Algebra handout 1

$\log_A(Rhythm)$

August 22, 2023

§1 Problems

Problem 1.1 (1983 AIME P.9)**.** Find the minimum value of $\frac{9x^2\sin^2 x+4}{x\sin x}$ for $0<x<\pi$.

Problem 1.2 (2016 AIME I P.1)**.** For $-1<r<1$, let $S(r)$ denote the sum of the geometric series

$$12+12r+12r^2+12r^3+\cdots.$$

Let a between -1 and 1 satisfy $S(a)S(-a)=2016$. Find $S(a)+S(-a)$.

Problem 1.3 (2002 AIME I P.6)**.** The solutions to the system of equations

$$\begin{aligned}\log_{225} x+\log_{64} y&=4\\ \log_x 225-\log_y 64&=1\end{aligned}$$

are (x_1,y_1) and (x_2,y_2). Find $\log_{30}(x_1y_1x_2y_2)$.

Problem 1.4 (2005 AIME II P.7)**.** Let $x=\frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

Problem 1.5 (1990 AIME P.15)**.** Find ax^5+by^5 if the real numbers a,b,x, and y satisfy the equations

$$\begin{aligned}ax+by&=3,\\ ax^2+by^2&=7,\\ ax^3+by^3&=16,\\ ax^4+by^4&=42.\end{aligned}$$

Problem 1.6 (2022 DIME P.2)**.** Let $P(x)=x^2-1$ be a polynomial, and let a be a positive real number satisfying

$$P(P(P(a)))=99.$$

The value of a^2 can be written as $m+\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 1.7 (2000 AMC 12 P.12)**.** Let A,M, and C be nonnegative integers such that $A+M+C=12$. What is the maximum value of $A\cdot M\cdot C+A\cdot M+M\cdot C+A\cdot C$?

Problem 1.8 (2021 DIME P.9)**.** Real numbers a, b, c, and d satisfy the system of equations

$$\begin{aligned} -a - 27b - 8d &= 1, \\ 8a + 64b + c + 27d &= 0, \\ 27a + 125b + 8c + 64d &= 1, \\ 64a + 216b + 27c + 125d &= 8. \end{aligned}$$

Find $12a + 108b + 48d$.

Problem 1.9 (2008 HMNT P.7)**.** Find all ordered pairs (x, y) such that

$$(x - 2y)^2 + (y - 1)^2 = 0.$$

Problem 1.10 (2013 AMC 12B P.17)**.** Let a, b, and c be real numbers such that

$$\begin{aligned} a + b + c &= 2, \text{ and} \\ a^2 + b^2 + c^2 &= 12 \end{aligned}$$

What is the difference between the maximum and minimum possible values of c?