

Algebra handout 1 answers

$$\log_A(Rhythm)$$

August 22, 2023

§1 Problems

Problem 1.1 (1983 AIME P.9). Find the minimum value of $\frac{9x^2 \sin^2 x + 4}{x \sin x}$ for $0 < x < \pi$.

Problem 1.2 (2016 AIME I P.1). For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Problem 1.3 (2002 AIME I P.6). The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$

$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2) . Find $\log_{30}(x_1 y_1 x_2 y_2)$.

Problem 1.4 (2005 AIME II P.7). Let $x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

Problem 1.5 (1990 AIME P.15). Find $ax^5 + by^5$ if the real numbers a, b, x , and y satisfy the equations

$$ax + by = 3,$$

$$ax^2 + by^2 = 7,$$

$$ax^3 + by^3 = 16,$$

$$ax^4 + by^4 = 42.$$

Problem 1.6 (2022 DIME P.2). Let $P(x) = x^2 - 1$ be a polynomial, and let a be a positive real number satisfying

$$P(P(P(a))) = 99.$$

The value of a^2 can be written as $m + \sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Problem 1.7 (2000 AMC 12 P.12). Let A, M , and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C$?

Problem 1.8 (2021 DIME P.9). Real numbers a , b , c , and d satisfy the system of equations

$$\begin{aligned} -a - 27b - 8d &= 1, \\ 8a + 64b + c + 27d &= 0, \\ 27a + 125b + 8c + 64d &= 1, \\ 64a + 216b + 27c + 125d &= 8. \end{aligned}$$

Find $12a + 108b + 48d$.

Problem 1.9 (2008 HMNT P.7). Find all ordered pairs (x, y) such that

$$(x - 2y)^2 + (y - 1)^2 = 0.$$

Problem 1.10 (2013 AMC 12B P.17). Let a, b , and c be real numbers such that

$$\begin{aligned} a + b + c &= 2, \text{ and} \\ a^2 + b^2 + c^2 &= 12 \end{aligned}$$

What is the difference between the maximum and minimum possible values of c ?