Mathematics

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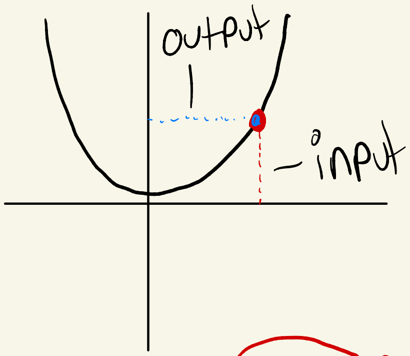
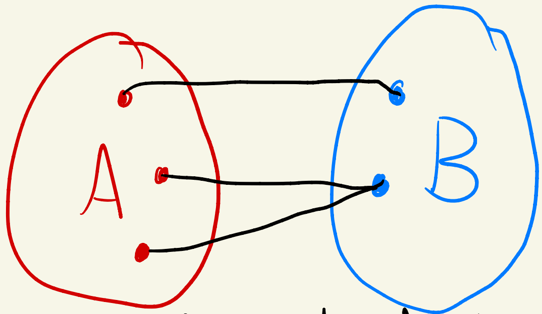
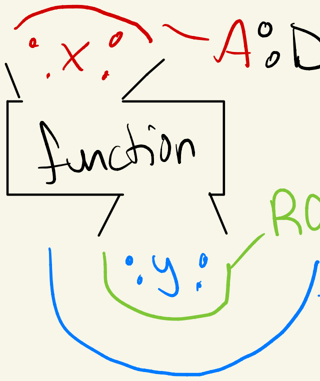
2.2 Limit of a function

$f(x)$: a relationship that assigns each element in set A
to exactly one element in set B







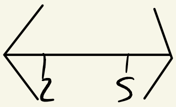
A: Domain - set of all eligible inputs

Range

B: Co-Domain

Notations $f: A \to B$
$x \mapsto y = f(x)$

$\mathbb{R}$ - set of all real #'s



$f: \boxed{I} \to \boxed{\mathbb{R}}$
integer rational #

Real valued functions

Note ① Classification of Real Valued Functions

(a) algebraic functions

ex) Polynomial functions
- Constant ex) $y=3$
- linear ex) $y=x-5$ highest power $=1$
- quadratic ex) $y=2x^2-3x+5$
- Cubic ex) $y=\frac{1}{3}x^3-2x^2+5x-2$

Rational # = $\frac{\text{integer}}{\text{integer} \neq 0}$

Rational function = $\frac{\text{Poly func}}{\text{Poly func} \neq 0}$

(b) Trancendental function = non algebraic function

ex) Trignometric, Exponential, Log

(C) Piecewise functions - Different functions at different parts of domain

ex) $y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Note ②

(a) Numerical representation using a table

ex) a.

x	0	1	-1	3
y	$\frac{1}{2}$	10	$\frac{1}{2}$	-5

(b) Symbolic representation

ex) $y = 3x - 7$

$f(x) = 3x - 7$

(C) Graphical representation

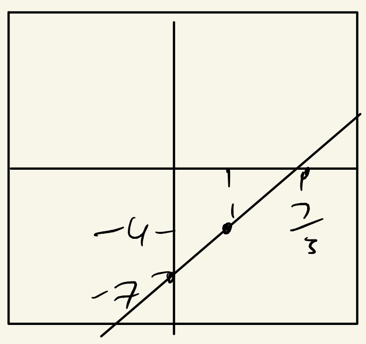
$y = f(x)$ = set of all solutions (x, y) satisfying $y = f(x)$

$y = 3x - 7$

One solution: $x = 0, x = -7 \rightarrow (0, -7)$

One solution: $x = 1, y = -4 \rightarrow (1, -4)$

One Solution: $x = \frac{7}{3}, y = 0 \rightarrow (\frac{7}{3}, 0)$



Note ① Limit of a function: Numerical Approach

ex) $f(x) = \frac{x^3-1}{x-1}$ $f(1)$: Undefined (UD) - can be used on test

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
y	2.313	2.71	2.97	2.997	UD	3.003	3.03	3.31	3.813

As x approaches 1 from the left, f(x) approaches 3

$$\lim_{x \to 1^-} f(x) = 3$$

Left-Handed Limit

As x approaches 1 from the right, f(x) approaches 3

$$\lim_{x \to 1^+} f(x) = 3$$

Right-handed Limit

★ ∃ = exist Definition

if $\exists \lim_{x \to c^-} f(x) = \exists \lim_{x \to c^+} f(x)$,

$\qquad \| \qquad \qquad \|$

$\qquad \quad L$

then the limit of f(x) as x approaches c, exists and write...

$$\exists \lim_{x \to c} f(x) = L$$

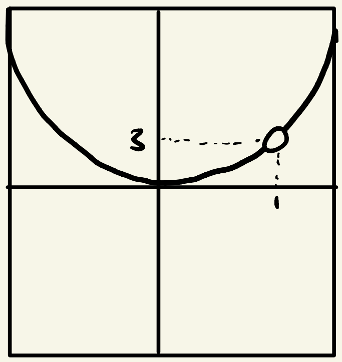
Two-Sided Limit

Note ① If either of the one-sided limits DNE (Does not exist) then limit DNE

② If $\exists \lim_{x\to c^-} f(x) \neq \exists \lim_{x\to c^+} f(x)$ then $\lim_{x\to c} f(x) = DNE$

Note Ⅱ Limit of function : Graphical approach

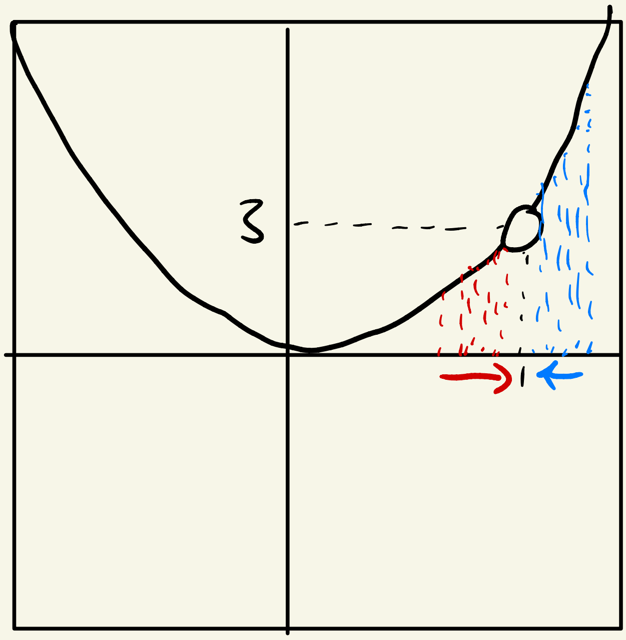
ex) $f(x) = \frac{x^3-1}{x-1}$ $f(1) = UD$



$\lim_{x\to 1^-} f(x) = 3$

$\lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1} f(x) = 3$



2.3 Limit Laws

Note III Limit of a function: Algebraic Approach

① Limit Laws

Let $\exists \lim_{x \to c} f(x)$ & $\exists \lim_{x \to c} g(x)$

① $\lim_{x \to c} f(x) \pm g(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$ || Sum/Difference Rule

② $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$ || Constant Multiple rule

③ $\lim_{x \to c} (f(x))(g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ || Product rule

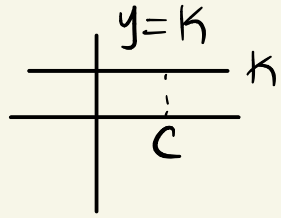
④ $\lim_{x \to c} (f(x))^k = (\lim_{x \to c} f(x))^k$ || Power rule

⑤ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ || Quotient rule

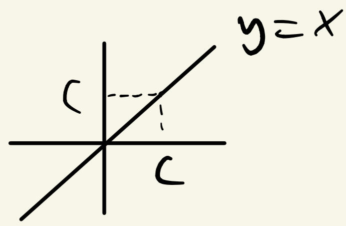
② Direct substitution

(a) $\lim_{x \to c} k = k$

constant



(b) $\lim_{x \to c} x = c$



(c) $\lim_{x \to c} x^k = (\lim_{x \to c} x)^k = c^k$

③ For a quotient of 2 function w/ indeterminate form $\frac{0}{0}$!

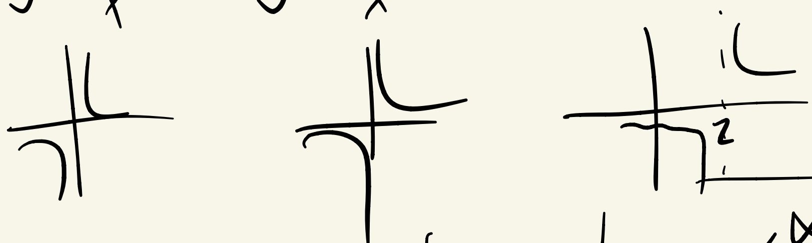
- Use dividing out technique

ex) factorization, rationalization, trig identities

④ For a quotient of two functions of indeterminate form $\frac{\text{constant}}{0}$!

ex) $\lim_{x \to 2} \frac{3}{x-2} = \frac{3}{0}$!

$y = \frac{1}{x} \to y = \frac{3}{x} \to y = \frac{3}{x-2}$



(C) algebraic approach

$\lim_{x \to 2} \frac{3}{x-2} = \frac{3}{0}$!

★ $<\begin{matrix} \infty \\ -\infty \end{matrix}$ are types of DNE

★ $\frac{\text{constant}}{0}$ form for one sided limit $<\begin{matrix} \infty \\ -\infty \end{matrix}$

two sided limit $<\begin{matrix} \infty \\ \text{DNE} \\ -\infty \end{matrix}$

$\lim_{x \to 2^-} \frac{3}{x-2} = \frac{3}{1-2} = \boxed{-} 3 = -\infty$

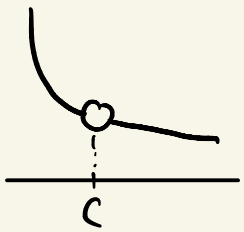
$\lim_{x \to 2^+} \frac{3}{x-2} = \frac{3}{3-2} = \boxed{+} 3 = +\infty$

$\lim_{x \to 2} \frac{3}{x-2} = <\begin{matrix} -\infty \\ \boxed{\text{DNE}} \\ +\infty \end{matrix}$ LH $\neq$ RH

2.4 Continuity

Note ① Types of discontinuity of f(x) at $x=c$

Removable Hole

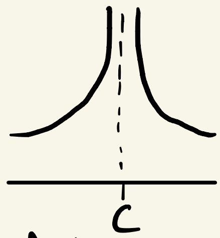


$f(c)$: UD

$\exists \lim_{x \to c} f(x)$

removable

Blown up

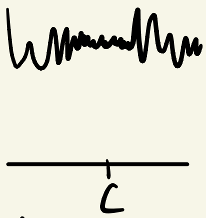


$f(c)$: UD

$\nexists \lim_{x \to c} f(x)$

non-removable

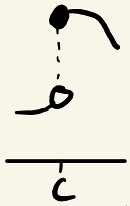
Oscillating



$f(c)$: UD

$\nexists \lim_{x \to c} f(x)$

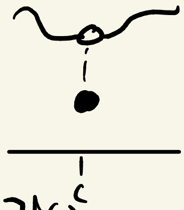
non-removable



$\exists f(c)$

$\nexists \lim_{x \to c} f(x)$

non-removable



$\exists f(c)$

$\exists \lim_{x \to c} f(x)$ $\Big\} \neq$

removable

Definition: $f(x)$ is continuous at $x=c$ if...

① $\exists \lim_{x \to c} f(x)$

② $\exists f(c)$

③ $\lim_{x \to c} f(x) = f(c)$

If one of 3 conditions are unsatisfied, then $f(x)$ is discontinuous at $x=c$

If function is continuous, direct substitution works

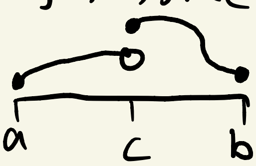
Note ②

① Non-Piecewise functions are continuous over the domain

$\parallel$

where the function is continuous

ex) $y=f(x)$: Piecewise function

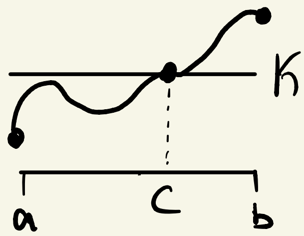


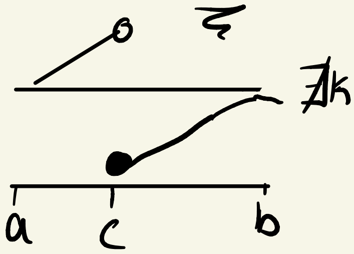
Domain $(a,c) \cup (c,b)$

2.5 Intermediate Value Theorem (IVT)

Theorem

If f: continuous on $[a,b]$, K: only real # between $f(a)$ and $f(b)$,
then there exists atleast one $\underline{c}$ in $[a,b]$ such that $f(c)=K$



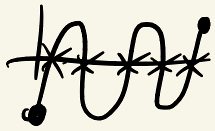


Def. A zero of fuction $y=f(x)$
$= $ A solution to $f(x)=0$
$= $ An x-intercept of $y=f(x)$

ex) $y=x^2-x$
$y=x(x-1)$
$y=0$
$x=0,1$

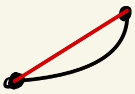
Note. Special case of IVT

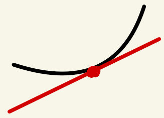
If f: continous on $[a,b]$ & $f(a)$ and $f(b)$ have opposite signs
then ther exists one "zero", c, in $[a,b]$



Chapter 3. Derivatives

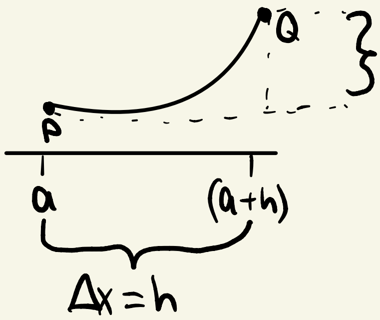
3.1 Defining the Derivative

Def. Secant Line : a line joining two points on a curve 

tangent Line : A line that skims a curve at a point 

3.2 The derivative as a function

Tangent Line Approximation and tangent line

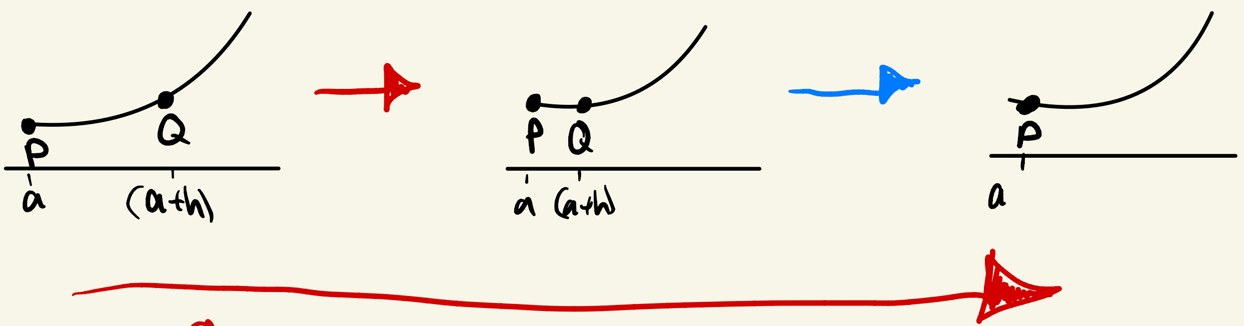


$\}\ \Delta y = f(a+h) - f(x)$

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} = m_{sec} \text{ over } [a, a+h]$$

= Difference quotient

Avg rate of change

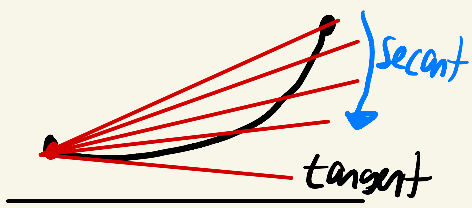


As Q approaches P = As Δx approaches 0

Def.

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



Note. Differentiability of $y=f(x)$ at $x=a$

① Algebraically:

$$\left(\exists f'(a) = \exists \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right) \longleftrightarrow$$

if A, then B

if $\sim B$, then $\sim A$

($\sim$ = not)

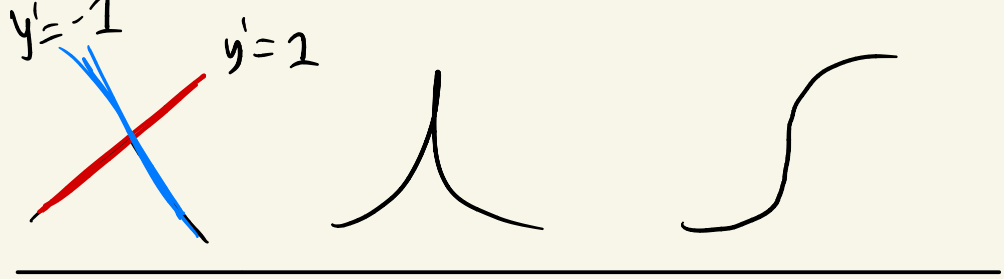
② Graphically

Theorem. If f is differentiable at $x=a$, then f is continuous at $x=a$

DIFFERENTIABILITY implies CONTINUITY

$\longleftrightarrow$ hole, blown-up, break, jump $\}$ not continuous $\}$ not differentiable

③ Even if conti. at $x=a$, may not be diff.



Sharp corner   Cusp   vertical tangent

Slope is ∞

Def. Differentiation: the process of finding a derivative

Def. higher order derivative of $f(x)$

2^{nd} order derivative $(f'(x))' = f''(x)$

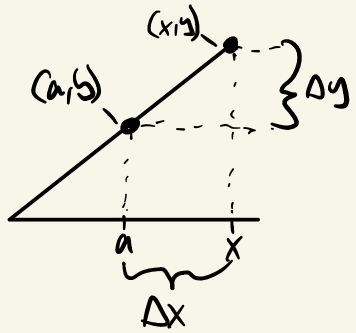
$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$$

n^{th} order derivative

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n}$$

Note. Equation of a line



Slope : $\frac{\Delta y}{\Delta x} = \frac{y-b}{x-a} = m$

$m(x-a) = y-b$

$\boxed{y = m(x-a) + b}$: point slope form

Slope : m

Point : (a,b)

3.3 Derivative Rules

1. Constant Rule: $(c)' = 0$
2. Constant multiple rule: $(cf(x))' = c(f(x))'$
3. Sum/Difference rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
4. Product rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + g'(x) \cdot f(x)$
5. Quotient rule: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
6. Chain rule: $[(f \circ g)(x)]' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$
7. Power rule: $(x^n)' = n \cdot x^{n-1}$ for $\mathbb{R}$

(7a) $\left[(u(x))^n\right]' = n(u(x))^{n-1} \cdot u'(x)$

8. Exponential rule: $[b^x]' = b^x \cdot \ln(b) \cdot x'$

$$[e^x]' = e^x \cdot \underbrace{\ln(e)}_{\cdot\, 1\, \cdot} \cdot x'$$

$$[b^{u(x)}]' = b^{u(x)} \cdot \ln b \cdot u'(x)$$

9. Logarithmic rule: $[\ln(x)]' = \frac{1}{x} \cdot x'$

$$[\ln|x|]' = \begin{cases} \frac{1}{x} \\ \frac{1}{-x} \cdot -1 = \frac{1}{x} \end{cases} \Big\rangle \frac{1}{x}$$

$$[\log_b(x)]' = \frac{1}{x} \cdot \frac{1}{\ln(b)}$$

10. Triganometric rules: $[\sin(x)]' = \cos(x)$

$$[\cos(x)]' = -\sin(x)$$
$$[\tan(x)]' = \sec^2(x)$$
$$[\sec(x)]' = \sec(x)\tan(x)$$
$$[\csc(x)]' = -\csc(x)\cot(x)$$
$$[\cot(x)]' = -\csc(x)$$

3.4 Derivatives as a Rate of Change

* $s(t)$: Position function
  $s'(t) = v(t)$: velocity, $|v(t)|$ = speed
  $s''(t) = v'(t) = a(t)$: acceleration

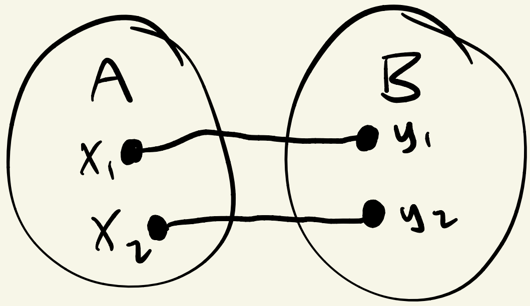
} time, $t \geq 0$

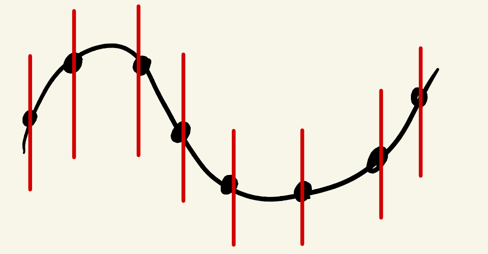
	$v(t) > 0$ moving forward	$v(t) = 0$ at rest	$v(t) < 0$ moving backward
$a(t) > 0$	Speeding up		Slowing down
$a(t) < 0$	Slowing down		Speeding up

3.7 Derivative of inverse function

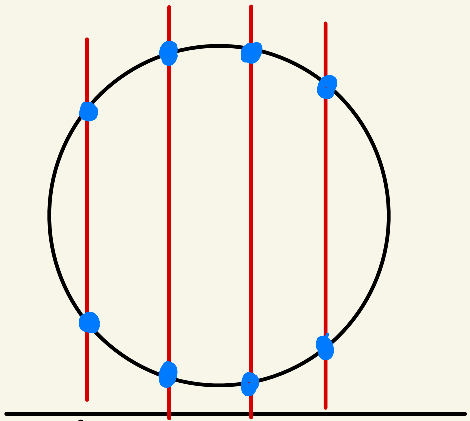
Reminder.

① One-to-one function

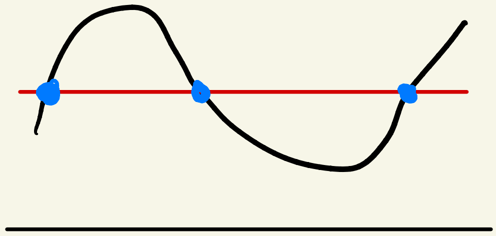




Vertical line test



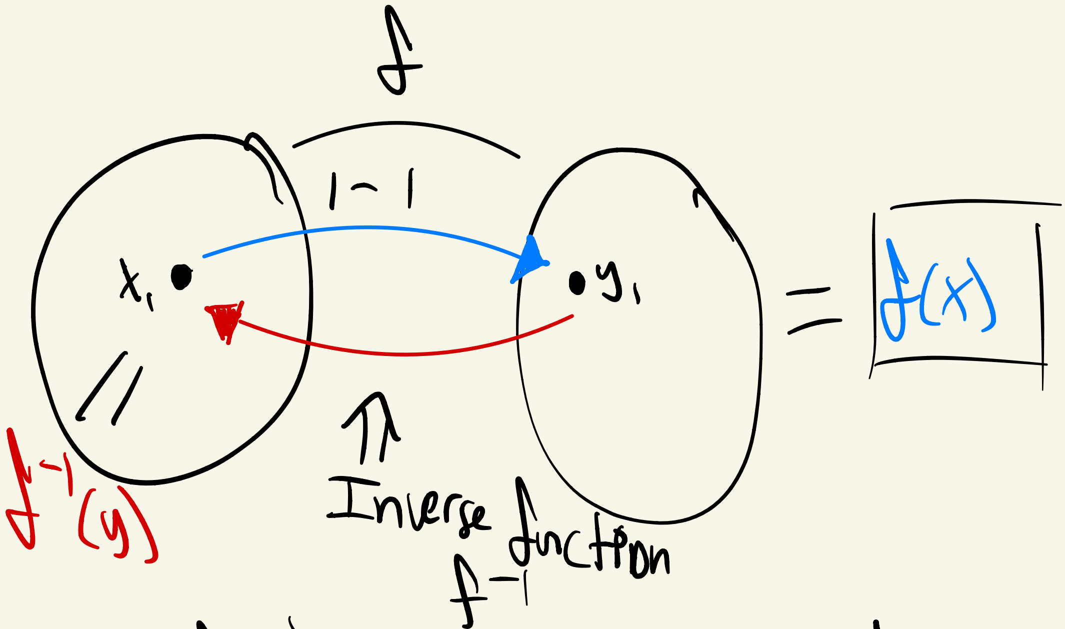
fails, intersection at 2 points



Fails horizontal line test

Not one-to-one

② from a 1-1 function, we define f^{-1}



$$y = f(x) \longleftrightarrow x = f^{-1}(y)$$

$$f(f^{-1}(y)) = y \Rightarrow y = f(x)$$
$$= f(f^{-1}(y))$$
$$= (f \circ f^{-1})(y)$$

$$x = f^{-1}(y) = f^{-1}(f(x))$$

Find $(f^{-1})'(x)$ w/o finding $f^{-1}(x)$

By definition, $f(f^{-1}(x)) = x' = 1$

$$f'(f^{-1}(x)) \circ (f^{-1}(x))' = 1$$

$$\boxed{[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}} \qquad \boxed{(f^{-1})'(b) = \frac{1}{f'(a)}}$$

Rule 10.

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}} \qquad (\arcsin(u(x)))' = \frac{1}{\sqrt{1-(u(x))^2}} \cdot u'(x)$$

$$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan(x))' = \frac{1}{1+x^2}$$

3.8 Implicit Differentiation

Note ① Important usage of chain rule $<$ implicit diff / related rates

② Equation in x and y

The eq. explicitly defines y as a function of x	The eq. implicitly defines y as a function of x
$y = f(x)$	$F(x,y) = 0$ w/ $y = f(x)$
ex) $y = 2x^3 - 1$	$7y^3x + 4e^{xy} = y^2 - 2x$
$y' = (2x^3-1)'$	$y = ???$
$y' = 6x^2$	$y' = ?$

④ Step 1) Take derivatives of both sides

2) Remove all ( )'s

3) Isolate all y' terms and solve for $y' / \frac{dy}{dx}$

3.9 Derivatives of exp/log functions

$2^x = 4$

$2^x = 2^2$

$x = 2$

$2^x = 5$

$x = \log_2(5) \Rightarrow \log_b a = c \longleftrightarrow b^c = a$

$\Rightarrow y = \ln(x) = \log_e x \longleftrightarrow e^y = x$

$\rightsquigarrow y = \ln(e^{y}) = y\underbrace{\ln(e)}_{1} = y$

$\cancel{e^{\ln}}(x) = x$

Log Properties

$\ln(A \cdot B) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln A^{3} = \ln(A \cdot A \cdot A) = \ln A + \ln A + \ln A = 3\ln A$

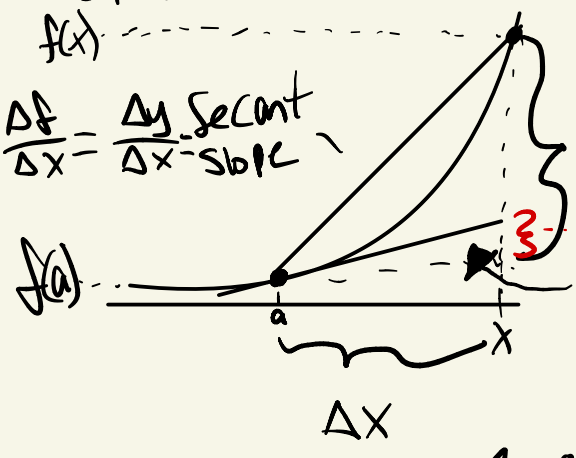
ex) $y = \dfrac{x^2 \sin^3(4x)}{\sqrt{1-5x^2}}$ $\quad \ln(y) = \ln(\sim) = \ln(x^2 \cdot \sin^3(4x)) - \ln(\sqrt{1-5x^2})$

$2\ln x + 3\ln(\sin(4x))$

$\frac{1}{2}\ln(1-5x^2)$

4.2 Linear approximation and differentials

Note.



$\Delta y = \Delta f = f(x) - f(a)$

tangent slope $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{df}{dx}\Big|_{x=a}$

$= f'(a)$

Δy : Exact change

dy : Estimated change using tangent line

$$\Delta y \neq dy$$

Def. differential of $x = dx = \Delta x = x - a$ = change in x

② differential of $y = dy = f'(x) \cdot dx$

$f'(x) = \frac{dy}{dx}$

Note. If $\Delta x = dx = x - a$: small enough,

$\boxed{\Delta y \approx dy} = f'(a)dx$

"Real" change = "Estimated" change

$$f(x) - f(a) \approx f'(a)(x-a)$$

$$f(x) \approx \underbrace{f'(a)(x-a) + f(a)}_{\text{tangent line}}$$

$L(x)$: Linear approximation of $f(x)$ at $x = a$

☆☆
To estimate value of $f(x)$
Use $f(x) \approx L(x)$

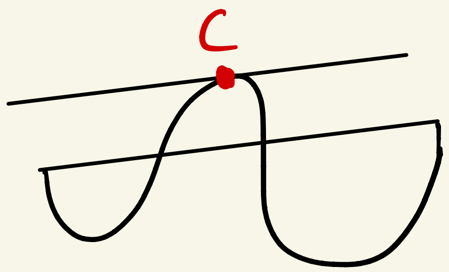
To estimate Δy
use $\Delta f \approx df$

4.4 Mean Value Theorem (MVT)

Note ① MVT

If f: continuous on [a,b] &
f: differentiable on (a,b) then,
exists c on [a,b] such that

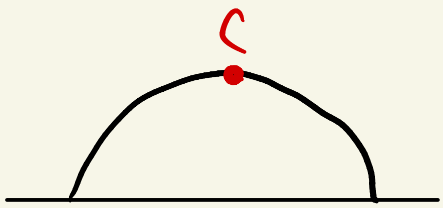
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



② Rolle's Theorem

$$f(a) = f(b)$$

$$f'(c) = 0$$



4.4 Limits at Infinity (Asymtotic Limits)

Reminder $\lim\limits_{x \to a} f(x)$

- Direct substitution
- $\frac{0}{0}$ form
  - factor
  - rationalize
  - simplify
- $\frac{\text{Constant}}{0}$
  - $+\infty$
  - DNE
  - $-\infty$
- limits at infinity

Note ① ∞: Endless, unbound, concept
not real number
does not grow

② Operations on ∞

$\infty + \infty = \infty$

$-\infty - \infty = -\infty$

$\infty \cdot \infty = \infty$

$-\infty \cdot (-\infty) = \infty$

$c + \infty = \infty$

$c - \infty = -\infty$

$c \cdot \infty = \infty$

$-c \cdot \infty = -\infty$

③ Indeterminate Operations

$\infty - \infty = ?$

$\infty \cdot 0 = ?$

$\frac{\infty}{\infty} = ?$ ex) $\frac{\infty}{\infty} = \frac{\infty + \infty}{\infty} = \frac{2\cancel{\infty}}{\cancel{\infty}} = \cancel{2}$ wrong

$= \frac{(\infty+\infty)+(\infty+\infty)}{\infty} = \frac{4\cancel{\infty}}{\cancel{\infty}} = \cancel{4}$

$\infty^0 = ?$

$1^\infty = ?$

Note. lim at infinity

ex) $f(x) = \dfrac{3x^2}{x^2 - 4}$

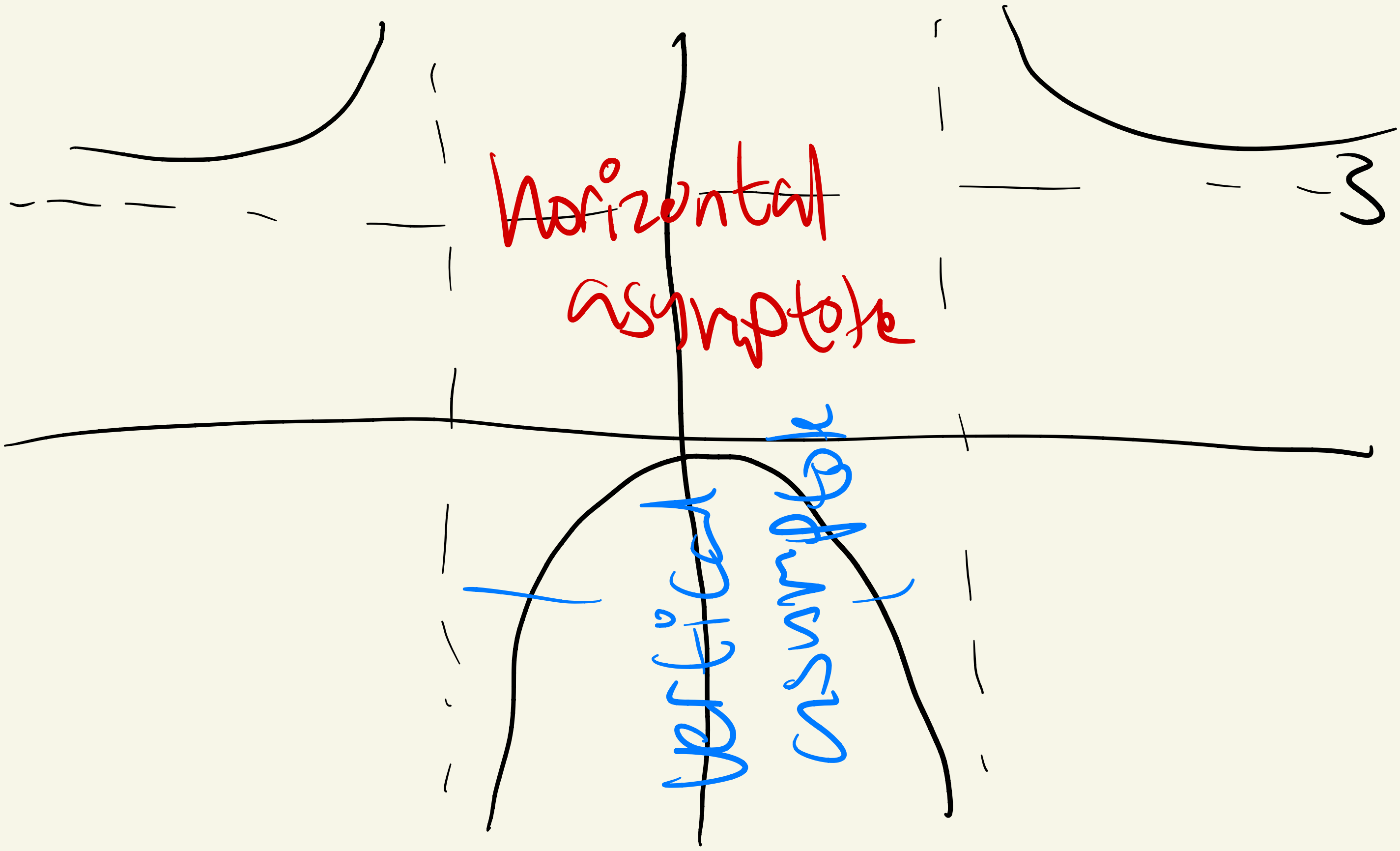
(a) Numerical Approach

x	100	-10	0	10	100
f(x)	3.0001	3.125	0	3.125	3.0001

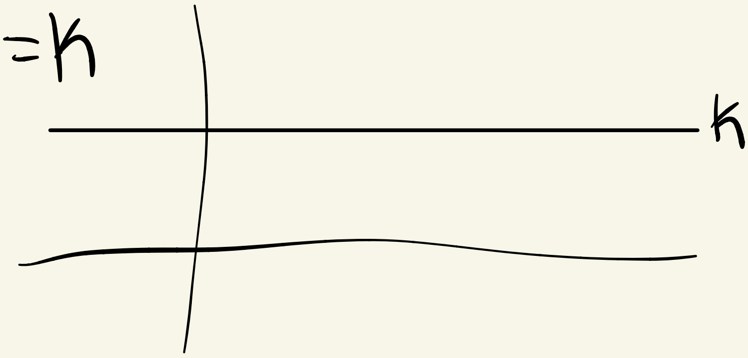
$\lim_{x \to -\infty} f(x) = 3$

$\lim_{x \to \infty} f(x) = 3$

(b) Graphical Approach

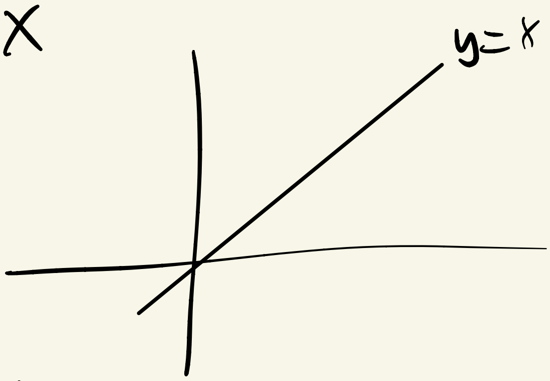


$f(x) = K$



$\lim_{x \to \pm\infty} K = K$

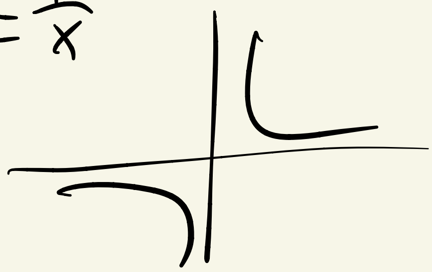
$f(x) = x$



$\lim_{x \to \infty} x = \infty$

$\lim_{x \to -\infty} x = -\infty$

$f(x) = \frac{1}{x}$



$\lim_{x \to \pm\infty} \frac{1}{x} = 0$

Note ① Limit at infinity: for a rational function or quotient involving radical

- Consider only highest power of x

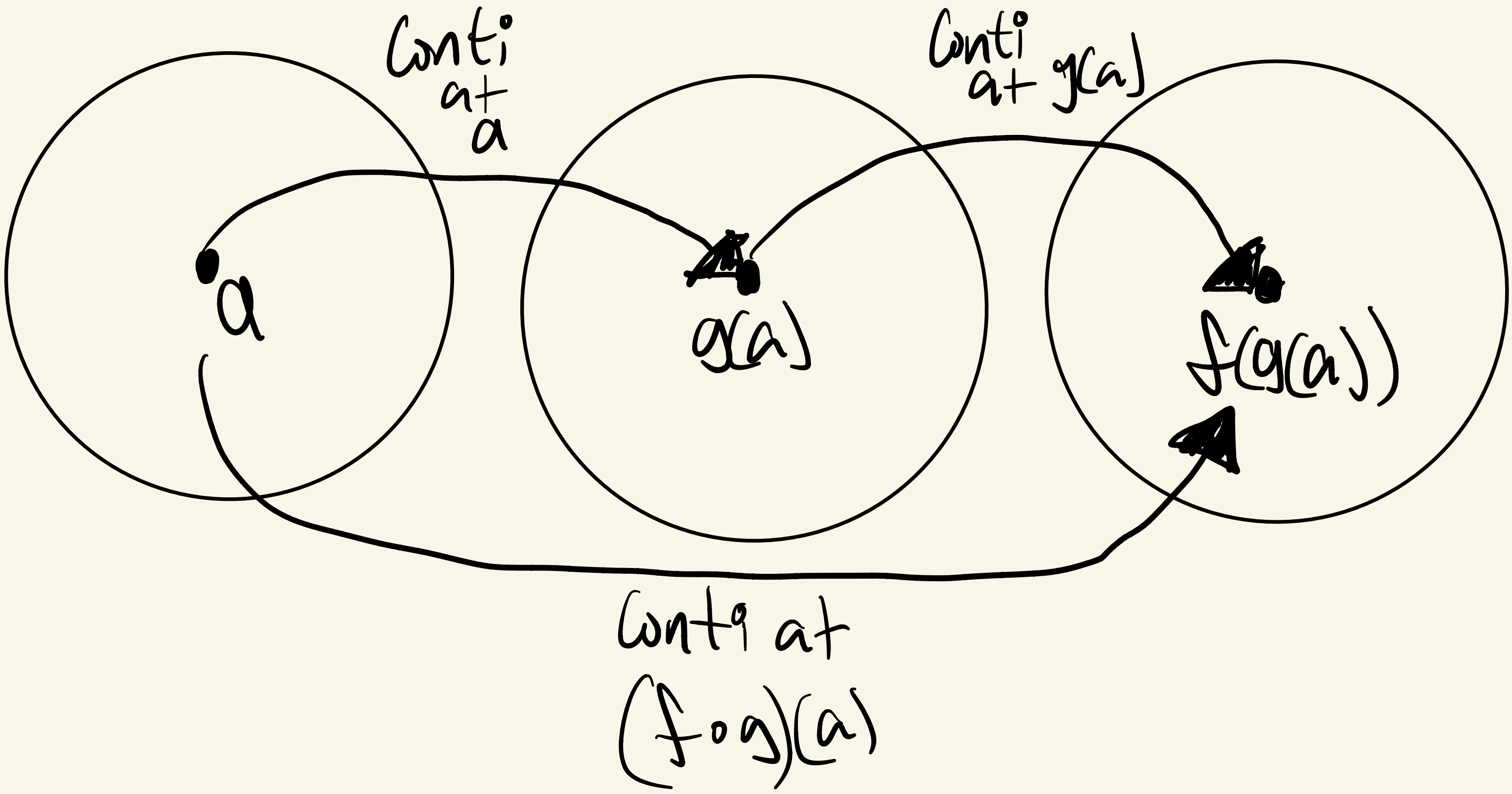
408 L'Hopital's rule only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

Note. $\lim_{x \to a} \frac{f(x)}{g(x)} \overset{LH}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)} = \ldots$

Stop at when numerator or denominator has a non-zero finite number

ex $\lim_{x \to 0} \frac{x+5}{\cos(x)} \overset{LH}{=} \frac{5}{1}$

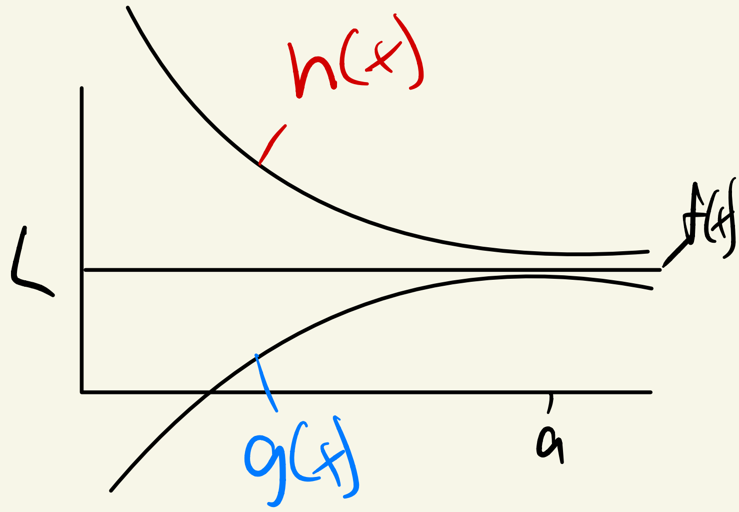
Note 2 Continuous function Theorem



$$\lim_{x \to a} f(g(x)) = f(g(a)) = f\left(\lim_{x \to a} g(x)\right)$$

Note 3 Squeeze Theorem

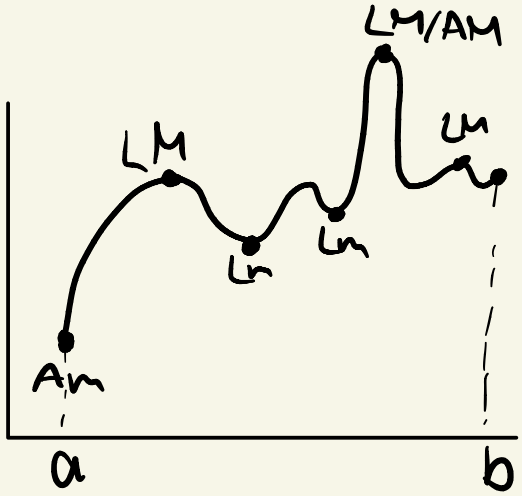
$$\underset{\downarrow\, L}{g(x)} \le \underset{\downarrow\, L}{f(x)} \le \underset{\downarrow\, L}{h(x)}$$



4.3 Maxima/Minima

Def. Extrema = Extreme values

- Absolute (global): Max, Min
- Local (relative): Max, Min



Absolute extrema includes endpoints
Local doesn't

$f(c)$: Abs/Rel extrema
c: Critical point

When,

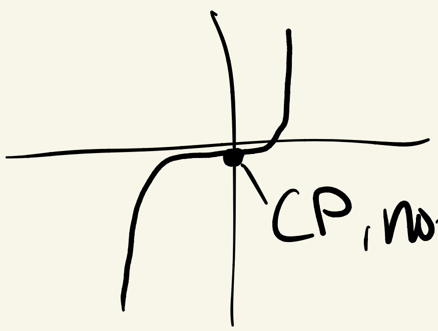
$$f'(c) = \begin{cases} 0 \\ UD \end{cases} \Bigg\} \text{ Critical Points}$$

Fermat's Theorem

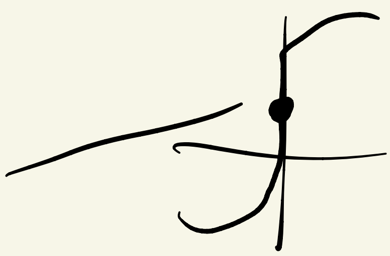
If $f(c)$ is a local extreme value, then c must be a CP

- However, if we dont know extremes

Even if c is a CP, $f(c)$ may not be an extreme



CP, not a extreme



Extreme Value Theorem

If f: continuous on $[a,b]$
then $f(x)$ has both AM and Am

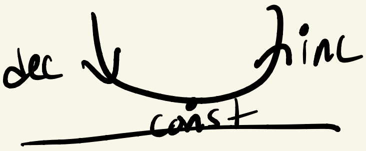
Step 1) Find all CP's in $[a,b]$

2) Find $f(a)$, $f(b)$

3) Largest - AM
Smallest - Am

4.5 Derivatives and Shape of graph

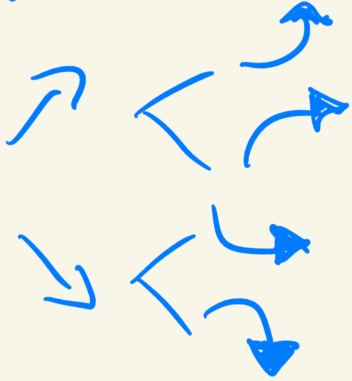
Note ①



f	↘	no change	↗
f'	−	0	+

1st derivative shows inc/dec of f

However if inc/dec

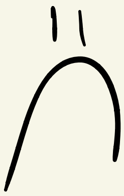


Concavity

②

Concave up

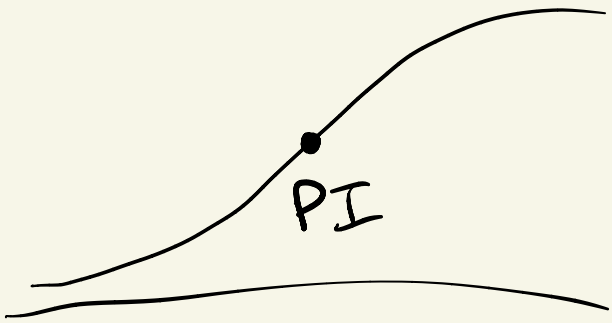
$f'' > 0$

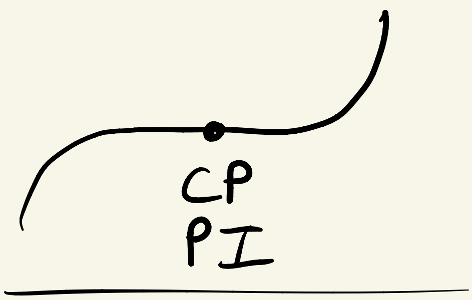


Concave down

$f'' < 0$

Def. Point of Inflection: a point that changes concavity





Theorem If $(c, f(c))$ is a PI, then $f''(c) = \begin{cases} UD \\ 0 \end{cases}$

* 2^{nd} derivative test for LM/Lm

f: conti on open interval

$\underbrace{f'(c)}_{CP} = 0$

If $f''(c) = 0$, 1^{st} Derivative test

$f''(c) > 0$, Lm

$f''(c) < 0$, LM

4.7 Applied Optimization Problems

AM/Am : $[a,b]$ | Dont use y
LM/Lm : (a,b) | "y" = f(x)

1. Find variable (use picture)
2. Find constraint equation
   - relationship between values given and variables
3. Primary equation
   - Relationship between 2-3 variables only
4. Feasible Domain
   - Set constraint to fixed chosen variable

   ex) $0 \leq x \leq max$
5. Classification : Extrema
   - Primary derivative, set to 0, find CP
6. Find remaining variables

4.10 Antiderivatives

ex) $4^2 = 16$ $(-4)^2 = 16$

sqrt of 16   sq of 4   root   sq of (-4)

= positive sqrt   = neg root of 16

$= \sqrt{16}$ $= -\sqrt{16}$

$x^2 = 16 \rightarrow x = \pm\sqrt{16} \neq \sqrt{16}$

$(x^3)' = 3x^2$

Def. $F'(x) = f(x)$

antiderivative of $f(x)$ derivative of $F(x)$

$(x^3 + [\infty])' = (x^3 + c)' = 3x^2$

infinitely many antiderivatives of $3x^2$
differs by constant

$(x^3 + c)' \underset{\text{integration}}{\overset{\text{differentiation}}{=}} 3x^2$

Def $F'(x) = f(x)$

$F(x) + C$ = set of all integrals of $f(x)$

= general integral

= indefinite integral of $f(x)$

$= \int f(x)\,dx$

sum   integrand

Note. Basic integration rules

① Constant multiple rule: $\int k f(x)\,dx = k\int f(x)\,dx$

② +/− rule: $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

③ 0 rule: $(C)' = 0 \longleftrightarrow \int 0\,dx = C$

④ Power rule: $\int x^n\,dx = \boxed{\frac{1}{n+1}x^{n+1} + C}$

⑤ Log rule: $\int \frac{1}{x}\,dx = \ln\underline{|x|} + C$

abs value

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

⑥ Trig rules: $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$

⑦ Exp rules: $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

⑧ inv trig rules: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

Chapter 5. Integration

5.1 Approximations Area

Note. $\boxed{\text{Precalc}} \longrightarrow \boxed{\text{Limit}} \longrightarrow \boxed{\text{Calculus}}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Difference quotient

secant line

Derivative

tangent line

Definite integral ~ finite sum

exact area (signed)

Estimated area

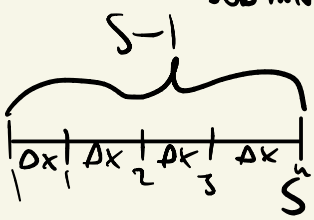
Note. Estimating area under curve on $[a,b]$

① Right End Point Approximation

R_n of A = Right Riemann's sum

Partition $[a,b]$ into n subintervals with equal width Δx

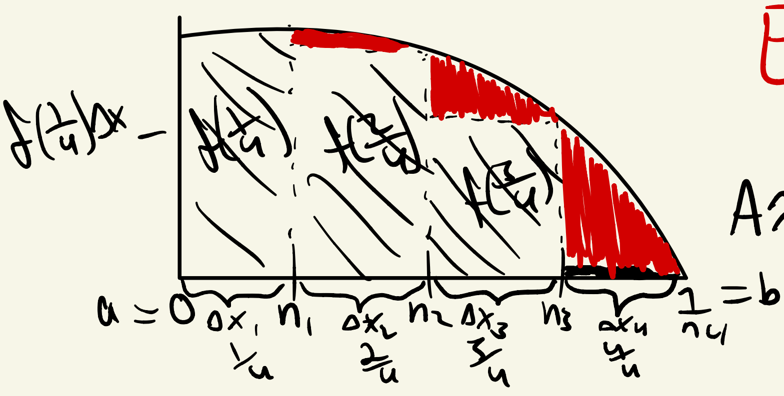
$$\Delta x = \frac{b-a}{\text{subintervals}(n)}$$



$$\Delta x = \frac{5-1}{4} = 1$$

4 subintervals

Use right endpoint as height of rectangle



Error exists but shows estimate

$$A \approx R_4 = \underset{\frac{1}{4}}{\Delta x}\left(f(\tfrac{1}{4}) + f(\tfrac{2}{4}) + f(\tfrac{3}{4}) + f(1)\right)$$

$$= \Delta x\, f(n)$$

② Left Riemann's sum is sum but left endpoints

$$A \approx L_4 = \Delta x \left(f(0) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{2}{4}\right) + f\left(\tfrac{3}{4}\right) \right)$$

③ Midpoint Riemann's

$$A \approx M_4 = \Delta x \left(f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right) \right)$$

Note.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

Def. $a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i$

Riemann's Sum

n – upper bound

$i=1$ – lower bound

index of summation

Note. ① Constant Multiple rule: $\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$

② +/− rule: $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

③ Constant rule: $\sum_{i=1}^{n} c = c\underbrace{\sum_{i=1}^{n} 1}_{1_1 + 1_2 + 1_3 + \dots + 1_n} = cn$

④ Index rule $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 1 + 2 + 3 + \dots 98 + 99 + 100$

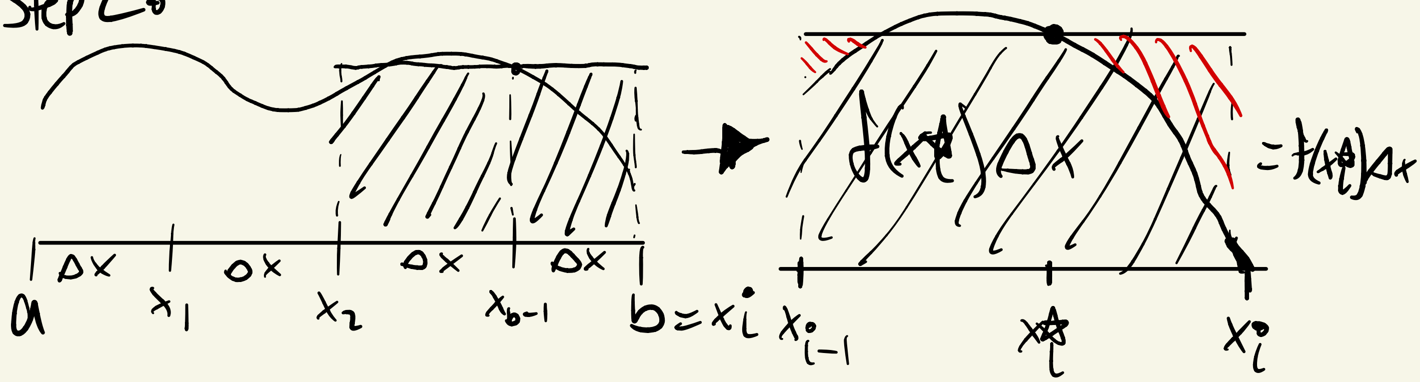
$= \frac{100 + 101}{2}$

Note. Find Area A under $y=f(x)\ge 0$ over $[a,b]$

Step 1) Divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$

with equal $\Delta x = \frac{b-a}{n} = x_i - x_{i-1}$

Step 2.



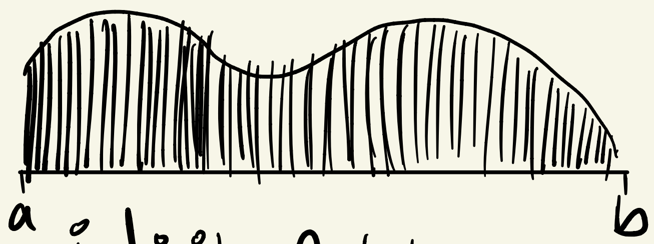
Step 3. Choose any point in $[x_{i-1}, x_i]$

$$A \approx S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad f(x_i^*)\Delta x \approx A[x_{i-1}, x_i]$$

Step 4. take lim as $\Delta x = \frac{b-a}{n} \to 0$

n has to become greater

$$\underbrace{\int_a^b f(x)dx = \lim_{\Delta x\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}}\, f(x_i^*)\Delta x \overset{f\ge 0}{=} A$$



infinite Partitions

5.2 Definite Integral

Def. $f(x)$: any function on $[a,b]$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \;\underset{=}{\text{if lim exists}}\; \int_a^b f(x)\,dx$$

upper bound (b)

lower bound (a)

variable of integration (dx)

Called the definite integral

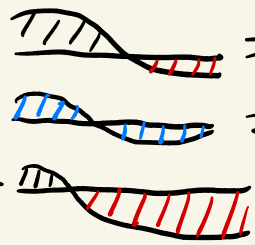
f: any function

= (net) "signed" area enclosed by f & x-axis

$f(x)\geq 0$ = Area under curve

$= A_1 - A_2$ with three cases:

- $+$ = + area
- 0 = 0 area
- $-$ = − area



Note:

Signed area $= \int_a^b f(x)\,dx = A_1 - A_2$

Total Area $= \int_a^b |f(x)|\,dx = A_1 + A_2$

Def. ① If $\exists \int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$,

it is integrable over $[a,b]$

continuous $\Rightarrow$ integrable

$\not\Leftarrow$

ex

Note. Indefinite integral

set of all possible integrals $\boxed{\int f(x)\,dx} = F(x) + C$

$$\int_a^b f(x)\,dx = \text{Number!}$$

$$\int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

"dummy" variable

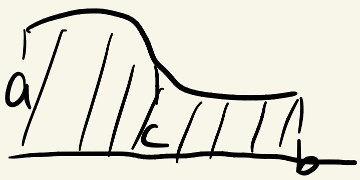
Def. Average value of f over $[a,b]$

$$= \frac{\int_a^b f(x)\,dx}{b-a}$$

(a) $\int_a^a f(x)\,dx = 0 = \left[\int_a^b f(x)\,dx - \left(-\int_b^a f(x)\,dx\right)\right]$

(b) $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$

(c) $\int_a^b \left[f(x) \pm g(x)\right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(d) 

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(e) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Note. To evaluate $\int_a^b f(x)\,dx$

1. Use Riemann's sum
2. Use geometric formula
3. Use Fundamental theorem of Calculus (FTC)

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx \overset{FTC\,II}{=\!=} F(b) - F(a) = \left[F(x)\right]_a^b$$

$$\int_a^b F'(x) \overset{FTC\,II}{=\!=} F(x)\Big|_a^b = \int_a^b \frac{d}{dx} F(x)\,dx$$

If not continuous, area is negative

Note. ② Proving FTC I using FTC II.

$$\int_a^b \cancel{\frac{d}{dx}} F(x)\,dx \overset{\text{FTC II}}{=\!=} F(x)\Big|_a^b$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt \overset{\text{FTC II}}{=\!=} \frac{d}{dx}\left(F(t)\right)\Big|_{a(x)}^{b(x)}$$

$F'(t) = f(t)$

$$= \frac{d}{dx}\Big(F(b(x)) - F(a(x))\Big)$$

$$\overset{\text{FTC I}}{=\!=} F'(b(x)) \circ b'(x) - F'(a(x)) \circ a'(x)$$

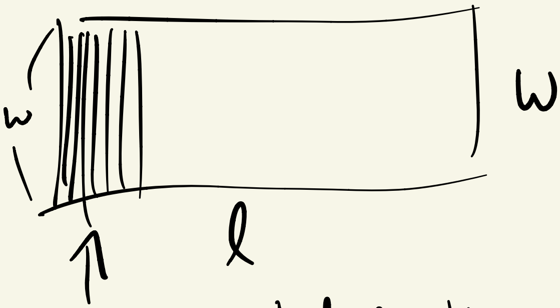
$$= f(b(x)) \circ b'(x) - f(a(x)) \circ a'(x)$$

Reminder

FTC II $\int_a^b \frac{d}{dx} F(x)\,dx = [F(x)]_a^b$

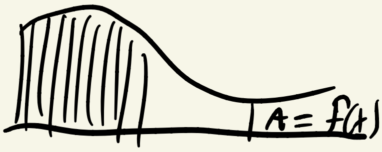
FTC I $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(a(x)) \circ a'(x) - f(b(x)) \circ b'(x)$

Note 1



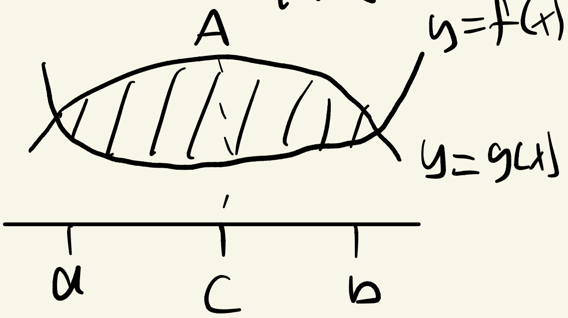
line segment of length w repeated l times

2



$$A = \int_a^b f(x)\,dx$$

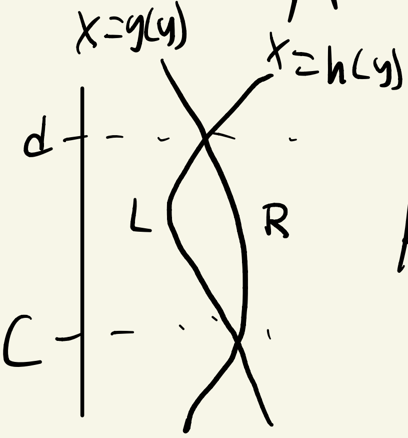
3



$$A = \int_a^b [f(x) - g(x)]\,dx$$

$$A = \int_a^b \left[\text{Top Curve} - \text{Bottom Curve}\right] dx$$

4



$$A = \int_c^d \left[\overset{R}{g(y)} - \overset{L}{h(y)}\right] dy$$

$f(c) > g(c)$ ∴ $f(x) = T$

∴ $g(x) = B$

5.5/5.6/5.7 Substitution

☆ $\int_a^b f(x)\,dx$ ← R.S / geo formula / FTC → area

Note ① Composite function

$\underbrace{f}_{\text{outer func}}(\underbrace{g(x)}_{\text{inner func}})$ $\qquad \sqrt{x^2-5} \qquad \dfrac{1}{1+2x^2} = (1+2x^2)^{-1}$

e^{x^2+4}

② (Algebraic) Substitution for integration

= Reverse chain rule for differentiation

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

(a) Indefinite integral: $\underbrace{\quad}_{\text{how do we get this}} = \int [f(g(x))]'\,dx$

$= f(g(x)) + C$

Note: U-Substitution

$$\frac{df}{dx} = f'(x)$$
$$df = f'(x)\,dx$$

$$\begin{pmatrix} u = g(x) \\ du = g'(x)\,dx \end{pmatrix} = \frac{f'(g(x))}{u} \cdot \frac{g'(x)\,dx}{du}$$

$$= \int f'(u)\,du$$

$$= f(u) + C = f(g(x)) + C$$

(b) Definite Integral: $\int_a^b f'(g(x)) \cdot g'(x)\,dx$

$$\begin{pmatrix} u = g(x) \\ du = g'(x)\,dx \\ x = a \rightarrow a = g(a) \\ x = b \rightarrow b = g(b) \end{pmatrix} = \int_{g(a)}^{g(b)} f'(u)\,du$$

$$= f(u)\Big|_{g(a)}^{g(b)}$$

$$= f(g(b)) - f(g(a))$$

Notations - Calculus I

Colon	:	is
Bar-Arrow	$A \mapsto B$	Set A : Domain is to Codomain : set B
Doublebar R	$\mathbb{R}$	Set of all Real Numbers
	UD	Undefined
	$\exists$	Exists
	lim	Limit
	DNE	Does not Exist
Delta	Δy	Change in y
Prime	$'$	f(x) derivative
	CP	Critical Point