NUMBER THEORY handout 1 answers

$\log_A(Rhythm)$

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§1 Problems

Problem 1.1 (2020 AIME II P.1). Find the number of ordered pairs of positive integers (m, n) such that $m^2n = 20^{20}$.

answer. 231
Because $20^{20} = 2^{40}5^{20}$ (Prime factorisation), if $m^2n = 20^{20}$, there must be nonnegative integers a, b, c, and d such that $m = 2^a5^b$ and $n = 2^c5^d$. Then

$$2a + c = 40$$

and

$$2b + d = 20$$

The first equation has 21 solutions corresponding to $a = 0, 1, 2, \ldots, 20$, and the second equation has 11 solutions corresponding to $b = 0, 1, 2, \ldots, 10$. Therefore there are a total of $21 \cdot 11 = 231$ ordered pairs (m, n) such that $m^2n = 20^{20}$. □

Problem 1.2 (2019 AIME I P.1). Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{321 \text{ digits}}.$$

Find the sum of the digits of N.

answer. 342
Write

$$\begin{aligned} N &= (10 - 1) + (10^2 - 1) + \cdots + (10^{321} - 1) \\ &= 10 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6 + \cdots 10^{321} - 321 \\ &= 1110 - 321 + 10^4 + 10^5 + 10^6 + \cdots + 10^{321} \\ &= 789 + 10^4 + 10^5 + 10^6 + \cdots + 10^{321} \end{aligned}$$

The sum of the digits of N is therefore equal to $7 + 8 + 9 + (321 - 3) = 342$. □

Problem 1.3 (1995 AIME P.6). Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

answer. 589
We know that $n^2 = 2^{62}3^{38}$ must have $(62+1)\times(38+1)$ factors by its prime factorization. If we group all of these factors (excluding n) into pairs that multiply to n^2, then one factor per pair is less than n, and so there are $\frac{63\times39-1}{2} = 1228$ factors of n^2 that are less than n. There are $32\times20-1 = 639$ factors of n, which clearly are less than n, but are still factors of n. Therefore, using complementary counting, there are $1228-639 = \boxed{589}$ factors of n^2 that do not divide n. □

Problem 1.4 (2021 DIME P.1)**.** Find the remainder when the number of positive divisors of the value

$$(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$$

is divided by 1000.

answer. 783
Let x be equal to 3^{2020}. Thus $(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$ is equal to $(x+3x)(3x+9x)(9x+27x)(27x+81x)$, or

$$(3+1)x\cdot(3+9)x\cdot(9+27)x\cdot(27+81)x = 4\cdot12\cdot36\cdot108\cdot3^{8080}.$$

Finding the prime factorization of our expression gives

$$4\cdot12\cdot36\cdot108\cdot3^{8080} = 2^2\cdot(2^2\cdot3)\cdot(2^2\cdot3^2)\cdot(2^2\cdot3^3)\cdot3^{8080} = 2^8\cdot3^{8086}.$$

For each divisor, we can choose a power of 2 from 0 to 8 and a power of 3 from 0 to 8086, so there are $(8+1)(8086+1)$ divisors of $2^8\cdot3^{8086}$. Thus our answer is congruent to

$$9\cdot8087 \equiv 9\cdot87 \equiv \boxed{783} \pmod{1000}.$$

□

Problem 1.5 (2023 MBMT P.7)**.** What is the largest integer n such that 3^n is a factor of $18!+19!+20!$?

answer. 8
We first notice that each of the terms is divisible by 18! so we can factor it out. This gives us $18!+19!+20! = 18!(1+19+19\cdot20) = 18!\ \cdot(400)$. 400 has no factors of 3, so we just count the factors of 3 in 18!. There is 1 factor of 3 from 3, 6, 12, 15, and 2 factors of 3 from 9, 18. This adds to a total of 8 factors of 3, thus the answer is 8. □

Problem 1.6 (2022 BMT P.2)**.** Compute the number of positive integer divisors of 100000 which do not contain the digit 0.

answer. 11
Note that $100000 = 2^5\cdot5^5$. Any multiple of 10 ends in a 0, so a divisor of $2^5\cdot5^5$ that does not contain a 0 is either not divisible by 2 or not divisible by 5. We consider the cases separately.
• A divisor of $2^5\cdot5^5$ that is not divisible by 5 will not contain the prime factor 5, and thus it is either 1 or only contains the prime factor 2, so it must be a power of 2. The powers of 2 up to 25 are 1, 2, 4, 8, 16, and 32.
• A divisor of $2^5\cdot5^5$ that is not divisible by 2 will not contain the prime factor 2, so it is either 1 or only contains the prime factor 5, and thus it must be a power of 5. The powers of 5 up to 5^5 are 1, 5, 25, 125, 625, and 3125.
Since 1 appears in both lists, the total number of divisors of $2^5\cdot5^5$ that do not contain a 0 is $6+6-1 = 11$.

□

Problem 1.7 (2022 IOQM P.6)**.** Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

answer. 43

$$a^3 - b^3 - ab = 25$$
$$a^3 - b^3 = 25 + ab$$
$$(a - b)(a^2 + ab + b^2) = 25 + ab$$
$$(a - b)(a^2 + b^2) + (a - b)ab = 25 + ab$$

By observing second term from LHS & RHS we conclude that $a - b = 1$ $(a, b) = (4, 3)$ only possible which leads the answer to 43 □

Problem 1.8 (2007 PAMO P.5)**.** For which positive integers n is $231^n - 222^n - 8^n - 1$ divisible by 2007?

answer. $2007 = 9 \times 223$. So taking mod 9, we have $6^n - 6^n - (-1)^n - 1$. So n is odd. Taking mod 223, we have $8^n - (-1)^n - 8^n - 1$. So n is odd again □

Problem 1.9 (1983 AIME P. 8)**.** What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

answer. 61
Expanding the binomial coefficient, we get $\binom{200}{100} = \frac{200!}{100!100!}$. Let the required prime be p; then $10 \le p < 100$. If $p > 50$, then the factor of p appears twice in the denominator. Thus, we need p to appear as a factor at least three times in the numerator, so $3p < 200$. The largest such prime is $\boxed{61}$, which is our answer. □

Problem 1.10 (1984 AIME P.2)**.** The integer n is the smallest positive multiple of 15 such that every digit of n is either 8 or 0. Compute $\frac{n}{15}$.

answer. 592
Any multiple of 15 is a multiple of 5 and a multiple of 3.

Any multiple of 5 ends in 0 or 5; since n only contains the digits 0 and 8, the units digit of n must be 0.

The sum of the digits of any multiple of 3 must be divisible by 3. If n has a digits equal to 8, the sum of the digits of n is $8a$. For this number to be divisible by 3, a must be divisible by 3. We also know that $a > 0$ since n is positive. Thus n must have at least three copies of the digit 8.

The smallest number which meets these two requirements is 8880. Thus the answer is $\frac{8880}{15} = \boxed{592}$. □