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演習８　勾配降下法

課題

行列Aおよびベクトルbが既知のとき，以下の最小二乗法

$x^* = \arg\min_x ||Ax - b||_2^2$

を勾配降下法を用いて未知のx^*を求めるアルゴリズムを実装せよ．初期値は$x = (3,3)^\top$とし，勾配降下のステップサイズは$\tau = 0.1$とする．

追加課題

余裕のあるものは，反復ごとのx_1, x_2を保存しておいて，反復によるこれらの変数の変化をグラフにせよ．さらに，τを変えてこのグラフがどうなるか観察せよ．

For nonlinear programming problem, it is difficult to find a optimal solution by finite computation. The optimal solution is expected to be obtained by generating a sequence of points $\{x^{(k)}\}$ that converge to the optimal one.

For a convex function f, the global optimum can be obtained by Gradient Descent Method, which the gradient vector $\nabla f(x^{(k)})$ leads to the direction that the value of the object function will increase or decrease to the optimum point.

Here, the parameter α is proposed as the step to control the speed of converging. By choosing an appropriate starting point $x(0)$, and generating the next point by

$$x^{(k+1)} = x^{(k)} - \alpha^{(k)}\nabla f(x^{(k)})$$

iteratively, the optimal solution can be obtained.

In machine learning, this problem can be considered as linear regression with one variable. Thus, the matrix A must contains one more column of 1 for convenience of computation.

In [1]:
```python
import numpy as np
import matplotlib.pyplot as plt

a = np.c_[np.asarray([21.3, 22, 26.9, 32.3, 33.1, 38.2])/100]
b = np.c_[np.asarray([116.5, 125.5, 128.1, 132, 141, 145.2])/100]
A = np.concatenate([np.ones([6,1]), a], axis=1)

ATA = np.matmul(A.T, A)
ATb = np.matmul(A.T, b)
# ソルバで解いた解x(デバッグ用)
tx = np.linalg.solve(ATA, ATb)
print(tx)

tau = 0.1 #勾配降下のステップサイズ
#tau == alpha
Niter = 5000 # 反復回数

# xの初期値
x = np.c_[np.asarray([3, 3])]
```

```
[[0.88743422]
 [1.47203377]]
```

In this case, the objective function is $f(x) = ||Ax - b||_2^2$.

In [2]:
```python
# In machine learning, the function usually is called as cost function or loss function.
def computeCost(A, b, x):
  return sum((A.dot(x)-b)**2)
```

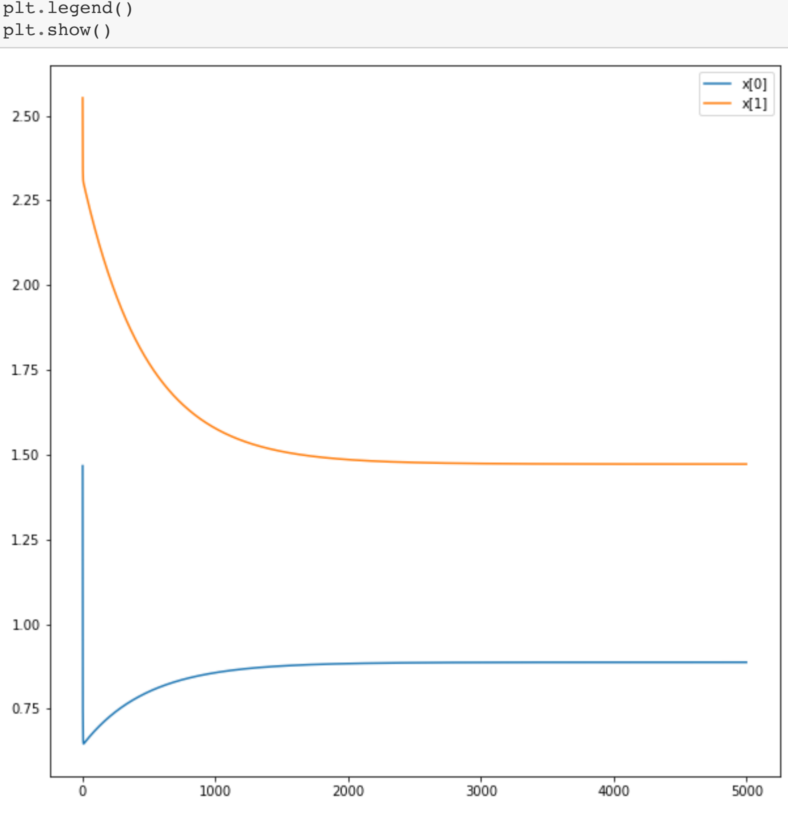
In [3]:
```python
# The parameters are the observed variables, the predictions, the starting point
# step, and number of iterations.

# The parameters that could be used to predict are expected to be obtained.

def gradientDescent(A, b, x, alpha, Niter):
  xtmp = x
  J_history = np.zeros((Niter, 1))
  x_history = np.zeros((Niter, 2, 1))

  for iter in range(Niter):
    #tmp_x0 = x[0] - alpha * sum(A.dot(x)-b)
    tmp_x0 = xtmp[0] - alpha * sum(A.dot(xtmp)-b)
    #tmp_x1 = x[1] - alpha * sum(np.multiply(A.dot(x) - b, A[:,1].reshape(-1,1)))
    tmp_x1 = xtmp[1] - alpha * sum(np.multiply(A.dot(xtmp) - b, A[:,1].reshape(-1,1)))

    xtmp = np.array([tmp_x0, tmp_x1])
    x_history[iter] = xtmp
    #print(xtmp)
    J_history[iter] = computeCost(A, b, xtmp)

  return xtmp, J_history, x_history
```

In [4]:
```python
x_final, JHistory, xHistory = gradientDescent(A, b, x, tau, Niter)
print(x_final)
```

```
[[0.88742668]
 [1.47205971]]
```

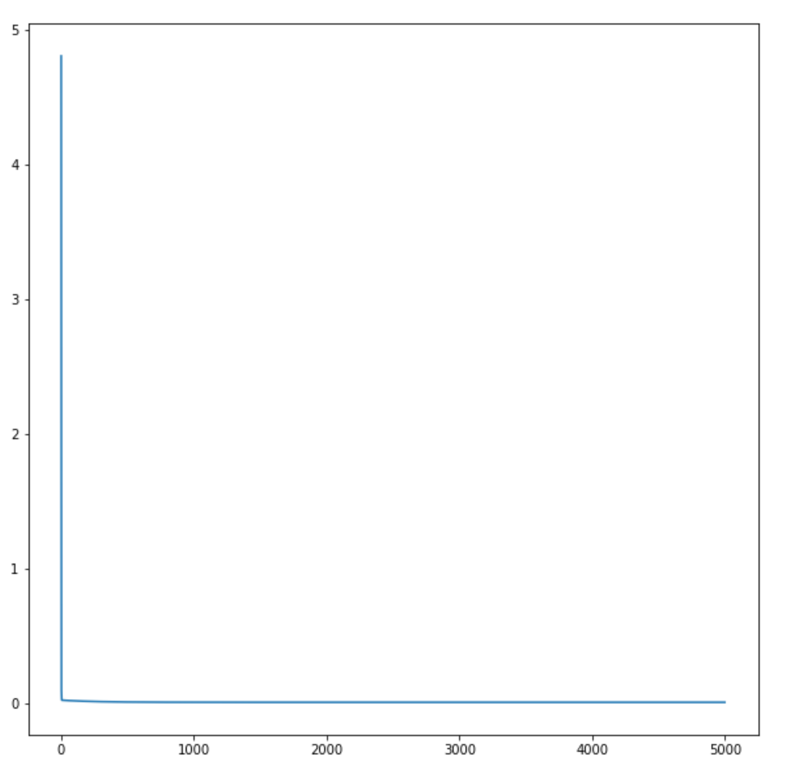
The final result of iteration is very close to direct solution, meaning it is correct.

In [5]:
```python
# Plot the x[0], x[1] changes as the iterations increase.
# They finally converge to a number.

plt.figure(figsize=(10,10))
l1 = plt.plot(list(range(Niter)),list(xHistory[:,0]), label = "x[0]")
l2 = plt.plot(list(range(Niter)),list(xHistory[:,1]), label = "x[1]")
plt.legend()
plt.show()
```



In [ ]:
```python
# Plot the object function changes as the iterations increase.

plt.figure(figsize=(10,10))
plt.plot(list(range(Niter)),list(JHistory[:,0]))
```

Out[ ]: `[<matplotlib.lines.Line2D at 0x7f0203241240>]`

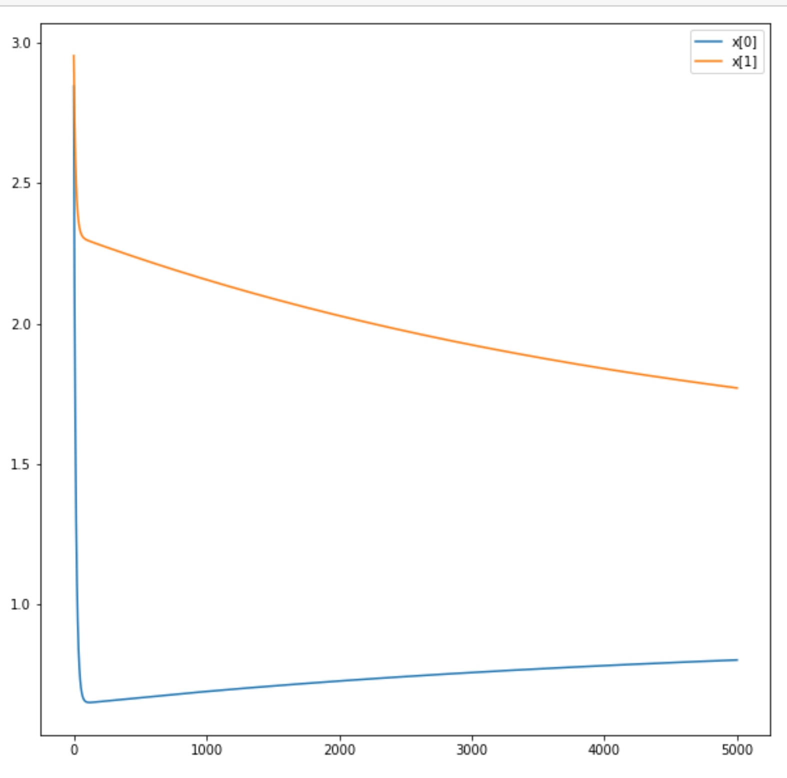


In [6]:
```python
# Change the step to a smaller number, meaning the object function may converge slowly.

alpha = 0.01

xfinal_, JHistory_, xHistory_ = gradientDescent(A, b, x, alpha, Niter)
print(xfinal_)
```

```
[[0.80067302]
 [1.77051845]]
```

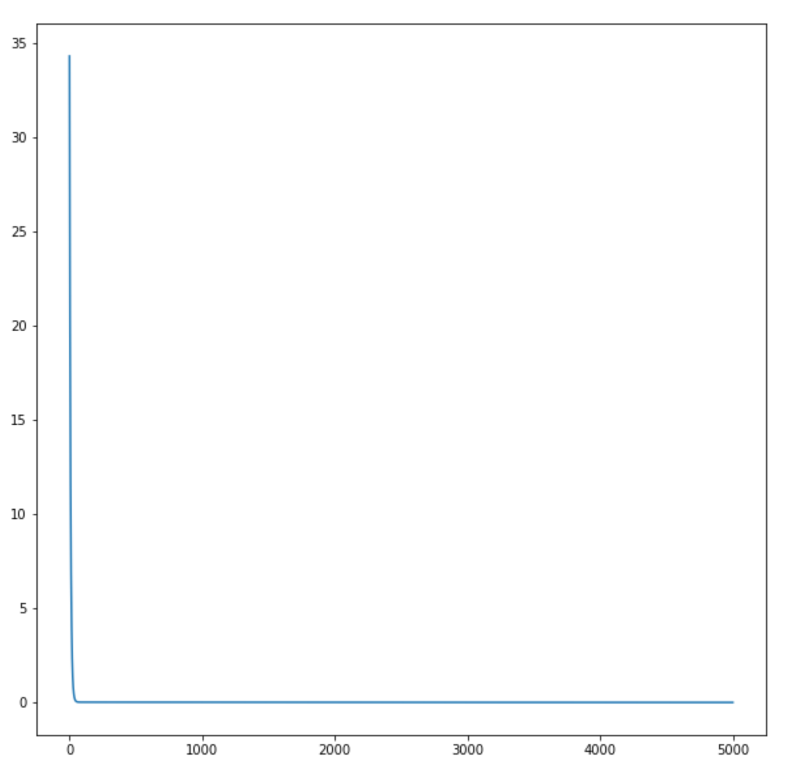
The result is not as good as that when using a larger step.

In [8]:
```python
#Plot of x[0], x[1]

plt.figure(figsize=(10,10))
l3 = plt.plot(list(range(Niter)),list(xHistory_[:,0]), label = "x[0]")
l4 = plt.plot(list(range(Niter)),list(xHistory_[:,1]), label = "x[1]")
plt.legend()
plt.show()
```



In [ ]:
```python
plt.figure(figsize=(10,10))
plt.plot(list(range(Niter)),list(JHistory_[:,0]))
```

Out[ ]: `[<matplotlib.lines.Line2D at 0x7f386d0ffda0>]`



演習１０　Chambolleによるtotal variation(TV)最小化アルゴリズムの実装

課題

観測モデル$u_0 = \hat{u} + n$に従う観測画像u_0が与えられているとする．Chambolleらの提案したTV最小化の数値計算アルゴリズムにより，ノイズ除去を行い（u_0に近いことが期待できる）推定画像uを求める手法を実装せよ．

Image denosing is a typical inverse problem, of which the selection of regularization terms is very important. For a unknown image $\hat{\boldsymbol{u}}$, the observed model is $\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = K\hat{\boldsymbol{u}} + \boldsymbol{n}$$

Total Variation is a regularization introduced by Rudin, Osher, and Fatemi, and the following formula is called as ROF model.

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \frac{1}{2\lambda}||\boldsymbol{u_0} - \boldsymbol{u}||_2^2 + ||D\boldsymbol{u}||_{1,2}$$

$D : \mathbb{R}^N \rightarrow \mathbb{R}^{2N}$ is the discrete gradient operator for position (i, j) in the image. And $||D\boldsymbol{u}||_{1,2}$ is called as TV norm. In this view, the image denoising is performed as an infinite-dimensional minimization problem.

Chambolle proposed a dual-formulation-based projection algorithm for TV-denoising. The ROF model can be represented as

$$\min_{\boldsymbol{u}} \frac{1}{2\lambda}||\boldsymbol{u_0} - \boldsymbol{u}||_2^2 + \max_{\boldsymbol{p}} < D^T\boldsymbol{p}, \boldsymbol{u} >$$

where the $D^T : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$ is the conjugate operator that forms the following equation:

$$< \boldsymbol{p}, D\boldsymbol{u} > = < D^T\boldsymbol{p}, \boldsymbol{u} >$$

To simplify, with gradient direction $-D(D^T\boldsymbol{p} - \frac{\boldsymbol{u_0}}{\lambda})$, the pixels in the denoised image can be obtained using projected gradient method:

$$\boldsymbol{p}_{i,j}^{k+1} = \frac{\boldsymbol{p}_{i,j}^k - \tau((D(D^T\boldsymbol{p}^k - \frac{\boldsymbol{u_0}}{\lambda})))_{i,j}}{max\{1, |\boldsymbol{p}_{i,j}^k - \tau((D(D^T\boldsymbol{p}^k - \frac{\boldsymbol{u_0}}{\lambda})))_{i,j}|\}}$$

where $|\boldsymbol{z}| := \sqrt{(\boldsymbol{z}_1)^2 + (\boldsymbol{z}_2)^2}$

The denoised image is obtained by:

$$\hat{\boldsymbol{u}} = \boldsymbol{u}_0 - \lambda D^T\boldsymbol{p}$$

In [10]:
```
!git clone https://github.com/mdipcit/standard_images/
```

```
Cloning into 'standard_images'...
remote: Enumerating objects: 10, done.
remote: Counting objects: 100% (10/10), done.
remote: Compressing objects: 100% (8/8), done.
remote: Total 10 (delta 0), reused 0 (delta 0), pack-reused 0
Unpacking objects: 100% (10/10), done.
```

In [11]:
```
cd standard_images/
```

```
/content/standard_images
```

In [12]:
```python
import numpy as np
import matplotlib.pyplot as plt
from PIL import Image
from numpy import *
```

Define operator D and D^T.

In [13]:
```python
def D(im):
    h, w = im.shape
    dh = np.concatenate([im[1:, :] - im[0:-1, :], \
        np.zeros([1, w])], axis=0)
    dw = np.concatenate([im[:, 1:] - im[:, 0:-1], \
        np.zeros([h, 1])], axis=1)
    return np.dstack([dh, dw])

def Dt(p):
    h, w, _ = p.shape
    return np.concatenate([-p[0, :, 0].reshape([1, w]), -p[1:-1, :, 0]\
         + p[0:-2, :, 0], p[-2, :, 0].reshape([1, w])], axis = 0) \
         + np.concatenate([-p[:, 0, 1].reshape([h, 1]), -p[:, 1:-1, 1]\
         + p[:, 0:-2, 1], p[:, -2, 1].reshape([h, 1])], axis = 1)
```

Define the calculation of signal-to-noise ratio. High signal-to-noise ratio will be obtained if denoising is well done.

In [14]:
```python
def PSNR(im1, im2):
    return 10*np.log10(1/np.mean((im1-im2)**2))
```

In [15]:
```python
uh= np.asarray(Image.open('pepper.png')).astype(float)/255.0
h, w = uh.shape
sig = 10/255
u0 = uh + np.random.randn(h, w) * sig
print(PSNR(uh, u0))
```

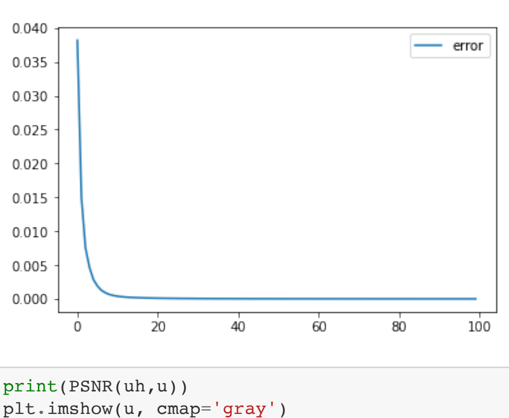
```
28.100653200424873
```

In [16]:
```python
# initialization
tau=0.9/4 #反復法の勾配降下のパラメータτ
lamb = 0.03 #目的関数（式(15))中の正則化項の重みλ
Niter = 100 #反復回数
# 反復中で使う変数を初期化
p = np.zeros([h, w, 2])

# Make a list of error to record the difference of updated U and previous U
error = list(range(Niter))

# The parameters include the obeseved image, the initializaed p,
# the initial U, the gradient parameter tau, the weight of regularization,
# and the number of iterations
def denoise(im, p, U_init, tau, lamb, Niter):
  h, w = im.shape

  U = U_init
  Px = p[:,:,0]
  Py = p[:,:,1]

  for i in range(Niter):
  # Chambolle's iteration
  #for i in range(Niter):
  # ここにpの更新の式を書く
    Uold = U

    # The gradient of rows and columns is calculated separately
    GradUx = D(Dt(p) - im / lamb)[:,:,0]
    GradUy = D(Dt(p) - im / lamb)[:,:,1]

    PxNew = Px - tau * GradUx
    PyNew = Py - tau * GradUy

    NormNew = maximum(1, sqrt(PxNew**2+PyNew**2))

    Px = PxNew/NormNew
    Py = PyNew/NormNew

    p = np.dstack([Px, Py])


    #---dual→primal
    #u= u0 - lamb*Dt(p)
    U = im - lamb * Dt(p)

    error[i] = linalg.norm(U-Uold) / sqrt(h*w)

  return U, error
```

In [17]:
```python
u, error = denoise(u0, p, u0, tau, lamb, Niter)
```

In [18]:
```python
plt.plot(list(range(Niter)),error, label = 'error')
plt.legend()
```

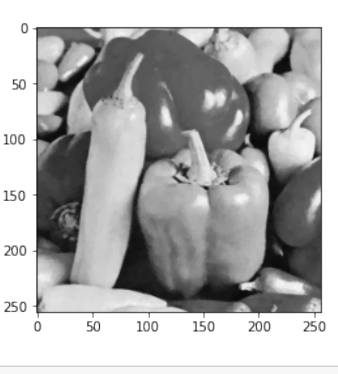
Out[18]: `<matplotlib.legend.Legend at 0x7f0608d84978>`



In [19]:
```python
print(PSNR(uh,u))
plt.imshow(u, cmap='gray')
```

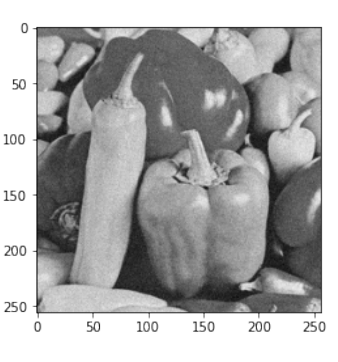
```
33.42813939954505
```

Out[19]: `<matplotlib.image.AxesImage at 0x7f0605d75908>`



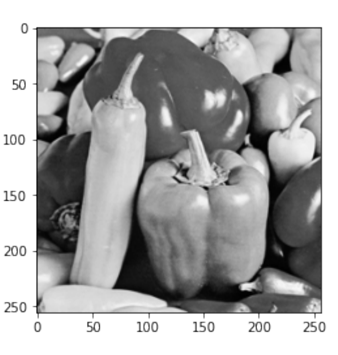
In [ ]:
```python
plt.imshow(u0, cmap='gray')
```

Out[ ]: `<matplotlib.image.AxesImage at 0x7fb3e018a4a8>`



In [20]:
```python
plt.imshow(uh, cmap='gray')
```

Out[20]: `<matplotlib.image.AxesImage at 0x7f0605d5fda0>`



The image u0 is well denoised.