	11
M_2	15	∞	13	20
M_3	5	7	10	6
M_4	0	0	0	0

Step 1 : Subtract the smallest element of each row from every element in that row :

Machines	Places			
	A	B	C	D
M_1	3	0	2	1
M_2	2	∞	0	7
M_3	0	2	5	1
M_4	0	0	0	0

Step 2 : Since the smallest element in each column is zero, there is no need to subtract anything from columns. Therefore, the resultant matrix is as given in the above table.

Step 3 : Since the number of straight lines covering all zeros is equal to the number of row/columns, the optimal solution is reached. The optimal assignment can be made as follows :

Machines	Places			
	A	B	C	D
M_1	3	$\boxed{0}$	2	1
M_2	2	∞	$\boxed{0}$	7
M_3	$\boxed{0}$	2	5	1
M_4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

The following is optimal solution obtained :

Machines	Places	Cost (in hundred ₹)
M_1	B	10
M_2	C	13
M_3	A	5
	Total	28

Total minimum cost = ₹ 2800.

Examples for Practice	**3 or 4 marks each**

1. A job production unit has four jobs P, Q, R, S which can be manufactured on each of the four machines I, II, III and IV. The processing cost of each job for each machine is given in the following table :

Jobs	Processing cost (in ₹)			
	Machines			
	I	II	III	IV
P	31	25	33	29
Q	25	24	23	21
R	19	21	23	24
S	38	36	34	40

Find the optimal assignment to minimize the total processing cost.

(March '22)

2. Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at 5 stations I, II, III, IV and V. The mileage between various stations are given in the table below. How should the wagons be transported so as to minimize the mileage covered?

	I	II	III	IV	V
1	10	5	9	18	11
2	13	9	6	12	14
3	7	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

3. Solve the following assignment problem to maximize sales :

Salesmen	Territories				
	I	II	III	IV	V
A	11	16	18	15	15
B	7	19	11	13	17
C	9	6	14	14	7
D	13	12	17	11	13

4. Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost matrix is given below :

Machines	Places				
	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	–	5	4
M_3	–	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule. ***(July '22–'24)***

5. A marketing manager has list of salesmen and towns. Considering the capabilities of the salesmen and the nature of town, the marketing manager estimates amounts of sales per month (in thousand ₹) for each salesman in each town. Suppose these amounts are as follows :

Salesmen	Towns				
	T_1	T_2	T_3	T_4	T_5
S_1	37	43	45	33	45
S_2	45	29	33	26	41
S_3	46	32	38	35	42
S_4	27	43	46	41	41
S_5	34	38	45	40	44

Find the assignment of salesmen to towns that will result in maximum sale.

Answers

1. $P \to II$, $Q \to IV$, $R \to I$, $S \to III$. Minimal Cost = ₹ 99.
2. $1 \to I$, $2 \to III$, $3 \to IV$, $4 \to II$, $5 \to V$. Minimum mileage covered = 39 miles.
3. $A \to V$, $B \to II$, $C \to IV$, $D \to III$, $E \to I$. Maximum sales = 65 units.
4. $M_1 \to A$, $M_2 \to B$, $M_3 \to E$, $M_4 \to D$, $M_5 \to C$. Total cost = 12.
5. $S_1 \to T_2$, $S_2 \to T_5$, $S_3 \to T_1$, $S_4 \to T_4$, $S_5 \to T_3$. Maximum sales = ₹ 2,16,000.

15.2 SEQUENCING

Remember :

Sequencing problem is used to determine the sequence (order) for a series of jobs to be done on a finite number of service facilities, in some pre-assigned order, so as to optimize the total time involved.

(1) Total Elapsed Time : It is the time required to complete all the jobs, i.e. the entire task.
Thus, total elapsed time is the time between the beginning of the first job on first machine till the completion of the last job on the last machine.

(2) Idle Time : Idle time is the time when a machine is available but not being used. Thus, it is the time that the machine is available but is waiting for a job to be processed.

Solved Examples	3 or 4 marks each

Ex. 5. Find the sequence that minimizes the total elapsed time to complete the following jobs in the order AB. Find the total elapsed time and idle time for machine B :

Jobs	I	II	III	IV	V	VI	VII
Machine A	7	16	19	10	14	15	5
Machine B	12	14	14	10	16	5	7

(March '25)

Solution :

Here, Min. $(M_{i1}, M_{i2}) = 5$, which corresponds to both machines A and B.

Therefore, job VII is processed first and job VI is processed last.

VII						VI

The problem now reduces to jobs I, II, III, IV and V.

Here, Min. $(M_{i1}, M_{i2})=7$, which corresponds to machine A.

Therefore, job I is processed next to job VII.

VII	I					VI

The problem now reduces to jobs II, III, IV and V.

Here, Min. $(M_{i1}, M_{i2}) = 10$, which corresponds to both machines A and B.

Therefore, job IV is processed next to job I.

VII	I	IV				VI

The problem now reduces to jobs II, III and V.

Here, Min. $(M_{i1}, M_{i2}) = 14$, which corresponds to both machines A and B.

Therefore, job V is processed next to job IV.

VII	I	IV	V			VI

The problem now reduces to jobs II and III.

Here, Min. $(M_{i1}, M_{i2}) = 14$, which corresponds to machine B.

Therefore, job II is processed in the last next to job VI and job III is processed last next to job II.

VII	I	IV	V	III	II	VI

Total elapsed time is obtained as follows :

Jobs Sequence	**Machine A**		**Machine B**		**Idle time for Machine B**
	Time in	**Time out**	**Time in**	**Time out**	
VII	0	5	5	12	5
I	5	12	12	24	0
IV	12	22	24	34	0
V	22	36	36	52	2
III	36	55	55	69	3
II	55	71	71	85	2
VI	71	86	86	91	1
Total idle time for Machine B					13

Optimal sequence of jobs is

VII → I → IV → V → III → II → VI

Total elapsed time $T = 91$ units

Idle time for Machine B = 13 units.

Ex. 6. A toy manufacturing company produces five types of toys. Each toy has to go through three machines A, B, C in the order ABC. The time required in hours for each process is given in the following table :

Type	**1**	**2**	**3**	**4**	**5**
Machine A	16	20	12	14	22
Machine B	10	12	4	6	8
Machine C	8	18	16	12	10

Find the total elapsed time and also find idle time for machine B.

(March '24)

Solution :

Here Min. (A) = 12, Min. (C) = 8 and Max. (B) = 12.

Since Min. (A) $\geqslant$ Max. (B) is satisfied, the problem can be converted into 5 types – 2 machines problem and two fictitious machines are,

$G = A + B$ and $H = B + C$

The problem now can be written as follows :

Types of toys	**Processing time (in hours)**	
	G = A + B	**H = B + C**
1	26	18
2	32	30
3	16	20
4	20	18
5	30	18

Here, Min. (G, H) = 16, which corresponds to G.

Therefore, type 3 toy is processed at first.

3				

The problem now reduces to type 1, 2, 4, 5 toys.

Here, Min. (G, H) = 18, which corresponds to H.

Therefore, type 1 toy is processed in the last, type 4 toy is processed at the last next to type 1 toy and type 5 toy is processed at last next to type 4 toy.

3		5	4	1

Now, type 2 toy is processed at last next to type 5 toy and the optimal sequence is obtained as follows :

3	2	5	4	1

Total elapsed time is obtained as follows :

Sequence of type of toy	Machine A		Machine B		Machine C		Idle time for Machine C
	Time in	Time out	Time in	Time out	Time in	Time out	
3	0	12	12	16	16	32	16
2	12	32	32	44	44	62	12
5	32	54	54	62	62	72	0
4	54	68	68	74	74	86	2
1	68	84	84	94	94	102	8
Total idle time for Machine C							38

Total elapsed time T = 102 hours

Idle time for machine B

= T – Sum of processing time for all jobs on machine B

= 102 – 40 = 62 hours.

Examples for Practice	3 or 4 marks each

1. There are five jobs, each of which must go through two machines in the order XY. Processing time (in hours) are given below :

Determine the optimal sequence for the jobs that will minimize the total elapsed time. Also, find the total elapsed time and idle time for machine Y.

Job	A	B	C	D	E
Machine X	10	2	18	6	20
Machine Y	4	12	14	16	8

(July '23)

2. Determine the optimal sequence of job that minimizes the total elapsed time for the data given below (processing time on machines is given in hours). Also, find total elapsed time T and the idle time for three machines.

Job	I	II	III	IV	V	VI	VII
Machine A	3	8	7	4	9	8	7
Machine B	4	3	2	5	1	4	3
Machine C	6	7	5	11	5	6	12

3. A publisher produces 5 books on Mathematics. The books have to go through composing, printing and binding done by 3 machines A, B, C. The time schedule for the entire task in proper unit is as follows :

Books → / Machines ↓	I	II	III	IV	V
Machine A	4	9	8	6	5
Machine B	5	6	2	3	4
Machine C	8	10	6	7	11

Determine the total elapsed time and idle time for machines A, B and C.

(Sept. '21)

4. Five jobs are performed first on machine M_1 and then on machine M_2. Time taken in hours by each job on each machine is given below :

Jobs → / Machines ↓	1	2	3	4	5
M_1	6	8	4	5	7
M_2	3	7	6	4	16

Determine the optimal sequence of jobs and total elapsed time. Also, find the idle time for two machines. ***(March '22)***

5. Five jobs must pass through a lathe and a surface grinder, in that order. The processing times in hours are shown below. Determine the optimal sequence of the jobs. Also, find the total elapsed time.

Jobs	I	II	III	IV	V
Lathe	4	1	5	2	5
Surface grinder	3	2	4	3	6

(July '24)

6. Find the sequence that minimizes the total elapsed time to complete the following jobs. Each job is processed in order AB :

Machines	Jobs (Processing times in minutes)						
	I	II	III	IV	V	VI	VII
Machine A	12	6	5	11	5	7	6
Machine B	7	8	9	4	7	8	3

Determine the sequence for the jobs so as to minimize the processing time. Find the total elapsed time and the idle times for both the machines.

(July '22)

Answers

1. B → D → C → E → A

   Total elapsed time = 60 hours

   Idle time for machine Y = 6 hours

2. I → IV → VII → VI → II → III → V

   Total elapsed time = 59 hours

   Idle time for machine A = 13 hours

   Idle time for machine B = 37 hours

   Idle time for machine C = 7 hours

3. I → IV → V → II → III

   Total elapsed time = 51 hours

   Idle time for machine A = 19 hours

   Idle time for machine B = 31 hours

   Idle time for machine C = 9 hours

4. 3 → 5 → 2 → 4 → 1

   Total elapsed time = 41 hours

   Idle time for machine M_1 = 11 hours

   Idle time for machine M_2 = 5 hours

5. II → IV → V → III → I

   Total elapsed time = 21 hours

6. III → V → II → VI → I → IV → VII

   Total elapsed time = 55 minutes

   Idle time for machine A = 3 minutes

   Idle time for machine B = 9 minutes

ACTIVITIES	4 marks each

1. A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below :

Salesmen	Districts			
	1	**2**	**3**	**4**
A	16	10	12	11
B	12	13	15	15
C	15	15	11	14
D	13	14	14	15

Find the assignment of salesmen to various districts which will yield maximum profit.

It is a maximization problem. Subtract all the elements from $\boxed{}$.

$$\begin{array}{cc} & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 6 & 4 & 5\\4 & 3 & 1 & 1\\1 & 1 & 5 & 2\\3 & 2 & 2 & 1\end{bmatrix}\end{array}$$

Subtract the smallest element of each row from the elements of that row :

$$\begin{array}{cc} & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 6 & 4 & 5\\3 & 2 & 0 & 0\\0 & 0 & 4 & 1\\2 & 1 & 1 & 0\end{bmatrix}\end{array}$$

Subtract the smallest element of each column from the elements of that column :

$$\begin{array}{cc} & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 6 & 4 & 5\\3 & 2 & 0 & 0\\0 & 0 & 4 & 1\\2 & 1 & 1 & 0\end{bmatrix}\end{array}$$

Since the number of lines covering zeros is equal to the order of matrix, the optimal solution has reached :

$$\begin{array}{cc} & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 6 & 4 & 5\\3 & 2 & 0 & 0\\0 & 0 & 4 & 1\\2 & 1 & 1 & 0\end{bmatrix}\end{array}$$

The optimal solution is obtained :

Salesmen	Districts	Profits (₹)
A	1	16
B	$\boxed{}$	$\boxed{}$
C	$\boxed{}$	$\boxed{}$
D	4	15

Total profit = ₹ $\boxed{}$. ***(July '23)***

Solution :

It is maximization problem. Subtract all the elements from $\boxed{16}$.

Salesmen	Districts			
	1	**2**	**3**	**4**
A	0	6	4	5
B	4	3	1	1
C	1	1	5	2
D	3	2	2	1

Step 1 : Subtract the minimum (smallest) element of each row from the elements of that row :

Salesmen	Districts			
	1	**2**	**3**	**4**
A	0	6	4	5
B	3	2	0	0
C	0	0	4	1
D	2	1	1	0

Step 2 : Subtract the smallest element of each column from the elements of that column :

Salesmen	Districts			
	1	**2**	**3**	**4**
A	0	6	4	5
B	3	2	0	0
C	0	0	4	1
D	2	1	1	0

Step 3 : Since the number of lines covering zeroes is 4 equal to the order of matrix 4, the optimal solution has reached.

Salesmen	**Districts**			
	1	**2**	**3**	**4**
A	$\boxed{0}$	6	4	5
B	3	2	$\boxed{0}$	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	4	1
D	2	1	1	$\boxed{0}$

The following optimal solution is obtained :

Salesmen	**Districts**	**Profits (₹)**
A	1	16
B	3	15
C	2	15
D	4	15

Total profit = ₹ $\boxed{61}$.

2. Solve the following assignment problem for minimization :

	I	**II**	**III**	**IV**	**V**
1	18	24	19	20	23
2	19	21	20	18	22
3	22	23	20	21	23
4	20	18	21	19	19
5	18	22	23	22	21

Step I : Subtract the smallest element of each row from every element of that row :

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array}
\begin{array}{c} \begin{array}{ccccc} \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} & \mathbf{V} \end{array} \\ \begin{bmatrix} 0 & 6 & 1 & 2 & 5 \\ 1 & 3 & 2 & 0 & 4 \\ 2 & 3 & 0 & 1 & 3 \\ 2 & 0 & 3 & 1 & 1 \\ 0 & 4 & 5 & 4 & 3 \end{bmatrix} \end{array}$$

Step II : Subtract the smallest element of each column from every element of that column :

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 0 & 6 & 1 & 2 & 4\\ 1 & 3 & 2 & 0 & 3\\ 2 & 3 & 0 & 1 & 2\\ 2 & 0 & 3 & 1 & 0\\ 0 & 4 & 5 & 4 & 2 \end{bmatrix}\end{array}$$

Step III : Draw minimum number of lines covering all zeros :

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 0 & 6 & 1 & 2 & 4\\ 1 & 3 & 2 & 0 & 3\\ 2 & 3 & 0 & 1 & 2\\ 2 & 0 & 3 & 1 & 0\\ 0 & 4 & 5 & 4 & 2 \end{bmatrix}\end{array}$$

Here, minimum number of lines (4) < order of matrix (5).

Step IV : The smallest uncovered element is 1, which is to be subtracted from all uncovered elements and add it to all elements which lie at the intersection of two lines :

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 0 & 5 & 0 & \square & 3\\ 2 & 3 & 2 & 0 & 3\\ 3 & 3 & 0 & \square & 2\\ 3 & 0 & 3 & 1 & 0\\ 0 & 3 & 4 & 3 & \square \end{bmatrix}\end{array}$$

Step V : Draw minimum number of lines that are required to cover all zeros :

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 0 & 5 & 0 & 1 & 3\\ 2 & 3 & 2 & 0 & 3\\ 3 & 3 & 0 & 1 & 2\\ 3 & 0 & 3 & 1 & 0\\ 0 & 3 & 4 & 3 & 1 \end{bmatrix}\end{array}$$

Here, minimum number of lines $\neq$ order of matrix.

Step VI : Find the smallest uncovered element 1. Subtract this number from all uncovered elements and add it to all elements which lie at the intersection of two lines :

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 0 & 4 & 0 & 0 & 2\\ 3 & 3 & 3 & 0 & 3\\ 3 & \square & 0 & 0 & \square\\ 4 & 0 & 4 & 1 & 0\\ 0 & 2 & 4 & 2 & 0 \end{bmatrix}\end{array}$$

Now, minimum number of lines = order of matrix.

The optimal assignment can be made.

Optional solution is

$1 \to \mathrm{I}$, $2 \to \mathrm{IV}$, $3 \to \boxed{}$, $4 \to \boxed{}$, $5 \to \mathrm{V}$

Minimum value = $\boxed{}$ ***(March '25)***

Solution : Step IV :

	I	II	III	IV	V
1	0	5	0	$\boxed{1}$	3
2	2	3	2	0	3
3	3	3	0	$\boxed{1}$	2
4	3	0	3	1	0
5	0	3	4	3	$\boxed{1}$

Step VI :

	I	II	III	IV	V
1	0	4	0	0	2
2	3	3	3	0	3
3	3	$\boxed{2}$	0	0	$\boxed{1}$
4	4	0	4	1	0
5	0	2	4	2	0

Now, minimum number of lines = order of matrix.

The optimal assignment can be made.

The assignment is made as follows :

	I	II	III	IV	V
1	$\boxed{0}$	4	$\cancel{0}$	$\cancel{0}$	2
2	3	3	3	$\boxed{0}$	3
3	3	2	$\boxed{0}$	$\cancel{0}$	1
4	4	$\boxed{0}$	4	1	$\cancel{0}$
5	$\cancel{0}$	2	4	2	$\boxed{0}$

Optimal solution is

$1 \to \mathrm{I}$, $2 \to \mathrm{IV}$, $3 \to \boxed{\mathrm{III}}$, $4 \to \boxed{\mathrm{II}}$, $5 \to \mathrm{V}$

Minimum value $= 18 + 18 + 20 + 18 + 21 = \boxed{95}$.

---

3. A plant manager has four subordinates and four tasks to perform. The subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Estimates of the times subordinates would take to perform tasks are given in the matrix below :

	I	II	III	IV
A	3	11	10	8
B	13	2	12	2
C	3	4	6	1
D	4	15	4	9

Complete the following activity to allocate tasks to subordinates to minimize total time.

Step I : Subtract the smallest element of each row from every element of that row :

	I	II	III	IV
A	0	8	7	5
B	11	0	10	0
C	2	3	5	0
D	0	11	0	5

Step II : Since all columns' minimums are zero, no need to subtract anything from columns.

Step III : Draw the minimum number of lines to cover all the zeros.

	I	II	III	IV
A	0	8	7	5
B	11	0	10	0
C	2	3	5	0
D	0	11	0	5

Since [Minimum number of lines] = [Order of matrix], optimal solution has been reached.

$\therefore$ optimal assignment schedule is A → $\boxed{}$

B → $\boxed{}$

C → IV

D → $\boxed{}$

Total minimum time = $\boxed{}$ hours. ***(Sept. '21)***

Solution : Optimal solution is

	I	II	III	IV
A	$\boxed{0}$	8	7	5
B	11	$\boxed{0}$	10	$\cancel{0}$
C	2	3	5	$\boxed{0}$
D	$\cancel{0}$	11	$\boxed{0}$	5

$\therefore$ optimal assignment is A $\rightarrow$ $\boxed{\text{I}}$, B $\rightarrow$ $\boxed{\text{II}}$, C $\rightarrow$ IV, D $\rightarrow$ $\boxed{\text{III}}$

Total minimum time $= 3 + 2 + 1 + 4 = \boxed{10}$ hours.

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4. Six jobs are performed on machines M_1 and M_2 respectively. Time in hours taken by each job on each machine is given below :

Jobs $\rightarrow$ / Machines $\downarrow$	A	B	C	D	E	F
M_1	3	12	5	2	9	11
M_2	8	10	9	6	3	1

Complete the following activity to determine the optimal sequence of jobs and find total elapsed time. Also, find the idle time for machines M_1 and M_2.

Given jobs can be arranged in optimal sequence as

D	A	C	B	E	F

Jobs	Machine M_1		Machine M_2	
	In	Out	In	Out
D	0	2	$\boxed{}$	8
A	2	5	8	16
C	5	10	16	25
B	10	22	25	35
E	22	31	35	38
F	31	42	$\boxed{}$	43

Total elapsed time = $\boxed{}$ hours.

Idle time for machine $M_1 = 43 - 42 = 1$ hour

Idle time for machine $M_2 = \boxed{}$ hours. *(March '23)*

Solution : Given jobs can be arranged in optimal sequence as

D	A	C	B	E	F

Jobs	Machine M_1		Machine M_2	
	In	Out	In	Out
D	0	2	$\boxed{2}$	8
A	2	5	8	16
C	5	10	16	25
B	10	22	25	35
E	22	31	35	38
F	31	42	$\boxed{42}$	43

Total elapsed time = $\boxed{43}$ hours.

Idle time for machine $M_1 = 43 - 42 = 1$ hour

Idle time for machine $M_2 = \boxed{6}$ hours.

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The objective of an assignment problem is to assign

(a) number of jobs to equal number of persons at maximum cost

(b) number of jobs to equal number of persons at minimum cost

(c) only to maximize cost

(d) only to minimize cost *(March '25)*

2. The objective of sequencing problem is

(a) to find the order in which jobs are to be made

(b) to find the time required for completing all the jobs on hand

(c) to find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs

(d) to maximize the cost

3. The assignment problem is said to be balanced, if it is a

(a) square matrix (b) rectangular matrix

(c) unit matrix (d) triangular matrix ***(March '22)***

4. To use the Hungarian method, a profit maximization assignment problem requires

(a) converting all profits to opportunity losses

(b) a dummy person or job

(c) matrix expansion

(d) finding the maximum number of lines to cover all the zeros in the reduced matrix ***(March '24)***

5. In an assignment problem, if number of rows is greater than number of columns, then

(a) dummy column is added (b) dummy row is added

(c) row with cost 1 is added (d) column with cost 1 is added

6. In sequencing, an optimal path is one that after minimizes give ....

(a) Elapsed time (b) Idle time

(c) Both (a) and (b) (d) Ready time ***(July '22)***

7. If there are n jobs and m machines, then there will be .... sequences of doing the jobs.

(a) mn (b) $m(n!)$ (c) n^m (d) $(n!)^m$

8. The processing times required for four jobs A, B, C and D on Machine M_1 are 5, 8, 10 and 7 hours and on Machine M_2, it requires 7, 4, 3, and 6 hours respectively. The jobs are processed in the order M_1 M_2. The sequence that minimizes total elapsed time is ....

(a) ABCD (b) BCDA (c) ADBC (d) ABDC ***(Sept. '21)***

9. If jobs A to D have processing times 5, 6, 8, 4 on first machine and 4, 7, 9, 10 on second machine, then the optimal sequence is ....

(a) CDAB (b) DBCA (c) BCDA (d) ADBC ***(July '23)***

10. If jobs I, II, III have processing times 8, 6, 5 on machine M_1 and 8, 3, 4 on Machine M_2 in the order $M_1 - M_2$. Then the optimal sequence is ....
(a) I II III (b) I III II (c) II I III (d) III II I ***(July '24)***

Answers

1. (b) number of jobs to equal number of persons at minimum cost
2. (c) to find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs
3. (a) square matrix
4. (a) converting all profits to opportunity losses
5. (a) dummy column is added
6. (c) Both (a) and (b)
7. (d) $(n!)^m$
8. (c) ADBC
9. (b) DBCA **10.** (b) I III II.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. The Hungarian method operates on the principle of matrix reduction, whereby the cost table is reduced to a set of opportunity costs.
2. The purpose of dummy row or column in an assignment problem is to obtain balance between total number of activities and total number of resources.
3. To convert maximization type assignment problem into a minimization problem, the smallest element in the matrix is deducted from all elements of matrix. ***(March '22)***
4. In an assignment problem, if number of columns is greater than number of rows, then a dummy column is added. ***(July '22–'24; March '23)***
5. In a sequencing problem, the processing times are dependent of order of processing the jobs on machines.

Answers

1. True **2.** True **3.** False **4.** False **5.** False.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. An unbalanced assignment problem can be balanced by adding dummy rows or columns with .... cost.
2. An assignment problem is said to be unbalanced when ....
3. The time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines is called .... ***(March '24)***
4. A dummy row(s) or column(s) with the cost elements as .... the matrix of an unbalanced assignment problem as a square matrix.
5. In sequencing problem, the time which is required to complete all the jobs on machine is called ....
6. The time for which a machine j does not have a job to process to the start of job is called ....

Answers

1. zero **2.** number of rows is not equal to number of columns

3. total elapsed time **4.** zero **5.** total elapsed time **6.** idle time.

Question Set 16

PROBABILITY DISTRIBUTION

(Marks with option : 11)

16.1 PROBABILITY DISTRIBUTION OF A DISCRETE RANDOM VARIABLE

Remember :

1. **Probability Distribution :** If X is a discrete random variable assuming the values in the range $\{x_1, x_2, x_3, ..., x_n\}$ with corresponding probabilities p_i ($i = 1, 2, 3, ..., n$), then the set of ordered pairs (x_i, p_i), $i = 1, 2, 3, ..., n$ is called a probability distribution of a random variable X.

2. **Probability Mass Function (p.m.f.) :** If there is a function f such that $f(x_i) = p_i = P[X = x_i]$, for all possible values of X, then 'f' is called the probability mass function (p.m.f.) of X.
   $0 \leqslant p_i \leqslant 1$, $i = 1, 2, 3, ..., n$; $\Sigma p_i = 1$.

3. **Cumulative Distribution Function :** The cumulative distribution function (c.d.f.) of X is defined as follows and is denoted by $F(X)$.

   $F(X) = P[X \leqslant x]$, $x \in \mathbb{R}$

   $$= \sum_{x_i \leqslant x} f(x_i)$$

4. **Expected Value and Variance :** If the discrete random variable X take the values $x_1, x_2, x_3, ..., x_n$ with respective probabilities $p_1, p_2, p_3, ..., p_n$, then

   (1) Expected values $= E(X) = \mu = \sum_{i=1}^{n} x_i p_i$

   (2) Variance of X is denoted by σ^2 or Var (X) is given by

   $$\sigma^2 = \text{Var}(X) = \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i \right)^2$$

   $$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

5. $E(X) = \bar{X}$ and S.D. $= +\sqrt{V(X)}$.

Solved Examples	3 or 4 marks each

Ex. 1. Find the probability distribution of the number of tails in the simultaneous tosses of three coins.

Solution :

When three coins are tossed simultaneously, then the sample space is

{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

Let X denotes the number of tails.

Then X can take the value 0, 1, 2, 3.

$\therefore \quad P[X=0]=P(0)=\frac{1}{8}$

$P[X=1]=P(1)=\frac{3}{8}$

$P[X=2]=P(2)=\frac{3}{8}$

$P[X=3]=P(3)=\frac{1}{8}$

$\therefore$ the required probability distribution is :

$X=x_i$	0	1	2	3
$P[X=x_i]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Ex. 2. The probability distribution of a discrete r.v. X is as follows :

$X=x$	1	2	3	4	5	6
$P[X=x]$	k	$2k$	$3k$	$4k$	$5k$	$6k$

(i) Determine the value of k.

(ii) Find $P(X \leqslant 4)$, $P(2<X<4)$, $P(X \geqslant 3)$. *(March '22)*

Solution :

(i) For the probability distribution of a discrete r.v. X, we have

$\Sigma P(X=x)=1$

$\therefore k+2k+3k+4k+5k+6k=1$

$\therefore 21k=1 \qquad \therefore k=\frac{1}{21}.$

(ii) Now, the probability distribution is written as follows :

X	1	2	3	4	5	6
$P[X=x]$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$

$$P[X\leqslant 4]=P[X=1]+P[X=2]+P[X=3]+P[X=4]$$

$$=\frac{1}{21}+\frac{2}{21}+\frac{3}{21}+\frac{4}{21}=\frac{10}{21}$$

$$P[2<X<4]=P[X=3]=\frac{3}{21}=\frac{1}{7}$$

$$P[X\geqslant 3]=P[X=3]+P[X=4]+P[X=5]+P[X=6]$$

$$=\frac{3}{21}+\frac{4}{21}+\frac{5}{21}+\frac{6}{21}=\frac{18}{21}=\frac{6}{7}$$

Hence, $P[X\leqslant 4]=\frac{10}{21}$, $P[2<X<4]=\frac{1}{7}$, $P[X\geqslant 3]=\frac{6}{7}$.

Ex. 3. A random variable X has the following probability distribution :

x	1	2	3	4	5	6	7
$P(x_i)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine

(i) k ***(March '23–'24)*** **(ii)** $P(X<3)$ ***(March '23–'24)***

(iii) $P(X>4)$ ***(March '23)*** **(iv)** $P(X>6)$ ***(March '24)***

Solution :

(i) For the probability distribution, $\Sigma P(X)=1$

$\therefore\ k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\therefore\ 10k^2+9k=1$

$\therefore\ 10k^2+9k-1=0$

$\therefore\ 10k^2+10k-k-1=0$

$\therefore\ 10k(k+1)-1(k+1)=0$

$\therefore\ (10k-1)(k+1)=0$

$\therefore\ 10k-1=0$ *OR* $k+1=0$

$\therefore\ k=\frac{1}{10}$ *OR* $k=-1$

$\therefore k = -1$ is not possible because $P(X)$ cannot be negative.

$\therefore k = \dfrac{1}{10}$.

(ii) $P(X < 3) = P(X = 1) + P(X = 2) = k + 2k = 3k$

By putting $k = \dfrac{1}{10}$, we get

$P(X < 3) = 3 \times \dfrac{1}{10} = \dfrac{3}{10}$.

(iii) $P(X > 4) = P(X = 5) + P(X = 6) + P(X = 7)$

$$= k^2 + 2k^2 + 7k^2 + k = 10k^2 + k$$

By putting $k = \dfrac{1}{10}$, we get

$$P(X > 4) = 10 \times \left(\frac{1}{10}\right)^2 + \frac{1}{10} = 10 \times \frac{1}{100} + \frac{1}{10}$$

$$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5} = 0.2.$$

(iv) $P(X > 6) = P(X = 7) = 7k^2 + k$

$$= 7\left(\frac{1}{10}\right)^2 + \frac{1}{10} \qquad \text{... } [\because k = 10]$$

$$= \frac{7}{100} + \frac{1}{10} = \frac{17}{100}.$$

Ex. 4. Find the mean of number of heads in three tosses of a fair coin.

(March '25)

Solution : Three tosses of a fair coin are done.

$\therefore$ S = { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

Let X be the number of heads.

The possible values of X are 0, 1, 2, 3.

The probability distribution of X is obtained as follows :

$X = x_i$	p_i	$x_i p_i$
0	$\frac{1}{8}$	$0\times\frac{1}{8}=0$
1	$\frac{3}{8}$	$1\times\frac{3}{8}=\frac{3}{8}$
2	$\frac{3}{8}$	$2\times\frac{3}{8}=\frac{6}{8}$
3	$\frac{1}{8}$	$3\times\frac{1}{8}=\frac{3}{8}$
Total	$\Sigma p_i = 1$	$\Sigma x_i p_i = \frac{12}{8}$

Mean number of heads :

$$\mu = E(X) = \Sigma x_i p_i = \frac{12}{8} = 1.5.$$

Ex. 5. Let the p.m.f. of the r.v. X be $P(x) = \begin{cases} \dfrac{3-x}{10}, & \text{for } x = -1, 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$

Calculate $E(X)$ and $\mathrm{Var}(X)$.

Solution :

$X = x_i$	$P(x_i) = \frac{3-x}{10}$	$x_i P(x_i)$	$x_i^2 P(x_i) = x_i P(x_i)\, x_i$
-1	$\frac{4}{10}$	$-\frac{4}{10}$	$\frac{4}{10}$
0	$\frac{3}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
1	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
Total	$\Sigma P(x_i) = 1$	$\Sigma x_i P(x_i) = 0$	$\Sigma x_i^2 P(x_i) = \frac{10}{10} = 1$

$E(X) = \Sigma x_i P(x_i) = 0$

$\text{Var}(X) = \Sigma x_i^2 P(x_i) - [E(X)]^2$

$= 1 - 0 = 1.$

Ex. 6. The following is the c.d.f. of a r.v. X :

X	-3	-2	-1	0	1	2	3	4
$F(X)$	0.1	0.3	0.5	0.65	0.75	0.85	0.9	1

Find the probability distribution of X and $P(-1 \leqslant X \leqslant 2)$.

Solution : We are given c.d.f. and a r.v. X. We have to find the probability distribution of X.

We know that, $F(x) = P(X \leqslant x)$

i.e. $P(X \leqslant x) = F(x)$

$\therefore P(X \leqslant -3) = F(-3)$

$\therefore P(X = -3) = 0.1$

Now, $P(X = -2) = F(-2) - F(-3) = 0.3 - 0.1 = 0.2$

$P(X = -1) = F(-1) - F(-2) = 0.5 - 0.3 = 0.2$

$P(X = 0) = F(0) - F(-1) = 0.65 - 0.5 = 0.15$

$P(X = 1) = F(1) - F(0) = 0.75 - 0.65 = 0.10$

$P(X = 2) = F(2) - F(1) = 0.85 - 0.75 = 0.10$

$P(X = 3) = F(3) - F(2) = 0.9 - 0.85 = 0.05$

$P(X = 4) = F(4) - F(3) = 1 - 0.9 = 0.1$

$\therefore$ probability distribution of X is obtained as shown in the following table :

$X = x$	-3	-2	-1	0	1	2	3	4
$P(x_i)$	0.1	0.2	0.2	0.15	0.10	0.10	0.05	0.1

From the above table,

$P(-1 \leqslant X \leqslant 2) = P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$

$= 0.2 + 0.15 + 0.10 + 0.10$

$= 0.55.$

Examples for Practice	3 or 4 marks each

1. Two cards are randomly drawn, with replacement, from a well-shuffled deck of 52 playing cards. Find the probability distribution of the number of aces drawn.

2. The probability distribution of discrete r.v. X is as follows :

x	1	2	3	4	5	6
$P(X=x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

(i) Determine the value of k

(ii) Find $P(X \leqslant 4)$, $P(2 < X < 4)$, $P(X \geqslant 3)$.

3. 70% of members favour and 30% oppose a proposal in a meeting. The random variable X takes the value 0, if a member opposes the proposal and the value 1, if a member is in favour. Find $E(X)$ and Var (X).

4. A player tosses two coins. He wins ₹ 10 if 2 heads appear, ₹ 5 if 1 head appears and ₹ 2 if no head appears. Find the expected value and variance of winning amount.

5. Find expected value and variance of X using the following p.m.f. :

x	-2	-1	0	1	2
$P(x)$	0.2	0.3	0.1	0.15	0.25

(July '23)

6. The following is the probability distribution of r.v. X :

x	1	2	3	4	5
$P(x)$	$\frac{1}{20}$	$\frac{3}{20}$	a	$2a$	$\frac{1}{20}$

Find a and obtain the c.d.f. of X.

Answers

1.

$X=x$	0	1	2
$P(X=x)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

2. (i) $\frac{1}{21}$ (ii) $\frac{10}{21}, \frac{1}{7}, \frac{6}{7}$

3. $E(X)=0.7$, $\text{Var}(X)=0.21$ 4. $E(X)=₹\ 5.5$, $\text{Var}(X)=₹\ 8.25$

5. $E(X)=-0.05$, $\text{Var}(X)=2.2475$

6. $a=\frac{1}{4}$

x_i	1	2	3	4	5
$F(x_i)$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{9}{20}$	$\frac{19}{20}$	1

16.2 PROBABILITY DISTRIBUTION OF CONTINUOUS RANDOM VARIABLE

Remember :

1. **Probability Density Function (p.d.f.) :** Let X be a continuous random variable taking the values in the interval (a, b). The probability density function (p.d.f.) of X is an integrable function f that satisfies the following conditions :

(1) $f(x) \geq 0$, for all $x \in (a, b)$ (2) $\int_a^b f(x)\,dx = 1$

2. **Cumulative Distribution Function (c.d.f.) :** The cumulative distribution function (c.d.f.) of a continuous random variable X is denoted by F and is defined by

$F(x) = 0$, for all $x < a$

$= \int_a^x f(x)\,dx$, for all $x \geq a$

3. $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$, $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left[\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right]^2$

Solved Examples	**3 or 4 marks each**

Ex. 7. $f(x) = \begin{cases} \frac{x+2}{18}, & \text{for } -2 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$

Find (i) $P(X < 1)$ (ii) $P(|X| < 1)$.

Solution :

(i) $P(X<1)=\int_{-\infty}^{1} f(x)\,dx=\int_{-\infty}^{-2} f(x)\,dx+\int_{-2}^{1} f(x)\,dx$

$$=0+\int_{-2}^{1}\left(\frac{x+2}{18}\right)dx=\frac{1}{18}\int_{-2}^{1}(x+2)\,dx$$

$$=\frac{1}{18}\left[\frac{x^2}{2}+2x\right]_{-2}^{1}=\frac{1}{18}\left[\left(\frac{1}{2}+2\right)-(2-4)\right]$$

$$=\frac{1}{18}\left(\frac{5}{2}+2\right)=\frac{1}{18}\times\frac{9}{2}=\frac{1}{4}.$$

(ii) $P(|X|<1)=P(-1<X<1)$

$$=\int_{-1}^{1} f(x)\,dx=\int_{-1}^{1}\left(\frac{x+2}{18}\right)dx$$

$$=\frac{1}{18}\int_{-1}^{1}(x+2)\,dx=\frac{1}{18}\left[\frac{x^2}{2}+2x\right]_{-1}^{1}$$

$$=\frac{1}{18}\left[\left(\frac{1}{2}+2\right)-\left(\frac{1}{2}-2\right)\right]$$

$$=\frac{1}{18}\left(\frac{5}{2}+\frac{3}{2}\right)=\frac{1}{18}\times 4=\frac{2}{9}.$$

Ex. 8. Find k, if the following function represents the p.d.f. of a r.v. X :

$$f(x)=\begin{cases} kx(1-x), & \text{for } < x<1 \\ 0, & \text{otherwise.}\end{cases}$$

Also, find (i) $P\left[\frac{1}{4}<X<\frac{1}{2}\right]$ (ii) $P\left[X<\frac{1}{2}\right]$.

Solution :

$f(x)$ is p.d.f. of a r.v. X, if $\int_{-\infty}^{\infty} f(x)\,dx=1$

$\therefore \int_{-\infty}^{0} f(x)\,dx+\int_{0}^{1} f(x)\,dx+\int_{1}^{\infty} f(x)\,dx=1$

$\therefore 0+\int_{0}^{1} f(x)\,dx+0=1$

$\therefore \int_{0}^{1} kx(1-x)\,dx=1$

$\therefore \int_0^1 kx\,dx - \int_0^1 kx^2 dx = 1$

$\therefore \left[\frac{kx^2}{2}\right]_0^1 - \left[\frac{kx^3}{3}\right]_0^1 = 1 \qquad \therefore \frac{k}{2} - \frac{k}{3} = 1$

$\therefore 3k - 2k = 6 \qquad \therefore k = 6$

(i) $P\left[\frac{1}{4} < X < \frac{1}{2}\right] = \int_{1/4}^{1/2} 6x(1-x)\,dx$

$$= \int_{1/4}^{1/2} 6x\,dx - \int_{1/4}^{1/2} 6x^2\,dx$$

$$= \left[\frac{6x^2}{2}\right]_{1/4}^{1/2} - \left[\frac{6x^3}{3}\right]_{1/4}^{1/2}$$

$$= \left[\frac{6}{8} - \frac{6}{32}\right] - \left[\frac{6}{24} - \frac{6}{192}\right]$$

$$= \frac{24-6}{32} - \frac{48-6}{192} = \frac{18}{32} - \frac{42}{192}$$

$$= \frac{108-42}{192} = \frac{66}{192} = \frac{11}{32}$$

Hence, $P\left[\frac{1}{4} < X < \frac{1}{2}\right] = \frac{11}{32}$.

(ii) $P\left[X < \frac{1}{2}\right] = \int_0^{1/2} 6x\,dx - \int_0^{1/2} 6x^2\,dx$

$$= \left[\frac{6x^2}{2}\right]_0^{1/2} - \left[\frac{6x^3}{3}\right]_0^{1/2}$$

$$= \frac{6}{8} - \frac{6}{24} = \frac{18-6}{24} = \frac{12}{24} = \frac{1}{2}.$$

Ex. 9. The p.d.f. of the r.v. X is given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{x}}, & \text{for } 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Determine k, the c.d.f. of X and hence find $P(X \leqslant 2)$ and $P(X \geqslant 1)$.

Solution :

$$f(x)=\frac{k}{\sqrt{x}}, \quad 0<x<4$$

$$=0, \qquad \text{otherwise.}$$

To determine k :

For the p.d.f. of a r.v. X, we have

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx+\int_{0}^{4} f(x)\,dx+\int_{0}^{\infty} f(x)\,dx=1$$

$$\therefore 0+\int_{0}^{4} f(x)\,dx+0=1$$

$$\therefore \int_{0}^{4}\frac{k}{\sqrt{x}}dx=1$$

$$\therefore k[2\sqrt{x}]_0^4=1 \qquad \therefore k[2\sqrt{4}-0]=1$$

$$\therefore 4k=1$$

$$\therefore k=\frac{1}{4}$$

c.d.f. of X :

$$F(x_i)=\int_{-\infty}^{x_i} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$

$$=0+\int_{0}^{x}\frac{k}{\sqrt{x}}dx, \text{ where } k=\frac{1}{4}$$

$$=\int_{0}^{x}\frac{1}{4\sqrt{x}}dx=\frac{1}{4}[2\sqrt{x}]_0^x$$

$$F(x_i)=\frac{\sqrt{x}}{2}$$

$$P[X\leqslant 2]=F(2)=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$$P[X\geqslant 1]=1-P[X<1]=1-F(1)$$

$$=1-\frac{\sqrt{1}}{2}=1-\frac{1}{2}=\frac{1}{2}.$$

Ex. 10. Following is the p.d.f. of a continuous r.v. X.

$$f(x)=\begin{cases}\dfrac{x}{8}, & \text{for } 0<x<4,\\ 0, & \text{otherwise.}\end{cases}$$

Find (a) $P(X<1.5)$ (b) $P(1<X<2)$ (c) $P(X>2)$. ***(Sept. '21)***

Solution :

(a) $P(X<1.5)=\int_0^{1.5}\frac{x}{8}\,dx$

$$=\frac{1}{8}\left[\frac{x^2}{2}\right]_0^{1.5}=\frac{1}{16}[2.25-0]$$

$$=\frac{2.25}{16}=0.1406.$$

(b) $P[1<X<2]=\int_1^{2}\frac{x}{8}\,dx$

$$=\frac{1}{8}\left[\frac{x^2}{2}\right]_1^{2}=\frac{1}{16}[4-1]$$

$$=\frac{3}{16}=0.1875.$$

(c) $P(X>2)=\int_2^{4}\frac{x}{8}\,dx$

$$=\frac{1}{8}\left[\frac{x^2}{2}\right]_2^{4}=\frac{1}{16}[16-4]$$

$$=\frac{12}{16}=0.75.$$

Examples for Practice	**3 or 4 marks each**

1. Suppose error involved in making a certain measurement is a continuous r.v. X with p.d.f.

$$f(x)=\begin{cases}k(4-x^2), & \text{for } -2\leqslant x\leqslant 2,\\ 0, & \text{otherwise.}\end{cases}$$

Compute (i) $P(X>0)$, (ii) $P(-1<X<1)$.

2. It is felt that error in measurement of reaction temperature (in Celsius) in an experiment is a continuous r.v. with p.d.f.

$$f(X)=\begin{cases}\dfrac{x^3}{64}, & \text{for } 0\leqslant X\leqslant 4,\\ 0\ , & \text{otherwise.}\end{cases}$$

(i) Verify whether $f(X)$ is a p.d.f.

(ii) Find $P(0<X\leqslant 1)$.

3. Suppose X is the waiting time (in minutes) for a bus and its p.d.f. is given by

$$f(X)=\begin{cases}\dfrac{1}{5}, & \text{for } 0\leqslant X\leqslant 5,\\ 0\ , & \text{otherwise.}\end{cases}$$

Find the probability that (i) waiting time is between 1 and 3 minutes. (ii) waiting time is more than 4 minutes.

4. The p.d.f. of a continuous r.v. X is

$$f(X)=\begin{cases}\dfrac{3x^2}{8}, & \text{for } 0<X<2\\ 0, & \text{otherwise.}\end{cases}$$

Determine the c.d.f. of X and hence find

(i) $P(X>0)$ (ii) $P(1<X<2)$.

Answers

1. $k=\dfrac{3}{32}$ (i) $\dfrac{1}{2}$ (ii) $\dfrac{11}{16}$

2. (i) Yes (ii) $\dfrac{1}{256}$

3. (i) $\dfrac{2}{5}$ (ii) $\dfrac{1}{5}$

4. $F(X)=\dfrac{x^3}{8}, 0<X<2$ (i) 1 (ii) $\dfrac{7}{8}$.

16.3 BINOMIAL DISTRIBUTION AND POISSON DISTRIBUTION

Remember :

1. **Binomial Distribution :**

   A discrete random variable X is said to follow a Binomial distribution with parameters n and p, if its probability mass function (p.m.f.) is given by

   $$P[X=x]=p(x)$$
   $$={}^{n}C_{x}p^{x}q^{n-x}, \quad x=0, 1, 2, ..., n$$
   $$0<p<1, q=1-p$$
   $$=0, \quad \text{otherwise.}$$

2. **Mean and Variance of Binomial Distribution :**

   Let $X \sim B(n, p)$

   Mean : $\mu=E(X)=np$

   Variance : $\sigma^{2}=\text{Var}(X)=npq$

3. **Poisson Distribution :**

   A discrete random variable X is said to follow Poisson distribution with parameter m, if its probability mass function $p(x)$ is given by

   $$P[X=x]=p(x)=\frac{e^{-m}m^{x}}{x!}, \quad x=0, 1, 2, ...$$
   $$m=np>0$$
   $$=0, \quad \text{otherwise.}$$

   where $e=$ constant $=2.7183$

4. Mean $=E(X)=m$ and Variance $=\text{Var}(X)=m$

   $\therefore$ Mean $=$ Variance $=m$

Solved Examples	3 or 4 marks each

Ex. 11. The eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg in the lot of 10 eggs. ***(March '24)***

Solution : Let $X=$ Number of defective eggs

$p=$ probability of defective eggs

$\therefore p=10\%=\frac{10}{100}=\frac{1}{10}$ and $q=1-p=1-\frac{1}{10}=\frac{9}{10}$

Given : $n=10$

$\therefore X \sim B\left(10, \frac{1}{10}\right)$

The p.m.f. of X is given by

$P(X=x) = {}^nC_x p^x q^{n-x}$

i.e. $p(x) = {}^{10}C_x\left(\frac{1}{10}\right)^x\left(\frac{9}{10}\right)^{10-x}$, $x = 0, 1, 2, \ldots, 10$

$P(\text{at least one defective egg}) = P(X \geqslant 1)$

$= 1 - P(X=0) = 1 - p(0)$

$= 1 - {}^{10}C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{10-0}$

$= 1 - 1 \times 1 \times (0.9)^{10} = 1 - (0.9)^{10}$.

Ex. 12. Defects on plywood sheet occur at random with the average of one defect per 50 sq ft. Find the probability that such a sheet has (i) no defect, (ii) at least one defect. Use $e^{-1} = 0.3678$. ***(July '22; March '25)***

Solution :

X = Number of defects on a plywood sheet

$m = 1, e^{-1} = 0.3678$

$\therefore X \sim P(m=1)$

Hence, $p(x) = \dfrac{e^{-m} m^x}{x!}$ $\quad \therefore p(x) = \dfrac{e^{-1} 1^x}{x!} = (0.3678)\dfrac{1}{x!}$

(i) P[no defect]

$= P[X=0] = p(0) = 0.3678 \times \frac{1}{0!} = 0.3678 \times 1$

$\therefore P[X=0] = 0.3678$

Hence, the probability that sheet will have no defect is 0.3678.

(ii) P[At least one defect]

$= P[X \geqslant 1] = 1 - p(0)$

$= 1 - 0.3678 = 0.6322$

Hence, the probability that sheet will have at least one defect is 0.6322.

Ex. 13. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability that

(i) all the five cards are spades.

(ii) only 3 cards are spades. ***(March '25)***

Solution : Here, $n=5$, X = card of spade

$p =$ Probability that card is spade

$= \frac{13}{52} = \frac{1}{4}$ ... [$\because$ spade cards = 13]

$\therefore\ q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Now, $X \sim B\left(5, \frac{1}{4}\right)$

$\therefore\ P(X=x) = {}^nC_x p^x q^{n-x} = {}^5C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{5-x}$

(i) P[All the five cards are spades]

$= P[X=5] = {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$

$= 1 \times \frac{1}{1024} \times 1$

$\therefore\ P(X=5) = \frac{1}{1024}$.

(ii) P[Only 3 cards are spades]

$= P[X=3] = {}^5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2$

$= 10 \times \frac{1}{64} \times \frac{9}{16} = \frac{45}{512}$.

Ex. 14. If X follows Poisson distribution such that $P(X=1) = 0.4$ and $P(X=2) = 0.2$, find variance of X. ***(March '23)***

Solution : $X \sim P(m)$

Given : $P(X=1) = 0.4$ and $P(X=2) = 0.2$,

$$P(X=x) = p(x) = \frac{e^{-m} m^x}{x!}$$

$\therefore\ P(X=1) = p(1) = \frac{e^{-m} m^1}{1!}$

$\therefore\ 0.4 = e^{-m} \times m$ ... (1)

$\therefore\ P(X=2) = p(2) = \frac{e^{-m} m^2}{2!}$

$$\therefore\ 0.2=\frac{e^{-m}m^2}{2} \qquad \text{... (2)}$$

From (1), $e^{-m}=\frac{0.4}{m}$.

Put this result in (2), we get

$$0.2=\frac{\frac{0.4}{m}\times m^2}{2}$$

$\therefore\ 0.4=0.4\,m$

$\therefore\ m=1$

Hence, variance of $X=m=1$.

Ex. 15. Given that $X\sim B(n,p)$. If $n=25$, $E(X)=10$, find p and Var (X).

Solution : Given : $n=25$, $E(X)=10$

$\therefore\ np=10$

$\therefore\ 25p=10$

$\therefore\ p=\frac{10}{25}=0.4$

$\therefore\ q=1-p=1-0.4=0.6$

$\text{Var}(X)=npq=25\times0.4\times0.6=10\times0.6=6$

Hence, $p=0.4$ and Var $(X)=6$.

Examples for Practice	**3 or 4 marks each**

1. If a fair coin is tossed 10 times, find the probability of obtaining
   (i) exactly six heads (ii) at least six heads (iii) at most six heads.
2. A die is thrown 4 times. If getting 'an odd number' is a success, find the probability of
   (i) 2 successes (ii) at least 3 successes (iii) at most 2 successes.
3. The probability that a bomb will hit the target is 0.8. Find the probability that, out of 5 bombs, exactly 2 will miss the target.
4. Let $X\sim B(n,p)$. If $E(X)=6$ and Var $(X)=4.2$, find n and p.

5. Let the p.m.f. of r.v. X be $P(X=x)={}^4C_x\left(\frac{5}{9}\right)^x\left(\frac{4}{9}\right)^{4-x}$, for $x=0, 1, 2, 3, 4$. Find $E(X)$ and Var (X).

6. There are 10% defective items in a large bulk of items. What is probability that a sample of 4 items will include not more than one defective item? ***(July '22–'24)***

7. In a town, 10 accidents take place in the span of 50 days. Assuming that the number of accidents follows Poisson distribution, find the probability that there will be 3 or more accidents on a day. [Given : $e^{-0.2}=0.8187$]

8. If X has Poisson distribution with parameter m and $P(X=2)=P(X=3)$, then find $P(X\geqslant 2)$.

9. If a fair coin is tossed 6 times, find the probability of obtaining :
(a) exactly 4 heads (b) at least 4 heads (c) at most 2 heads. ***(Sept. '21)***

10. If $X \sim P(m)$ with $m=3$ and $e^{-3}=0.0497$, then find
(a) $P(X=3)$ (b) $P(X\geqslant 2)$. ***(Sept. '21)***

Answers

1. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$ 2. (i) 0.375 (ii) 0.3125 (iii) 0.6875

3. 0.2084 4. $n=20$, $p=0.3$ 5. $E(X)=\frac{20}{9}$, Var $(X)=\frac{80}{81}$

6. $1.3(0.9)^3$ 7. 0.0012 8. 0.8012

9. (a) $\frac{15}{64}$ (b) $\frac{11}{32}$ (c) $\frac{11}{32}$ 10. (a) 0.2237 (b) 0.8012.

ACTIVITIES	4 marks each

1. If $X\sim P(m)$ with $P(X=1)=P(X=2)$, then find the mean and $P(X=2)$.
[Given : $e^{-2}=0.1353$]

Since $P(X=1)=P(X=2)$

$$\therefore \frac{e^{\boxed{}}m^1}{1!}=\frac{e^{-m}m^2}{\boxed{}}$$

$$\therefore m=\boxed{}$$

$\therefore P(X=2)=\frac{e^{-2}.2^2}{2!}=\boxed{}$ *(March '24)*

Solution : Since $P(X=1)=P(X=2)$

$\therefore \frac{e^{\boxed{-m}}m^1}{1!}=\frac{e^{-m}m^2}{\boxed{2!}}$

$\therefore \frac{m}{1}=\frac{m^2}{2}$ $\quad\therefore m=\boxed{2}$

$\therefore P(X=2)=\frac{e^{-2}.2^2}{2!}$

$=\frac{0.1353\times 4}{2}$ ... [Given : $e^{-2}=0.1353$]

$=\boxed{0.2706}$.

---

2. The probability distribution of X is as follows :

x	0	1	2	3	4
$P(X=x)$	0.1	k	$2k$	$2k$	k

Find : (a) k (b) $P(X<2)$ (c) $P(1\leqslant X<4)$ (d) $F(2)$.

The table gives a probability distribution

$\therefore \Sigma p_i=1$

$\therefore 0.1+k+2k+2k=1$

(a) $k=\boxed{}$

(b) $P(X<2)=P(X=0)+P(X=1)=\boxed{}$

(c) $P(1\leqslant X<4)=P(1)+P(2)+P(3)=\boxed{}$

(d) $F(2)=P(X\leqslant 2)$

$=P(0)+P(1)+P(2)$

$=\boxed{}$ *(July '24)*

Solution : The table gives a probability distribution

$\therefore \Sigma p_i=1$

$\therefore 0.1+k+2k+2k+k=1$

$\therefore 6k=1-0.1=0.9$

(a) $\therefore k=\frac{0.9}{6}=\boxed{0.15}$

(b) $P(X<2)=P(X=0)+P(X=1)$

$$=0.1+k=0.1+0.15=\boxed{0.25}$$

(c) $P(1\leqslant X<4)=P(1)+P(2)+P(3)$

$$=k+2k+2k=5k=5\,(0.15)=\boxed{0.75}$$

(d) $F(2)=P(X\leqslant 2)$

$$=P(0)+P(1)+P(2)$$

$$=0.1+k+2k=0.1+3k$$

$$=0.1+3(0.15)=0.1+0.45=\boxed{0.55}.$$

---

3. If X has Poisson distribution with parameter m and $P[X=2]=P[X=3]$, then find $P[X\geqslant 2]$. **[Given : $e^{-3}=0.0497$]**

$X\sim P(m)$ $\quad P[X=x]=\dfrac{e^{-m}m^x}{x!}$

$\therefore P[X=2]=\dfrac{e^{-m}m^2}{2!}$ and $P[X=3]=\boxed{}$

Now, $P[X=2]=P[X=3]$

$\therefore \dfrac{e^{-m}m^2}{2!}=\dfrac{e^{-m}m^3}{3!}$ $\qquad \therefore m=\boxed{}$

Now, $P[X\geqslant 2]=1-\boxed{}=1-(P[X=0]+P[X=1])$

$$=1-\left[\frac{e^{-3}\cdot 3^0}{0!}+\frac{e^{-3}\cdot 3^1}{1!}\right]$$

$$=1-e^{-3}[1+3]=1-0.0497\times 4$$

Hence, $P[X\geqslant 2]$ is $\boxed{}$ *(July '23)*

Solution : $X\sim P(m)$ $\quad P[X=x]=\dfrac{e^{-m}m^x}{x!}$

$\therefore P[X=2]=\dfrac{e^{-m}m^2}{2!}$ and $P[X=3]=\boxed{\dfrac{e^{-m}m^3}{3!}}$

Now, $P[X=2]=P[X=3]$

$\therefore \dfrac{e^{-m}m^2}{2!}=\dfrac{e^{-m}m^3}{3!}$ $\qquad \therefore m=\boxed{3}$

Now, $P[X\geqslant 2]=1-\boxed{P[X<2]}=1-(P[X=0]+P[X=1])$

$$=1-\left[\frac{e^{-3}\cdot 3^0}{0!}+\frac{e^{-3}\cdot 3^1}{1!}\right]$$

$$= 1 - e^{-3}[1+3] = 1 - 0.0497 \times 4$$
$$= 1 - 0.1988 = 0.8012$$

Hence, $P[X \geqslant 2]$ is $\boxed{0.8012}$

---

4. An examination consists of 5 multiple choice questions, in each of which the candidate has to decide which one of 4 suggested answers is correct. A completely unprepared student guesses each answer completely randomly. Complete the following activity to find the probability that,

(a) the student gets 4 or more correct answers.

(b) the student gets less than 4 correct answers.

Let X = number of correct answers

p = probability of guessing a correct answer

$p = \boxed{}$, $q = \boxed{}$

Here, $n = 5$ $\quad\therefore X \sim B(n, p)$

For binomial distribution, $p(x) = {}^nC_x p^x q^{n-x}$

(a) Probability that the student gets 4 or more correct answers

$= P(X \geqslant 4) = P(X = 4) + P(X = 5)$

$= \boxed{}$

(b) Probability that the student gets less than 4 correct answers

$= P(X < 4) = 1 - P(X \geqslant 4)$

$= \boxed{}$

(March '22)

Solution : Let X = number of correct answers

p = probability of guessing a correct answer

$p = \boxed{\frac{1}{4}}$, $q = \boxed{\frac{3}{4}}$

Here, $n = 5$

$\therefore X \sim B(n, p)$

For binomial distribution,

$p(x) = {}^nC_x p^x q^{n-x}$

$\therefore p(x) = {}^5C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{5-x}$

(a) Probability that the student gets 4 or more correct answers

$$= P(X \geqslant 4) = P(X=4) + P(X=5)$$

$$= {}^5C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{5-4} + {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{5-5}$$

$$= 5 \times \frac{1}{4^4} \times \frac{3}{4} + 1 \times \frac{1}{4^5} \times 1 = \frac{15+1}{4^5} = \boxed{\frac{1}{64}}$$

(b) Probability that the student gets less than 4 correct answers

$$= P(X<4)$$

$$= 1 - P(X \geqslant 4) = 1 - \frac{1}{64} = \boxed{\frac{63}{64}}.$$

---

5. A pair of dice is thrown 3 times. If getting a doublet is considered a success, complete the following activity to find the probability of getting at least two successes.

A pair of dice is thrown 3 times.

$\therefore n = 3$

Let X = number of success (doublets)

p = probability of success (doublets)

$p = \boxed{\quad}$, $q = \boxed{\quad}$

$\therefore P(x) = {}^nC_x p^x q^{n-x}$

Probability of getting at least two successes means $x \geqslant 2$.

$\therefore P(x \geqslant 2) = P(x=2) + P(x=3)$

$$= \boxed{\quad} + \boxed{\quad} = \frac{2}{27}.$$

(March '23)

Solution :

A pair of dice is thrown 3 times.

$\therefore n = 3$

Let X = number of success (doublets)

p = probability of success (doublets)

$$p = \boxed{\frac{1}{6}}, \; q = 1 - p = 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$$

$$\therefore P(x) = {}^nC_x p^x q^{n-x} = {}^3C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{3-x}$$

Probability of getting at least two successes means $X \geqslant 2$.

$\therefore P(x \geqslant 2) = P(X=2) + P(X=3)$

$= {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{3-2} + {}^3C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{3-3}$

$= 3 \times \frac{1}{36} \times \frac{5}{6} + 1 \times \frac{1}{216} \times 1$

$= \boxed{\frac{15}{216}} + \boxed{\frac{1}{216}} = \frac{16}{216} = \frac{2}{27}.$

MULTIPLE CHOICE QUESTIONS	1 mark each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. Three coins are tossed simultaneously. X is the number of heads. Then the expected value of X is

(a) 1 (b) 1.5 (c) 1.9 (d) 1.017 ***(July '24)***

2. X is r.v. with p.d.f. $f(x) = \frac{k}{\sqrt{x}}$, $0 < x < 4$

$= 0$, otherwise,

then $E(x) =$ ....

(a) $\frac{1}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) 1

3. If $X \sim B\left(20, \frac{1}{10}\right)$, then $E(X) =$ ....

(a) 2 (b) 5 (c) 4 (d) 3

(March '22; July '22)

4. If $E(X) = m$ and $\text{Var}(X) = m$, then X follows

(a) Binomial distribution (b) Poisson distribution

(c) Normal distribution (d) (a) and (b) both ***(July '23)***

5. If $E(X) = 4$ and X follows Poisson distribution, then $V(X) =$ ....

(a) 2 (b) -2 (c) 4 (d) -4 ***(July '24)***

6. If X denotes the number of heads obtained when an unbiased coin is tossed twice, then the expected value of X is

(a) 1 (b) 0.5 (c) 1.5 (d) 2 ***(Sept. '21)***

7. The following function represents the p.d.f. of a r.v. X

$$f(X)=\begin{cases} kX, & \text{for } 0<X<2 \\ 0, & \text{otherwise,} \end{cases}$$

then the value of k is

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0 *(March '22)*

8. X is a number obtained on uppermost face when a fair dice is thrown, then $E(X)=$ ....

(a) 3 (b) 3.5 (c) 4 (d) 4.5 *(July '22)*

9. If $X \sim B\left(20, \frac{1}{10}\right)$, then $\text{Var}(X)=$ ....

(a) $\frac{9}{5}$ (b) 2 (c) $\frac{5}{9}$ (d) $\frac{1}{2}$ *(July '23)*

10. If the p.d.f. of continuous random variable X is

$$f(X)=kx^2(1-X), \quad 0<X<1$$
$$=0, \quad \text{otherwise,}$$

then the value of k is

(a) 12 (b) 10 (c) -12 (d) 8

11. The expected value of the sum of two numbers obtained when two fair dice are rolled is

(a) 5 (b) 6 (c) 7 (d) 8 *(March '25)*

Answers

1. (b) 1.5 **2.** (b) $\frac{4}{3}$ **3.** (a) 2 **4.** (b) Poisson distribution

5. (c) 4 **6.** (a) 1 **7.** (b) $\frac{1}{2}$ **8.** (b) 3.5

9. (a) $\frac{9}{5}$ **10.** (a) 12 **11.** (c) 7.

TRUE OR FALSE	1 mark each

State whether the following statements are True or False :

1. X is the number obtained on uppermost face when a die is thrown, then $E(X)=3.5$.

2. If p.m.f. of discrete r.v. X is

x	0	1	2
$P(X=x)$	q^2	$2pq$	p^2

then $E(x) = 2p$.

3. If $f(x) = kx(1-x)$, for $0 < x < 1$
$= 0$, otherwise,
then $k = 12$.

4. If $X \sim B(n, p)$ and $E(X) = 12$, $\text{Var}(X) = 4$, then the value of n is 18.

5. If $X \sim B(n, p)$ and $n = 6$ and $P(x=4) = P(x=2)$, then $p = \frac{1}{2}$.

6. If $X \sim P(m)$ with $P(X=1) = P(X=2)$, then $m = 1$. *(March '22)*

7. If r.v. X assumes values 1, 2, 3, ..., n with equal probabilities, then
$E(X) = \frac{n+1}{2}$. *(July '23)*

Answers

1. True 2. True 3. False 4. True 5. True 6. False 7. True.

FILL IN THE BLANKS	1 mark each

Fill in the following blanks :

1. The values of continuous r.v. are generally obtained by ....

2. If X is discrete random variable takes the values $x_1, x_2, x_3, \ldots x_n$, then
$\sum_{i=1}^{n} P(x_i) = \ldots\ldots\ldots$. *(July '22)*

3. Given p.d.f. of a continuous random variable X as :
$f(x) = \frac{x}{8}$, for $0 < x < 4$
$= 0$, otherwise.
Then $P(1 < x < 2) = \ldots\ldots\ldots$ *(July '24)*

4. If X is continuous r.v. and $F(x_i) = P(X \leqslant x_i) = \int_{-\infty}^{x} f(x)\,dx$, then $F(x)$ is called ....

5. If $X \sim P(m)$ with $m = 5$ and $e^{-5} = 0.0067$, then $P(X = 5) =$ ....

6. If $F(x)$ is a distribution function of discrete r.v. X with p.m.f. $p(x) = \frac{x-1}{3}$, for $x = 1, 2, 3$ and $p(x) = 0$, otherwise, then $F(4) =$ .... ***(July '22–'23)***

7. In a Binomial distribution, $n = 4$. If $2 \cdot P(X = 3) = 3 \cdot P(X = 2)$, then $p =$ ....

8. Let $X \sim B(10, 0.2)$, then $P(X = 1) =$ ....

Answers

1. measurement 2. 1 3. $\frac{3}{16}$ 4. distribution function

5. 0.1745 6. 1 7. $\frac{9}{13}$ 8. $2 \times (0.8)^9$.

Section 3

BOARD'S QUESTION PAPER : JULY 2025
(FOR PRACTICE)

Time : 3 Hours] **Solution on QR Code** [Max. Marks : 80

General Instructions :

(i) All questions are compulsory.

(ii) There are 6 questions divided into two sections.

(iii) Write answers of Section – I and Section – II in the same answer book.

(iv) Use of logarithmic tables is allowed. Use of calculator is not allowed.

(v) For LPP and Time series, graph paper is not necessary. Only rough sketch of graph is expected.

(vi) Start answer to each question on a new page.

(vii) For each objective type of questions (i.e. Q. 1 and Q. 4), only the first attempt will be considered for evaluation.

SECTION – I

Q. 1. (A) Select and write the correct answer of the following multiple choice type of questions : **[6]**

(i) Matrix $B=\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ p & 4 & 0 \end{bmatrix}$ is skew-symmetric, then value of p is

(a) 1 (b) -1 (c) 0 (d) -3 **(1)**

(ii) If $y=e^{\log x}$, then $\frac{dy}{dx}=$ ....

(a) $\frac{1}{x}$ (b) $\frac{1}{2}$ (c) $\frac{e^{\log x}}{x}$ (d) 0 **(1)**

(iii) $\int(1-x)^{-3}dx=$ ....

(a) $\frac{1}{2}(1-x)^{-2}+c$ (b) $\frac{1}{2}(1+x)^{-2}+c$

(c) $\frac{1}{2}(1-x)^{-2}+\frac{x}{2}+c$ (d) $\frac{1}{2}(1-x)^{-2}-\frac{x}{2}+c$ **(1)**

(iv) If $\int_0^a 3x^2dx=8$, then $a=$ ....

(a) 0 (b) 2 (c) 8 (d) $\frac{4}{3}$ **(1)**

(v) Area of the region bounded by the curve $y = x^2$, the X-axis and the lines $x = 1$ and $x = 3$ is ....

(a) $\frac{3}{26}$ sq units (b) 3 sq units

(c) 26 sq units (d) $\frac{26}{3}$ sq units **(1)**

(vi) The order and degree of $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$ are respectively ....

(a) 1, 1 (b) 1, 2 (c) 2, 1 (d) 2, 2 **(1)**

(B) State whether the following statements are True or False : [3]

(i) Dual of $(p \wedge \sim q) \vee t$ is $(p \vee \sim q) \vee c$. **(1)**

(ii) For $\int \frac{x-1}{(x+1)^3} e^x dx = e^x \cdot f(x) + c$, where $f(x) = (x+1)^2$. **(1)**

(iii) The integrating factor (I.F.) of $\frac{dy}{dx} + y = e^{-x}$ is e^x. **(1)**

(C) Fill in the following blanks : [3]

(i) Negation of 'Some men are animals' is .... **(1)**

(ii) If the average revenue is 45 and elasticity of demand is 3, then marginal revenue is .... **(1)**

(iii) To find the value of $\int \frac{10x^9 + 10^x \cdot \log 10}{10^x + x^{10}} dx$, the proper substitution is .... **(1)**

Q. 2. (A) Attempt *any TWO* of the following questions : [6]

(i) Determine whether following statement pattern is tautology, contradiction or contingency :

$[(p \wedge q) \vee (\sim p)] \vee [p \wedge (\sim q)]$ **(3)**

(ii) Find $\frac{dy}{dx}$, if $y = (\log x)^x + x^5$. **(3)**

(iii) Find the area of region bounded by the parabola $y^2 = 4x$ and line $x = 3$. **(3)**

(B) Attempt *any TWO* of the following questions : [8]

(i) Find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as

$E_c = (0.0003)\, I^2 + (0.075)\, I$, when $I = 1000$. **(4)**

(ii) Solve the following differential equation :

$x^2y\,dx-(x^3+y^3)\,dy=0.$ **(4)**

(iii) Express the following equations in matrix form and solve them by method of reduction :

$x+2y+z=8,\ 2x+3y-z=11,\ 3x-y-2z=5.$ **(4)**

Q. 3. (A) Attempt *any TWO* of the following questions : [6]

(i) Write the converse, inverse and contrapositive of the following statement :

'If a man is bachelor, then he is happy.' **(3)**

(ii) Find the inverse of $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$ by adjoint method. **(3)**

(iii) If $x^5\cdot y^7=(x+y)^{12}$, then show that $\dfrac{dy}{dx}=\dfrac{y}{x}$. **(3)**

(B) Attempt *any ONE* of the following questions : [4]

(i) Evaluate : $\int x^2e^{3x}dx.$ **(4)**

(ii) Evaluate : $\displaystyle\int_1^4 \frac{\sqrt[3]{x+6}}{\sqrt[3]{x+6}+\sqrt[3]{11+x}}\,dx.$ **(4)**

(C) Attempt *any ONE* of the following questions (Activity) : [4]

(i) Find the values of x, such that $f(x)$ is increasing function, where $f(x)=2x^3-15x^2-144x-7.$

Solution :

Given : $f(x)=2x^3-15x^2-144x-7$

$\therefore f'(x)=6x^2-30x-144$

Now, $f'(x)>0$, as $f(x)$ is increasing

$\therefore 6x^2-30x-144>0$

$\therefore x^2-5x-24>0$

$\therefore (x-8)(x+3)>0$

Case (I) : $x-8>0$ and $x+3>0$

$\therefore x>8$ and $x>-3$

$\therefore x>\boxed{}$

Case (II) : $x-8<0$ and $x+3<0$

$\therefore x<8$ and $x<-3 \qquad \therefore x<\boxed{}$

$\therefore f(x)=2x^3-15x^2-144x-7$ is increasing if and only if

$x\in\left(-\infty, \boxed{}\right)$ or $x\in\left(\boxed{}, \infty\right)$. **(4)**

(ii) Solve the following differential equation, hence find the particular solution when $x=0, y=1$:

$$y^3-\frac{dy}{dx}=x\frac{dy}{dx}$$

Solution :

$$y^3=x\frac{dy}{dx}+\frac{dy}{dx}$$

$$\therefore y^3=(x+1)\boxed{}$$

$$\therefore (x+1)\,dy=y^3dx$$

Separating the variables, we get

$$\frac{1}{y^3}\,dy=\frac{1}{x+1}\,dx$$

Now, integrating, we get

$$\int\frac{1}{y^3}\,dy=\int\frac{1}{x+1}\,dx$$

$$\therefore -\frac{1}{2y^2}=\boxed{}+c \qquad \text{... (1)}$$

which is required general solution.

Put $x=0$ and $y=1$ in (1), we get

$$-\frac{1}{2(1)^2}=\log|0+1|+c$$

$$\therefore \boxed{}=c$$

$\therefore \dfrac{1}{2y^2}=\boxed{}-\dfrac{1}{2}$ is the particular solution. **(4)**

SECTION – II

Q. 4. (A) Select and write the correct answer of the following multiple choice type of questions : **[6]**

(i) The sum due is also called as ....

(a) true discount (b) face value

(c) present value (d) cash value **(1)**

(ii) Following are different types of insurance

(I) Life insurance (II) Health insurance

(III) Liability insurance

(a) Only II (b) Only III (c) Only I (d) All the three **(1)**

(iii) $|b_{xy}+b_{yx}| \geqslant$ ....

(a) $2|r|$ (b) $2r$ (c) r (d) $|r|$ **(1)**

(iv) If P_{01} (L) = 120.4 and P_{01} (P) = 130.6, then P_{01}(D-B) is ....

(a) 25.1 (b) 60.2 (c) 125.5 (d) 65.3 **(1)**

(v) $F(x)$ is c.d.f. of discrete r.v. X whose distribution is ....

X_i	-2	-1	0	1	2
P_i	0.2	0.3	0.15	0.25	0.1

then $F(-3) =$ ....

(a) 1 (b) 0.2 (c) 0.15 (d) 0 **(1)**

(vi) Given p.d.f. of a continuous r.v. X as

$f(X)=\frac{x^2}{3}$, for $-1<x<2$

$=0$, otherwise,

then $F(1)=$ ....

(a) $\frac{3}{9}$ (b) $\frac{4}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$ **(1)**

(B) State whether the following statements are True or False : [3]

(i) The banker's discount is also called as commercial discount. **(1)**

(ii) The optimum value of the objective function of LPP occurs at the centre of the feasible region. **(1)**

(iii) If $E(X) > \text{Var}(X)$, then X follows Binomial distribution. **(1)**

(C) Fill in the following blanks : [3]

(i) The region represented by the inequality $y \leqslant 0$ lies in .... quadrants. **(1)**

(ii) The time required for printing of four books A, B, C and D is 5, 8, 10 and 7 hours, while its data entry requires 7, 4, 3 and 6 hours respectively. The sequence that minimizes total elapsed time is .... **(1)**

(iii) If X has Poisson distribution with parameter m and $P(X=3)=P(X=4)$, then $m=$ .... **(1)**

Q. 5. (A) Attempt *any TWO* of the following questions : **[6]**

(i) A person wants to create fund of ₹ 6,96,150 after 4 years at the time of his retirement. He decides to invest a fixed amount at the end of every year in a bank that offers him interest of 10% p.a. compounded annually. What amount should he invest every year? [Given $(1.1)^4 = 1.4641$] **(3)**

(ii) Following are given information about advertising expenditure and sales :

	Advertisement expenditure (₹ in lakh) (X)	**Sales (₹ in lakh) (Y)**
Arithmetic mean	10	90
Standard deviation	3	12

Correlation coefficient between X and Y is 0.8.

(a) Obtain the regression of Y on X.

(b) What is the likely sales when the advertising budget is ₹ 15 lakh? **(3)**

(iii) Find y, if the cost of living index is 200 for the following data :

Group	**Food**	**Clothing**	**Fuel and Lighting**	**House Rent**	**Miscellaneous**
I	180	120	160	300	200
W	4	5	3	y	2

(3)

(B) Attempt *any TWO* of the following questions : **[8]**

(i) A bill of ₹ 5475 drawn on 19th January 2015 for 8 months was discounted on 28th February 2015 at 8% p.a. interest. What is the banker's discount? What is the cash value of the bill? **(4)**

(ii) Following table shows that the all India Infant Mortality Rates (per '000) for years 1980 to 1986 :

Year	1980	1981	1982	1983	1984	1985	1986
IMR	10	6	5	3	3	1	0

Fit a trend line to the above data by the method of least squares. **(4)**

(iii) Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 can not be placed at A. The cost matrix is given below :

Machines	**Places**				
	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	–	5	4
M_3	–	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule. **(4)**

Q. 6. (A) Attempt *any TWO* of the following questions : [6]

(i) Obtain 4-yearly centred moving averages for the following time series :

Years	1981	1982	1983	1984	1985	1986
Number of crimes ('000)	40	42	43	42	44	44
Years	1987	1988	1989	1990	1991	
Number of crimes ('000)	43	46	47	45	46	

(3)

(ii) Calculate Walsch's Price Index Number for the following data :

Commodity	**Base Year**		**Current Year**	
	Price	**Quantity**	**Price**	**Quantity**
L	4	8	3	2
M	6	16	8	9
N	8	18	7	32

(3)

(iii) A pair of dice is thrown 3 times. If getting a doublet is considered a success, find the probability of two successes. **(3)**

(B) Attempt *any ONE* of the following questions : [4]

(i) The equations of two regression lines are $8x - 10y + 66 = 0$ and $40x - 18y = 214$. Find

(a) the mean values of X and Y.

(b) correlation coefficient between X and Y. **(4)**

(ii) Maximize : $z = 60x + 50y$ subject to :
$x + 2y \leqslant 40,\ 3x + 2y \leqslant 60,\ x \geqslant 0,\ y \geqslant 0.$ **(4)**

(C) Attempt *any ONE* of the following questions (Activity) : [4]

(i) Find the sequence that minimizes the total elapsed time to complete the following jobs in the order AB. Find the total elapsed time and idle times for both the machines.

Jobs	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**
Machine A	**7**	**16**	**19**	**10**	**14**	**15**	**5**
Machine B	**12**	**14**	**14**	**10**	**16**	**5**	**7**

Solution :

Using the optimal sequence algorithm, the following optimal sequence can be obtained.

$\square$	$\square$	IV	V	III	$\square$	$\square$

Total elapsed time is obtained as follows :

Jobs	**Machine A**		**Machine B**	
	Time in	**Time out**	**Time in**	**Time out**
$\square$	0	5	5	12
$\square$	5	12	12	24
IV	12	22	24	34
V	22	36	36	52
III	36	55	55	69
$\square$	55	71	71	85
$\square$	71	86	86	91

$\therefore$ total elapsed time T = 91 units

Idle time for machine A = $\square$ units

Idle time for machine B = $\square$ units. **(4)**

(ii) The probability distribution of *X* is as follows :

x	**0**	**1**	**2**	**3**	**4**
$P[X = x]$	**0.1**	k	$2k$	$2k$	k

Find (a) k (b) $P(X > 2)$ (c) $P(1 < X \leqslant 4)$.

Solution :

(a) The table gives a probability distribution and therefore

$\Sigma P(X=x)=1$

$\therefore P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$

$\therefore 0.1+k+2k+2k+k=1$

$\therefore 6k=1-0.1$

$\therefore 6k=0.9$

$\therefore k=\boxed{\quad}$

(b) $P(X>2)=P(X=3)+P(X=4)$

$$=2k+k$$

$$=\boxed{\quad}$$

(c) $P(1<X\leqslant 4)+P\left(X=\boxed{\quad}\right)+P(X=3)+P(X=4)$

$$=\boxed{\quad}+2k+k$$

$$=0.75$$

(4)

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