Algebra handout 1 answers

$\log_A(Rhythm)$

September 16, 2023

§1 Problems

Problem 1.1 (1983 AIME P.9)**.** Find the minimum value of $\frac{9x^2\sin^2 x+4}{x\sin x}$ for $0 < x < \pi$.

answer. 12
Let $y = x\sin x$. We can rewrite the expression as $\frac{9y^2+4}{y} = 9y + \frac{4}{y}$.

Since $x > 0$, and $\sin x > 0$ because $0 < x < \pi$, we have $y > 0$. So we can apply AM-GM:

$$9y + \frac{4}{y} \geq 2\sqrt{9y \cdot \frac{4}{y}} = 12$$

The equality holds when $9y = \frac{4}{y} \iff y^2 = \frac{4}{9} \iff y = \frac{2}{3}$.

Therefore, the minimum value is $\boxed{12}$. This is reached when we have $x\sin x = \frac{2}{3}$ in the original equation (since $x\sin x$ is continuous and increasing on the interval $0 \leq x \leq \frac{\pi}{2}$, and its range on that interval is from $0 \leq x\sin x \leq \frac{\pi}{2}$, this value of $\frac{2}{3}$ is attainable by the Intermediate Value Theorem). $\square$

Problem 1.2 (2016 AIME I P.1)**.** For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

answer. 336
The sum of an infinite geometric series is $\frac{a}{1-r} \to \frac{12}{1\mp a}$. The product $S(a)S(-a) = \frac{144}{1-a^2} = 2016$. $\frac{12}{1-a} + \frac{12}{1+a} = \frac{24}{1-a^2}$, so the answer is $\frac{2016}{6} = \boxed{336}$. $\square$

Problem 1.3 (2002 AIME I P.6)**.** The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$
$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1y_1x_2y_2)$.

answer. 12
Let $A = \log_{225} x$ and let $B = \log_{64} y$.

From the first equation: $A + B = 4 \Rightarrow B = 4 - A$.

Plugging this into the second equation yields $\frac{1}{A} - \frac{1}{B} = \frac{1}{A} - \frac{1}{4-A} = 1 \Rightarrow A = 3 \pm \sqrt{5}$ and thus, $B = 1 \pm \sqrt{5}$.

So, $\log_{225}(x_1x_2) = \log_{225}(x_1) + \log_{225}(x_2) = (3+\sqrt{5}) + (3-\sqrt{5}) = 6 \Rightarrow x_1x_2 = 225^6 = 15^{12}$.

And $\log_{64}(y_1y_2) = \log_{64}(y_1) + \log_{64}(y_2) = (1+\sqrt{5}) + (1-\sqrt{5}) = 2 \Rightarrow y_1y_2 = 64^2 = 2^{12}$.

Thus, $\log_{30}(x_1y_1x_2y_2) = \log_{30}\left(15^{12}\cdot 2^{12}\right) = \log_{30}\left(30^{12}\right) = \boxed{12}$.

□

Problem 1.4 (2005 AIME II P.7)**.** Let $x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

answer. 125

We note that in general,

$(\sqrt[2^n]{5}+1)(\sqrt[2^n]{5}-1) = (\sqrt[2^n]{5})^2 - 1^2 = \sqrt[2^{n-1}]{5} - 1$.

It now becomes apparent that if we multiply the numerator and denominator of $\frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$ by $(\sqrt[16]{5}-1)$, the denominator will telescope to $\sqrt[1]{5} - 1 = 4$, so

$x = \frac{4(\sqrt[16]{5}-1)}{4} = \sqrt[16]{5} - 1$.

It follows that $(x+1)^{48} = (\sqrt[16]{5})^{48} = 5^3 = \boxed{125}$.

□

Problem 1.5 (1990 AIME P.15)**.** Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

answer. 20

Set $S = (x+y)$ and $P = xy$. Then the relationship

$$(ax^n + by^n)(x+y) = (ax^{n+1} + by^{n+1}) + (xy)(ax^{n-1} + by^{n-1})$$

can be exploited:

$$\begin{aligned} (ax^2 + by^2)(x+y) &= (ax^3 + by^3) + (xy)(ax + by) \\ (ax^3 + by^3)(x+y) &= (ax^4 + by^4) + (xy)(ax^2 + by^2) \end{aligned}$$

Therefore:

$$\begin{aligned} 7S &= 16 + 3P \\ 16S &= 42 + 7P \end{aligned}$$

Consequently, $S = -14$ and $P = -38$. Finally:

$$\begin{aligned} (ax^4 + by^4)(x+y) &= (ax^5 + by^5) + (xy)(ax^3 + by^3) \\ (42)(S) &= (ax^5 + by^5) + (P)(16) \\ (42)(-14) &= (ax^5 + by^5) + (-38)(16) \\ ax^5 + by^5 &= \boxed{20} \end{aligned}$$

□

Problem 1.6 (2022 DIME P.2)**.** Let $P(x) = x^2 - 1$ be a polynomial, and let a be a positive real number satisfying

$$P(P(P(a))) = 99.$$

The value of a^2 can be written as $m + \sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

answer. 12
First, we obtain $P(a) = a^2 - 1$. Upon plugging in this value into the polynomial again, we obtain

$$P(P(a)) = (a^2 - 1)^2 - 1 = (a^2 - 1 + 1)(a^2 - 1 - 1) = a^2(a^2 - 2) = a^4 - 2a^2.$$

Finally, upon plugging in this value into the polynomial again, we obtain

$$\begin{aligned} P(P(P(a))) &= (a^4 - 2a^2)^2 - 1 \\ &= (a^4 - 2a^2 + 1)(a^4 - 2a^2 - 1) \\ &= (a^2 - 1)^2((a^2 - 1)^2 - 2) \\ &= (a^2 - 1)^4 - 2(a^2 - 1)^2. \end{aligned}$$

Setting this equal to 99 and letting $y = a^2 - 1$, we get $y^4 - 2y^2 = 99$. Adding 1 to both sides of the equation gives us

$$y^4 - 2y^2 + 1 = 100 \implies (y^2 - 1)^2 = 100 \implies (a^4 - 2a^2)^2 = 100 \implies a^4 - 2a^2 = \pm 10.$$

Next, by the quadratic formula, we obtain

$$a^2 = \frac{2 \pm \sqrt{4 \pm 40}}{2} = 1 \pm \sqrt{11}.$$

Since a^2 is positive, we have that $a^2 = 1+\sqrt{11}$, so the requested answer is $1+11 = \boxed{12}$. □

Problem 1.7 (2000 AMC 12 P.12)**.** Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C$?

answer. It is not hard to see that

$$(A + 1)(M + 1)(C + 1) =$$

$$AMC + AM + AC + MC + A + M + C + 1$$

Since $A + M + C = 12$, we can rewrite this as

$$(A + 1)(M + 1)(C + 1) =$$

$$AMC + AM + AC + MC + 13$$

So we wish to maximize

$$(A + 1)(M + 1)(C + 1) - 13$$

Which is largest when all the factors are equal (consequence of AM-GM). Since $A + M + C = 12$, we set $A = M = C = 4$ Which gives us

$$(4 + 1)(4 + 1)(4 + 1) - 13 = 112$$

so the answer is $\boxed{112}$. □

Problem 1.8 (2021 DIME P.9)**.** Real numbers a, b, c, and d satisfy the system of equations

$$\begin{aligned} -a - 27b - 8d &= 1, \\ 8a + 64b + c + 27d &= 0, \\ 27a + 125b + 8c + 64d &= 1, \\ 64a + 216b + 27c + 125d &= 8. \end{aligned}$$

Find $12a + 108b + 48d$.

answer. 12
Notice that all four equations satisfy

$$f(x) = ax^3 + b(x+2)^3 + c(x-1)^3 + d(x+1)^3 = (x-2)^3.$$

Then, we expand to get

$$a + b + c + d = 1$$

$$6b - 3c + 3d = -6$$

$$12b + 3c + 3d = 12$$

$$8b - c + d = -8.$$

Simplify the middle two equations to get

$$a + b + c + d = 1$$

$$2b - c + d = -2$$

$$4b + c + d = 4$$

$$8b - c + d = -8.$$

Solving this gives $(a, b, c, d) = (-6, -1, 4, 4)$, so the answer is $12(-6) + 108(-1) + 48(4) = \boxed{12}$. □

Problem 1.9 (2008 HMNT P.7)**.** Find all ordered pairs (x, y) such that

$$(x - 2y)^2 + (y - 1)^2 = 0.$$

answer. (2,1)
The square of a real number is always nonnegative. Hence, $y - 1$ and $x - 2y$ must both equal 0 for $(x - 2y)^2 + (y - 1)^2 = 0$ to be true. The only pair that satisfies the condition is $\boxed{(2, 1)}$. □

Problem 1.10 (2013 AMC 12B P.17)**.** Let a, b, and c be real numbers such that

$$a + b + c = 2, \text{ and}$$

$$a^2 + b^2 + c^2 = 12$$

What is the difference between the maximum and minimum possible values of c?

answer. $\frac{16}{3}$
From the given, we have

$$a + b = 2 - c$$

$$a^2 + b^2 = 12 - c^2$$

This immediately suggests use of the Cauchy-Schwarz inequality. By Cauchy, we have

$$2\left(a^2 + b^2\right) \geq (a + b)^2$$

Substitution of the above results and some algebra yields

$$3c^2 - 4c - 20 \leq 0$$

This quadratic inequality is easily solved, and it is seen that equality holds for $c = -2$ and $c = \frac{10}{3}$.

The difference between these two values is $\boxed{\dfrac{16}{3}}$. $\square$