Stochastic Simulation
Spring 2025 March 4, 2025

Homework #3: Input and Output Analysis
Due: March 20, noon

Part I: Theoretic Questions
#1 **(Input Analysis)** Consider the shifted (two-parameter) exponential distribution, which has density function

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-(x-\gamma)/\beta} & \text{if } x \geq \gamma \\ 0 & \text{otherwise} \end{cases},$$

for $\beta > 0$ and any real number γ. Given a sample $X_1, X_2, \ldots, X_n$ of i.i.d. random values from this distribution, find formulas for the joint MLEs $\hat{\gamma}$ and $\hat{\beta}$. *Hint: Remember that γ cannot exceed any X_i.*

#2 **(Output Analysis)** Let p be the probability of success of some random experiment. We want to estimate the mean number of experiments to obtain the first success, i.e. $1/p$. Suppose we have n i.i.d. outcomes $X_1, ..., X_n$ of the random experiment via simulation. In detail, $X_i = 1$(the ith experiment succeeds). Derive an estimate for $1/p$ from the simulation data and a 95% confidence interval using Delta method.

Part II: Numerical Experiments
For the following problem, report your numerical results with properly designed tables or graphs and clearly explain the results with a few sentences. Put your .py file (or Jupyter notebook files) in a single zip file and submit it via Blackboard. In case you use any data input, you should also include your .csv files or other type of files in your submission. Please name your file `Lastname_StudentID.zip`. Also, give meaningful names to your decision variables and constraints, and add comments to your code liberally.

#3 **(Confidence Interval)** Recall that in the lecture we have derive the variance of AR(1) time series:

$$Z_n = aZ_{n-1} + \varepsilon_n, \quad \varepsilon_n \overset{i.i.d.}{\sim} N(0,1) \text{ and } Z_1 \sim N(0, 1/(1-a^2)).$$

We will use this model to test and compare the following different methods for constructing confidence intervals (CI) for $\bar{Z}_n$:

(1). Construct CI using the *true value of the limiting variance* $\sigma^2 = \frac{1+a}{1-a} \cdot \frac{1}{1-a^2}$.

(2). Construct CI using *sample variance*, which is actually wrong according to our discussion on the lecture.

(3). Construct CI using *sectioning method* without using information of the limiting variance.

Let's take $a = 0.5$ and repeat the following procedure for 1000 rounds:

1. Generate a sequence $Z_1, ..., Z_n$ with $n = 500$;
2. Compute the 95% CIs for $\bar{Z}_n$ using the three approaches. (For sectioning method, you might need to choose the batch size by some trial runs.)
3. Count if the CI cover the true value $z = 0$ for each approach.

Report the proportion of times when the true value is covered by the CI for each approach in 1000 rounds of simulation, along with your analysis and evaluation on the three approaches.

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