The mathematics foundation of Navier-Stokes Equations

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1 Introduction

1.1 Notations

The well-posedness of Navier-Stokes equations

- $\Omega \subset \mathbb{R}^d$
- $T > 0$
- velocity field $u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix}, \Omega \times [0,T] \to \mathbb{R}^d$
- Initial value $u(x,0) \equiv u_0(x) : \Omega \to \mathbb{R}^d$

1.2 Incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f & \qquad \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0 & \qquad \text{in } \Omega \times (0,T), \\ u = 0 & \qquad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \qquad \text{in } \Omega. \end{aligned}$$

Question:
When $d = 3, f \equiv 0$ given u_0 there exists a (u,p), both smooth and globally defined and solve the N-S equations?

The goal of the lecture is as following:

1. The well-posedness (weak solution, strong solution, regularity)
2. Proof of unique existence.
   (a) Stationary problem $\begin{cases} \text{Stokes} \\ \text{Navier-Stokes,} \end{cases}$
   (b) Time-dependent $\begin{cases} \text{Stokes} \\ \text{Navier-Stokes} \end{cases}$ $\leftarrow$ compactness theorem.

2 The steady-state Stokes equation

Assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with sufficient regularity (e.g. bounded Lipschitz). Let $\Gamma = \partial\Omega$. The steady Stokes equations read

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \quad (2.1a) \\ \nabla \cdot u = 0 & \text{in } \Omega, \quad (2.1b) \\ u = 0 & \text{on } \Gamma. \quad (2.1c) \end{cases}$$

where $\nu > 0$ is the kinematic viscosity and

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} : \Omega \to \mathbb{R}^d, \qquad f \in H^{-1}(\Omega)^d = \left(H_0^1(\Omega)^d\right)^*.$$

Observation. Suppose (2.1) admits a smooth solution. Testing (2.1a) by $v \in C_0^\infty(\Omega)^d$,

$$\int_\Omega -\nu\Delta u \cdot v \,\mathrm{d}x + \int_\Omega \nabla p \cdot v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x.$$

By integrating by parts, the left-hand side becomes

$$\underbrace{\int_{\partial\Omega} -\nu\frac{\partial u}{\partial n} \cdot v \,\mathrm{d}s}_{=0} + \int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} p\,(v\cdot n)\,\mathrm{d}s}_{=0} - \int_\Omega p\,\nabla\cdot v \,\mathrm{d}x.$$

Thus, (2.1a)–(2.1b) can be transformed into

$$\begin{cases} \displaystyle\int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x - \int_\Omega p\,\nabla\cdot v \,\mathrm{d}x = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)^d, & (2.2a) \\ \displaystyle\int_\Omega q\,\nabla\cdot u \,\mathrm{d}x = 0 \qquad \forall q \in L_0^2(\Omega), & (2.2b) \end{cases}$$

where $\langle f, v\rangle$ denotes the duality pairing between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$ (which reduces to $\int_\Omega f \cdot v\,dx$ when $f \in L^2(\Omega)^d$). The weak formulation is

$$\text{Find} \quad u \in H_0^1(\Omega)^d, \quad p \in L_0^2(\Omega) \quad \text{s.t.} \quad (2.2) \text{ holds},$$

where

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q\,dx = 0 \right\}.$$

Consider the solenoidal test functions

$$\mathcal{V} = \{\, v \in C_0^\infty(\Omega)^d : \nabla\cdot v = 0 \,\}.$$

Testing (2.2a) with $v \in \mathcal{V}$ eliminates the pressure term:

$$\nu \int_\Omega \nabla u : \nabla v\,dx = \langle f, v\rangle \qquad \forall\, v \in \mathcal{V}.$$

Define the velocity space and its divergence-free subspace

$$X := H_0^1(\Omega)^d, \qquad V := \{\, v \in X : \nabla\cdot v = 0 \,\}.$$

Question:

$$V = \overline{\mathcal{V}}^{H_0^1(\Omega)^d} \;?$$

(For bounded Lipschitz domains, the above density property holds; hence testing on $\mathcal{V}$ is equivalent to testing on V.)

Weak form

$$\text{Find} \quad u \in V, \quad \text{s.t.} \quad a(u,v) = \langle f, v\rangle, \quad \forall v \in V, \tag{2.3}$$

where

$$a(u,v) = \nu \int_\Omega \nabla u : \nabla v \, dx.$$

Since $\mathcal{V}$ is dense in V (in the H_0^1-norm), (2.3) is equivalently characterized by $a(u,v) = \langle f, v\rangle$ for all $v \in \mathcal{V}$.

Equation (2.3) is well-posed (existence + uniqueness):

- $a(\cdot,\cdot)$ is an inner product on V;
- Lax–Milgram:
  - $a(\cdot,\cdot)$ is continuous on $V \times V$:
    $$|a(u,v)| \leqslant \nu \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)};$$
  - $a(\cdot,\cdot)$ is coercive on V:
    $$a(v,v) = \nu \|\nabla v\|^2_{L^2(\Omega)} \geq C\|v\|^2_{H^1(\Omega)},$$
    by Poincaré's inequality $\|v\|_{L^2(\Omega)} \leq C_P \|\nabla v\|_{L^2(\Omega)}$ for $v \in H_0^1(\Omega)^d$.

$$\begin{tikzcd}
\begin{array}{c}\text{PDE Stokes}\\ (2.1)\end{array} \arrow[d] \arrow[dr, "\nabla\cdot v = 0"] & \\
\boxed{\begin{array}{c}\text{Weak form}\\ (2.2)\end{array}} \arrow[r, shift left, "\nabla\cdot v = 0"] & \boxed{\begin{array}{c}\text{Weak form}\\ (2.3)\end{array}} \arrow[l, shift left, "\begin{array}{c}\exists\, p \in L^2_0(\Omega)\\ \text{``de Rham / inf--sup''}\end{array}"]
\end{tikzcd}$$

Theorem 1 (de Rham Theorem)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and let $h \in \left(C_0^\infty(\Omega)^d\right)^* = \mathcal{D}'(\Omega)^d$. Then there exists $p \in \mathcal{D}'(\Omega)$ such that $h = \nabla p$ in $\mathcal{D}'(\Omega)^d$ if and only if*

$$\langle h, v\rangle = 0, \qquad \forall\, v \in \mathcal{V}.$$

From (2.3), define the functional

$$F(v) := \langle f, v\rangle - \nu \int_\Omega \nabla u : \nabla v \, dx, \qquad v \in C_0^\infty(\Omega)^d.$$

Then

$$F(v) = 0, \qquad \forall v \in \mathcal{V}.$$

By de Rham's theorem, there exists $p \in \mathcal{D}'(\Omega)$ such that

$$F(v) = \langle \nabla p, v\rangle, \qquad \forall v \in C_0^\infty(\Omega)^d.$$

Equivalently,

$$\nu \int_\Omega \nabla u : \nabla v \, dx - \langle \nabla p, v\rangle = \langle f, v\rangle, \qquad \forall v \in C_0^\infty(\Omega)^d.$$

Question: $p \in L_0^2(\Omega)$?

Proposition 2. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded, Lipschitz.*

1. *If $p \in \mathcal{D}'(\Omega)$ and $\nabla p \in L^2(\Omega)^d$, then $p \in L^2(\Omega)$.*
2. *If $p \in \mathcal{D}'(\Omega)$ and $\nabla p \in H^{-1}(\Omega)^d$, then $p \in L^2(\Omega)$ and*
$$\frac{\|p\|_{L^2(\Omega)}}{R} \le C\|\nabla p\|_{H^{-1}(\Omega)},$$
where R denotes a characteristic length scale of Ω (e.g. its diameter).

Moreover,

$$\|\nabla p\|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)^d} \frac{\langle \nabla p, v\rangle}{\|v\|_{H^1(\Omega)}} = \sup_{v\in H_0^1(\Omega)^d} \frac{\langle f, v\rangle - \nu \int_\Omega \nabla u : \nabla v\, dx}{\|v\|_{H^1(\Omega)}} \le C\big(\|f\|_{H^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\big).$$

In particular, since p is determined up to an additive constant, we may choose the representative with zero mean, i.e. $p \in L_0^2(\Omega)$.

(2.2) $\iff$ Find $(u,p) \in X \times Q$ such that

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle, & \forall v \in X = H_0^1(\Omega)^d, \\ b(u,q) = 0, & \forall q \in Q = L_0^2(\Omega). \end{cases}$$

and

$$a(u,v) = \nu \int_\Omega \nabla u : \nabla v\, dx, \qquad b(v,p) = -\int_\Omega (\nabla\cdot v)\, p\, dx, \qquad \langle f, v\rangle = \langle f, v\rangle.$$

Moreover,

- $a(\cdot,\cdot)$ is continuous and coercive on V (and on X);
- $b(\cdot,\cdot)$ satisfies the inf–sup condition on $X \times Q$.

Definition 3 (Inf-sup condition)**.**

$$\inf_{p\in Q} \sup_{v\in X} \frac{b(v,p)}{\|v\|_{H_0^1(\Omega)^d}\, \|p\|_{L^2(\Omega)}} \ \ge\ \beta > 0.$$

Theorem 4. *If $a(\cdot,\cdot)$ is continuous and coercive on V (or on X), and $b(\cdot,\cdot)$ satisfies the above inf–sup condition, then problem* (2.2) *admits a unique solution $(u,p) \in X \times Q$.*

Proof. 1. **(Velocity in the divergence-free subspace).** Find $u \in V$ such that
$$a(u,v) = \langle f, v\rangle \qquad \forall v \in V.$$
Since $a(\cdot,\cdot)$ is continuous and coercive on V, by the Lax–Milgram lemma, this problem admits a unique solution $u \in V$.

2. **(Recovery of the pressure).** Define the functional
$$F(v) := \langle f, v\rangle - a(u,v), \qquad v \in X.$$
By construction, $F(v) = 0$ for all $v \in V$ (in particular for all $v \in \mathcal{V}$).
By the inf–sup condition, there exists a unique $p \in Q = L_0^2(\Omega)$ such that
$$b(v,p) = F(v) = \langle f, v\rangle - a(u,v) \qquad \text{for all } v \in X.$$
Hence (u,p) satisfies
$$a(u,v) + b(v,p) = \langle f, v\rangle, \qquad \forall v \in X,$$
and since $u \in V$, we also have
$$b(u,q) = 0, \qquad \forall q \in Q.$$
Therefore (u,p) solves (2.2).
Uniqueness follows from coercivity of a on V and the inf–sup condition for b.

□

3 Steady Navier–Stokes equations

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \qquad \text{in } \Omega, \tag{1.5a}$$
$$\nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{1.5b}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.5c}$$

- Nonlinear system; a key tool is fixed–point theory.

Observation. Suppose (1.5) admits a smooth solution (u,p). Test (1.5a) by $v \in C_0^\infty(\Omega)^d$:

$$\int_\Omega \big(-\nu\Delta u + (u\cdot\nabla)u + \nabla p\big)\cdot v\,dx = \int_\Omega f\cdot v\,dx =: \langle f, v\rangle.$$

Integrating by parts, the left-hand side becomes

$$\underbrace{\int_\Omega \nu\nabla u : \nabla v\,dx}_{=:a(u,v)} + \underbrace{\int_\Omega (u\cdot\nabla)u\cdot v\,dx}_{=:a_1(u,u,v)} - \underbrace{\int_\Omega p\,(\nabla\cdot v)\,dx}_{=:b(v,p)} + \underbrace{\int_{\partial\Omega}\big(-\nu\tfrac{\partial u}{\partial n}\cdot v - p\,v\cdot n\big)\,ds}_{=0}.$$

Here the boundary term vanishes since $v\in C_0^\infty(\Omega)^d$.

Remark.

$$u\cdot\nabla = \begin{pmatrix} u_1\\ \vdots \\ u_d\end{pmatrix}\cdot\begin{pmatrix}\partial_{x_1}\\ \vdots\\ \partial_{x_d}\end{pmatrix} = u_1\frac{\partial}{\partial x_1} + \cdots + u_d\frac{\partial}{\partial x_d}.$$

Define the trilinear form

$$a_1(u,v,w) := \int_\Omega (u\cdot\nabla)v\cdot w\,dx.$$

Divergence equation. Test (1.5b) by $q\in L^2(\Omega)$:

$$\int_\Omega (\nabla\cdot u)\,q\,dx = 0 \quad\Longrightarrow\quad b(u,q)=0.$$

Well-definedness of the convection term. Assume $u,v\in H_0^1(\Omega)^d$. Then for $d\le 4$,

$$\int_\Omega (u\cdot\nabla u)\cdot v\,dx \quad \text{is well-defined and finite.}$$

Indeed, using Hölder's inequality,

$$\left|\int_\Omega (u\cdot\nabla u)\cdot v\,dx\right| \le \int_\Omega |u|\,|\nabla u|\,|v|\,dx \le \|\nabla u\|_{L^2(\Omega)}\,\|uv\|_{L^2(\Omega)}.$$

Moreover,

$$\|uv\|^2_{L^2(\Omega)} = \int_\Omega |uv|^2\,dx \le \left(\int_\Omega |u|^4\,dx\right)^{\frac12}\left(\int_\Omega |v|^4\,dx\right)^{\frac12} = \|u\|^2_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)}.$$

Question. Does $v \in H_0^1(\Omega)^d$ imply $v \in L^4(\Omega)^d$? That is,

$$H_0^1(\Omega)^d \subset L^4(\Omega)^d \ ?$$

Sobolev embeddings (bounded Lipschitz domains).

- $d = 2:$ $\quad H_0^1(\Omega) \subset L^p(\Omega)$ for all $1 \le p < \infty$;
- $d = 3:$ $\quad H_0^1(\Omega) \subset L^6(\Omega)$;
- $d = 4:$ $\quad H_0^1(\Omega) \subset L^4(\Omega)$;
- $d > 4:$ $\quad H_0^1(\Omega) \subset L^{\frac{2d}{d-2}}(\Omega)$.

Hence for $d = 2, 3, 4$ we have $H_0^1(\Omega)^d \subset L^4(\Omega)^d$, and the convection term $a_1(u, u, v)$ is well-defined for $u, v \in H_0^1(\Omega)^d$.

Weak form for $d \le 4$. For $d \le 4$: find $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$a(u, v) + a_1(u, u, v) + b(v, p) = \langle f, v \rangle, \qquad \forall\, v \in H_0^1(\Omega)^d, \tag{1.6a}$$

$$b(u, q) = 0, \qquad \forall\, q \in L_0^2(\Omega). \tag{1.6b}$$

Assume $\Omega \subset \mathbb{R}^d$ is bounded, Lipschitz and connected. Define

$$X := H_0^1(\Omega)^d, \qquad V := \{v \in X : \nabla\cdot v = 0 \text{ in } \Omega\}, \qquad \mathcal{V} := \{v \in C_0^\infty(\Omega)^d : \nabla\cdot v = 0\},$$

so that (for bounded Lipschitz domains)

$$V = \overline{\mathcal{V}}^{H_0^1(\Omega)^d}.$$

From (1.6a)–(1.6b) we are led to the reduced problem:

$$\text{Find } u \in V \text{ such that } \ a(u, v) + a_1(u, u, v) = \langle f, v \rangle \qquad \forall v \in V. \tag{1.7}$$

We have "(1.6) $\Rightarrow$ (1.7)" and conversely.

Suppose (1.7) has a solution $u \in V$. Define a functional $h \in H^{-1}(\Omega)^d \equiv X^*$ by

$$\langle h, v \rangle := \langle f, v \rangle - a(u, v) - a_1(u, u, v), \qquad \forall v \in X.$$

From (1.7) we obtain

$$\langle h, v \rangle = 0 \qquad \forall v \in V.$$

By de Rham's theorem (equivalently, the Helmholtz decomposition in H^{-1}), there exists $p \in L^2(\Omega)/\mathbb{R}$, and choosing the zero–mean representative we have a unique $p \in L_0^2(\Omega)$ such that

$$h = \nabla p \quad \text{in } H^{-1}(\Omega)^d.$$

Hence, for any $v \in X$,

$$\langle h, v \rangle = \langle \nabla p, v \rangle = -\int_\Omega p\, \nabla\cdot v \, dx =: b(v, p),$$

and therefore, by the definition of h,

$$a(u, v) + a_1(u, u, v) + b(v, p) = \langle f, v \rangle \qquad \forall v \in X,$$

i.e. (1.6a) holds. Since $u \in V$, i.e. $\nabla\cdot u = 0$, we also have (1.6b). Thus

$$(1.7) \Longrightarrow (1.6), \qquad \text{and hence } (1.7) \Longleftrightarrow (1.6).$$

Therefore, it suffices to prove the well-posedness of (1.7).

Lemma 5. *The trilinear form $a_1(\cdot,\cdot,\cdot)$ is continuous on*

$$H_0^1(\Omega)^d \times H_0^1(\Omega)^d \times \big(H_0^1(\Omega)^d \cap L^d(\Omega)^d\big),$$

i.e. there exists $C > 0$ such that

$$|a_1(u,v,w)| \le C\,\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}\,\big(\|w\|_{H^1(\Omega)} + \|w\|_{L^d(\Omega)}\big).$$

Proof. First consider $d = 2$. Using Hölder's inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$,

$$|a_1(u,v,w)| = \left|\int_\Omega (u\cdot\nabla v)\cdot w\,dx\right| \le \|u\|_{L^4}\,\|\nabla v\|_{L^2}\,\|w\|_{L^4} \le C\|u\|_{H^1}\|v\|_{H^1}\|w\|_{H^1}.$$

For $d \ge 3$, by Hölder's inequality,

$$\left|\int_\Omega (u\cdot\nabla v)\cdot w\,dx\right| \le \|u\|_{L^{\frac{2d}{d-2}}}\,\|\nabla v\|_{L^2}\,\|w\|_{L^d}.$$

Note that

$$\frac{d-2}{2d} + \frac{1}{2} + \frac{1}{d} = 1.$$

By Sobolev embeddings, $H_0^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$, hence

$$|a_1(u,v,w)| \le C\|u\|_{H^1}\|v\|_{H^1}\|w\|_{L^d},$$

which proves the continuity on the stated product space. □

Lemma 6. *For $d \le 4$, the trilinear form $a_1(\cdot,\cdot,\cdot)$ is continuous on $H_0^1(\Omega)^d \times H_0^1(\Omega)^d \times H_0^1(\Omega)^d$.*

Indeed, for $d \le 4$ we have $H_0^1(\Omega) \subset L^d(\Omega)$, hence $H_0^1(\Omega)^d \cap L^d(\Omega)^d = H_0^1(\Omega)^d$.

Lemma 7. *For $d \le 4$ and $u \in V$ (i.e. $\nabla\cdot u = 0$ in Ω),*

$$a_1(u,v,v) = 0 \qquad \forall v \in H_0^1(\Omega)^d \cap L^d(\Omega)^d,$$

and in fact

$$a_1(u,v,w) = -\,a_1(u,w,v) \qquad \forall v,w \in H_0^1(\Omega)^d \cap L^d(\Omega)^d.$$

Proof. We compute

$$2\,a_1(u,v,v) = 2\int_\Omega (u\cdot\nabla v)\cdot v\,dx = \int_\Omega u\cdot\nabla(|v|^2)\,dx = \int_\Omega \nabla\cdot\big(u\,|v|^2\big)\,dx - \int_\Omega (\nabla\cdot u)\,|v|^2\,dx.$$

Since $u \in V$, $\nabla\cdot u = 0$ and $u \in H_0^1(\Omega)^d$ implies that the trace vanishes on $\partial\Omega$, we have $u|_{\partial\Omega} = 0$ and hence $u\cdot n = 0$ on $\partial\Omega$. Therefore,

$$\int_\Omega \nabla\cdot\big(u\,|v|^2\big)\,dx = \int_{\partial\Omega} (u\cdot n)\,|v|^2\,ds = 0,$$

which implies $a_1(u,v,v) = 0$. The skew-symmetry then yields $a_1(u,v,w) = -a_1(u,w,v)$. □

Theorem 8 (Compactness theorem)**.** *If Ω is bounded, open and Lipschitz, then*

$$W^{1,p}(\Omega) \hookrightarrow L^{q_1}(\Omega)$$

is a compact embedding for any exponent q_1 satisfying

$$\begin{cases} 1 \le q_1 < \infty, & p \ge d,\\[4pt] 1 \le q_1 < q < \dfrac{dp}{d-p}, & 1 \le p < d. \end{cases}$$

Remark. *In particular, if* $\{u_n\}_{n\geq 1} \subset W^{1,p}(\Omega)$ *with* $\|u_n\|_{W^{1,p}} \leq C$*, then there exists a subsequence* u_{n_k} *and* $u \in L^{q_0}(\Omega)$ *such that*

$$u_{n_k} \to u \quad \text{in } L^{q_0}(\Omega).$$

For $d \leq 4$ and $f \in H^{-1}(\Omega)^d$ we consider again

$$\text{Find } u \in V \text{ such that } a(u,v) + a_1(u,u,v) = \langle f, v\rangle \qquad \forall v \in V. \tag{1.7}$$

Theorem 9. *Problem* (1.7) *admits a solution* $u \in V$*.*

Proof. The basic idea is as follows.

- Step 1: construct a Galerkin approximation of (1.7) in a finite-dimensional subspace of V;
- Step 2: use compactness to extract a convergent subsequence and pass to the limit, obtaining a solution $u \in V$ of (1.7).

□

Step 1. Construction of finite–dimensional subspaces. The space V is a closed subspace of $H_0^1(\Omega)^d$, hence it is separable. Therefore there exists a basis $\{w_1, w_2, \dots\}$ of V and we can set

$$V_m := \operatorname{span}\{w_1, \dots, w_m\}, \qquad m = 1, 2, \dots,$$

with $\dim V_m = m$, $V_m \subset V$ and

$$V = \overline{\bigcup_{m=1}^{\infty} V_m}^{H_0^1(\Omega)^d}.$$

We seek an approximate solution of (1.7) in V_m.

Step 2. Galerkin problem. For each $m \in \mathbb{N}$, find

$$u_m = \sum_{k=1}^{m} \xi_k^{(m)} w_k \in V_m, \qquad \xi_k^{(m)} \in \mathbb{R},$$

such that

$$a(u_m, w_k) + a_1(u_m, u_m, w_k) = \langle f, w_k\rangle, \qquad k = 1, \dots, m. \tag{1.11}$$

Equivalently,

$$\text{find } u_m \in V_m \text{ such that } a(u_m, v) + a_1(u_m, u_m, v) = \langle f, v\rangle \quad \forall v \in V_m.$$

Problem (1.11) admits at least one solution $u_m \in V_m$ (by a finite-dimensional fixed point theorem).

Step 3. A priori estimate. Taking $v = u_m$ in the Galerkin equation and using Lemma 7 (so $a_1(u_m, u_m, u_m) = 0$), we obtain

$$a(u_m, u_m) = \langle f, u_m\rangle.$$

By coercivity of $a(\cdot,\cdot)$ and the estimate $\|u_m\|_{H^1(\Omega)^d} \simeq \|\nabla u_m\|_{L^2(\Omega)}$ (Poincaré inequality), we deduce

$$\|u_m\|_{H^1(\Omega)^d} \leq C\|f\|_{H^{-1}(\Omega)^d}.$$

Hence $\{u_m\}_{m=1}^{\infty}$ is bounded in $H_0^1(\Omega)^d$, and thus also bounded in V.

By the compact embedding (Theorem 8), we may extract a subsequence (still denoted u_m) and a function $u \in H_0^1(\Omega)^d$ such that

$$u_m \rightharpoonup u \quad \text{in } H_0^1(\Omega)^d, \qquad u_m \to u \quad \text{in } L^{q_1}(\Omega)^d,$$

where one may choose, for instance,

$$\begin{cases} 1 \leq q_1 < \infty, & d = 2, \\ 1 \leq q_1 < 6, & d = 3, \\ 1 \leq q_1 < 4, & d = 4. \end{cases}$$

Step 4. Passage to the limit. For each m and for all $v \in V_m$ we have

$$a(u_m, v) + a_1(u_m, u_m, v) = \langle f, v \rangle.$$

Fix any $k \in \mathbb{N}$ and take $v = w_k$. For all sufficiently large $m \geq k$ we have $w_k \in V_m$ and hence

$$a(u_m, w_k) + a_1(u_m, u_m, w_k) = \langle f, w_k \rangle.$$

Passing to the limit along the subsequence, using $u_m \rightharpoonup u$ in H_0^1, the continuity of $a(\cdot, \cdot)$, and the strong convergence $u_m \to u$ in L^{q_1} together with $\nabla u_m \rightharpoonup \nabla u$ in L^2, we obtain

$$a(u, w_k) + a_1(u, u, w_k) = \langle f, w_k \rangle.$$

Since $\{w_k\}$ spans a dense subspace of V, this identity extends to all $v \in V$, i.e.

$$a(u, v) + a_1(u, u, v) = \langle f, v \rangle \qquad \forall v \in V.$$

Finally, as V is weakly closed in $H_0^1(\Omega)^d$ and each $u_m \in V$, we conclude $u \in V$. Hence u solves (1.7).

Remark (convergence of the nonlinear term). For each fixed $v \in V$, one may justify

$$\int_\Omega (u_m \cdot \nabla u_m) \cdot v \, dx \longrightarrow \int_\Omega (u \cdot \nabla u) \cdot v \, dx$$

by writing

$$\int_\Omega (u_m \cdot \nabla u_m) \cdot v \, dx - \int_\Omega (u \cdot \nabla u) \cdot v \, dx = \int_\Omega \big((u_m - u) \cdot \nabla u_m\big) \cdot v \, dx + \int_\Omega \big(u \cdot \nabla(u_m - u)\big) \cdot v \, dx,$$

and then using $u_m \to u$ strongly in $L^{q_1}(\Omega)^d$ (with $q_1 \geq 4$ in $d = 2$, or $q_1 > 3$ in $d = 3$, or $q_1 < 4$ in $d = 4$ together with interpolation), together with ∇u_m bounded in $L^2(\Omega)^{d \times d}$ and $v \in H_0^1(\Omega)^d \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)^d$ for $d \geq 3$ (or $v \in L^p$ for any $p < \infty$ when $d = 2$).

Uniqueness. Suppose $u_1, u_2 \in V$ are two solutions of (1.7). Let

$$w := u_1 - u_2 \in V.$$

For any $v \in V$ we have

$$a(u_1, v) + a_1(u_1, u_1, v) = \langle f, v \rangle = a(u_2, v) + a_1(u_2, u_2, v),$$

hence

$$a(w, v) + a_1(u_1, u_1, v) - a_1(u_2, u_2, v) = 0.$$

Using the trilinearity of a_1,

$$a_1(u_1, u_1, v) - a_1(u_2, u_2, v) = a_1(u_1, w, v) + a_1(w, u_2, v),$$

so

$$a(w, v) + a_1(u_1, w, v) + a_1(w, u_2, v) = 0 \qquad \forall v \in V.$$

Choosing $v = w$ and using Lemma 7 (so $a_1(u_1, w, w) = 0$), we obtain

$$a(w, w) + a_1(w, u_2, w) = 0, \qquad \text{i.e.} \qquad a(w, w) = -a_1(w, u_2, w).$$

By the continuity of a_1 (Lemma 6),

$$|a_1(w, u_2, w)| \leq C_N \|w\|_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} = C_N \|u_2\|_{H^1(\Omega)} \|w\|^2_{H^1(\Omega)}.$$

On the other hand, by coercivity of a (Poincaré inequality),

$$a(w,w) = \nu \|\nabla w\|^2_{L^2(\Omega)} \geq c\,\|w\|^2_{H^1(\Omega)},$$

for some $c > 0$ depending on ν and Ω. Hence

$$c\|w\|^2_{H^1(\Omega)} \leq |a(w,w)| = |a_1(w,u_2,w)| \leq C_N \|u_2\|_{H^1(\Omega)} \|w\|^2_{H^1(\Omega)}.$$

If

$$c - C_N \|u_2\|_{H^1(\Omega)} > 0,$$

for instance if $\|u_2\|_{H^1(\Omega)}$ (hence $\|f\|_{H^{-1}}$) is small enough or the viscosity ν is large enough, then necessarily $\|w\|_{H^1(\Omega)} = 0$, so $u_1 = u_2$. Thus the steady Navier–Stokes problem has at most one solution under a small–data (or large–viscosity) condition.

4 Nonstationary Stokes equations

Let $u(x,t)$ be the velocity and $p(x,t)$ the pressure. Consider

$$u_t - \nu\Delta u + \nabla p = f \qquad \text{in } \Omega \times (0,T), \tag{4.1}$$
$$\nabla\cdot u = 0 \qquad \text{in } \Omega \times (0,T), \tag{4.2}$$
$$u = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.3}$$
$$u(x,0) = u_0(x) \qquad \text{in } \Omega. \tag{4.4}$$

Assume

$$f \in L^2(0,T;H^{-1}(\Omega)^d), \qquad u_0 \in H := \overline{\{v \in C_0^\infty(\Omega)^d : \nabla\cdot v = 0\}}^{L^2(\Omega)^d}.$$

Let

$$X := H_0^1(\Omega)^d, \qquad V := \{v \in X : \nabla\cdot v = 0 \text{ in } \Omega\}.$$

Weak form (S). Find

$$u \in L^2(0,T;V), \qquad u_t \in L^2(0,T;V'),$$

such that

$$\begin{cases} \dfrac{d}{dt}(u(t),v) + a(u(t),v) = \langle f(t),v\rangle, \quad \forall v \in V, \text{ a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{S}$$

Here

$$a(u,v) = \nu \int_\Omega \nabla u : \nabla v \, dx, \qquad (\cdot,\cdot) \text{ denotes the } L^2(\Omega)^d \text{ inner product.}$$

Using integration by parts one checks as before that for smooth solutions

$$(\nabla p, v) = -\int_\Omega p \, \nabla\cdot v \, dx + \int_{\partial\Omega} p \, v \cdot n \, ds = 0 \qquad \forall v \in V.$$

Theorem 10. *There exists a unique solution*

$$u \in L^2(0,T;V) \cap C([0,T];H)$$

satisfying the weak formulation (S).

Sketch of proof.
- Step 1: Galerkin approximation;
- Step 2: a priori estimates;
- Step 3: passage to the limit.

Step 1 (Galerkin approximation). Let $\{w_i\}_{i=1}^\infty$ be an orthonormal basis of H with $w_i \in V$. Set $V_m = \mathrm{span}\{w_1,\dots,w_m\}$ and $u_{0m} := P_{V_m} u_0$. Seek an approximate solution of the form

$$u_m(x,t) = \sum_{j=1}^m g_j(t) w_j(x) \in V_m,$$

such that

$$\frac{d}{dt}(u_m(t),v) + a(u_m(t),v) = \langle f(t),v\rangle, \qquad \forall v \in V_m, \text{ a.e. } t \in (0,T),$$

with $u_m(0) = u_{0m}$.

Taking $v = w_i$ $(i = 1,\dots,m)$ and using $(w_j,w_i) = \delta_{ij}$, we obtain the linear ODE system

$$g_i'(t) + \sum_{j=1}^m a(w_j,w_i)\, g_j(t) = \langle f(t), w_i\rangle, \qquad i = 1,\dots,m,$$

with initial condition $g_i(0) = (u_{0m}, w_i)$. Classical ODE theory yields a unique solution $g(\cdot) \in C^1([0,T];\mathbb{R}^m)$, hence a unique $u_m \in C^1([0,T];V_m)$.

Step 2 (A priori estimates). Take $v = u_m(t)$ in the Galerkin equation. Since

$$\frac{d}{dt}(u_m, u_m) = 2(u_{m,t}, u_m),$$

we get the energy identity

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + a(u_m(t), u_m(t)) = \langle f(t), u_m(t)\rangle.$$

Using Cauchy–Schwarz in $V' \times V$ and Young's inequality,

$$\langle f(t), u_m(t)\rangle \le \|f(t)\|_{V'}\|u_m(t)\|_V \le \frac{1}{2c}\|f(t)\|^2_{V'} + \frac{c}{2}\|u_m(t)\|^2_V,$$

where $c > 0$ is the coercivity constant of $a(\cdot,\cdot)$ on V, i.e. $a(v,v) \ge c\|v\|^2_V$ (here $\|v\|_V := \|\nabla v\|_{L^2(\Omega)}$). Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + \frac{c}{2}\|u_m(t)\|^2_V \le \frac{1}{2c}\|f(t)\|^2_{V'}.$$

Integrating from 0 to T yields

$$\|u_m(T)\|^2_{L^2(\Omega)} + c\int_0^T \|u_m(t)\|^2_V\, dt \le \|u_m(0)\|^2_{L^2(\Omega)} + \frac{1}{c}\int_0^T \|f(t)\|^2_{V'}\, dt.$$

In particular, $\{u_m\}$ is bounded in

$$L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V).$$

Step 3 (Passage to the limit). By the bounds above, we may extract a subsequence (still denoted u_m) and a function u such that

$$u_m \rightharpoonup u \quad \text{in } L^2(0,T;V), \qquad u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2(\Omega)^d).$$

Moreover, from the Galerkin equation one also obtains a uniform bound of $\{u_{m,t}\}$ in $L^2(0,T;V')$, so (by standard compactness arguments) we may pass to the limit in the variational formulation: for all $v \in V$ and all $\varphi \in C_c^\infty(0,T)$,

$$-\int_0^T (u(t), \varphi'(t)v)\, dt + \int_0^T a(u(t), \varphi(t)v)\, dt = \int_0^T \langle f(t), \varphi(t)v\rangle\, dt.$$

This shows that u satisfies (S) in the distributional sense in time, with $u(0) = u_0$. Finally, the regularity $u \in C([0,T];H)$ follows from the standard Lions–Magenes lemma (for $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V')$). Uniqueness follows from the energy estimate applied to the difference of two solutions. □