

# Kalman Filters

For the linear case, we have two models.

The linear dynamics model (how the state evolves over time)

## Linear Dynamics Model

Dynamics model: State undergoes linear transformation plus Gaussian noise

$$\underline{\mathbf{x}}_t \sim N(D_t \mathbf{x}_{t-1}, \Sigma_{d_t})$$

The linear measurement mode

## Linear Measurement Model

Observation model: Measurement is linearly transformed state plus Gaussian noise

$$\mathbf{y}_t \sim N(M_t \mathbf{x}_t, \Sigma_{m_t})$$

## Constant Velocity (1D example)

### Example: Constant velocity (1D)

State vector is position and velocity

$$x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix} \quad p_t = p_{t-1} + (\Delta t)v_{t-1} + \varepsilon$$
$$v_t = v_{t-1} + \xi$$

$$x_t = D_t x_{t-1} + \text{noise} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ v_{t-1} \end{bmatrix} + \text{noise}$$

## Example: Constant velocity (1D)

Measurement is position only

$$y_t = Mx_t + \text{noise} = [1 \quad 0] \begin{bmatrix} p_t \\ v_t \end{bmatrix} + \text{noise}$$

## Constant Acceleration (1D example)

### Example: Constant acceleration (1D)

State vector is position, velocity & acceleration

$$\begin{aligned} x_t = \begin{bmatrix} p_t \\ v_t \\ a_t \end{bmatrix} &\quad p_t = p_{t-1} + (\Delta t)v_{t-1} + \varepsilon \\ &\quad v_t = v_{t-1} + (\Delta t)a_{t-1} + \xi \\ x_t = D_t x_{t-1} + \text{noise} &= \begin{bmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ v_{t-1} \\ a_{t-1} \end{bmatrix} + \text{noise} \end{aligned}$$

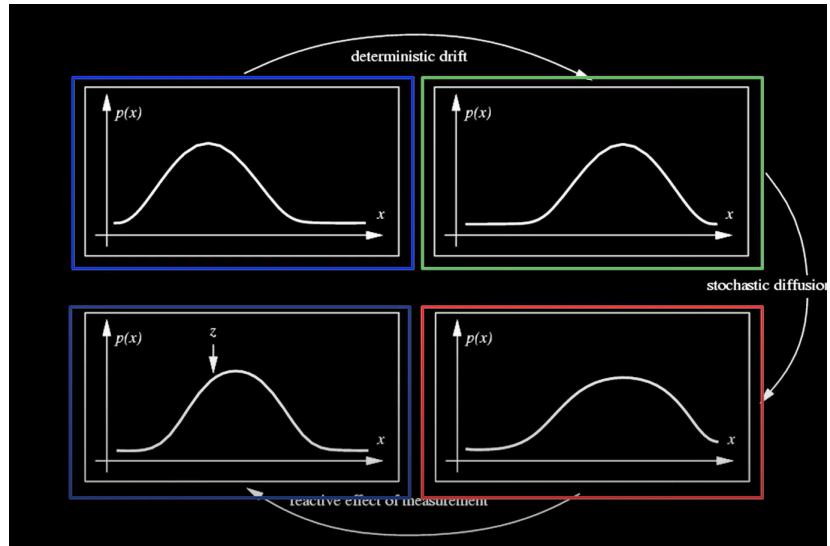
### Example: Constant acceleration (1D)

Measurement is position only

$$y_t = Mx_t + \text{noise} = [1 \quad 0 \quad 0] \begin{bmatrix} p_t \\ v_t \\ a_t \end{bmatrix} + \text{noise}$$

## Kalman Filter

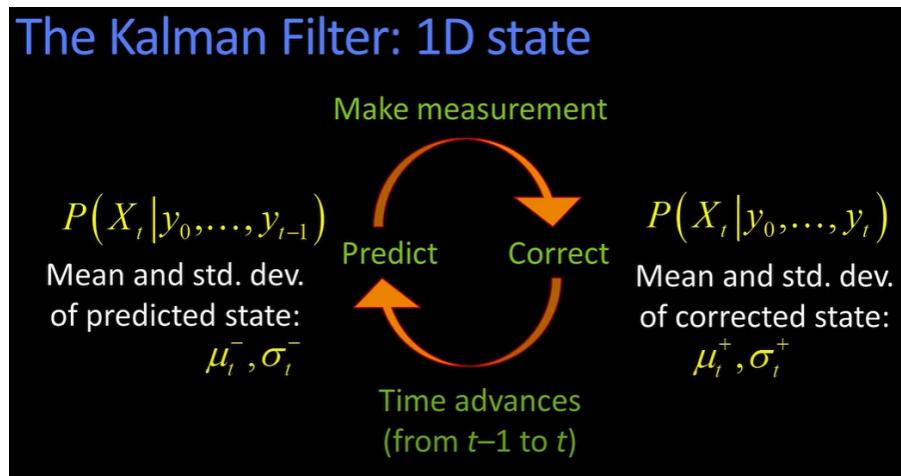
Method to track linear dynamical systems with GAUSSIAN NOISE. Both predicted and corrected states are both gaussian.



So you shift (scale mean) + expand (add noise) + measure + correct + shrink (because new information will only reduce uncertainty)

The first 3 images correspond to prediction.

## Kalman Filter 1D State



Note that prediction uses mu minus and sigma minus (mean and covariance BEFORE I take the measurement, this is based on mu plus and sigma plus at  $t-1$ )

Then after you make measurement and correct mean to mu plus and sigma plus at  $t$

## 1D Kalman Filter Prediction

In this linear dynamics model the next state is gotten by multiplying the previous state by a factor of  $d$  and adding some noise modeled by a gaussian.

## 1D Kalman Filter: Prediction

Linear dynamics model defines predicted state evolution, with noise

$$X_t \sim N(dx_{t-1}, \sigma_d^2)$$

So, if that's the case, the distribution of the next predicted state is also gaussian:

The distribution for next predicted state is also a Gaussian

$$P(X_t | y_0, \dots, y_{t-1}) = N(\mu_t^-, (\sigma_t^-)^2)$$

Crucially note that they are mu and sigma minus because this prediction was made before measurement

And remember that we need to update mu and sigma

Update the mean:

$$\mu_t^- = d\mu_{t-1}^+$$

Update the variance:

$$(\sigma_t^-)^2 = \sigma_d^2 + (d\sigma_{t-1}^+)^2$$

## 1D Kalman Correction

In this linear dynamics model, the measurement is gotten by multiplying the previous state by a factor of m and adding some gaussian noise

## 1D Kalman Filter: Correction

Mapping of state to measurements:

$$Y_t \sim N(mx_t, \sigma_m^2)$$

Now, given our predicted state (that we got above), we need to correct it using the measurement!

Predicted state:  $P(X_t | y_0, \dots, y_{t-1}) = N(\mu_t^-, (\sigma_t^-)^2)$

Want to estimate corrected distribution:

$$P(X_t | y_0, \dots, y_t) = \frac{P(y_t | X_t) P(X_t | y_0, \dots, y_{t-1})}{\int P(y_t | X_t) P(X_t | y_0, \dots, y_{t-1}) dX_t}$$

Now, the corrected distribution is defined to be a new Gaussian with new mean mu plus and sigma plus.

**Kalman:** With linear, Gaussian dynamics and measurements, the corrected distribution to be:

$$P(X_t | y_0, \dots, y_t) \equiv N(\mu_t^+, (\sigma_t^+)^2)$$

Formula for mu plus and sigma plus

Update the mean:

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

Update the variance:

$$\underline{(\sigma_t^+)^2} = \frac{\sigma_m^2 (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

## 1D Kalman Intuition

# 1D Kalman Filter: Intuition

From:

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

What is this?

- *The weighted average of prediction and measurement based on variances!*

**Weights Based on Variances:** The crucial insight is how the weights are determined. They are inversely proportional to the uncertainty (variance) of each estimate:

- The weight for the prediction ( $\mu_t^-$ ) is proportional to the variance of the measurement ( $\sigma_m^2$ ).
- The weight for the measurement-derived estimate ( $y_t/m$ ) is proportional to the variance of the prediction ( $(\sigma_t^-)^2$ ).

This means:

- If the measurement is very certain (small  $\sigma_m^2$ ), the weight for the prediction is smaller, and the weight for the measurement is larger. The filter trusts the measurement more.
- If the prediction is very certain (small  $(\sigma_t^-)^2$ ), the weight for the measurement is smaller, and the weight for the prediction is larger. The filter trusts the prediction more.

## Special Cases

## Prediction vs. correction

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2} \quad (\sigma_t^+)^2 = \frac{\sigma_m^2 (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

What if there is no prediction uncertainty?  $(\sigma_t^- = 0)$

$$\mu_t^+ = \mu_t^- \quad (\sigma_t^+)^2 = 0$$

The measurement is ignored!

What if there is no measurement uncertainty?  $(\sigma_m = 0)$

$$\mu_t^+ = \frac{y_t}{m} \quad (\sigma_t^+)^2 = 0$$

## Simplification of Mu+ and Sigma+

### 1D Kalman Filter: Intuition

Also:  $\mu_t^+ = \frac{\frac{\mu_t^- \sigma_m^2}{m^2} + \frac{y_t}{m} (\sigma_t^-)^2}{\frac{\sigma_m^2}{m^2} + (\sigma_t^-)^2}$

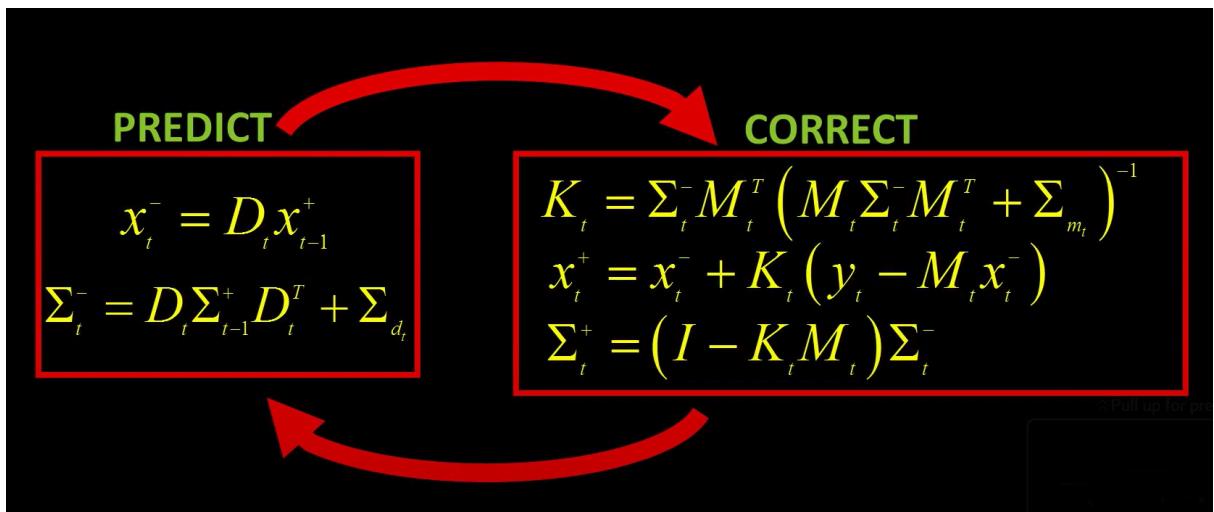
$$\mu_t^+ = \frac{a \mu_t^- + b \frac{y_t}{m}}{a + b} = \frac{(a + b) \mu_t^- + b \left( \frac{y_t}{m} - \mu_t^- \right)}{a + b}$$

$$\mu_t^+ = \frac{a\mu_t^- + b\frac{y_t}{m}}{a+b} = \frac{(a+b)\mu_t^- + b(\frac{y_t}{m} - \mu_t^-)}{a+b}$$

$$\mu_t^+ = \mu_t^- + \frac{b(\frac{y_t}{m} - \mu_t^-)}{a+b} = \mu_t^- + k(\frac{y_t}{m} - \mu_t^-)$$

$\mu_t^-$  is the prediction. The difference (red) is the residual. The residual is the difference between  $y_t/m$  (the measurements guess of  $x$ ) and  $\mu_t^-$ . Something like TD updates.

## N-Dimensional Kalman Filters (the real deal)



$\Sigma_t^-$ : This is the **predicted state covariance matrix** at time  $t$ . It represents the uncertainty in the predicted state  $x_t^-$ . The covariance matrix describes the variances of each state variable (diagonal elements) and the covariances between them (off-diagonal elements).

CORRECT

$$K_t = \Sigma_t^- M_t^T \left( M_t \Sigma_t^- M_t^T + \Sigma_{m_t} \right)^{-1}$$

$$x_t^+ = x_t^- + K_t \boxed{(y_t - M_t x_t^-)}$$

$$\Sigma_t^+ = (I - K_t M_t) \Sigma_t^-$$

Less weight on residual as a priori estimate error covariance approaches zero.

$K_t$  is Kalman Gain The green box is the residual

## Prof Amir's slide

### ALGORITHM EXPLAINED

- The prediction step involves two main equations:
  - State Prediction:  $\hat{x}_{k|k-1} = F \hat{x}_{k-1} + w_{k-1}$ , where  $\hat{x}_{k|k-1}$  is the predicted state at time k given the state at k-1, and  $w_k$  is the process noise, or a random vector drawn from a multivariate normal distribution with mean zero and covariance matrix Q,  $w_k \sim \mathcal{N}(0, Q)$
  - The error covariance prediction:  $P_{k+1} = FP_k F^T + Q$ , where Q is the process noise covariance matrix.
- $Q = \begin{bmatrix} q_x & 0 & q_{xu} & 0 \\ 0 & q_y & 0 & q_{yu} \\ q_{xu} & 0 & q_u & 0 \\ 0 & q_{yu} & 0 & q_v \end{bmatrix}$
- Where  $q_x, q_y, q_u, q_v$  are uncertainty in positions and velocities, and  $q_{xu}, q_{yu}$  are correlation between position's and velocity's uncertainty.

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