MA2104 Multivariable Calculus

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Vectors

Dot product

If a and b are non-zero vectors, then

$$\text{proj}_b a = \frac{a \cdot b}{||b||^2} b \quad \text{comp}_b a = \frac{a \cdot b}{||b||}$$

Proposition. Let θ be the angle between the vectors a and b with the same initial point. Then,

$$a \cdot b = ||a|| ||b|| \cos\theta$$

Note: Take $|a \cdot b|$ for angle between planes

Useful results:

1. $\text{comp}_c(a+b) = \text{comp}_c(a) + \text{comp}_c(b)$
2. Cauchy-Schwarz inequality: $|a \cdot b| \leq ||a|| ||b||$
3. Triangle inequality: $||a+b|| \leq ||a|| + ||b||$

Cross Product

If θ is the angle between a and b then

$$||a \times b|| = ||a|| ||b|| \sin\theta$$

Properties:

1. $a \times b = -b \times a$
2. $(da) \times b = a \times (db)$
3. $(a+b) \times c = (a \times c) + (b \times b)$

Scalar triple product

$|a \cdot (b \times c)|$ is the volume of the parallelepiped.

Properties:

1. Unchanged under a circular shift: $a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$
2. Swapping the positions of the operators without reordering the operands leaves the result unchanged.
3. Swapping any two operands **negates** the result.
4. If $|a \cdot (b \times c)| = 0$, then a, b and c are coplanar.

Useful result: Product rule works on cross product

Curves and Surfaces

Lines

If $f : R \to R$ is a one-to-one function whose image is all of R, then $R(t) = (a + f(t)x, b + f(t)y, c + f(t)z)$ is a parameterization of a line. An example of such functions is $R(t) = (a + t^3 x, b + t^3 y, c + t^3 z)$

Level sets

The k-level set of f is

$$\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = k\}$$

- If $(x(t), y(t), z(t))$ is a curve along a level set of f. the value of $f(x(t), y(t), z(t))$ does not vary with t

Quadric Surfaces

- Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ (Round-bottom cone)
- Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ (Saddleback)
- Elliptic Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ (Two inverted cones)

Theorem 1.6. If the limit of f and g both exist, then

1. $\lim(f \pm g) = \lim(f) \pm \lim(g)$
2. $\lim(fg) = (\lim(f))(\lim(g))$
3. $\lim(\frac{f}{g}) = \frac{\lim(f)}{\lim(g)}$, provided $\lim(g) \neq 0$

Note: The limit at a point of a function f is unique

Continuity

A function is continuous at (a, b) if the function is defined at (a, b) and the limit exists and equal to $f(a, b)$.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(\triangle x, \triangle y)\to(0,0)} f(a + \triangle x, b + \triangle y) = f(a, b)$$

From **Theorem 1.6**, the continuity of functions are similarly preserved under addition, multiplication, division, and composition.

- Same for differentiability.

Derivatives

Total derivatives

For function f defined on $D \subset \mathbb{R}^2$ and differentiable at (a, b) within the interior of D,

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a, b) - L(h, k)}{\sqrt{h^2 + k^2}} = 0$$

where $L : \mathbb{R}^2 \to \mathbb{R}$ is a linear map that assigns every vector v in $T_{(a,b)}\mathbb{R}^2$ a number $D_{f(a,b)}(v)$ in $\mathbb{R}$.

If f is differentiable at (a, b), the linear map L is given by:

$$L(h, k) = D_{f(a,b)}(h, k) = f_x(a, b)h + f_y(a, b)k.$$

Directional Derivatives

The directional derivative of f at (a, b) in the direction of the **unit vector** u with initial point (a, b) is defined as

$$D_f(u) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

or

$$D_{f(a,b)}(u) = \nabla f(a, b) \cdot u$$

Gradient Vector

The gradient vector is orthogonal to the level curves of a function. ∇f and $-\nabla f$ is the direction of the maximum and minimum rate of change of f respectively.

Notes about Differentiability

Consider f at a point (a, b):

- f_x and f_y exist does not implies differentiability
- f is differentiable $\implies$ f_x and f_y exist
- f is differentiable $\implies$ f is continuous

Implicit Differentiation

The equation of the tangent plane to the surface S at the point (a, b, c):

$$\frac{\partial z}{\partial x}(a, b, c)x + \frac{\partial z}{\partial y}(a, b, c)y - z = \frac{\partial z}{\partial x}a + \frac{\partial z}{\partial y}b - c$$

Defining Extremas

If the domain is **closed** and **bounded**, then the global extremas always exist (Extreme Value Theorem)

- Bounded if there exists a limit to how far you can go in **all** direction.
- Closed if there exists a well-defined boundary in **any** direction.

Note: if (a, b) is a local extremum of f, then all the directional derivatives of f at (a, b) are 0 ($f_x = f_y = 0$).

General Notes

- The continuity of a function necessarily implies that limits exists. But the converse is not true.
- If a point is differentiable, then it lies in the interior of the domain.

Integrating over planar domains

If $D \subset \mathbb{R}^2$ is closed and bounded, and f is continuous on D, then $\iint_D f(x, y)dA$ exists.

(Fubini's theorem) If f is continuous on the rectangle $D = [a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

Let f be a continous function on D, then

$$\iiint_D |f|dV \geq \iiint_D fdV$$

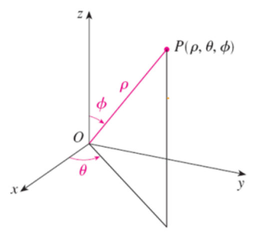
Change of Coordinates

Common Coordinate Systems

Polar Coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad \tan\theta = \frac{y}{x}$$

Spherical Coordinates



$$z = \rho\cos\phi \quad y = \rho\sin\phi\sin\theta \quad x = \rho\sin\phi\cos\theta$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \phi = \cos^{-1}\frac{z}{\rho}$$

$$\int_\alpha^\beta \int_\gamma^\delta \int_{\rho_1}^{\rho_2} f(x, y, z)\rho^2 \sin\phi d\rho d\theta d\phi$$

$$\rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

where

$$\theta = \begin{cases} \tan^{-1}(\frac{y}{x}) & \text{x} > 0 \\ \tan^{-1}(\frac{y}{x}) + \pi & \text{x} < 0 \text{ and } \text{y} \geq 0 \\ \tan^{-1}(\frac{y}{x}) - \pi & \text{x} < 0 \text{ and } \text{y} < 0 \\ \frac{\pi}{2} & \text{x = 0 and y} > 0 \\ -\frac{\pi}{2} & \text{x = 0 and y} < 0 \end{cases}$$

Notes

- The equation $\rho = c$ describes a sphere centered at origin.
- The equation $\theta = c$ describes a half-plane.
- The equation $\phi = c$ describes a cone with vertex at origin.

Plane Transformation

A plane transformation $T : S \to R$ is a differentiable map whose inverse is differentiable.

Jacobian

Let $T : S \to R$ be a planar transformation, where S lies in the uv plane and R lies in the xy-plane. Let A denote the area in the xy plane and A' denote the area in the uv plane. Let f be a two variable function from the xy-plane to R. Then,

$$\int\int_R f(x, y)dA = \int\int_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA'$$

- Derive the values of x and y in terms of u and v. If needed, use implicit differention and/or the inverse function theorem. For example, if $Y = f(x, u)$. Then make sure to calculate $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.
  Note that although the Jacobian can always be inverted in this manner, the resulting Jacobian is still in terms of the 'old' set of variables (x, y, z), so the inverse transformation may still be needed to rewrite the Jacobian in terms of the 'new' set of variables.
- The bounds in the new integral is derived from the uv-plane.
- The inverse map may not be invertible (i.e. not a 1-1 mapping – most likely due to squaring). In such cases, simply use one of the results (based on the bounds given and multiply by 2).

Line Integrals

Functions

Consider a function $f(x, y)$ and a curve C in $\mathbb{R}^2$. The line integral of f along C is defined as

$$\int_c f(x, y)ds = \int_a^b f(x(t), y(t))||R'(t)||dt = \lim_{n\to\infty} \sum_{i=1}^n f(x_i, y_i)\triangle s_i$$

- If F is continous along C, and C is closed and bounded, then the line integral exists.
- if $F = 1$, then the line integral is the arc length of the curve.

Vector Fields

Let **C** $= (C, o)$ be an **oriented** curve in $\mathbb{R}^2$ parameterized by a differentiable map $R(t) = (x(t), y(t)), a \leq t \leq b$, and let F be a continuous vector field defined along C. Then the line integral of $F(x, y)$ along C is:

$$\int_{\mathbf{C}} F(x, y) \cdot dr = \int_a^b F(x(t), y(t)) \cdot R'(t)dt = \int_a^b comp_{R'(t)}F||R'(t)||dt$$

Note that if $-C$ is the curve C with the opposite orientation, then

$$\int_C F(x, y) \cdot dr = -\int_{-C} F(x, y) \cdot dr$$

- Note that $||comp_{R'(t)}F(x(t), y(t))||^2 + ||comp_N F(x(t), y(t))||^2 = ||F||^2$
- If C is the union of finitely many oriented curves, then solve the line integral for each curve and add them up.
- **Double check the orientation**

Conservative Vector Fields

(**Gradient Theorem**) Let **C** = (C, o) be an oriented curve in D parameterized by $R(t) = (x(t), y(t), z(t)), a \leq t \leq b$, and f is a three variable function whose gradient vector is **continuous** along C, then

$$\int_{\mathbf{C}} \nabla f d\mathbf{r} = f(R(b)) - f(R(a))$$

The vector field is conservative $\iff$ every line integrals about **all** oriented loops are 0.

(Test for conservative fields) Suppose $F(x, y, z)$ is a vector field in an **open** and **simply-connected** region D in $\mathbb{R}^2$, and both X, Y and Z have continuous first-order partial derivatives on D. Then,

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}, \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}$$

iff F is a conservative on D.

- Break up the vector fields into conservative and divergence parts, and perform the line integral on each part separately.

(Green's Theorem) Let **C** = (C, o) be a piecewise differentiable, simple loop in $\mathbb{R}^2$ that is oriented counter-clockwise, and let **D be the region bounded by C**. Let $\mathbf{F}(x, y) = X(x, y)i + Y(x, y)j$ be a vector field such that $X(x, y)$ and $Y(x, y)$ have **continuous partial derivatives** on an open region that contains D. Then

$$\int_{\mathbf{C}} F \cdot dr = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dA$$

Note that the "macroscopic" circulation around C is equal to the total "microscopic" circulation if F is defined everywhere inside the domain.

To compute the **area of D**, let $Y = x, X = 0$ or $Y = 0, X = -y$ or $Y = \frac{1}{2}x, X = -\frac{1}{2}y$. We can convert the double integral into a line integral:

$$\text{Area of D} = \iint_D 1 dA = \int_{\mathbf{C}} \begin{pmatrix} X \\ Y \end{pmatrix} \cdot d\mathbf{r}$$

(Outwards Flux) The outwards flux of **F** across C is given by

$$\iint_D \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} dA = \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{\mathbf{C}} G \cdot dr$$

where $F = \begin{pmatrix} X \\ Y \end{pmatrix}$ and $G = \begin{pmatrix} -Y \\ X \end{pmatrix}$

Surface Integrals

A (differentiable) surface is *orientable* if it is possible to define a unit vector with initial point (x, y, z) such that n varies continuously with (x, y, z). The graph of any differentiable function is an orientable surface.

Functions

Let $R : D \to S$ be a (differentiable) parameterization of S.

$$\iint_S f(x, y, z) dS = \iint_D g(u, v) \|R_u \times R_v\| du dv$$
$$= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \triangle S_{ij}$$

where dA is a parameterization of the surface S. When S is the graph of a function, the surface integral is:

$$\iint_D f(x, y, g(x, y)) (\sqrt{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + 1}) dx dy$$

(Surface area) $A(S) = \iint_D \|R_u \times R_v\| du dv$

Vector Fields

If F is a continuous vector field defined on an oriented surface **S** = (S, **n**), then the surface integral of F over S is:

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_{\mathbf{S}} F \cdot dS = \iint_D F \cdot (R_u \times R_v) du dv$$

This integral is also called the flux of F across S.

When S is the graph of a function:

$$\iint_{\mathbf{S}} F \cdot dS = \iint_D (-X \frac{\partial g}{\partial x} - Y \frac{\partial g}{\partial y} + Z) dx dy$$

- Remember to evaluate all piecewise differentiable surfaces separately.
- D is the bounds of the projection
- **Project to the xy-plane, then to theta-r if needed**.

Guass Theorem

Let E be a **solid region** where the **boundary surface** S of E is piecewise smooth, and let S denote S equipped with the **outward orientation**. Let $F(x, y, z)$ be a vector field whose component functions have continuous partial derivatives on an open region that **contains E**. Then,

$$\iiint_E \text{div } \mathbf{F}\, dS = \iint_S \mathbf{F} \cdot \mathbf{n} dV = \iint_{\mathbf{S}} F \cdot dS$$

where

$$\text{div} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

More precisely, the theorem yields the approximation

flux across **S** $\approx$ volume of **E** $\times$ divergence of **F**

- If div $F(P) > 0$, then the outward flux across any small surface enclosing P is positive.
- If div $F(P) < 0$, then the outward flux across any small surface enclosing P is negative.
- Two vector fields, F and G must satisfy $div(F \times G) = G \cdot curlF - F \cdot curlG$.
- Divergence calculates the flux of E. Remember to substract **unneeded** parts.
- If there are holes in the domain, then the divergence of F is zero.
- If the domain is symmetric, then the integral is 0.

Stoke's Theorem

The curl of **F** is defined as

$$curlF = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z})i + (\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x})j + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})k$$

Let S be a **piecewise differentiable surface** bounded by a simple, closed, piecewise-smooth **boundary curve** C. Let n be an orientation on S, let $\mathbf{S} = (S, n)$, and let C be the curve C equipped with the induced orientation. If F is a vector field whose coordinate functions have continuous partial derivatives, the

$$\iint_{\mathbf{S}} \text{curl F} \cdot dS = \int_{\mathbf{C}} F \cdot dr$$

If S_1 and S_2 are two surfaces with the same boundary curve C, then

$$\iint_{\mathbf{S1}} \text{curl F} \cdot dS = \iint_{\mathbf{S2}} \text{curl F} \cdot dS = \int_{\mathbf{C}} F dr$$

Suppose $F = (X(x, y, z), Y(x, y, z), Z(x, y, z))$ and the functions X, Y and Z have continuous second order partial derivatives, then the function F must satisfy $div(curlF) = 0$.

To deal with the concatenation of two curves, look for a surface that fits within the boundaries. Most likely a sphere, hemisphere or a quartersphere, in which case, the flux is 0. For parameterization, **note where the curve lies**.

Parameterization

- Cones

$$R(r, \theta) = (r \cos \theta, r \sin \theta)$$

- Inverted Cone

$$z = -\frac{h}{a} \sqrt{x^2 + y^2} + h$$

- Ellipse

$$R(\theta) = (a \cos \theta, b \sin \theta)$$

- Ellipsoid

$$R(\theta, \phi) = (a \cos \theta \sin \phi, b \sin \theta \sin \phi, c \cos \phi)$$

or set $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$

- Paraboloid

$$R(r, \theta) = (ar \cos \theta, br \sin \theta, r^2)$$

- Cylinder

$$R(r, z) = (\cos \theta, \sin \theta, z)$$

Volume of solids

- Paraboloid

$$\pi(a)(b)$$

- Cones

$$\frac{1}{3} \pi(r^2)(h)$$

Notes

- When calculating actual area/density/volume, usage of $\int_0^{2\pi} f(x) dx$ is discouraged especially with $\sin$ functions.
- Take care of integrating functions in the negative height direction. Make sure to flip the integrals if necessary.
- $\int ye^y \to$ set $u = e^y$. $\int lnu \to ulnu - u$
- div F where $p = x^2 + y^2 + z^2$

$$F = (\frac{x}{(p)^{3/2}}, \frac{y}{(p)^{3/2}}, \frac{z}{(p)^{3/2}})$$

is 0. Instead, to calculate the surface integral, do the following:

Since $\mathbf{F_1}$ is undefined at $(0,0,0)$, we introduce a unit sphere T centered at $(0,0,0)$ and calculate a modified flux. First note that

$$\text{div}(\mathbf{F}) = \frac{3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$
$$= 0.$$

Then

$$\iint_S \mathbf{F_1} \cdot d\mathbf{S} - \iint_T \mathbf{F_1} \cdot d\mathbf{T} = \iiint_{E'} \text{div}(\mathbf{F_1})\ dV = 0$$
$$\therefore \iint_S \mathbf{F_1} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T}$$

A parametrisation of T is given by

$$R(\phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix},\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi.$$

and so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T} = \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} d\phi d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi} 1\ d\phi d\theta$$
$$= 4\pi.$$

Basic Trigonometric Identities		
$sin^2x + cos^2x = 1$	$1 + tan^2x = sec^2x$	$1 + cot^2x = cosec^2x$
Double Angles		
$sin2x = 2sinxcosx$	$cos2x = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$	$tan2x = \frac{2tanx}{1-tan^2x}$

Reduction formula	
$\int \cos^n x dx = \frac{n-1}{n} \int \cos^{n-2} x dx + \frac{\cos^{n-1} x \sin x}{n}$	$\int \sin^n x dx = \frac{n-1}{n} \int \sin^{n-2} x dx - \frac{\cos x \sin^{n-1} x}{n}$
Integration by parts	
$\int u v dx = u \int v dx - \int \frac{\partial u}{\partial x} (\int v dx) dx$	