# Real Analysis

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# Contents

# Chapter 0

# Introduction

AI is like how a chocolate croissant is... Irresistible

*Joseph H.G Fu*

The natural numbers are the work of God.

*Leopold Kronecker*

Sets are like machine code, its one of the ways you can have it, but you will never catch me talking about categories.

*Joseph H.G Fu*

You can't get anywhere in life without desire.

*Joseph H.G Fu*

Do not have a belabored proof. If you are able to convince yourself and show you understand what is happening, then there is nothing more that is needed. Many undergraduates write too much that it makes me less convinced.

*Joseph H.G Fu*

- First part: Undergraduate Studies. We follow Abbott Chapter 1-7.
- We follow Abbott 'Understanding Analysis':
  - Chap 1 'The Real Numbers'
  - Chap 2 'Sequences and Series'
  - Chap 3 'Basic Topology of $\mathbb{R}$'
  - Chap 4 'Functional Limits and Continuity'
  - Chap 5 'The Derivative'
  - Chap 6 'Sequences and Series of Functions'
  - Chap 7 'The Riemann Integral'
- Second part: Semi-Graduate Studies. We follow Abbott Chapter 8 only.
- 20 graduate problems (solve them during the semester) No feedback during the semester, but asking questions is allowed. Working together is allowed.
- Exam (1 paged cheat-sheet, closed book, fresh dissimilar questions)

# Part I

# Undergraduate Studies

# Chapter 1

# The Real Numbers

## 1.3 Some Preliminaries

### Lecture 1: Day 1

Though things arise naturally out of studies like Algebra, some claim that we need to go beyond such, and apply some intuition and "analytical" prowess to find ideas from what may seem abstract or uncorrelated. Real Analysis gave foundation and birth to many different fields we know today, including most of the applied math workforce, such as Chemists, Physicists, Biologists, Meteorologists, and so forth. But within math, also optimization, complex analysis, differential equations, differential geometry, and so forth. It is essentially a cornerstone of mathematics, regarded by many.

**Definition 1** (Real Numbers). **The complete ordered** field that contains $\mathbb{Q}$.

**Definition 2** (Rational Numbers).

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$$

**Definition 3** (Integers).

$$\mathbb{Z} = \{\pm n : n \in \mathbb{N}\} \cup \{0\}$$

**Definition 4** (Naturals).

$$\mathbb{N} = \{1, 2, \ldots\}$$

**Example** (Another Defintion of Rationals). It can be said that the rationals can be defined like the following.

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

**Remark.** Note that when were asserting $\mathbb{Q}$, then we were unable to use the defining operator : :=.

**Lemma 1.** Suppose we have $(a, b), (\alpha, \beta) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $(a, b) \sim (\alpha, \beta)$ if and only if $\alpha b = a\beta$.

**Definition 5** (Equivalence Relation)**.** Equivalence relations have three properties. Reflexivity, Symmetry, and Transitivity.

The reflexive property holds $(a, b) \sim (a, b)$ is true.

The symmetric property holds if $(a, b) \sim (\alpha, \beta)$, then $(\alpha, \beta) \sim (a, b)$.

The transitive property holds if $(a, b) \sim (\alpha, \beta)$ and $(\alpha, \beta) \sim (c, d)$, then $(a, b) \sim (c, d)$. Prove this property for Exercise 1.

**Property.** Check out Problem 0.1 for a related problem.

**Property.** Hence, we can define the rationals as the following.

$$\mathbb{Q} := \frac{\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})}{\sim}$$

**Definition 6** (Equivalence Class)**.**

$$[(a, b)] + [(\alpha, \beta)] = [(a\beta + \alpha b, b\beta)]$$

.

**Definition 7** (Equivalence Class)**.**

$$[(a, b)] + [(\alpha, \beta)] = [(a\beta + \alpha b, b\beta)]$$

.

**Proposition 1** (Addition is well-defined.)**.** Given $(a, b) \sim (c, d)$ and $(\alpha, \beta) \sim (\gamma, \delta)$, then $(a\beta + \alpha b, b\beta) \sim (c\delta + \beta d)$. Prove this for Exercise 2.

**Lemma 2** (Irrationality Proof)**.** There is no $q \in \mathbb{Q}$ such that $q^2 = 2$. We call this, $\sqrt{2}$ irrational.

**Proof.** Suppose for the sake of contradiction, if $q = \frac{a}{b}$, $a, b \in \mathbb{Z}$, and $a, b$ coprime, then $q^2 = \frac{a^2}{b^2} = 2$. Thus $a^2 = 2b^2$, then $a^2$ is even. Furthermore, $a = 2k$ for some $k \in \mathbb{Z}$, therefore $a^2 = 4k^2$. Hence $b^2$ is even, which contradicts our coprime assumption. ■

**Remark.** The direct statement of this lemma is:

> If $q^2 = 2$, then $q \notin \mathbb{Q}$.

A contrapositive of this lemma is:

> If $q \in \mathbb{Q}$, then $q^2 \neq 2$.

## 1.4 Axiom of Completeness

### Lecture 2: Day 2

**Definition 8** (Ordering Property)**.** Given $a, b \in \mathbb{R}$, either

$$a > b$$
$$a = b$$
$$a < b.$$

**Observe.** $\mathbb{R}$ is complete, which means it has the least upper bound (l.u.b) property.

**Definition 9** (l.u.b Property)**.** If $s \subset \mathbb{R}$ and $S$ is bounded above, i.e there is a $C \in \mathbb{R}$ such that $C \geq x$ for all $x \in S$, then there is some $\alpha \in \mathbb{R}$ which is the least upper bound for $S$. This means $\alpha \leq C$ for all upper bounds $C$ of $S$.

$$\alpha := \sup(S)$$

**Corollary.** The greatest lower bound property follows.

$$\beta := \inf(S)$$

**Corollary.**

$$\inf = -\sup\{-x : x \in S\}.$$

Prove this as an Exercise.

**Remark.** This is a convincing proof for Dr. Fu, but for now we must still be able to prove it.

**Corollary.** The Archimedean Property of $\mathbb{R}$ states:

> For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

**Proof.** Suppose for the sake of contradiction, there is no $n > x$. Let $\alpha := \sup(\mathbb{N})$. Since $\alpha - 1 < \alpha$, then $\alpha - 1$ cannot be the upper bound for $\mathbb{N}$, then there is an $m \in \mathbb{N}$ such that $\alpha - 1 < m$. Thus $\alpha < m + 1$. Hence a contradiction arose. ■

**Remark.** The direct statement of this corollary is:

> If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n > x$.

A contrapositive of this lemma is:

> If there is no $n \in \mathbb{N}$ such that $n > x$, then $x \notin \mathbb{R}$.

**Corollary.** For every real $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

**Proof.** By the archimedean principle, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ thus $\frac{1}{n} < \varepsilon$. ■

**Remark.** The direct statement of this corollary is:

> If $\varepsilon > 0$ is real, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

A contrapositive of this lemma is:

> If $\frac{1}{n} \geq \varepsilon$ for all $n \in \mathbb{N}$, then $\mathbb{R} \in \varepsilon \leq 0$.

**Lemma 3** (Ordering of Squares)**.** $a, b \in \mathbb{R}$, $a, b > 0$, $a < b$ if and only if $a^2 < b^2$.

**Proof.** Prove this as an Exercise. ■

**Corollary.** There is a real number $x > 0$ such that $x^2 = 2$.

**Proof.** Define $S := \{y \in \mathbb{R} : y^2 < 2\}$. By Lemma 1.16, $S$ is bounded above. Suppose $\alpha := \sup(S)$, and $\alpha^2 < 2$, then there is $\varepsilon > 0$ such that $(\alpha + \varepsilon)^2 < 2$. Then $\alpha^2 + 2\alpha\varepsilon + \varepsilon^2 < 2$. Since $\varepsilon$ is generally considered a very small value, then $\varepsilon^2 < \varepsilon$.

$$\begin{aligned}
\alpha^2 + 2\alpha\varepsilon + \varepsilon^2 &< \alpha^2 + 2\alpha\varepsilon + \varepsilon \\
&= \alpha^2 + \varepsilon(2\alpha + 1) \\
\alpha^2 + \varepsilon(2\alpha + 1) &< 2 \\
\varepsilon(2\alpha + 1) &< 2 - \alpha^2 \\
\varepsilon &< \frac{2 - \alpha^2}{2\alpha + 1}
\end{aligned}$$

Choose $\varepsilon = \min\left\{1, \dfrac{2-\alpha^2}{2(2\alpha+1)}\right\}$. We choose a half of what is above to have a strict inequality. Inputting this back into the problem:

$$\begin{aligned}
\alpha^2 + \varepsilon(2\alpha+1) &\leq \alpha^2 + \frac{2-\alpha^2}{2(2\alpha+1)}(2\alpha+1) \\
&< \alpha^2 + \frac{2-\alpha^2}{2\alpha+1}(2\alpha+1) \\
&= \alpha^2 + 2 - \alpha^2 \qquad\qquad = 2.
\end{aligned}$$

Hence a contradiction, therefore $\alpha \geq 2$. Suppose again, for a contradiction, $\alpha > 2$. Then there is $\varepsilon > 0$ such that $(\alpha - \varepsilon)^2 > 2$. Then:

$$\begin{aligned}
\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 &> 2 \\
\alpha^2 - 2\alpha\varepsilon + \overset{\varepsilon^2}{\cancel{> 0}} &> 2 \\
\alpha^2 - 2\alpha\varepsilon &> 2 \\
-2\alpha\varepsilon &> 2 - \alpha^2 \\
-(-2\alpha\varepsilon) &< -(2-\alpha^2) \\
2\alpha\varepsilon &< \alpha^2 - 2 \\
\varepsilon &< \frac{\alpha^2-2}{2\alpha}.
\end{aligned}$$

Thus choose $\varepsilon = \min\left\{\alpha, \dfrac{\alpha^2-2}{2\alpha}\right\}$. Then $\alpha^2 - 2\alpha\varepsilon = \alpha^2 - \alpha^2 + 2 = 2$. Therefore another contradiction, hence $\alpha^2 = 2$ as needed. Uniqueness comes from Lemma – Ordering of Squares. ■

### Lecture 3: Day 3

No substantial work.

## 1.5 Consequences of Completeness

### Lecture 4: Day 4

**Theorem 1** (Complete Exponential Theorem)**.** Define $S := \{x \in \mathbb{R} : x^n < c\}$, S is bounded above, then set $\alpha := \sup(S)$ and $\alpha^n = c$.

**Proof.** Given $S$ is bounded above, suppose $\alpha^n = \sup(S)$ and $\alpha^n < c$. Then there is a real $\varepsilon > 0$ such that $(\alpha+\varepsilon)^n < c$. Then:

$$\begin{aligned}(\alpha+\varepsilon)^n &= \sum_0^n \binom{n}{k}\alpha^{n-k}\varepsilon^k \\ &= \alpha^n + \sum_1^n \frac{n!}{k!(n-k)!}\alpha^{n-k}\varepsilon^k \\ &= \alpha^n + n\alpha^{n-1}\varepsilon + \frac{n(n-1)}{2}\alpha^{n-2}\varepsilon^2 + \sum_3^n \frac{n!}{k!(n-k)!}\alpha^{n-k}\varepsilon^k \\ &\leq \alpha^n + n\alpha^{n-1}\varepsilon + n\alpha^{n-1}\varepsilon + \ldots + n\alpha^{n-1}\varepsilon \\ &\leq \alpha^n + n^2\alpha^{n-1}\varepsilon \\ &< \alpha^n + n^2(\alpha+1)^{n-1}\varepsilon\end{aligned}$$

Now that we have reduced the equation $(\alpha+\varepsilon)^n \leq \alpha^n + n^2\alpha^{n-1}\varepsilon$, then:

$$\begin{aligned}\alpha^n + n^2(\alpha+1)^{n-1}\varepsilon < c \\ \varepsilon < \frac{c-\alpha^n}{n^2(\alpha+1)^{n-1}}.\end{aligned}$$

Choose $\varepsilon = \min\{1, \dfrac{c-\alpha^n}{n^2(\alpha+1)^{n-1}}\}$. Then:

$$\begin{aligned}\alpha^n + \cancel{n^2(\alpha+1)^{n-1}}\frac{c-\alpha^n}{\cancel{n^2(\alpha+1)^{n-1}}} &= \alpha^n + c - \alpha^n \\ c &= c.\end{aligned}$$

Hence a contradiction, therefore $\alpha \geq 2$. Suppose there is a $\delta > 0$ such that $(\alpha-\delta)^n > c$, then:

$$\begin{aligned}(\alpha-\delta)^n &= \sum_0^n \binom{n}{k}\alpha^{n-k}(-\delta)^k \\ &= \alpha^n - n\alpha^{n-1}\delta + \frac{n(n-1)}{2}\alpha^{n-2}\delta^2 + \sum_3^n \frac{n!}{k!(n-k)!}\alpha^{n-k}(-\delta)^k \\ &\geq \alpha^n - n\alpha^{n-1}\delta - (n-2)\alpha^{n-3}\delta - (n-4)\alpha^{n-5}\delta + \ldots + \delta^n \\ &\geq \alpha^n - n\alpha^{n-1}\delta - n\alpha^{n-1}\delta - n\alpha^{n-1}\delta + \ldots + \delta^n \\ &\geq \alpha^n - n^2\alpha^{n-1}\delta + \delta^n \\ &\geq \alpha^n - n^2\alpha^{n-1}\delta + \delta \\ &\geq \alpha^n - \delta(n^2\alpha^{n-1}+1). \qquad < \alpha^n - \delta(n^2(\alpha+1)^{n-1}+1).\end{aligned}$$

Thus $(\alpha-\delta)^n \geq \alpha^n - \delta(n^2(\alpha+1)^{n-1}+1) > c$. Then:

$$\begin{aligned}\alpha^n - \delta(n^2(\alpha+1)^{n-1}+1) > c \\ \delta < \frac{\alpha^n - c}{n^2(\alpha+1)^{n-1}+1}.\end{aligned}$$

Choose $\delta = \min\{1, \dfrac{\alpha^n - c}{n^2(\alpha+1)^{n-1}+1}\}$. Then:

$$\alpha^n - \delta(n^2(\alpha+1)^{n-1}+1) = \alpha^n - \frac{\alpha^n - c}{\cancel{n^2(\alpha+1)^{n-1}+1}}\cancel{(n^2(\alpha+1)^{n-1}+1)}$$
$$c = c$$

Therefore another contradiction, hence $\alpha^n = c$ as needed. ■

**Definition 10** (Cardinality of Sets)**.** It is the number of elements in set $S$.

**Definition 11** (Equal Cardinality)**.** Sets have the same cardinality if there is a bijective function $f : A \leftrightarrow B$.

**Definition 12** (Countably Infinite)**.** A set is countably infinite if has the same cardinality as $\mathbb{N}$.

**Definition 13** (Countable)**.** A set is countable if it's either finite, or countably infinite. $S$ is also countable if it's elements can be listed as a sequence.

**Theorem 2.** $\mathbb{Z}, \mathbb{Q}$ are both countably infinite.

**Proof.** $\mathbb{Z}$: For $n \in \mathbb{N}$, we can define any function, but an example would be

$$f(2n) = n - 1$$
$$f(2n+1) = -n.$$

$\mathbb{Q}$: We can do the criss-crossing across the diagonal. We claim that we can define $g : \mathbb{Z} \hookrightarrow \mathbb{Q}$ defined by $g(-n) = f(n)$ and $g(0) = 0$. and use $g \circ f : \mathbb{N} \hookrightarrow \mathbb{Q}$. ■

**Theorem 3** (Nested Interval Theorem)**.** Suppose $[a_1, b_1], [a_2, b_2], \ldots$ is a sequence of closed bounded intervals in $\mathbb{R}$, which is nested.

$$[a_n, b_n] \supset [a_{n+1}, b_{n+1}], \forall n \in \mathbb{N}$$

Then:

$$\bigcap_{i=1}^{\infty} [a_i, b_i] := \{x : x \in [a_i, b_i]\} \neq \varnothing$$

**Proof.** Let $A := \{a_1, a_2, \ldots\}, B := \{b_1, b_2, \ldots\}$. Note that for any $b \in B$, is an upper bound of $A$. Thus establishing $\sup(A) \leq \inf(B)$. Since $\sup(A) \geq a_i$ for all $i$ and $\sup(A) \leq \inf(B)$, thus $\sup(A) \in [a_i, b_i]$ for all $i$. ■

**Remark.** The direct statement of this corollary is:

> If $[a_1, b_1], [a_2, b_2], \ldots$ is a sequence of closed bounded intervals in $\mathbb{R}$ is nested, then $\bigcap_{i=1}^{\infty}[a_i, b_i] := \{x : x \in [a_i, b_i]\} \neq \varnothing$.

A contrapositive of this lemma is:

> If $\bigcap_{i=1}^{\infty}[a_i, b_i] := \{x : x \in [a_i, b_i]\} = \varnothing$, then all $[a_1, b_1], [a_2, b_2], \ldots$ are not nested.

## 1.6 Cardinality

**Theorem 4.** $\mathbb{R}$ is uncountable.

**Proof.** Suppose $\mathbb{R}$ is countable, then there is a sequence $(x_1, x_2, x_3, \ldots)$ such that $\{x_1, x_2, x_3, \ldots\} = \mathbb{R}$. Take $a_1 > x_1$ and $b_1 = a_1 + 1$. If $x_2 < a_1$ or $x_2 > b_2$, then take $[a_2, b_2]$ such that $x_2 \notin [a_2, b_2]$. Continue this until all $x_n \notin [a_i, b_i]$, so by nested interval theorem:

$$\bigcap_{1}^{\infty}[a_i, b_i] \neq \varnothing,$$

but $x_i \notin [a_i, b_i]$ for all $i \in \mathbb{N}$. Then $x_i \notin I$ for all $i \in \mathbb{N}$, where $I$ is an interval. Hence we have produced some real number that was not there. Hence $\mathbb{R}$ is uncountable. ■

**Theorem 5.** If $A \subseteq B$ and $B$ is countable, then $A$ is either countable or finite.

**Remark.** A contrapositive of this lemma is:

> If $A$ is uncountable and finite, then $A \nsubseteq B$ or $B$ is uncountable

**Theorem 6.** If $A_i$ are each countable sets for $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i$ is countable.

**Remark.** A contrapositive of this lemma is:

> If $\bigcup_{i \in \mathbb{N}} A_i$ is uncountable, then there exists an $A_i$ that is uncountable for $i \in \mathbb{N}$.

## 1.7 Cantor's Theorem

### Lecture 5: Day 5

**Theorem 7** (Cantor's Diagonalization Argument)**.** Interval $C \in \mathbb{R}$ is uncountable.

**Proof.** Suppose for the sake of contradiction, $C$ is countable, and let $c_1, c_2, \ldots$ enumerate $C$ such that:

$$\begin{aligned} c_1 &= (c_{11}, c_{12}, c_{13}, \ldots) \\ c_2 &= (c_{21}, c_{22}, \ldots) \\ c_3 &= (c_{31}, \ldots) \\ &\vdots \end{aligned}$$

To construct $\gamma \in C$ such that $\gamma \neq c_i$ for all $i$, take

$$\gamma_j = 1 - c_{jj} = \begin{cases} 0 & , c_{jj} = 1 \\ 1 & , c_{jj} = 0 \end{cases}$$

But $\gamma \neq c_k$ for any $k$ since the entries of $\gamma$ do not match the diagonals. Therefore $\gamma$ cannot be $C$. ■

# Chapter 2

# Sequences and Series

### Lecture 6: Day 6

**Definition 14** (Sequence). A sequence on $\mathbb{R}$ is in fashion of $\mathbb{N} \to \mathbb{R}$.

**Definition 15** (Sequence Convergence). A sequence $(x_n)$ converges to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|x_n - L| < \varepsilon.$$

**Corollary.** This is true if and only if for every $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|x_n - L| < \frac{1}{m}.$$

**Proof.** From the original definition of Sequence Convergence, it is easy to see the corollary is true. Let $\varepsilon > 0$ and let $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$, then the reverse holds true by the archimedian principle. ■

## 2.3 The Monotone Convergence Theorem and a First Look at Infinite Series

**Definition 16** (Monotonically Increasing/Decreasing). A sequence $(x_n)$ is monotonically increasing, if:

$$x_1 \leq x_2 \leq x_3 \leq \ldots$$

or $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Similarly for monotonically decreasing if $x_n \geq x_{n+1}$. A function is monotonic if any of these hold true.

**Definition 17** (Bounded). $(x_n)$ is bounded if there exists $C \in \mathbb{R}$ such that

$|x_n| \leq C$ for all $n \in \mathbb{N}$.

**Theorem 8** (Monotonic Convergence Theorem)**.** A bounded monotonic sequence converges.

**Proof.** Assume without loss of generality that $(x_n)$ is increasing. Let $S := \{x \in \mathbb{R} : x = x_n \text{ for some } n \in \mathbb{N}\}$. Then S is bounded above. Suppose C is a bound for $(x_n)$, then $x_n \leq |x_n| \leq C$ for all $n \in \mathbb{N}$. Set $L := \sup(S)$ to show $(x_n) \to L$. Let $\varepsilon > 0$, then there exists $x \in S$ such that $x_n = x > L - \varepsilon$, otherwise $L - \varepsilon$ is an upperbound for S. For all $n > N$, then $x_n \geq x_N > L - \varepsilon$, since $(x_n)$ is increasing, then $|L - x_n| < \varepsilon$.

On the otherhand, $x_n \leq L$ for all $n$, thus $|L - x_n| = L - x_n < \varepsilon$. ■

**Proposition 2.** If $(x_n) \to L$, then $(-x_n) \to L$.

**Proof.** Given $\varepsilon > 0$, for some $N \in \mathbb{N}$ such that $n > N$, then $|x_n - L| < \varepsilon$. For every $n > N$, then $|(-x_n) - (-L)| = |L - x_n| < \varepsilon$. ■

**Proposition 3.** If $(x_n) \to L$, and $(x_n) \to M$, then $L - M$.

**Proof.** Given $\varepsilon > 0$, for $N_1, N_2 \in \mathbb{N}$ such that if for all $n > N_1$ $|x_n - L| < \dfrac{\varepsilon}{2}$ and $n > N_2$ $|x_n - M| < \dfrac{\varepsilon}{2}$, then take $N := \max\{N_1, N_2\}$, then for $n > N$

$$\begin{aligned}
|x_n - L| &< \frac{\varepsilon}{2} \\
|x_n - M| &< \frac{\varepsilon}{2}; \\
|L - M| &= |(L - x_n) + (x_n - M)| \\
&\leq |L - x_n| + |x_n - M| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrarily chose, then $L = M$. ■

**Proposition 4.** If $(x_n) \to L$ and $(y_n) \to M$, then $(x_n + y_n) \to L + M$.

**Proof.** Let $N_1, N_2 \in \mathbb{N}$ such that $n > N_1, N_2$, then for all $n > N_1$ $|x_n - L| < \dfrac{\varepsilon}{2}$ and for all $n > N_2$ $|y_n - M| < \dfrac{\varepsilon}{2}$. Set $N := \max\{N_1, N_2\}$

such that for all $n > N$, then $|x_n - L| < \frac{\varepsilon}{2}$ and $|y_n - M| < \frac{\varepsilon}{2}$. Then:

$$\begin{aligned}|(x_n + y_n) - (L + M)| &= |(x_n - L) + (y_n - M)| \\ &\leq |x_n - L| + |y_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

■

**Definition 18** (Triangle Inequality).

$$\begin{aligned}|a + b| &\leq |a| + |b| \\ ||a| - |b|| &\leq |a - b|.\end{aligned}$$

**Proposition 5.** If $x_n \to L$, then $(|x_n|) \to |L|$.

**Proof.** Let $\varepsilon > 0$ for some $N \in \mathbb{N}$ such that for all $n > N$, then $|x_n - L| < \varepsilon$. For all $n > N$, then:

$$\begin{aligned}||x_n| - |L|| &\leq |x_n - L| \\ &< \varepsilon.\end{aligned}$$

■

## Lecture 7: Day 7

**Proposition 6.** Any convergent sequence is bounded.thm:2.3.2]

**Proof.** Prove this as an exercise. ■

**Proposition 7.** Convergence is a "tail phenomenon". If $(x_n) = (z_n)$ for $n > C$, there is $C \in \mathbb{N}$ such that $(x_n) \to L$ if and only if $(z_n) \to L$.

**Proof.** Prove this as an exercise. ■

**Proposition 8.** For $(x_n) \to L$ and $(y_n) \to M$, then $(x_n y_n) \to LM$.

**Proof.** Let $\varepsilon > 0$ be given, then:

$$\begin{aligned}x_n y_n - LM &= x_n y_n - x_n M + x_n M - LM \\ &= x_n(y_n - M) + M(x_n - L)\end{aligned}$$

Let $0 < C \in \mathbb{R}$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$. By the first proposition, take $N_1, N_2 \in \mathbb{N}$ such that $n > N_1$, then $|x_n - L| < \frac{\varepsilon}{2(|M| + 1)}$.

**Remark.** Since M may be 0, we add 1.

For all $n > N_2$, then $|y_n - M| < \dfrac{\varepsilon}{2(|C|+1)}$. For all $n > N := \max(N_1, N_2)$:

$$\begin{aligned}
|x_n y_n - LM| &= |x_n(y_n - M) + M(x_n - L)| \\
&\leq |x_n|\,|y_n - M| + |M|\,|x_n - L| \\
&< |C| \frac{\varepsilon}{2(|C|+1)} + |M| \frac{\varepsilon}{2(|M|+1)} \\
&= \varepsilon \frac{|C|}{2(|C|+1)} + \varepsilon \frac{|M|}{2(|M|+1)} \\
&< \varepsilon(\frac{1}{2} + \frac{1}{2}) \\
&= \varepsilon.
\end{aligned}$$

■

**Proposition 9.** For $(x_n) \to L, (y_n) \to M$, if $M \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$ then $\left(\dfrac{x_n}{y_n}\right) \to \dfrac{L}{M}$.

**Proof.** Prove this as an exercise. Suffices to show $\left(\dfrac{1}{y_n}\right) \to \dfrac{1}{M}$, then use the product property. ■

**Proposition 10.** If $(y_n) \to M \neq 0$, then there exists $N \in \mathbb{N}$ such that for some $n > N$, then $y_n \neq 0$.

**Proof.** Prove this as an exercise. ■

**Proposition 11.** If $(y_n)$ is a convergent sequence with $M \neq 0$, and $y_n > 0$ for all $n \in \mathbb{N}$, then there exists $\varepsilon > 0$ such that $|y_n| > \varepsilon$.

**Proof.** Let $N \in \mathbb{N}$ such that $n > N$ $|y_n - M| < \dfrac{|M|}{2} > 0$ since $M \neq 0$. Then $|y_n| > \dfrac{|M|}{2}$. Now let:

$$\varepsilon := \frac{1}{2} \min\{|y_1|, |y_2|, \ldots, |y_n|, \frac{|M|}{2}\},$$

which is a finite list of positive numbers. Hence the minimum is also positive. ■

## Lecture 8: Day 8

**Proposition 12.** If $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y_n \to M \neq 0$, then there is $\varepsilon > 0$ such that $|y_n| \geq \varepsilon$.

**Proof.** Let $N \in \mathbb{N}$ such that $n > N$, then:

$$|y_n - M| < \frac{|M|}{2} \Rightarrow$$
$$|y_n| > \frac{|M|}{2}.$$

Take $\varepsilon := \min\{|y_1|, \ldots, \frac{|M|}{2}\}$, which is a finite list of positive numbers. Hence the minimum is also positive. ∎

**Remark.** An aside from Dr. Fu, not verbatim:

> Something I have noticed upon teaching this class [...] there seems to be a struggle with understanding negation in mathematics. [...] It is fine if you have never seen it or properly learned it, best to come to office hours to quickly fix this for the rest of this course. ∼ *Joseph H.G Fu*

The negation of the statement $(x_n) \to 4$:

$$\begin{aligned}
\neg(x_n \to 4) &= \neg[(\forall \varepsilon > 0(\exists N \in \mathbb{N}(n > N \Rightarrow |x_n - 4| < \varepsilon)))] \\
&= \neg(\neg\exists \varepsilon > 0)\neg(\neg\forall N \in \mathbb{N})\neg(\neg(n > N) \wedge (|x_n - 4| \geq \varepsilon)) \\
&= \cancel{\neg(\neg}\exists \varepsilon > 0)\cancel{\neg(\neg}\forall N \in \mathbb{N})\cancel{\neg(\neg}(n > N) \wedge (|x_n - 4| \geq \varepsilon)) \\
&= \exists \varepsilon > 0(\forall N \in \mathbb{N}(\exists n \in \mathbb{N} \wedge |x_n - 4| \geq \varepsilon)).
\end{aligned}$$

**Theorem 9** (Bolzano-Weierstrass)**.** Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

We will show this proof next class.

## 2.4 Subsequences and the Bolzano–Weierstrass Theorem

### Lecture 9: Day 9

**Example** (Examples of Sequences).

$$x_n = n, \quad \text{or} \qquad 1, 2, 3, \ldots.$$

$$x_n = \frac{1}{n}, \quad \text{or} \qquad 1, \frac{1}{2}, \frac{1}{3}, \ldots. \tag{2.1}$$

$$x_n = (-1)^n, \quad \text{or} \qquad -1, 1, -1, 1, \ldots. \tag{2.2}$$

$$x_n = (-1)^n + \frac{1}{n}, \quad \text{or} \qquad 0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, \ldots. \tag{2.3}$$

$$x_n = n - \mu(n), \quad \text{or} \qquad 0, 1, 0, 1, 2, 0, 1, 2, 3, \ldots.$$

$$\mu(n) := \max\left\{\binom{m}{2} : \binom{m}{2} \leq n\right\}.$$

$$x_n = \frac{1}{n + 1 - \mu(n)}, \quad \text{or} \qquad 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \ldots$$

$$x_n = \frac{n - \binom{m}{2}}{m}, \binom{m-1}{2} < n \leq \binom{m}{2}, m \geq 2,$$

$$\text{where } \binom{m}{2} := \begin{cases} \frac{m(m-1)}{2}, & m \geq 2 \\ 0, & m = 1, \end{cases} \quad \text{or} \quad \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots. \tag{2.4}$$

Both lim sup and lim inf are convergent subsequences.

**Theorem 10** (Bolzano-Weierstrass). Any bounded sequence of real numbers admits a convergent subsequence

**Proof.** For each $n \in \mathbb{N}$, put $y_n := \sup\{x_m : m \geq n\}$. Then $(y_n)$ is monotonically decreasing and bounded below. Define

$$\limsup x_n := \lim y_n.$$

Define $\liminf x_n$ similarly.

> **Remark.** Note that in the previous example, the following are bounded: (1.1), (1.2), (1.3), and (1.4).

Suppose $L$ is the limit of some subsequence $(x_{n_k})_k$. Then:

$$\liminf x_n \leq L \leq \limsup x_n.$$

Given such $L$ set $\varepsilon := \dfrac{L - \limsup x_n}{2}$. Let $N \in \mathbb{N}$ be large enough that:

$$n \geq N \Rightarrow |y_n - \limsup x_n| < \varepsilon.$$

The triangle inequality implies that for such $n$

$$y_n < \limsup x_n + \varepsilon = L - \varepsilon$$

which in turn implies that

$$x_n \leq y_n < L - \varepsilon \Rightarrow |x_n - L| = L - x_n > \varepsilon,$$

for all $n \geq N$. Since any subsequence $(x_{n_k})_k$ must have $n_k \geq N$ for large enough $k$ it follows that such a subsequence cannot converge to $L$.

Let

$$L^* := \limsup x_n.$$

By construction of $y_k$, for each $k \in \mathbb{N}$, we may find $m_k \geq k$ such that

$$x_{m_k} > y_k - \frac{1}{k}.$$

Since $y_k \geq x_{m_k}$ and

$$\lim_{k\to\infty} \left( y_k - \frac{1}{k} \right) = \lim y_k - \lim \frac{1}{k} = L' = \lim y_k.$$

the Squeeze Theorem implies that $x_{m_k} \to L^*$ as $k \to \infty$.

Unfortunately $(x_{m_k})_k$ is not necessarily a subsequence of $(x_n)$ since it may not be the case that $m_1 < m_2 < \ldots$. This is easily rectified by the Lemma below. ■

**Lemma 4.** Let $m_1, m_2, \ldots$ be a sequence of natural numbers that diverges to $\infty$. Then there is a subsequence $(m_{n_k})_k$ that is strictly increasing.

**Proof.** Let $n_1 := 1$. Having selected $n_1 < n_2 < \ldots < n_j$ and define $n_{j+1}$ recursively as follows. Since $m_k \to \infty$, given any constant $C$ there exists $K \in \mathbb{N}$ such that

$$k \geq K \Rightarrow m_k > C.$$

Take $C$ to be $n_j$, and $n_{j+1} := K$ to be in this relation. ■

**Example.**
(1) For any enumeration $(x_n)$ of $\mathbb{Q}$, every real number is the limit of some subsequence of $(x_n)$.

(2) For any enumeration $(x_n)$ of $(0,1)$,

$$\limsup x_n = 1, \quad \liminf x_n = 0.$$

## Lecture 10: Day 10

**Theorem 11** (Squeeze Theorem). Suppose $(x_n)$, $(y_x)$, and $(z_n)$ are sequences such that

$$z_n \leq x_n \leq y_n \quad \forall n \geq N$$

and $z_n, y_n \to L$. Then $\lim x_n = L$.

**Proof.** Let $\varepsilon > 0$ be given and let $N \in \mathbb{N}$ such that $n > N$, then

$$|y_n - L| < \varepsilon$$
$$|z_n - L| < \varepsilon.$$

Since $y_n, z_n \in (L-\varepsilon, L+\varepsilon)$, then $[z_n, y_n] \subset [L-\varepsilon, L+\varepsilon]$, but $x_n \in [z_n, y_n]$. Thus $|x_n - L| < \varepsilon$. ■

**Theorem 12.** The sequence converges if and only if $\liminf x_n = \lim x_n = \limsup x_n$.

**Proof.** ($\Leftarrow$). Given $\limsup x_n = \liminf x_n$, then since $\liminf x_n \leq x_n \leq \limsup x_n$, then by squeeze theorem, $\lim x_n = \limsup x_n = \liminf x_n$.

($\Rightarrow$). Given $x_n \to L$, then let $\varepsilon > 0$, $N \in \mathbb{N}$, $n \geq N$, then $|x_n - L| < \varepsilon$. $L - \varepsilon < x_n < L + \varepsilon$ if and only if for all $n > N$, $\{\mathrm{x}_m : m \geq n\} \subset (L-\varepsilon, L+\varepsilon). Then take y_n := \sup\{x_m : m \geq n\} \leq \sup(L-\varepsilon, L+\varepsilon) = L+\varepsilon$. Similarly, take $z_n := \inf\{x_m : m \geq n\} \leq \inf(L-\varepsilon, L+\varepsilon) = L - \varepsilon$. For all $n > N, y_n, z_n \in [L-\varepsilon, L+\varepsilon]$ if and only if

$$|y_n - L| < \varepsilon$$
$$|z_n - L| < \varepsilon \Rightarrow$$
$$\lim y_n = \lim x_n = \lim z_n = L.$$

■

**Definition 19** (Diverges)**.** A sequence $(x_n)$ diverges to $+\infty$ if for every $C \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, then $x_n > C$ and $x_n \to +\infty$. Similarly for $x_n \to -\infty$.

**Lemma 5.** Let $(x_n)$ be any sequence in $\mathbb{R}$. Suppose that there is a sequence in $\mathbb{N}$, $(m_1, m_2, m_3, \ldots)$ such that $m_k \to +\infty$ and $(x_{m_k}) \to L$. Then there exists a subsequence of $(x_n)$ that converges to $L$.

**Example.** For $n_k \in \mathbb{N}$, $n_1 < n_2 < \ldots$, then $n_k \to +\infty$.

**Proof.** Choose that $n_k \geq k$ for all $k \in \mathbb{N}$. ■

**Property.** $x_n \to \infty$ if and only if $x_{n_k} \to \infty$.

## Lecture 11: Day 11

No substantial work.

## Lecture 12: Day 12

**Theorem 13** (Bolzano-Weierstrass)**.** If $(x_n)$ is bounded, then there exists a subsequence converging to $\limsup x_n = L$.

**Proof.** To construct a sequence $m_k \in \mathbb{N}$ for all $k$, $m_k \to \infty$, and $\limsup x_{m_k} = L$. Let

$$y_k := \sup\{x_m : m \geq k\},$$

then $y_k \to L$, by definition. We have $m_k \geq k$ uch that $y_k \geq x_{m_k} \geq y_k - \frac{1}{k}$. Now since $y_k \to L$, then

$$\begin{aligned} \lim y_k - \frac{1}{k} &= \lim y_k - \lim \frac{1}{k} \\ &= L - 0 \\ &= L \end{aligned}$$

Hence, $x_{m_k} \to L$, as needed. ∎

## 2.5 The Cauchy Criterion

**Definition 20** (Cauchy Sequences)**.** A sequence $(x_n)$ in $\mathbb{R}$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$, then $|x_m - x_n| < \varepsilon$.

**Proposition 13.** Any convergent sequence is Cauchy.

**Proof.** Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that:

$$\begin{aligned} n > N &\Rightarrow |x_n - L| < \frac{\varepsilon}{2} \\ m > N &\Rightarrow |x_m - L| < \frac{\varepsilon}{2}. \end{aligned}$$

Then $m, n > N$ implies:

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |x_n - L| \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

∎

**Lemma 6.** Any Cauchy Sequence is bounded.

**Proof.** Let $N \in \mathbb{N}$ such that $m, n \geq N$, then $|x_m - x_n| < 1$ for $\{x_n : n \geq 1\}$. Then for $m \geq N$, then $|x_m| < |x_N| + 1$. This is infact implied by:

$$\begin{aligned} |x_N - x_m| &\geq ||x_m| - |x_N|| \\ &> |x_m| - |x_N| \end{aligned}$$

Hence, for any $m \in \mathbb{N}$, we have

$$|x_m| \leq \max\{|x_N| + 1, |x_1|, |x_2|, \ldots, |x_N|\}.$$

■

**Theorem 14.** Any Cauchy Sequence Converges.

**Proof.** Define $y_n := \{x_m : m \geq n\}$ and $z_n := \{x_m : m \geq n\}$. Let $\varepsilon > 0$, for some $N \in \mathbb{N}$ such that $m, n \geq N$, then $|x_n - x_m| < \varepsilon$. Then set $S_N := \{x_n : n \geq N\}$, thus:

$$\begin{aligned}
y_N - z_N &= \sup(S_N) - \inf(S_N) \\
&= \sup(S_N) - (-\sup(-S_N)) \\
&= \sup(S_N) + \sup(-S_N) \\
&= \sup(S_N - S_N) \\
&= \sup\{x_m - x_n : m, n \geq N\} \\
&= \sup\{|x_m - x_n| : m, n \geq N\} < \varepsilon.
\end{aligned}$$

Thus $\lim y_n = \lim z_n$. Hence, by a previous theorem, or a corollary of the squeeze theorem, our cauchy sequence converges. ■

## 2.6 Properties of Infinite Series

### Lecture 13: Day 13

**Definition 21** (Infinite Series). Given a sequence, $(a_1, a_2, a_3, \ldots)$, consider:

$$\sum_0^\infty a_n.$$

**Definition 22** (Convergence of Series). $\sum_0^\infty$ converges if the sequence of partial sums:

$$S_N := \sum_0^N a_n.$$

If $S_N \to L$ for $N \to \infty$, then $\sum_0^\infty a_n = L$.

**Definition 23** (Cauchy Criterion). The series $\sum a_n$, then for all $\varepsilon > 0$ then there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$, then:

$$\left|\sum_{n+1}^m a_n\right| < \varepsilon.$$

**Theorem 15.** If $\sum a_n$ converges, then $a_n \to 0$.

**Definition 24** (Harmonic Series).

$$\sum_1^\infty \frac{1}{n},$$

Diverges.

**Theorem 16.** If all $a_n \geq 0$, then $\sum a_n$ converges if and only if $(\sum a_n)$ is bounded.

**Definition 25** (Geometric Series). A series is geometric if for some $r \neq 0$, $a_0 \neq 0$, then:

$$\frac{a_{n+1}}{a_n} = r$$

**Remark.** That is

$$a_0 + ra_0 + r^2 a_0 + \ldots = a_0(1 + r + r^2 + \ldots.$$

Also take $r = \frac{1}{2}$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + = 2$$

**Lemma 7.** If $|r| < 1$, then $r^n \to 0$.

**Proof.** This is equivalent to $|r|^n \to 0$, so then for $r > 0$, then $r^0 = 1 > r > r^2 > r^3 > \ldots \geq 0$. ■

**Remark.** It is a bounded and monotonic sequence, so $r^n \to L$ But also $r^{n+1} \to L$ then $rL = L$ if and only if $(1 - r)L = 0$. Thus $L = 0$.

**Lemma 8.**

$$\sum_0^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

**Proof.**

$$\left(\sum_0^N r^n\right)(1-r) = (1+r+r^2+\ldots+r^N)(1-r)$$
$$= (1+r+r^2+\ldots+r^N) - (r+r^2+\ldots+r^N+r^{N+1})$$
$$= 1-r^{N+1}.$$

Then $\sum_0^n r^n(1-r) = 1-r^{N+1}$. Hence:

$$\sum_0^N r^n = \frac{1-r^{N+1}}{1-r}.$$

■

**Theorem 17** (Geometric Series Convergence). $\sum r^n$ converges if and only if $|r| < 1$.

**Proof.** If $|r| \geq 1$, then we have by the contrapositive of $\sum a_n$ converges then $a_n \to 0$, $\sum r^n$ diverges. If $|r| < 1$, then:

$$S_N = \frac{1-r^{N+1}}{1-r} \to \frac{1}{1-r} - \frac{\lim\limits_{N\to\infty} r^{N+1}}{1-r}$$
$$= \frac{1}{1-r}.$$

Hence, since all $S_N$ converges to $\dfrac{C}{1-r}$, then we have the series itself converge. ■

**Definition 26** (Absolute Convergence). If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.

**Definition 27.** If $\sum a_n$ converges but $\sum |a_n|$ diverges, then this series converges conditionally.

**Theorem 18.** If $\sum a_n$ converges absolutely, then itself must converge.

**Proof.** Apply the Cauchy Criterion. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for $m > n > N$ for

$$\sum_n^m |a_k| < \varepsilon.$$

Thus $|\sum_m^n a_k| \leq \sum_n^m |a_k|$ by the triangle inequality, for such $m, n$. ■

**Example.**

$$\sum_1 \frac{(-1)^n}{n} \quad \text{converges conditionally}$$

$$\sum_1 \frac{(-1)^n}{n^2} \quad \text{converges absolutely}$$

Any convergent geometric series, converges absolutely.

$$\sum |r^n| = \sum |r|^n = \frac{1}{1-|r|}$$

## Lecture 14: Day 14

**Theorem 19** (Comparison Test)**.** Given $|a_n| \leq |b_n|$ for all $n$, if $\sum b_n$ converges then $\sum a_n$ converges absolutely.

**Proof.** Given $\varepsilon > 0$, then for some $N \in \mathbb{N}$ such that $m \geq n \geq N$, then:

$$\sum_n^m b_k < \varepsilon \quad \text{(N exists by Cauchy Criterion).}$$

But $|\sum_n^m a_k| \leq \sum_n^m |a_k| \leq \sum_n^m b_k < \varepsilon$. ∎

**Lemma 9** (Lemma A)**.** If $a < \limsup x_n < b$, then there exists $N \in \mathbb{N}$ such that $n > N$ implies $x_n < b$.

**Proof.** Let $N \in \mathbb{N}$ such that for $n > N$ then $|y_n - L| < \dfrac{b-L}{2}$. Thus $y_n < \dfrac{b+L}{2} < b$. But $m \geq n$, hence $x_m \leq y_n$. ∎

**Lemma 10** (Lemma B)**.** For all $N \in \mathbb{N}$ there exists $n > N$ such that $a < x_n$.

**Proof.** Let $N \in \mathbb{N}$ such that for $n > N$, $|y_n - L| < \dfrac{L-a}{2}$. Thus $y_n > \dfrac{a+L}{2} > a$. Hence, there is $m \geq N$ such that $x_m > \dfrac{a+L}{2}$. ∎

**Theorem 20** (Root Test)**.** A series $\sum a_n$ converges absolutely if:

1. $$L := \limsup |a_n|^{\frac{1}{n}} < 1.$$

2. Diverges if
$$\limsup |a_n|^{\frac{1}{n}} > 1.$$

**Proof.** Let $L < \dfrac{1+L}{2} = r < 1$, then $0 \le L < r < 1$. By Lemma A, there is $N \in \mathbb{N}$ such that $n \ge N$ implies

$$|a_n|^{\frac{1}{n}} r \Leftrightarrow |a_n| < r^n$$

Then $\sum |a_n|$ converges by comparison test. Hence $\sum |a_n| \to \sum^{N+1} |a_n| + \sum_N |a_n|$.

By Lemma B, for all $N \in \mathbb{N}$ there exists $n \ge N$ such that

$$|a_n|^{\frac{1}{n}} > \frac{1+L}{2} = R.$$

Thus $|a_n| > R > 1$. Hence $a_n \not\to 0$. ■

**Theorem 21** (Ratio Test)**.** If

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| = L < 1,$$

then $\sum a_n$ converges absolutely.

If

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| = M > 1$$

then $\sum a_n$ diverges.

**Proof.** By Lemma A, there exists $N$ such that $n \ge N$ implies $\left| \dfrac{a_{n+1}}{a_n} \right| < \dfrac{1+L}{2} < 1$. Then $|a_{N+1}| < r\,|a_N|$ implies $|a_{N+2}| < r\,|a_{N+1}| < r^2\,|a_N|$. Thus $n \ge N$, therefore $|a_n| \le r^{n-N}\,|a_N|$. By comparison test, $\sum |a_N|\, r^{n-N}$ converges, thus $\sum |a_n|$ converges. Hence $\sum a_n$ converges.

A modified Lemma A implies there exists $N \in \mathbb{N}$ for $n \ge N$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > \frac{1+M}{2} = R > 1$$

Now $|a_{N+1}| > R\,|a_N|$. Thus for $k \ge N$ implies $|a_k| > R^{k-N}\,|a_N| > |a_N| > 0$. Hence $a_n \not\to 0$. ■

## Lecture 15: Day 15

No substantial work.

## Lecture 16: Day 16

Review for Test.

### Lecture 17: Day 17

Test One Day.

### Lecture 18: Day 18

No substantial work.

## 2.7 Double Summations and Productsof Infinite Series

### Lecture 19: Day 19

**Definition 28** (Cauchy Product)**.** The Cauchy product of two series is

$$\sum_{i+j=N} a_i b_j$$

**Theorem 22** (Cauchy Product Convergence)**.** If both series converge absolutely, then the cauchy product converges absolutely:

$$\sum_{i+j=N} c_N = \left(\sum a_n\right)\left(\sum b_m\right) = AB$$

**Proof.**

$$\left|\sum_{\substack{i+j>N \\ 0\le i,j\le N}} a_i b_j\right| \le \sum |a_i|\,|b_j|$$

$$\le \sum_{N/2<i\le N} |a_i| \sum_{j=0}^{N} |b_j| + \sum_{i=0}^{N} |a_i| \sum_{N/2<j\le N} |b_j|\,. \tag{2.5}$$

Let $A' = \sum |a_i|, B' = \sum |b_j|$. Given $\varepsilon > 0$, choose $M + N$ such that $m > n > M$ implies $\sum_n^M < \frac{\varepsilon}{2B'}, \sum_m^m |b_j| = \frac{\varepsilon}{2A'}$. If $A', B' = 0$, then there is nothing to prove. Else, if $N > 2M$, then (1.5) holds less than $\varepsilon$. ∎

### Lecture 20: Day 20

No substantial work.

# Chapter 3

# Topology of $\mathbb{R}$

## 3.2 Open and Closed Sets

### Lecture 21: Day 21

**Definition 29** ($\varepsilon$-Neighbourhood)**.** Let $\varepsilon > 0$. The neighbourhood of $x \in \mathbb{R}$ is:

$$\vee_\varepsilon(x) := \{y : |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

**Definition 30** (Open Set)**.** A set is open if it contains an epsilon of all of its points.

**Example** (Any open interval is open)**.** If $x \in (a, b)$, take $\varepsilon := \min\{b - x, x - a\} > 0$. Then $\vee_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

**Theorem 23** (Union of Open Sets)**.** The union of any family of open sets is open.

**Proof.** Given a family $\{U_i\}$ of open sets, suppose $x \in \bigcup_\alpha U_\alpha$. Then $x \in U_{\alpha_0}$ for some $\alpha_0 \in I$. Since $U_{\alpha_0}$ is open, then by definition, there exists $\varepsilon > 0$ such that $\vee_\varepsilon(x) \subset U_{\alpha_0} \subset \bigcup_\alpha U_\alpha$. ■

**Example** (Examples of Open Sets)**.** $\{0\}$ is not open.

$(0, 1]$ is not open.

$\varnothing$ is open and is vacuously true.

**Example.** Let $I = \mathbb{N}$, and $\{q_1, q_2, \ldots\}$ be an invocation of $\mathbb{Q}$. Deduce for $n \in \mathbb{N}$, $U_n := \vee_{1/2^n}(q_n) = (q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n})$. $U := \bigcup_1 U_n$ is open. The total length of $U < \sum \frac{1}{2^{n-1}} = 4$.

**Theorem 24** (Intersection of Open Sets)**.** The intersection of any finite collection of open sets is open.

**Proof.** Let $U_1, \ldots, U_N$ be open and $x \in \bigcap_1^N U_i$, then since $U_i$ is open, then there exists $\varepsilon_i > 0$ such that $x \in \vee_{\varepsilon_i}(x) \subset U_i$. Put $\varepsilon := \min\{\varepsilon_i\} > 0$. Then for each $i = 1, \ldots, N$, $\vee_\varepsilon \subseteq \vee_{\varepsilon_1} \subseteq \ldots \subseteq \vee_{\varepsilon_N} \subseteq U_i$. Hence, $\vee_\varepsilon \subseteq \bigcap_1^N U_i$. ■

**Definition 31** (Limit Point)**.** Let $E \subset \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is a limit point of $E$ if for every $\varepsilon > 0$,

$$(\vee_\varepsilon(x_0) \cap E) \setminus \{x_0\} \neq \varnothing$$

**Example.** If $E = (0, 1)$, then $x_0 \in [0, 1]$.

**Example.** $\{c\}$ is a singleton set, which has no limit points since there cannot be an epsilon-neighbourhood within the same set.

**Definition 32** (Isolated Point)**.** Given $E \subset \mathbb{R}$, $x_0 \in E$, $x_0$ is an isolated point of $E$ if $x_0$ is not a limit point.

**Proposition 14.** Given a set $E \subset \mathbb{R}$, there exists $x_0 \in \mathbb{R}$ is a limit point of $E$ if and only if there exists $(x_n)$ in $E$ such that $x_1 = x_0$ and $x_n \to x_0$.

**Proof.** Suppose $x_0$ is a limit point of $E$, then for every $n \in \mathbb{N}$, $\vee_{\frac{1}{n}}(x_0) \cap E$ contains atleast one point of $E$ other than $x_0$, call it $x_n$. Let $x_1, x_2, \ldots$ be the sequence constructed in that way. We see all $x_n \in E$, then to see $x_n \to x_0$: Let $\varepsilon > 0$, for some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ if and only if $\frac{1}{N} < \varepsilon$. Then $x \geq N$ if and only if $|x_n - x_0| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. ■

**Definition 33** (Closed)**.** $E$ is closed if it contains all its limit points.

**Example.** $(0, 1)$ is not closed, $0, 1 \notin (0, 1)$.
$\{0\}$ is not closed.
$\varnothing$ is closed and open.
$\mathbb{Z}$ is closed, all isolated points, no limit points.
$\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed. 0 is not a limit point contained.

**Corollary.** $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed. 0 is not a limit point contained.

**Proof.** Given $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$. $S \cap \left(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n} + \frac{1}{n+1}\right) = S \cap \left(\frac{1}{n+1}, \frac{2n+1}{n(n+1)}\right) = \left\{\frac{1}{n}\right\}$. ∎

**Definition 34** (Closure). The closure of $E$, labelled $\overline{E}$ or $\mathrm{cl}(E)$ in the union of $E$ with all its limit points.

**Example.** $\overline{(0,1)} = [0,1]$.
$\overline{\{0\}} = \{0\}$.

**Proposition 15.** $E$ is closed if and only if $E = \mathrm{cl}(E)$.

**Proof.** Suppose $y$ is a limit point of $\mathrm{cl}(E)$, then let $\varepsilon > 0$. If $x \in \vee_\varepsilon(y) \cap \mathrm{cl}(E)$. If $x \in E$, we are done as $\mathrm{cl}(E) = E \cup L :=$ { all the limit points of $E$}. If $x \notin E$, then $\vee_{\varepsilon - |y-x|}(x) \cap E \neq \varnothing$. Any epsilon neighbourhood of $x$ meets $E$, $\vee_\varepsilon(x) \supset \vee_{\varepsilon-|y-x|}(x) \cap E \neq \varnothing$. Take $\varepsilon$ to lie in its set. ∎

### Lecture 22: Day 22

**Theorem 25.** $U$ is closed if and only if $U^c$ is open.

**Proof.** Suppose $U$ is open. For all $\varepsilon > 0$, $\vee_\varepsilon(x) \cap U^c \setminus \{x\} \neq \varnothing$. If $x \in U$, then there exists $\varepsilon_0 > 0$ such that $\vee_{\varepsilon_0}(x) \subset U$. Take $\varepsilon = \varepsilon_0$, we want $\vee_\varepsilon(x) \cap U \neq \varnothing$, but $\vee_\varepsilon(x) \subset U$, so a contradiction arises.

Suppose $U^c$ is closed, then let $x \in (U^c)^c = U$. Then $x \notin U^c$ implies there is not a point of $U$ such that there is an $\varepsilon > 0$ such that $\vee_\varepsilon(x) \cap U^c \setminus \{x\} = \varnothing$. Hence, $\vee_\varepsilon(x) \subset U$, since $x \in U$ to begin with. ∎

## 3.3 Compact Sets

### Lecture 23: Day 23

**Definition 35** (Open Cover). An open cover of $C$ is a family of sets such that all $U_\alpha$ is open set such that

$$C \subset \bigcup_{\alpha \in A}.$$

**Example.** Is $\mathbb{R}$ a subcover of $\mathbb{R}$. No. $\{\{x\}\}$ a singleton set is not closed and bounded.

**Example.** If $A = \mathbb{Z}$, then $U_n := (n-1, n+1)$.

**Example.** If $U_0 := (-\infty, 1)$ and $U_1 := (-1, \infty)$, these are open covers of $\mathbb{R}$.

**Example.** $C := \{\frac{1}{n} : n \in \mathbb{N}\}$ has no finite subcover.

**Theorem 26** (Heine-Borel). Given $C \subset \mathbb{R}$, the following are equivalent.

1. $C$ is closed and bounded.
2. Every sequence $(x_n)$ in $C$ admits a convergent subsequence in $C$.
3. Every open cover of $C$ admits a finite subcover.
4. $C$ is compact.

**Example.** $C' := \{0\} \cup C$ is closed and bounded and has finite subcovers.

### Lecture 24: Day 24

Restatement of Heine-Borel.

**Theorem 27** (Heine-Borel). Let $C \subset \mathbb{R}$. The following are equivalent:

1. $C$ is closed and bounded.
2. Bolzano-Weierstrass holds.
3. **Every** open cover admits a finite subcover.
4. $C$ is compact.

**Proof.** **((1)⇒(2)).** Given $(x_n)$ a sequence in $C$, it is bounded, hence Bolzano-Weierstrass holds since there is a convergent subsequence $x_{n_k} \to x^*$.

> **Remark.** Recall that in a bounded interval, there will always be a subsequence that will atleast go to $\sup(C)$!

Either $x^* = x_{n_k}$ for some $k$ or $x^*$ is a limit point of $C$, since $C$ is closed.

**((2)⇒(1)).** If $C$ is unbounded, then for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $|x_n| > n$. So $(|x_n|)$ diverges, therefore $(x_n)$ cannot converge. If $z$ is a limit point of $C$, then there exists a sequence $(x_n)$ in C such that $x_n \to z$ and so does every subsequence, so $z \in C$.

> **Remark.** Here we take a break from connecting (3).

■

**Example.** If $C = [0,1]$, for each $x \in C$, let $\varepsilon > 0$, then put $U_x = \vee_\varepsilon(x)$. Hence, $\{U_x\}$ is an open cover of $[0,1]$.

**Lemma 11** (How to find an open cover of $C$.)**.** Let $\{U_x\}$ be some family of open sets in $\mathbb{R}$. Then there exists a countable set $\{\alpha_1, \alpha_2, \dots\} \subset A$, such that
$$\bigcup_1^\infty U_{\alpha_i} = \bigcup_{\alpha \in A} U_\alpha.$$

**Remark.** This lemma was confusing to the majority of the class, however, I will try my best to explain as I feel confident in my understanding.

Let $\mathcal{Q} := \{(p,q) : p,q \in \mathbb{Q}, p < q\}$. Then $\mathcal{Q}$ is countable. This follows from the properties of rationals. Let $\mathcal{Q}' := \{(p,q) \in \mathcal{Q} : \exists \alpha \in A, (p,q) \subset U_\alpha\}$, thus countable and open. Given $(p,q) \in \mathcal{Q}'$, select $\phi(p,q) \in A$, which is a subcount of the original $\alpha \in A$ such that $(p,q) \subset U_{\phi(p,q)}$. Thus $\{U_{\phi(p,q)}\}_{(p,q)\in\mathcal{Q}'}$ is countable.

**Remark.** Given that $\mathcal{Q}$ is countable, we are trying to find an open cover that can be comprised of subcovers. Since we have found a countable set of intervals, then we have to take a subset of $\mathcal{Q}$ with more restrictions to show that it is infact open, hence we now have countable subcovers, which means we can also take finitely many of them, $\phi(p,q)$ as our subcounts, which we will show next.

If $x \in \bigcup_{\alpha\in A} U_\alpha$ then $x \in U_{\alpha_0}$ for some $\alpha_0 \in A$. Since $U_{\alpha_0}$ is open, then there exists $\varepsilon > 0$, such that $\vee_\varepsilon(x) \subset U_{\alpha_0}$. Select $p = \mathbb{Q} \cap (x - \varepsilon, x)$ and $q = \mathbb{Q} \cap (x, x+\varepsilon)$, then $x \in (p,q) \subset \vee_\varepsilon(x) \subset U_{\alpha_0}$. Thus $(p,q) \in \mathcal{Q}'$ and $x \in (p,q) \subset U_{\phi(p,q)}$. Hence,
$$x \in \bigcup_{(p,q)\in\mathcal{Q}'} U_{\phi(p,q)}.$$

**Remark.** We now will take an open interval around some element $x$, which is contained within the open set $U_x$. We can then select $p,q \in \mathbb{Q}$, since this way we can easier select these subcovers, instead of choosing $p,q \in \mathbb{R} \setminus \mathbb{Q}$ for the sake of simplicty. This will also let us stay within the assumptions of $\mathcal{Q}$ and therefore $\mathcal{Q}'$. Finally, we are able to show we have a subcover that contains all $x$ in the open cover, which is also included in the finite subcovers of such open cover!

■

## Lecture 24: Day 24

No substantial work.

## Lecture 25: Day 25

**Proof.** **((2)⇒(3)).** Let $\{U_\alpha\}$ be an open cover of $C$. By the lemma, we may assume $A = \mathbb{N}$. Then we want to show that $\bigcup_1^N U_i \supset C$ for some $N \in \mathbb{N}$. Otherwise, for each $n \in \mathbb{N}$, there exists $C \setminus \bigcup_1^n U_i \neq \varnothing$. Take $x_n \in C$, a sequence such that $x_n \in C$, but $x_n \notin \bigcup_1^n U_i$. Let $x \in C$ such that $x_n \to x$. But $x \in C \subset \bigcup^\infty U_i$. Hence $x \in U_M$ for some $M \in \mathbb{N}$. Since $U_M$ is given, then there exists $k \in \mathbb{N}$ such that $k \geq K$ implies $x_{n_k} \in U_n$. But $n_k > M$ for some $k$ sufficiently large, then $x_{n_k} \in \bigcup_1^M U_i \supset U_M$, which is a contradiction.

**((3)⇒(1)).** If $C$ is unbounded, let $U_n := (-n, n), n \in \mathbb{N}$. Then $\bigcup^\infty U_n = \mathbb{R}$. If there is a finite subcover, then there exists $N \in \mathbb{N}$ such that $(-N, N) = U_N \supset C$. If $C$ is not closed, then there exists a limit point that is not in $C$. Let $x \in \mathbb{R}, x \notin C$, and $x$ is a limit point. Let $U_n := \mathbb{R} \setminus (x - \frac{1}{n}, x + \frac{1}{n})$. Then $\{U_n\}$ covers $C$ if $y \in C, y \neq x$, so take $n \in \mathbb{N}$ such that $\frac{1}{n} < |y - x|$. Then $y \in U_n$, but $c \notin U_n$ for any $N \in \mathbb{N}$. Since $x$ is a limit point, for any $N \in \mathbb{N}$, there exists a point $z \in C \cap \vee_{1/N}(x)$ implying $z \notin U_N$. Since $U_1 \subset U_2 \subset \cdots$, it is enough to show there is no finite subcover. ■

**Theorem 28** (Nested Compact Sets Property)**.** If $C_1 \supset C_2 \supset \cdots$ are all compact, non-empty, then $\bigcap_1^\infty C_i \neq \varnothing$.

**Proof.** Let $x_n \in C_n, n \in \mathbb{N}$. Then $x_m \in C_m$ whenever $m > n$, for each $n$, the tails of the sequence lies in $C_n$. By B-W, there is a sequence $(x_{n_k})$ converging to $x \in C_n$ and since $n_k \geq k$, there is a finite subsequence $x \in C_N$ for every $N$. As such

$$x \in \bigcap_1^\infty C_i.$$

■

## 3.4 Perfect Sets and Connected Sets

### Lecture 26: Day 26

**Theorem 29** (Nested Compact Sets Property)**.** If $C_1 \supset C_2 \supset \cdots$ are all compact, non-empty, then $\bigcap_1^\infty C_i \neq \varnothing$.

**Proof.** If $\bigcap_1^\infty C_i = \varnothing$, then $C_1 \cap (\bigcap_2^\infty C_i) = \varnothing$. Thus $C_1 \subset (\bigcap_2^\infty C_i)^c$ thus $\{C_i^c\}_2^\infty$ is an open cover of $C$. Since $C_1$ is compact, there is $N \in \mathbb{N}$ such that $C_1 \subset \bigcap_2^N C_i^c = (\bigcap_2^\infty C_i)^c$. Hence, $C_1 \cap \bigcap_2^N C_i = \varnothing$. ■

**Definition 36** (Cantor Set)**.** Take $C_1 = [0, 1], C_2 = [0, \frac{1}{2}] \cup [\frac{2}{3}, 1], C_3 =$

$[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Hence, $C_{n+1} = \frac{1}{3} C_n \cup \left(\frac{2}{3} + \frac{1}{3} C_n\right)$. All $C_i$ are compact since they are closed and bounded. $C_i \neq \varnothing$, then $\mathcal{C} = \bigcap_1^\infty C_i \neq \varnothing$.

**Remark.**

1. $\mathcal{C} = \frac{1}{3} C \cup \left(\frac{2}{3} + \frac{1}{3} C\right)$.
2. Length of $[0,1] \setminus \mathcal{C}$ is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = \sum_0^\infty \frac{1}{3} \left(\frac{2}{3}\right)^n = 1.$$

**Definition 37** (Perfect). $S \subset \mathbb{R}$ is perfect if $S$ is closed and every if $S$ is a limit point of $S$.

**Remark.** Every point of $S$ is a limit point of $S$ if and only if $S$ contains no isolated points.

**Property.** Cantor Set is perfect.

**Example** (Non-example and Example of Perfect Sets).

1. $\{\frac{1}{n}\} \cup \{0\}$ is closed, not perfect.
2. $[a, b]$ is perfect.
3. The Cantor Set is perfect.

## Lecture 27: Day 27

**Theorem 30.** Any non-empty perfect sets is uncountable.

**Proof.** Suppose $P \neq \varnothing$ is not perfect and let $x_1, x_2, \ldots \in P$. To show: $P \setminus \{x_1, x_2, \ldots\} \neq \varnothing$. To select closed, bounded $J_1 = [a_1, b_1] \supset J_2 = [a_2, b_2] \supset J_3$ such that $x_n \notin J_n$ and $J_n \cap P$ is infinite. If this can be done, then every $J_n \cap P$ is compact. Implies

$$\bigcap_{n=1}^{\infty} (J_n \cap P) \neq \varnothing,$$

which will conclude the proof. Take any point $z_1 \in P, z_1 \neq x$ and $\varepsilon := |z_1 - x_1|$ and set $J_1 := \overline{V_{\varepsilon/2}}(z_1) = [z - \frac{\varepsilon}{2}, z_1 + \frac{\varepsilon}{2}]$. Then $J_1 \cap P$ is infinite where $x_1 \neq \bigcup V_{\varepsilon/2}(z_1), z_1 \in P$. Let $z_2 \in J_1 \cap P, z_2 \neq x_2, z_2$ is not a compact of $J_1$. Take $\varepsilon$ such that $\overline{\vee_\varepsilon(z_1)} \subset J_1$ where $[z_2 - \varepsilon, z_2 + \varepsilon] = J_2$.

$$\varepsilon := \frac{1}{2}\min\{|x_1 - z_1|, |z_2 - J_2|, |z_n - J_n|\}.$$

Continuing in this way, we can unite at sequence of $J$. ■

**Definition 38** (Limit Point). Let $A \subset \mathbb{R}, A \neq \varnothing, f : A \to \mathbb{R}$, then $x_0$ is a limit point if and only if

$$\lim_{x \to x_0} f(x) = L.$$

This means for any $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < |x - x_0| < \delta$ implying $|f(x) - L| < \varepsilon, x \in A$.

# Chapter 4

# Functional Limits and Continuity

## 4.2 Functional Limits

### Lecture 28: Day 28

**Example.**

$$\lim_{x\to 3} x^2 = 9$$

**Proof.** Let $\varepsilon > 0$ be given. We want to show that $\left|x^2 - 9\right| = |x-3|\,|x+3| < \varepsilon$. Take $\delta := \min\{1, \frac{\varepsilon}{7}\}$. Then $|x-3| < \delta$ implies $2 < x < 4$. Thus:

$$\begin{aligned}\left|x^2 - 3\right| &= |x-3|\,|x+3| \\ &< 7\,|x-3| \\ &< \frac{7\varepsilon}{7} \\ &= \varepsilon\end{aligned}$$

■

**Theorem 31.**

$$\lim_{x\to x_0} f(x) = L$$

if and only if for any sequence $x_1, x_2, \ldots \in A, x_n \neq x_0$ such that $x_n \to x_0$, we have

$$\lim_{n\to 0} f(x_n) = L.$$

**Proof.** **(⇒).** Assume $x_1, x_2, \ldots, x_n \in A, x_n \to x, \forall n$. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N$ implying $|x_n - x_0| < \delta$. Thus $|f(x_n) - L| < \varepsilon$. Since $x_n \in A, x_n \neq x_0$.

**(⇐).** We will prove the contrapositive. Assume that if it is not true that $\lim_{x \to x_0} f(x) = L$ and we will construct a sequence $x_1, x_2, \ldots \in A, x_1 \neq x_0$ such that $f(x_0) \not\to L$. This assumption means there is $\varepsilon > 0$ such that for every $\delta > 0$ there is $x \in A, x \neq x_0, 0 < |x - x_0| < \delta$ such that $|f(x) - L| \geq \varepsilon$. For each $n \in \mathbb{N}$, let $x_n \in A$ such that $0 < |x_n - x_0| < \frac{1}{n}$, then $x_n \neq x_0$. Thus $|f(x_n) - L| \geq \varepsilon$. Thus $f(x_n) \not\to L$. ■

**Theorem 32.** Suppose $f, g : A \to \mathbb{R}, c \in \mathbb{R}$

$$\lim_{x \to x_0} f(x) = L,$$

$$\lim_{x \to x_0} g(x) = M.$$

Then,

1. $\lim f(x) + g(x) = L + M$.
2. $\lim f(x)g(x) = LM$.
3. $\lim cf(x) = cL$.
4. $\lim \frac{f(x)}{g(x)} = \frac{L}{M}$.

**Definition 39** (One-Sided Limits). Given $A \subset \mathbb{R}$, $c$ is a right limit point of $A$ if $\forall \varepsilon : (c, c + \varepsilon) \cap A \neq \varnothing$. Similarly left limit point is$(c - \varepsilon, c) \cap A \neq \varnothing$.

**Definition 40.** Suppose $c$ is a right limit point of $A$, then we write

$$\lim_{x \to c^+} f(x) = L$$

or more tangibly, we have for all $\varepsilon > 0$, there is a $\delta > 0$ such that $c < x < c + \varepsilon$ implies $|f(x) - L| < \varepsilon$. Then $x \in (c, c + \varepsilon) \cap A$.

**Proposition 16.**

$$\lim_{x \to c^+} f(x) = L$$

if and only if for every sequence $x_1, x_2, \ldots \in A$ for $x_n > c$ and $x_n \to c$, we have

$$\lim_{n \to \infty} f(x_n) = L.$$

**Proposition 17.** Suppose $c$ is a right and left limit point of $A$, then

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L.$$

## 4.3 Continuous Functions

**Definition 41** (Continous). Suppose $A \subset \mathbb{R}, f : A \to \mathbb{R}, c \in A$. We say that $f$ is continuous at $c$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| > \varepsilon$.

**Remark.** Any $f$ is continous at any isplated point of $A$.

**Definition 42** (One-Sided Continuity). $f$ is continuous from the right of $c$, if for all $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < x - c < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

**Example.**

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Not left continuous at $x = 0$, but is right continuous at 0.

**Example.**

$$\{1/n\} =: S$$

0 is a right limit point of $S$, and not left.

**Remark.** If $c$ is a limit point of $A$ and $c \in A$, then $f : A \to \mathbb{R}$ is continuous at c if and only if

$$\lim_{x \to c} f(x) = f(c)$$

**Example.**

$$f(x) = \lfloor x \rfloor = n$$

if $n \in \mathbb{Z}$ and $n \leq x < n + 1$. It is continuous at all $x \notin \mathbb{Z}$, and right continuous at all points of $\mathbb{R}$ but not left continuous at $n \in \mathbb{Z}$.

**Example.**

$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

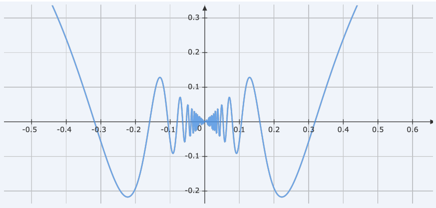



Figure 4.1: Graph of $x \sin(1/x)$.

Continuous at all $c \in \mathbb{R}$.

**Proposition 18.** This function is continuous at 0.

**Proof.** Let $\varepsilon > 0$, then

$$\begin{aligned} |x| &= |x - 0| < \varepsilon \\ \Rightarrow |x \sin(1/x)| &= |x|\,|\sin(1/x)| \\ &\leq |x| \qquad , x \neq 0. \end{aligned}$$

■

**Theorem 33.** Suppose $f : A \to \mathbb{R}$ and $f(A) \subset B$ such that $f(A) := \{f(a) : a \in A\}$ and $g : B \to \mathbb{R}$ and $c \in A$ and $c$ is continuous at $c$ and $g$, then $g \circ f$ is continuous at $c$.

**Proof.** Let $\varepsilon > 0$. We want to show tht there exists a $\delta > 0$ such that $x \in A$ and $|x - c| < \delta$ implies $|g(f(x)) - g(f(c))| < \varepsilon$. First, since $g$ is continuous at $f(c)$, then there ies a $\delta' > 0$ such that $g \in (B)$ and $|y - f(c)| < \delta'$ implies $|g(y) - g(f(c))| < \varepsilon$.

Given some $\delta'$, there is $\delta > 0$ such that $x \in A$, there is some $\delta < \varepsilon$ such that $|f(x) - f(c) < \delta'|$. Implying $|g(f(x)) - g(f(c))| < \varepsilon$. By taking $y = f(x)$. ■

**Definition 43** (Continuous)**.** A function $f : A \to \mathbb{R}$ is continuous if $f$ is continuous at every $c \in A$.

## 4.4 Continuous Functions on Compact Sets

**Theorem 34** (Continuous Functions on Compact Sets)**.** Suppose $C \subset \mathbb{R}$ is compact and $f : C \to \mathbb{R}$ is continuous, then $f(C)$ is compact.

**Proof.** We want to show there is a subsequence that converges in $f(C)$. Let $x_n \in C$ such that $f(x_n) = y_n$. Since $C$ is compact, there is $x \in C$ and $x_{n_k} \to x \in C$ as $k \to \infty$. THen by continuity of $f$, $f(x_{n_k}) \to f(x)$ as $k \to \infty$, all in $f(C)$. ■

**Corollary.** $f$ is continuous at $c$ if and only if whenever $A \ni x_n \to c \in A$, we have $f(x_n) \to f(c)$.

## 4.5 The Intermediate Value Theorem

### Lecture 29: Day 29

**Lemma 12.** Suppose $S \subseteq \mathbb{R}$ is nomempty, closed, bounded, then $S$ contains $\sup(S)$.

**Proof.** $\sup(S)$ is a limit point, since $S$ is closed ■

**Corollary.** $f$ is continuous at $c$ if and only if whenever $A \ni x_n \to c \in A$, we have $f(x_n) \to f(c)$.

**Proof.** Let $A = [a, b]$ a closed and bounded interval. Let $V = f([a, b])$, $V$ is compact since $f$ is continuous and $[a, b]$ is compact. By the lemma, $V$ contains $\sup(V)$, closed $[a, b] \ni x$, with $\sup(V) = f(x_1)$. Then, we need $f(x_1) \geq f(x)$ for all $x \in (a, b)$. Take any $x \in [a, b]$, then $f(x) \in V$, do $f(x) \leq \sup(V) = f(x_1)$. Next, we do the same thing by with the infimum. ■

**Theorem 35** (Intermediate Value Theorem)**.** Let $(X, d)$ connected and $f : X \to \mathbb{R}$ continuous. If $a, b \in (X)$ and $a < r < b$, then $r \in f(X)$.

> **Remark.** I like this proof better

Suppose $r \in f(x)$. Let $A = (-\infty, r)$ and $B = (r, \infty)$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a disjoint union of open sets, hence a contradiction. ■

**Theorem 36.** Let $C \subseteq \mathbb{R}$ be compact. Suppose $f : C \to \mathbb{R}$ is continuous at each point of $C$. Then $f$ is uniformly continuous on $C$.

**Proof.** Assume for the sake of contradiction that the uniform continuity condition fails at $\varepsilon > 0$. Then $\forall n \in \mathbb{N}, \exists x_n, y_n \in C$ with $|x_n - y_n| < \frac{1}{n}$, $|f(x_n) - f(y_n)| \geq \varepsilon$. We can extract a convergent subsequence $(x_{n_k})$ from $(x_n)$, say $x_{n_k} \to x \in C$. Notice $y_{n_k} = x_{n_k} + (y_{n_k} - x_{n_k}) \to x$, so $y_{n_k} \to x$. Since $f$ is continuous, $f(x_{n_1}) \to f(x)$, and $f(y_{n_k}) \to f(x)$, then $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(x) = 0$. Then $|f(x_{n_k}) - f(y_{n_k})| \to 0$, but our assumption is $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$, hence a contradiction arose. ■

### Lecture 30: Day 30

**Definition 44** (Uniform Continuity)**.** Let $S \subset \mathbb{R}$, a function $f : S \to \mathbb{R}$ is uniformly continuous on $S$ if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in S)(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon)$.

**Theorem 37.** Let $C \subseteq \mathbb{R}$ be compact. Suppose $f : C \to \mathbb{R}$ is continous at each point of $C$. Then, $f$ is uniformly continous on $C$.

**Proof.** Assume for the sake of contradiction that the uniform continuity condition fails at $\varepsilon > 0$. Then $\forall n \in \mathbb{N}, \exists x_n, y_n \in C$ with $|x_n - y_n| < \dfrac{1}{n}$, then $|f(x_n) - f(y_n)| \geq \varepsilon$. We can extract a convergent subsequence $(x_{n_k})$ from $(x_n)$, say $x_{n_k} \to x \in C$. Notice $y_{n_k} = x_{n_k} + (y_{n_k} - x_{n_k}) \to x$, so $y_{n_k} \to x$. Since $f$ is continuous, then $f(x_{n_k}) \to f(x)$, $f(y_{n_k}) \to f(x)$, so $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(x) = 0$. Then $|f(x_{n_k}) - f(y_{n_k})| \to 0$, however, $|f(x_n) - f(y_n)| \geq \varepsilon$. Hence, a contradiction. ■

## 4.6 Sets of Discontinuity

### Lecture 31: Day 31

**Definition 45** (Removable Discontinuity)**.** If $\lim_{x \to x_0} f(x)$ exists, but $\lim f(x) \neq f(x_0)$.

**Definition 46** (Jump Discontinuity)**.** If $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist, but are not equal.

**Definition 47** (Essential Discontinuity)**.** If atleast one of $f(x_0^{\pm})$ doesn't exist.

**Theorem 38.** A monotone function has only jump discontinuities. Furthermore, there are only countably many jump discontinuities.

### Lecture 32: Day 32

**Theorem 39.** If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then

1. Any discontinuity is jump.
2. There are only countably many

**Proof. 1.** Assume without loss of generality, $f$ is increasing. We want to show for every $x_0 \in \mathbb{R}$, $f(x_0^{\pm})$ both exists point implying

$$f(x_0^-) \leq f(x_0) \leq f(x_0^+).$$

In fact, we claim that $f(x_0^+) = \inf\{f(x) : x > x_0\}$ and $f(x_0^- = \sup\{f(x) : x < x_0\})$. Since $f(x_0) \leq f(x)$ for $x > x_0$, then $\{f(x) : x > x_0\}$ is bounded below. So the infinimum exists for all $\iota$.

**Remark.** ~Yes, iota was used.~

Let $\varepsilon > 0$, then there is $x^* > x_0$ such that

$$\iota \leq f(x^*) < \iota + \varepsilon.$$

But if $x_0 < x' < x^*$, then

$$\iota \leq f(x') \leq f(x^*) < \iota + \varepsilon.$$

So take $\delta := x^* - x_0$, then

$$\lim_{x \to x_0^+} f(x) = \delta$$

**2.** Let $J \subset \mathbb{R}$ be the set of discontinuities of $f$, such that $x_0 < x_1$ for $x_0, x_1 \in J$, then $f(x_0^+) \leq f(x_1^-)$, since if $x_0 < x_2 < x_1$, then $f(x_0^+) \leq f(x_2) \leq f(x_1^-)$. Therefore, if $x_0, x_1 \in J, x_0 \neq x_1$, then

$$(f(x_0^-), f(x_0^+)) \cap (f(x_1^-), f(x_1^+)) = \varnothing.$$

If we take $q(x) \in (f(x_0^-), f(x_0^+)) \cap \mathbb{Q}$, a rational, and define $q : J \to \mathbb{Q}$ for $x \in J$, then $q$ is injective. So we have that infact, $q$ is also bijective to some subset of $\mathbb{Q}$, which is countable. ∎

**Lemma 13.** Let $S \subset \mathbb{R}$. Then the set

$$S' := \{x \in S : x \text{ is isolated in } S\},$$

is countable.

**Proof.** For $x \in S'$, let

$$d(x) := \inf\{|x - x'| : x' \in S, x' \neq x\} > 0.$$

For each $x \in S'$, assume

$$q(x) \in \mathbb{Q} \cap (x - \frac{d(x)}{2}, x + \frac{d(x)}{2}).$$

This assignment must be injective, since $x, x' \in S', x \neq x'$ implies

$$\vee_{\frac{d(x)}{2}}(x) \cap \vee_{\frac{d(x)}{2}}(x') = \varnothing.$$

Since $|x - x'| \geq \min\{d(x), d(x')\}$, it follows that

$$\begin{aligned} g(x) \in \vee_{\frac{d(x)}{2}}(x) \Leftrightarrow |y - x| < \frac{d(x)}{2} \leq \frac{|x - x'|}{2} \\ |y - x'| \geq |x - x'| - |y - x| \\ \geq \frac{|x - x'|}{2} > d(x) \end{aligned}$$

■

**Theorem 40.** If $f : \mathbb{R} \to \mathbb{R}$ is any function, then the sets of removable jump discontinuities are countable.

**Proof.** For $\alpha > 0$, define

$$D_\alpha := \{x : f(x^\pm) \to L \in \mathbb{R}\},$$

and either:

$$\begin{aligned} \left|f(x^+) - f(x^-)\right| > \alpha \\ \left|f(x^+) - f(x)\right| > \alpha \end{aligned}$$

Then:

$$\bigcup_{n=1}^{\infty} D_{\frac{1}{n}} = \bigcup_{\alpha > 0} D_\alpha,$$

is the set of all jump discontinuities.

**Observe.** Every point of $D_\alpha$ is isolated.

**Proof.** If $x_0 \in D_\alpha$, then $f(x_0^\pm)$ both exists. Let $\delta > 0$ such that

$$\begin{aligned} 0 < x - x_0 < \delta \Rightarrow \left|f(x_0^+) - f(x)\right| < \frac{\alpha}{3} \\ -\delta < x - x_0 < 0 \Rightarrow \left|f(x_0^-) - f(x)\right| < \frac{\alpha}{3}. \end{aligned}$$

Thus

$$\left.\begin{aligned} 0 < x' - x_0 < \delta \\ 0 < x - x_0 < \delta \end{aligned}\right\} \Rightarrow |f(x') - f(x)| \leq \left|f(x_0^+) - f(x')\right| + \left|f(x) - f(x_0^+)\right| < \frac{2\alpha}{3}$$

■

In particular, if $0 < x - x_0 < \delta$, then this and $f(x^\pm)$ both exists. Hence:

$$\begin{aligned} \left|f(x^+) - f(x^-)\right| \leq \frac{2\alpha}{3} \\ \left|f(x^+) - f(x)\right| \leq \frac{2\alpha}{3}. \end{aligned}$$

■

# Chapter 5

# Derivatives

## 5.2 Derivatives and the Intermediate Value Property

### Lecture 33: Day 33

**Definition 48** (Derivative)**.** Let $f : A \to \mathbb{R}$, $c \in A$. Then:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If this exists, then $f$ is differentiable at $c$.

**Example.** $f(x) = |x|$ is differentiable at all $c \neq 0$.

**Proof.**

$$\lim_{x \to c} \frac{|x| - |c|}{x - c} = \begin{cases} \lim\limits_{x \to c} \dfrac{x - c}{x - c} = 1, & c > 0 \\ \lim\limits_{x \to c} \dfrac{c - x}{x - c} = -1, & c < 0 \\ \lim\limits_{x \to c^+} \neq \lim\limits_{x \to c^-}, & c = 0. \end{cases}$$

■

**Proposition 19.** If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

**Proof.**

$$\begin{aligned}
f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
f'(c)(x-c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c) \\
&= \lim_{x\to c} f(x)-f(c) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} * \lim_{x\to c}(x-c) \\
&= f'(c) * 0 \\
&= 0.
\end{aligned}$$

■

**Theorem 41.** Derivative arithmetic is well defined with our known conventions.

**Theorem 42** (Linear Approximation Formula)**.** Suppose $f : A \to \mathbb{R}, c \in A$, then $f$ is differentiable at $c$. With derivative $L$ if and only if $f(x) = f(c) + L(x-c) + \mathcal{E}(x)$, where $\mathcal{E}$ is the error term, where:

$$\lim_{x\to c} \frac{\mathcal{E}(x)}{x-c} = 0.$$

**Proof.**

$$\begin{aligned}
f(x) &= f(c) + L(x-c) + \mathcal{E}(x) \\
\mathcal{E}(x) &= f(x) - f(c) - L(x-c) \\
\frac{\mathcal{E}(x)}{x-c} &= \frac{f(x)-f(c)}{x-c} - L \\
\lim_{x\to c} \frac{\mathcal{E}(x)}{x-c} &= 0 \\
&\Rightarrow f'(c) = L.
\end{aligned}$$

■

**Corollary.** $f$ is differentiable at $c$ with derivative $L$ if and only if

$$f(x) = f(c) + d(x)(x-c),$$

where $d(x) = L$.

**Theorem 43** (Chain Rule)**.** Suppose $f : A \to \mathbb{R}, g : B \to \mathbb{R}$, $A, B$ are intervals. Suppose $f \circ g$ is differentiable, then $f \circ g$ is differentiable at $c$ with

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

**Proof.** We will use the corollary, where $\lim d(x) = \lim \delta(x) = ML$.

$$\begin{aligned} g(x) &= g(c) + d(x)(x-c), \quad d(x) \to L, x \to c \\ f(g) &= f(g(c)) + \delta(y)(y - g(c)), \quad \delta(y) \to M, x \to c \\ f(g(x)) &= f(g(c)) + \delta(g(x))(g(x) - g(c)) \\ &= f(g(c)) + [\delta(g(x))d(x)](x-c). \end{aligned}$$

Where:

$$\lim_{x\to c} \delta(g(x))d(x) = \lim_{x\to c} \delta(g(x)) \lim_{x\to c} d(x).$$

■

## 5.3 Mean Value Theorems

**Theorem 44** (Mean Value Theorem). Suppose $f$ is continuous on $[a,b]$, differentiable on $(a,b)$, then there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

**Theorem 45** (Cauchy MVT). $f, g$ as above, then there exists $c \in (a,b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

**Theorem 46** (Rolle's Theorem). Suppose $f$ is as above, and $f(b) = f(a)$, then there is $c \in (a,b)$ such that $f'(c) = 0$.

**Proof.** We claim $c \in (a,b)$ where $f$ attains either it's maximum or it's minimum. Since $g$ is continuous, and $[a,b]$ is compact, then the max/min are attainable. If both are at the end points, then the function is constant. Say $f$ achieves it's max at $c \in (a,b)$, then

$$f'(c) = \left\{ \begin{array}{l} \lim\limits_{x\to c^+} \dfrac{f(x)-f(c)}{x-c} \leq 0 \\ \lim\limits_{x\to c^-} \dfrac{f(x)-f(c)}{x-c} \geq 0 \end{array} \right\} \Rightarrow f'(c) = 0.$$

■

## Lecture 34: Day 34

**Theorem 47** (Cauchy MVT). $f, g$ as above, then there exists $c \in (a, b)$ such that
$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

**Proof.** Consider

$$\begin{aligned} h(x) &:= f(x)(g(b) - g(x)) - g(x)(f(b) - f(a)) \\ h'(x) &:= f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)) \\ h(b) &= f(a)g(b) - g(a)f(b) \\ h(a) &= f(b)g(a) - g(b)f(a). \end{aligned}$$

Thus by Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. ■

-

**Theorem 48** (L'Hopital's Rule).

1. Suppose $f, g : [a, b] \to \mathbb{R}$ continuous, differentiable on $(a, b)$, $g'(x) \neq 0$ for all $x$. If $f(a) = g(a) = 0$, then
$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

   **Remark.** $g(x) \neq 0$ for all $x$ near $a$.

   By MVT, for all $x \neq a$, there is a $c$ between $a$ and $x$ such that $g(x) = g(x) - g(a) = g'(c)(x - a) \neq 0$.

2. Suppose $f, g$ as above, $lim_{x \to a} g(x) = \pm\infty$, then if
$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

**Proof.** **1.** Apply the Cauchy MVT to $[a, x], x \to a^+$. Then there exists $c_x \in (a, x)$ such that $f'(c_x)g(x) = g'(c_x)f(x)$ implies
$$\frac{f'(c_x)}{g'(c_x)} = \frac{f(x)}{g(x)}.$$

Since $c_x \to a$ as $x \to g$:
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c_x)}{g'(c_x)} = L$$

**2.** For $a < x < b$, apply Cauchy MVT, then there exists $c_x \in (x, b)$

such that:

$$\begin{aligned} f'(c_x)(g(x)-g(b)) &= g'(c_x)(f(x)-g(b)) \\ \Rightarrow \frac{f'(c_x)}{g'(c_x)} &= \frac{f(x)-f(b)}{g(x)-g(b)} \\ \Rightarrow \lim_{x\to a}\frac{f(x)-f(b)}{g(x)-g(b)} &= \lim_{x\to a}\frac{f'(c_x)}{g'(c_x)} = L. \end{aligned}$$

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(b)}{g(x)-g(b)}\right| &\le \left|\frac{f(b)}{g(x)-g(b)}\right| + |f(x)|\left|\frac{1}{g(x)} - \frac{1}{g(x)-g(b)}\right| + \left|\frac{f(x)}{g(x)}\right|\left|\frac{g(b)}{g(a)-g(b)}\right| \\ |\alpha(x)-\beta(x)| &\le \phi(x) + |f(x)|\left|\frac{1}{g(x)} - \frac{1}{g(x)-g(b)}\right| + \left|\frac{f(x)}{g(x)}\right|\psi(x) \\ |\alpha(x)-\beta(x)| &\le \phi(x) + |\alpha(x)|\,\psi(x). \end{aligned}$$

Let $\varepsilon > 0$, then there exists $\delta$ such that $0 < x - a < \delta$ implies

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

For any $c_x \in (a,x)$, then $0 < c_x - x_n < \delta$. ■

## Lecture 35: Day 35

**Theorem 49.** Suppose $f, g : (a, b] \to \mathbb{R}$ are differentiable, $g(x) \to \infty$ as $x \to a^+$, and $g' \neq 0$. If

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L,$$

, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

**Proof.** Let $0 < \varepsilon < 1$ be given. let $\delta > 0$ be such that $0 < x - a < \delta$ imply

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

Take $y \in (a, a + \delta)$, so

$$\left|\frac{f'(y)}{g'(y)} - L\right| < \varepsilon.$$

For $a < x < y$, Cauchy MVT implies if $c = c_{xy}$ such that $x < c < y$ and

$$f'(c)(g(x) - g(y)) = g'(c)(f(x) - f(y)),$$

if and only if

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(y)}{g(x) - g(y)}.$$

Thus

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right| < \varepsilon \quad x \in (a, y).$$

Consider

$$\left|\frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)}\right| \le \left|\frac{f(y)}{g(x) - g(y)}\right| + \left|\frac{f(x)}{g(x)}\right| \left|\frac{g(y)}{g(x) - g(y)}\right|$$

$$0 < x - a < \delta' < \delta \Rightarrow \begin{cases} \left|\dfrac{f(y)}{g(x) - g(y)}\right| < \varepsilon \\ \left|\dfrac{g(y)}{g(x) - g(y)}\right| < \varepsilon \end{cases}$$

$$\begin{aligned}
|\alpha(x) - \beta(x)| &\le \varepsilon + |\alpha(x)|\,\varepsilon \\
|(\alpha(x) - L) - (\beta(x) - L)| &\Rightarrow |\alpha(x) - L| \\
|\alpha(x) - L| &\le |\beta(x) - L| + \varepsilon + |L|\,\varepsilon + |\alpha(x)|\,\varepsilon \\
\Rightarrow (1 - \varepsilon)\,|\alpha(x) - L| &\le (2 + |L|)\varepsilon \\
&\le \varepsilon + |L|\,\varepsilon + |\alpha(x) - L|\,\varepsilon \\
\Rightarrow |\alpha(x) - L| &\le \frac{\varepsilon'}{1 - \varepsilon'}(2 + |L|) \to 0 \le \varepsilon.
\end{aligned}$$

■

## Lecture 36: Day 36

**Theorem 50.** $f, g$ continuous implies $\max\{f(x), g(x)\}$ is continuous.

**Proof.**

$$\max(a,b) = \frac{|a-b| + |a+b|}{2}$$
$$[\max\{f,g\}](x) = \frac{|f(x)-g(x)| + |f(x)+g(x)|}{2}.$$

Then a corollary is that $f_1, f_2, \ldots, f_N$ being continuous implies $\max\{f_1, \ldots, f_N\}$ is continuous. ■

**Example.** If $f_1, f_2, \ldots$ is a sequence of continuous functions, with all $f_i(x) \leq C$ for all $x$. It follow that

$$(\sup(f_i))(x) := \sup_{i \in I} f_i(x)$$

**Example.** Let $f_m : [0,1] \to \mathbb{R}$ such that $f_m(x) = x^m$. Then

$$\inf_{m \in \mathbb{N}} x^m = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1. \end{cases}$$

**Example.** What does

$$\lim_{x \to \infty} f(x) = \infty$$

mean?

**Example.** Suppose $|r| < 1$. Prove that the series

$$1 + 2r + 3r^2 + \cdots = \sum_{n=0}^{\infty} (n+1) r^n$$

converges absolutely, with sum $\dfrac{1}{(1-r)^2}$.

**Proof.**

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{|(n+2)r^{n+1}|}{|(n+1)r^n|}\right| &= \lim_{n\to\infty}\frac{(n+2)\,|r|^{n+1}}{(n+1)\,|r|^n} \\
&= \lim_{n\to\infty}\frac{(n+2)\,|r|}{n+1} \\
&= \lim_{n\to\infty}\frac{n\,|r|}{n+1} + \lim_{n\to\infty}\frac{2\,|r|}{n+1} \\
&= |r|\lim_{n\to\infty}\frac{n}{n+1} + 0 \\
&= |r| < 1
\end{aligned}$$

Thus by the ratio test, it converges absolutely.

$$\begin{aligned}
1 + 2r + 3r^2 + 4r^3 + \ldots &= 1 + r + r^2 + r^3 + \ldots \\
&\quad + r + r^2 + r^3 + \ldots \\
&\quad + r^2 + r^3 + \ldots \\
&\quad + r^3 + \ldots \\
&\quad + \ldots \\
&= 1 + r + r^2 + r^3 + \ldots \\
&\quad + r(1 + r + r^2 + \ldots) \\
&\quad + r^2(1 + r + \ldots) \\
&\quad + r^3(1 + \ldots) \\
&\quad + \ldots \\
&= (1 + r + r^2 + r^3 + \ldots)^2
\end{aligned}$$

$$\left[\sum_{n=0}^{\infty} r^n\right]^2 = \frac{1}{(1-r)^2}$$

■

**Example.** Prove that there exists $c \in (1,2)$ such that $c^3 - c^2 = 1$.

**Proof.** We can rearrange this equation for perspective:

$$c^3 - c^2 - 1 = 0$$

Define a function $f(x) = x^3 - x^2 - 1$. Since $f$ is a polynomial, it is continuous, then:

$$\begin{aligned}
f(1) &= -1 \\
f(2) &= 3.
\end{aligned}$$

Hence, by the Intermediate Value Theorem, there exists $0 \in (f(1), f(2))$. In otherwords, there exists a $c \in (1,2)$ such that $f(c) = 0$ or $c^3 - c^2 = 1$. ■

**Example.** Prove that the set of irrational numbers is dense in $\mathbb{R}$, that is, given any $a < b$

$$(a, b) \setminus \mathbb{Q} \neq \varnothing.$$

**Proof.** Given $\dfrac{a+b}{2} \in (a, b)$ is a midpoint, then

$$s = a + \frac{b-a}{\sqrt{2}} \in (a, b),$$

and $s \notin \mathbb{Q}$. Hence $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. ∎

**Example.** The interior of a set $A \subset \mathbb{R}$, denoted $A^\circ$, is the set of all $x \in A$ such that there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subset A$. Show that

$$\mathbb{R} \backslash A^\circ = \overline{\mathbb{R} \backslash A}$$

**Proof.** **Case 1.** $\mathbb{R} \setminus \operatorname{Int}(A) \subset \overline{\mathbb{R} \setminus A}$. Given $x \in \mathbb{R} \setminus \operatorname{Int}(A)$, then $x \notin \operatorname{Int}(A)$ implies $\nexists \varepsilon > 0$ such that $\vee_\varepsilon(x) \subset A$. Thus there exists $y \in \mathbb{R}$ such that $y \in \vee_\varepsilon(x)$ but $y \notin A$, therefore $y \in \mathbb{R} \setminus A$,

**Case 2.** $\overline{\mathbb{R} \setminus A} \subset \mathbb{R} \setminus \operatorname{Int}(A)$. Given $x \in \overline{\mathbb{R} \setminus A}$, then $\vee_\varepsilon(x) \neq \varnothing$ for all $\varepsilon > 0$. There exists $y \in \vee_\varepsilon(x)$ such that $y \in \mathbb{R} \setminus A$. Thus $\vee_\varepsilon(x) A$ implies $x \notin \operatorname{Int}(A)$. Therefore $x \in \mathbb{R} \setminus \operatorname{Int}(A)$. Hence

$$\mathbb{R} \setminus \operatorname{Int}(A) = \overline{\mathbb{R} \setminus A}$$

∎

**Example.** What does

$$\lim_{x \to \infty} f(x) = \infty$$

mean?

**Proof.** A typical function limit statement, $\varepsilon - \delta$ argument, states:

> For all $\varepsilon > 0$, then there exists $\delta > 0$ for $|x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

Instead, suppose the opposite for divergence to infinity:

> For all sequences $x_n \to \infty \Rightarrow f(x_n) \to \infty$

∎

**Example.** Let $A \subset \mathbb{R}, f : A \to \mathbb{R}$, and $a$ a limit point of $A$. Show that $\lim\limits_{x \to a} f(x) = L$ iff

$$\lim_{n \to \infty} f(x_n) = L$$

for every sequence $x_1, x_2, \ldots \in A$ such that $x_n \to a$.

**Example.** Let $f$ be a continuous function defined on all of $\mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

Prove that $f$ attains its minimum, i.e. there exists $c \in \mathbb{R}$ such that

$$f(c) \leq f(x) \quad \text{for all } x \in \mathbb{R}.$$

**Example.** Suppose $f : [a, b] \to \mathbb{R}$ be differentiable everywhere on the closed bounded interval $[a, b]$, and suppose further that the derivative $f'$ is continuous. Show that $f$ is uniformly differentiable, which is to say:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $c \in [a, b]$,

$$0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

**Example.** Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and $c \in \mathbb{R}$, with $f'(c) = 0$ and $f''(c) > 0$. Prove that $c$ is a local minimum for $f$, i.e. there exists $\varepsilon > 0$ such that,

$$0 < |x - c| < \varepsilon \Rightarrow f(c) < f(x).$$

**Example.** Suppose $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Let $c \in [a, b]$ be a global maximum for $f$ on $[a, b]$, i.e. $f(c) \geq f(x)$ for every $x \in [a, b]$. Prove that either $c = a$, or $c = b$, or $f'(c) = 0$.

# Chapter 6

# Sequences and Series of Functions

## 6.2 Uniform Convergence of a Sequence of Functions

### Lecture 37: Day 37

No substantial work.

### Lecture 38: Day 38

**Definition 49** (Pointwise Convergence)**.** Given such $(f_n), f_0 : A \to \mathbb{R}$, then $(f_n)$ converges pointwise if for every $x \in A$, $f_n(x) \to f_0(x)$.

**Example.**

$$f_n(x) = x^n$$

$A = \mathbb{R}, [0, 1]$.

**Proof.**

$$x^n \to f_n(x) := \begin{cases} 0, & x \in (-1, 1) \\ 1, & x = 1 \end{cases}$$

Pointwise on $(-1, 1]$. ■

**Example.**

$$f_n(x) = x^{1/n}$$

$A = [0, \infty]$.

**Proof.**

$$x^{1/n} \to g_0(x) := \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Pointwise on $[0, \infty)$. ■

**Example.**

$$f_n(x) = \frac{x^2 + nx}{n}$$

$A \to \mathbb{R}$.

**Proof.**

$$\frac{x^2 + nx}{n} \to x$$

Pointwise on $\mathbb{R}$. ■

**Example** (Can the limits flip-flop)**.**

$$\lim_{x \to a} \lim_{n \to \infty} f_n(x) \not f_0(a) = \lim_{n \to \infty} \lim_{x \to a} f_n(x).$$

**Proof.** No, take a upper right triangular matrix with 1s on the top right. Then

$$a_{m,n} = \begin{cases} 1, & m \geq n \\ 0, & m < n \end{cases}$$

$$\lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{m \to \infty} 0 = 0$$
$$\lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{m \to \infty} 1 = 1.$$

So, no. ■

**Definition 50** (Pointwise Convergence)**.** $f_n \to f_0$ in pointwise if for any $x \in A$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|f_n(x) - f_0(x)| < \varepsilon.$$

**Definition 51** (Uniform Convergence)**.** $(f_n)$ converges to $f_0$ uniformly, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in A$, $n > N$ implies

$$|f_n(x) - f_0(x)| < \varepsilon.$$

Uniform convergence implies Pointwise convergence. Not conversely.

**Theorem 51.** If $f_n \to f_0$ unifromly convergent on $A$, and all $f_n$ are continuous, then so is $f_0$.

**Proof.** Next Time. ■

**Example.**

$$f_n(x) = x + \frac{x^2}{n}$$
$$f_0(x) = x.$$

$f_n \to f_0$ is uniformly convergent in each $[-C, C]$, but not in $\mathbb{R}$ or any unbounded set. It is also unformly convergent on any subset of the set.

**Proof.** Given $\varepsilon > 0$, to find $N$ such that $n > N$, then

$$|f_n(x) - f_0(x)| \leq \frac{x^2}{n} \leq \frac{C^2}{n} < \varepsilon.$$

Take $N := \dfrac{C^2}{\varepsilon}$. If $|x| > n$, then

$$|f_n(x) - f_0(x)| = \frac{x^2}{n} > \frac{n^2}{n} = n.$$

■

**Example.** $x^n$ does not converge uniformly on $[0, 1]$,

$$f_0(x) := \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1. \end{cases}$$

**Proof.** There exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $x \in [0, 1]$ such that

$$|f_n(x) - f_0(x)| \geq \varepsilon.$$

Take $\varepsilon = 1/2$, then for all $N \in \mathbb{N}$, there exists $0 < C < 1$ such that $C^N > 1/2$. On the other hand, $x^n \to 0$ uniformly on ever $[0, \gamma]$ for $\gamma < 1$. ■

## Lecture 39: Day 39

**Theorem 52** (Uniform Convergence Theorem)**.** If $f_n$ are continuous for all $n$, then so si $f_0$.

**Proof.** Let $c \in A$.

**Problem.**

$$\lim_{x \to c} f_0(x) = f(c)$$

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $|f_N(x) - f_0(x)| < \varepsilon/3$ for all $x \in A$. Now choose $\delta > 0$ such that $|x - c| < \delta$ implies $|f_N(x) - f_N(c)| < \varepsilon/3$. Then $|x - c| < \delta$ implies

$$\begin{aligned} |f_0(x) - f_0(c)| &\le |f_0(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f_0(c)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

■

## 6.5 Power Series

**Definition 52** (Power Series)**.**

$$\sum_{m=0}^{\infty} a_n (x-c)^n$$

with $c, a_0, a_1, \dots$ as constants.

**Example.**

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad c = 0, R = \infty$$

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

**Definition 53** (Uniform Cauchy)**.** A sequence of functions $f_1, f_2, \dots : A \to \mathbb{R}$ is unifromly Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \ge N$, then

$$|f_m(x) - f_n(x)| < \varepsilon \quad \forall x \in A.$$

**Proposition 20.** A uniformly cauchy sequence of functions converges uniformly to some function $g(x)$.

**Proof.** For all $x \in A$, the sequence $(f(x))$ is Cauchy, hence converges to some $g(x)$. To see that $f_n \to g$ uniformly, given $\varepsilon > 0$, let $N$ be as above such for $n \geq N$, then

$$|f_n(x) - g(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

■

**Theorem 53.** Let

$$\mathbb{R} := \frac{}{\limsup_{x\to\infty} \sqrt[n]{|a_n|}} \in [0, \infty).$$

Then, whenever $0 \leq p < R$, the series, $\sum a_n(x-c)^n$ converges uniformly on $[c-p, c+p]$. And if $|x-c| > R$, then it diverges.

**Proof.** Apply the Root Test for a fixed $x$. This series converges if

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-c)^n|} < 1.$$

diverges otherwise.

> **Remark.** This is true since if the nth root is greater than 1, then the radicand is greater than 1, hence it cannot go to 0, diverges by series convergence rule.

This thm converges if $|x-c| < R$.

> **Remark.** But what about uniformly convergence?

**Problem.** The sum of partial sums is uniformly cauchy over $[x-p, x+p], p < R$.

$p < R$ if and only if $p/R < 1$ which is true if and only if

$$p \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \Leftrightarrow \limsup_{n\to\infty} \sqrt[n]{\sqrt{|a_n|}\, p^n} < 1.$$

Therefore, there exists $0 \leq \gamma < 1$ and $N \in \mathbb{N}$ such that for $n \geq N$, then

$$\sqrt[n]{|a_n| p^n} \leq \gamma \quad \gamma \in (limsup_{n\to\infty} \sqrt[n]{\sqrt{|a_n|}\, p^n}, 1).$$

Thus

$$\begin{aligned}|a_n|\,p^n &\le \gamma^n\\ \left|\sum_{k=n}^{m} a_k(x-c)^k\right| &\le \sum_{k=n}^{m} |a_k|\,p^k \quad m \ge n \ge N\\ &\le \sum_{k=n}^{m} \gamma^k\\ &= \frac{\gamma^n-\gamma^m}{1-\gamma}.\end{aligned}$$

For $|x-c|<p$, fill in the rest after the first assumption. ■

## Lecture 40: Day 40

**Theorem 54.** Let $f_1, f_2, \ldots : [n,b] \to R$, $f_n' \to g$ uniform and $f_n \to f$ polnt wise. Then $f' = g$.

**Proof.** Let $c \in [a,b]$. Let $\varepsilon > 0$ be given.

$$\begin{aligned}\underbrace{\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right|}_{\mathrm{I}} + \underbrace{\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right|}_{\mathrm{II}} + \underbrace{|f_n'(c)-g(c)|}_{\mathrm{I}}\\ \ge \left|\frac{f(x)-f(c)}{x-c} - g(c)\right|\end{aligned}$$

$$\begin{aligned}\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} &= \lim_{m\to\infty} \frac{f_m(c)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\\ &= \lim_{m\to\infty} \frac{f_m(c)-f_m(c)-f_n(x)-f_n(c)}{x-c}\\ &= \lim_{m\to\infty} \frac{(f_m-f_n)(x)-(f_m-f_n)(c)}{x-c}\end{aligned}$$

Then by MVT, we have $(f_m'-f_n')(z)$ for $z \in (x,c)$. Since $f_n'$ is uniformly cuachy, then there exists $N \in \mathbb{N}$ such that for $m, n \ge N$, then

$$\frac{f_m'(z)-f_n'(z)}{<}\varepsilon/3$$

for all $z \in [a,b]$. Therefore, for $n > N$,

$$|g(z)-f_n'(z)| = \lim_{m\to\infty} |f_m'(z)-f_n'(z)| \le \varepsilon/3.$$

for all $z \in [a,b]$. Leaving chosen $N$, let $\delta > 0$ such that $0 < |x-c| < \delta$ implies

$$\left|\frac{f_N(x)-f_N(c)}{x-c} - f_N'(c)\right| < \varepsilon/3.$$

Let $n := N$, then $I \leq \varepsilon/3$, $II \leq \varepsilon/3$, $III \leq \varepsilon/3$ if $|x-c| < \delta$. So $|x-c| < \delta$, implies

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

■

# Chapter 7

# The Riemann Integral

## 7.1 Discussion

### Lecture 41: Day 41

**Example** (The Fundamental Theorem of Calculus).

$$\textbf{(i)} \quad \int_a^b F'(x)dx = F(b) - F(a).$$

$$\textbf{(ii)} \ \text{If} G(x) = \int_a^x f(t)dt, \ \text{then } G'(x) = f(x).$$

This is a statement about the inverse relationship between differentiation and integration.

**Proof.** The best way to think of integration is not as the inverse process of differentiation as this limits us and reduces the amount and types of integrals we can pursue. Historically, it was seen as the inverse by mathematicians such as Leibniz, Fermat, Newton, and other founders. However, instead defining it as an adaptation of anti-differentiation is ideal. But this does have some consequences of it's own. For example:

$$h(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } 1 \leq x < 2, \end{cases}$$

is not integrable on $[0, 2]$. Which this shift started from Cauchy and especially with Riemann. This evolved into Fourier Series. We learned from calculus that we have Riemann sums, usually drawn graphically as Rectangular Areas

$$\sum_{k=1}^{n} \underbrace{f(c_k)}_{\text{length}} \underbrace{(x_k - x_{k-1})}_{\text{width}}$$

Then it logically follows that the should the rectangles get thinner, then the width goes practically to 0, which just happens to be similar to a partition of $A$. Hence:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \int_a^b f.$$

## 7.2 The Definition of the Riemann Integral

### Lecture 42: Day 42

**Notation.** For $f : [a, b] \to \mathbb{R}$, $f$ is bounded on $(a, b]$.

**Definition 54** (Partition)**.** A partition $P$ on $[a, b]$ is a finite subset of $[a, b]$, $a, b \in P$.

$$P = \{x_0 = a < x_2 < \ldots < x_n = b\}.$$

**Definition 55** (Upper Riemann Sum)**.**

$$\mathcal{U}(f, P) := \sum_{i=1}^{n} \underbrace{\sup_{x \in [x_{i-1}, x_i]}}_{M_i} f(x)(x_i - x_{i-1})$$

**Definition 56** (Lower Riemann Sum)**.**

$$\mathcal{L}(f, P) := \sum_{i=1}^{n} \underbrace{\inf_{x \in [x_{i-1}, x_i]}}_{m_i} f(x)(x_i - x_{i-1})$$

**Definition 57** (Refinement)**.** A partition $Q$ is a refinement of $P$ if $Q$ contains all of the point of $P$, that is $P \subseteq Q$.

**Lemma 14.** If $P \subseteq Q$ partitions, then

$$\mathcal{U}(f, P) \geq \mathcal{U}(f, Q) \geq \mathcal{L}(f, Q) \geq \mathcal{L}(f, P).$$

**Proof.** Suffices to show that for $A = P \cup \{X\}$ if $Q$ contains only one element. Say $x \in (x_{i-1}, x_i)$,

$$\begin{aligned}\mathcal{U}(f,Q) - \mathcal{U}(f,P) = &\sup_{x\in[x_{i-1},x_i]} f(x)(x - x_{i-1}) \\ &+ \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x) \\ &- \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})\end{aligned}$$

$$\left(\sup_{x\in[x_{i-1},x]} f(x) - \sup_{x\in[x_{i-1},x]} f(x)\right)\underbrace{(x - x_{i-1})}_{\geq 0} + \left(\sup_{x\in[x,x_i]} f(x) - \sup_{x\in[x,x_i]} f(x)\right)\underbrace{(x - x_{i-1})}_{\geq 0}$$

Thus

$$\mathcal{U}(f,P\cup Q) \leq \mathcal{U}(f,P) \quad \mathcal{U}(f,P\cup Q) \geq \mathcal{L}(f,P\cup Q) \geq \mathcal{L}(f,Q).$$

Then by above, since $P \cup Q$ is a refinement of $P, Q$. ■

**Definition 58** (Upper Integral)**.** Let $P$ be the collection of all possible partitions of the interval $[a,b]$. The upper integral of $f$ is defined to be

$$\mathcal{U}(f) := \inf_P \mathcal{U}(f,P)$$

**Definition 59** (Lower Integral)**.** Let $P$ be the collection of all possible partitions of the interval $[a,b]$. The upper integral of $f$ is defined to be

$$\mathcal{L}(f) := \sup_P \mathcal{L}(f,P)$$

**Definition 60** (Riemann Integrable)**.** A bounded function $f$ defined on $[a,b]$ is Riemann-Integrable if $\mathcal{U}(f) = \mathcal{L}(f)$. We define $\int_a^b f$ to be this value such that

$$\int_a^b f dx = \mathcal{U}(f) = \mathcal{L}(f)$$

**Theorem 55.** If $f$ is continuous on $[a,b]$, then $f$ is Riemann Integrable on $[a,b]$.

**Proof.** Let $\varepsilon > 0$ be given. We want to find a partition such that

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon.$$

Recall that $f$ is uniformly continous, in particular, there exists $\delta > 0$ such that

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon/(b-a).$$

Let $P$ be every point such that $x_i - x_{i-1} < \delta$. Then for each $i$,

$$\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x]} f(x) = f(z) - f(w),$$

for $w, z \in (x_{i-1}, x_i)$ by EVT, then

$$|z - w| < x_i - x_{i-1} < \delta_i.$$

And so

$$\begin{aligned}\mathcal{U}(f,P) - \mathcal{L}(f,P) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \frac{\varepsilon}{b-a}(b-a) = \varepsilon.\end{aligned}$$

■

**Property.** $f$ is riemann integrable if and only if

$$\lim_{n\to\infty} \mathcal{U}(f, f_n) = \lim_{n\to\infty} \mathcal{L}(f, f_n).$$

## 7.4 Properties of the Integral

**Theorem 56.** Let $f : [a, b] \to \mathbb{R}, c \in (a, b)$. Then $f$ is riemann integrable if and only if $f$ is riemann integrable on both $[a, c], [c, b]$.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

**Proof.** $(\Leftarrow)$. $f$ is Riemann Integrable on both $[a, c]$ and $[c, b]$. Let $\varepsilon > 0$ be given. Want a partition $P$ of $[a, b]$ such that

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon.$$

But there are partitions $P_1, P_2$ of $[a, c]$ and $[c, b]$ respectively such that

$$\mathcal{U}(f,P_1) - \mathcal{L}(f,P_1) < \varepsilon/2.$$

$$\mathcal{U}(f,P_2) - \mathcal{L}(f,P_2) < \varepsilon/2.$$

Put $P := P_1 \cup P_2$, thus is a partition of $[a, c] \cup [c, b] = [a, b]$ with

$$\mathcal{U}(f,P) = \mathcal{U}(f,P_1) + \mathcal{U}(f,P_2)$$

and likewise for $\mathcal{L}(f, P)$.

($\Rightarrow$).

$$\begin{aligned}
\mathcal{U}(f,P)-\mathcal{L}(f,P) &= (\mathcal{U}(f,P_1)+\mathcal{U}(f,P_2))-(\mathcal{L}(f,P_1)+\mathcal{L}(f,P_2)) \\
&< \varepsilon/2+\varepsilon/2=\varepsilon.
\end{aligned}$$

■

## Lecture 43: Day 43

**Property.**

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$$

**Proof.** Given $P := \{x_0 < \ldots < x_n\}$,

$$\mathcal{U}(f,P)\sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$M_i = \sup_{[x_{i-1},x_i]} f(x).$$

$$R_i = \sup_{[x_{i-1},x_i]} |f|\,(x).$$

$$\Rightarrow |M_i| \leq R_i.$$

Likewise,

$$m_i = \inf_{[x_{i-1},x_i]} f(x)$$

$$r_i = \inf_{[x_{i-1},x_i]} |f|\,(x)$$

■

## Lecture 44: Day 44

**Theorem 57.** Suppose $f : [a, b] \to \mathbb{R}$ is Riemann Integrable, then so is $|f|$ and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx$$

**Proof.** Let $\varepsilon > 0$ be given and that $P = \{x_i\}$ be a partition of $[a, b]$ such that

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

Let $M_i, m_i$ as before, then

$$\begin{aligned} M_i - m_i &\geq \sup\{|f(z)| - |f(w)|\} \\ &= \overline{M_i} - \overline{m_i}, \\ \therefore \mathcal{U}(|f|, P) - \mathcal{L}(|f|, P) &= \sum_i (\overline{M_i} - \overline{m_i})(x_i - x_{i-1}) \\ &\leq \sum_i (M_i - m_i)(x_i - x_{i-1}) \\ &< \varepsilon. \end{aligned}$$

Observe $\overline{M_i} := \max\{|M_i|, |m_i|\}$. Given a bounded set $S \subset \mathbb{R}$, then

$$\begin{aligned} |\mathcal{U}(f, P)| &= \left| \sum_i M_i(x_i - x_{i-1}) \right| \\ &\leq \sum_i |M_i|\,(x_i - x_{i-1}) \\ &\leq \sum_i \overline{M_i}(x_i - x_{i-1}) \\ &= \mathcal{U}(|f|, P). \end{aligned}$$

Hence, both parts of the theorem holds. ■

## 7.5 The Fundamental Theorem of Calculus

**Theorem 58** (Fundamental Theorem of Calculus)**.** Let $f : [a, b] \to \mathbb{R}$ be Riemann Integrable. Define

$$F(x) := \int_a^b f(t)dt.$$

Then,

1. $F$ is continous.
2. If $F$ is continuous at $c \in [a, b]$, then $F'(c) = f(c)$.

3. If $G : [a, b] \to \mathbb{R}$ and $G' = f$, then

$$\int_a^b f(x)dx = G(b) - G(a).$$

**Proof.** **(1).** Given $c \in [a, b], \varepsilon > 0$. If $|x - c| < \varepsilon/C$, then

$$\begin{aligned} F(x) - F(c) &= \int_c^x f(t)dt \\ &= \int_a^x f(t)dt - \int_a^c f(t)dt \\ &\leq \int_c^x |f(t)|\, dt \\ &\leq \int_c^d C dt \\ &= C(x - c) < \varepsilon. \end{aligned}$$

**(2).** Let $\varepsilon > 0$ be given, and let $\delta > 0$ be such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. Then for $0 < x - c < \delta$,

$$\begin{aligned} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \left( \frac{1}{x - c} \int_c^x f(t)dt \right) - f(c) \right| \\ &= \left| \frac{1}{x - c} \int_c^x f(t) - f(c)dt \right| \quad f(c) \text{ constant} \\ &\leq \frac{1}{x - c} \int_c^x |f(t) - f(c)|\, dt \\ &< \frac{1}{x - c}(x - c)\varepsilon = \varepsilon. \end{aligned}$$

Similarly, if $0 < c - x < \delta$. Note that

$$\frac{1}{x - c} \int_c^x f(c)dt = \frac{1}{x - c}(f(c)(x - c)) = f(c).$$

**(3).** Let $P = \{x_i\}$ be any partition of $[a, b]$, a riemann sum for $f_w/P$ is

$$\sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}),$$

where $y \in [x_{i-1}, x_i]$, then

$$\mathcal{L}(f, P) \leq S \leq \mathcal{U}(f, P).$$

For $\varepsilon > 0$, choose $P$ such that

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon.$$

And

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^{n} G(x_i) - G(x_{i-1}) \\ &= G(x_1) - G(a) + G(x_2) - G(x_1) + \ldots + G(b) - G(x_{n-1}) \\ &= G(b) - G(a). \end{aligned}$$

By the MVT, there are $y_1, \ldots, y_n, y_i \in (x_{i-1}, x_i)$ such that

$$\begin{aligned} G(x_i) - G(x_{i-1}) &= G'(y_i)(x_i - x_{i-1}) \\ &= f(y_i)(x_i - x_{i-1}). \end{aligned}$$

So

$$G(b) - G(a) = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}) =: S,$$

which is a Riemann Sum with $P$. Therefore,

$$\mathcal{U}(f, P) - S = \mathcal{U}(f, P) - (G(b) - G(a)),$$

since

$$\left| \mathcal{U}(f, P) - \int_a^b f(t)dt \right| < \varepsilon.$$

We did that

$$\left| G(b) - G(a) - \int_a^b f(t)dt \right| < 2\varepsilon.$$

■

# Part II

# Semi-Graduate Studies

# Chapter 8

# Metric Spaces

## 8.1 More Preliminaries

### Lecture 45: Day 45

**Definition 61** (Euclidean Distance). Given $x, y \in \mathbb{R}^n$, then

$$\|x - y\| := \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

**Definition 62** (Metric Space). Set $X$ with a function $d : X \times X \to [0, \infty)$ called a metric that satisfies the triangle inequality, symmetry, and is positive definite.

**Example** ($\ell^p$-metric).

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}.$$

**Example** ($L^p$-metric).

$$I_p(x, y) := \left(\int_0^1 |f(x) - f(y)|^p \, dx\right)^{1/p}.$$

**Definition 63** (Set of Continuous Function).

$$C^0([a, b]) := \{\text{continuous functions from } [a, b] \to \mathbb{R}\}$$

$$C^k([a, b]) := \{\text{continuous k-differentiable functions from } [a, b] \to \mathbb{R}\}$$

**Definition 64** (Smooth Functions).

$$C^\infty([a,b])$$

**Proposition 21.**

$$\frac{d}{dx} : C^1([a,b]) \to C^0([a,b])$$

where $\frac{d}{dx}$ is continuous as a function between metric spaces.

**Proof.** Let $f, g \in C^1([a,b])$, then

$$d_{C^1}(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| + \sup_{x\in[a,b]} |f'(x) - g'(x)| < \delta,$$

implies

$$d_{C^0}(f',g') = \sup_{x\in[a,b]} |f'(x) - g'(x)| < \varepsilon.$$

Let $\delta = \varepsilon$, thus $d_{C^1}(f(x), g(x)) < \delta = \varepsilon$. ∎

## Lecture 46: Day 46

**Lemma 15** (Lebesgue Big Number Lemma). $(X,d)$ sequentially compact, open cover $\{U_i\}$ of $X$, then there exists $r > 0$ such that $\forall x \in X$, $B_r(x) \subseteq U_i$ for some $i$.

**Proof.** Assume not true, for all $r > 0$, there exists $x \in X$ such that $B_r(x) \nsubseteq U_{i_0}$ for each $r = \frac{1}{n}$. Choose $x_n$ to satisfy $B_r(x) \subseteq U_{i_0}$. We have a convergent subsequence $x_{n_k} \to x, x \in U_{i_0}$ for $i_0$. Then there exists $r_0 > 0$ such that $B_r(x) \subseteq U_{i_0}$, suppose $N$ sufficiently large so that

$$\frac{1}{N} < \frac{r_0}{2} \qquad d(x, x_n) < \frac{r_0}{2}.$$

Consider $B_{\frac{1}{N}}(x_n)$, for all $y \in B_{\frac{1}{N}}(x_n)$, $d(x,y) \leq d(x, x_N) + d(x_N, y)$. Thus $B_{\frac{1}{N}}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0}$, which is a contradiction. ∎

**Definition 65** (Totally Bounded). A metric space $X$ is totally bounded if for all $\varepsilon > 0$, there exists $y_1, \ldots, y_i$ such that

$$\bigcup_{i=1}^{K} B_\varepsilon(y_i) \supseteq X.$$

**Lemma 16.** If a set is sequentially compact, then it is totally bounded.

**Proof.** $\forall \varepsilon > 0$, then there does not exist finitely many $\varepsilon-$balls that cover $x$. ■

**Definition 66** (Norm). A norm $\|\cdot\| : V \to [0, \infty)$ is positive definite, has homogenuity: $\|\lambda v\| = |\lambda| \|v\|$, and has the triangle inequality.

**Definition 67** (Support of $f$). Closure of $\{x : f(x) \neq 0\} =: \overline{\{x : f(x) \neq 0\}}$.

**Theorem 59.** A set is sequentially compact if and only if it is topologically compact.

**Proof.** **(⇒).** Let $\{U_i\}$ be an open cover for $X$. By Lebesgue Number Lemma, there exists $r > 0$ such that $B_r(x) \subseteq U_i$ for some $i$. Let $r = \varepsilon$, then there are finitely many $y_1, \ldots, y_i$ such that:

$$X \subseteq \bigcup_{1} B_r(y_i).$$

**(⇐).** For the sake of contradiction, assume it is not sequentially compact, then there exists a sequence $(x_n)$ with no convergent subsequence. Then none of the $x_i$ can appear infinitely-many times. There also exists $\varepsilon_n > 0$ such that $U_j = B_{\varepsilon_n}(x_i) = (x_j)$. Let $U_0 = X \setminus \{x_i : i \in \mathbb{N}\}$.

$$X \subseteq U_0 \cap \bigcup_{1} U_j.$$

Every finite subcover omits infinitely many points, then there is no finite subcover, hence a contradiction. ■

**Definition 68** (Lipschitz). Given $f : X \to Y$, such that there exists $x, y \in \mathbb{R}$ then:

$$d_Y(f(x), f(y)) \leq k d_X(x, y).$$

**Proposition 22.** Lipschitz implies continuity.

**Proof.** Let $\varepsilon > 0$ and choose $\delta = \dfrac{\varepsilon}{K}$. Then $d_Y(f(x), f(y)) \leq k d_X(x, y) < k\delta = \varepsilon$. In fact, it is uniformly continuous. ■

**Definition 69** (Uniform Continuity). A $f : X \to Y$ is uniformly continuously if $\forall \varepsilon > 0, \exists \delta > 0$ such that:

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

## Lecture 47: Day 47

**Proposition 23.** If $f$ is continuous and $X$ compact, then $f$ is uniformly continuous.

**Proof.** $\forall \varepsilon > 0, \forall c \in X, \exists \delta_c > 0$ such that $d(x, c) < \delta_c$. Then $d(f(x), f(y)) < \frac{\varepsilon}{2}$. $B(c, \delta_c)$ covers $X$ by Lebesgue Number Lemma, there exists $\delta > 0$ such that for all $x \in X, \exists c \in X,\ B(x, \delta) \subseteq B(c, \delta_c)$. Thus $d(x, y) < \delta$ implies $y \in B(c, \delta_c)$. Thus $d(x, y) < \delta$ implies

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(c)) + d_Y(f(c), f(y)) \\ &< \frac{2\varepsilon}{2} = \varepsilon \end{aligned}$$

■

**Definition 70** (Contraction)**.** $f : X \to X$ is a contraction if it is Lipschitz for $0 \leq k < 1$.

**Definition 71** (Fixed Point)**.** $f : X \to X$, x is a fixed point if $f(x) = x$.

**Theorem 60** (Banach Fixed Point Theorem)**.** Let $X$ be a Cauchy Complete metric space and a $f : X \to X$ contraction implies there exists a fixed point which is unique.

**Proof.** Pick arbitrary $x_0 \in X$. Define $x_{n+1} = f(x_n)$, then $x_0, f(x_0), f(f(x_0)), \ldots$.

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{M-1} d(x_m, x_i) \\ &\leq \sum_{i=n}^{M-1} k^i d(x_1, x_0) \\ &\leq k^n d(x_1, x_0) \sum_{i=0}^{m-1-n} k^i \\ &= \frac{k^n}{1-k} d(x_1, x_0) \\ &< \varepsilon. \end{aligned}$$

We know there exists $x \in X$ such that $x_n \to x$. then

$$x = \lim_{n \to \infty} f(x_n) = f(\lim x_n) = f(x).$$

Suppose, for proving uniqueness, Let $y \in X$ such that $y = f(y)$. Then:

$$\begin{aligned} d(x,y) &= d(f(x), f(y)) \\ &\leq kd(x,y) \\ &\Rightarrow (1-k)d(x,y) \leq 0 \\ &\Rightarrow d(x,y) = 0 \\ &\Rightarrow x = y. \end{aligned}$$

■

## Lecture 48: Day 48

**Definition 72** (Completion)**.** Let $(M, d)$ be a metric space, then there exists $(\overline{M}, \overline{d})$ such that:

1. $M \subseteq \overline{M}$.
2. $\overline{d} = d$.
3. $\overline{M}$ is cauchy complete.
4. The closure of $M$ is $\overline{M}$.

**Definition 73** ($C_\infty(M)$)**.**

$$C_\infty(M) := \{f : f \text{ is continuous with } \sup_{m \in M} |f(m)| < \infty\}$$

is a metric space with $d_\infty(f,g) = \sup |f(m) - g(m)|$.

**Definition 74** (Banach Space)**.** A normed space that is Cauchy Complete with respect to the metric. Examples include $\mathbb{R}^n, \mathbb{C}^n, C_\infty(M), C^0([a,b])$.

**Definition 75** (Functional)**.** A linear map $T : X \to \mathbb{R}$ or $\mathbb{C}$ with

$$\|T\|_{op} = \sum_{\substack{x \in X \\ \|x\|=1}} |T_x| .$$

Then the set of functionals is Banach.

**Definition 76** (Inner Product Space)**.** A vector space $X$ with $< \cdot, \cdot >: X \times X \to \mathbb{R}$ such that

- $< x, y >=< y, x >$.
- $< ax + by, z >= a < x, z > +b < y, z >$.
- If $x \neq 0$, then $< x, x >> 0$.

**Definition 77** (Hilbert Space)**.** A Cauchy Completion of the Inner Product Space.

**Remark.** An example where Riemannian analysis fails and hence we must study metrics and measure theory would be:

$$\int \nvDash_{\mathbb{R}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

## Lecture 49: Day 49

**Definition 78** (Interior)**.**

$$\mathrm{Int}(A) = (\overline{A^c})^c.$$

If $x \in \mathrm{Int}(A)$ if and only if there exists $\varepsilon > 0$ such that $\vee_\varepsilon(x) \subset A$ which is open.

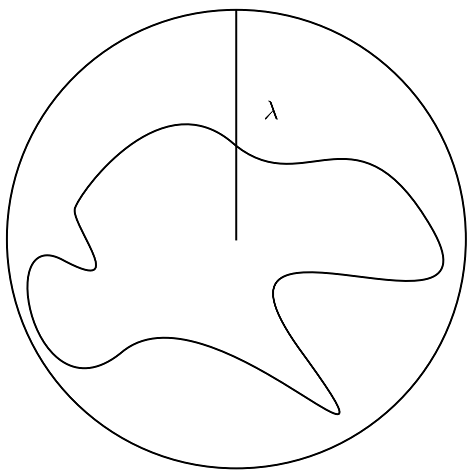
**Definition 79** (Exterior)**.**

$$(A) = (\mathrm{Int}(A))^c.$$

**Definition 80** (Boundary)**.**

$$(A) = X \setminus ((A) \cup \mathrm{Int}(A)).$$

**Definition 81** (Boundedness Revisited)**.** $A$ is bounded if and only if $A \subseteq \vee_\lambda(x_0)$ for some $\lambda$.



**Theorem 61.** Given $S$ is a compact subset of $X$ and $f : X \to \mathbb{R}$ be a continuous function, then $f$ has a maximum point in $S$.

**Proof.** Given $S$ is a compact subset, then $f(S)$ is also compact. Thus $f(S)$ is closed and bounded. Let $\alpha = \sup(f(S))$, then $\alpha \in f(S)$ since it is closed and there exists $x_0 \in S$ such that $f(x_0) = \alpha$. ∎

**Theorem 62.** If $f : X \to Y$ is continuous and $A$ compact subset of $X$, then $f$ is completely continuous on $A$.

**Proof.** Let $p \in A$, there exists $\delta_p > 0$ such that $|f(p) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in \vee_{\delta_p}(p)$. Consider $\{\vee_{\delta_p/2}(p) : p \in A\}$, an open cover of $A$. Then by H-B, there is a finite subcover $\{U_{\delta_{p_i}/2}(p) : 1 \leq i \leq N\}$. Choose $\delta \leq \min\{\frac{\delta_{p_i}/2}{\}}$.

We want to show that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Given $x$, choose $p$ such that $x \in \vee_{\delta_{p_i}/2}(p)$. So $d(p, x) < \frac{\delta_{p_i}/2}{}$ and $d(x, y) < \delta < \frac{\delta_{p_i}/2}{}$. By the triangle inequality, $d(p, y) < d_{p_i}$. Thus $x, y \in \vee_{\delta_{p_i}}(p)$ implies $|f(x) - f(p_i)| < \varepsilon/2$ and $|f(y) - f(p_i)| < \varepsilon/2$. Hence, $|f(x) - f(y)| < \varepsilon$. ∎

## Lecture 50: Day 50

**Definition 82** (Connectedness)**.** Given $(X, d)$ metric space is connected if it is impossible to write $X$ as a disjoint union $X = U_1 \cup U_2$ of nonempty open sets, such that $U_1 \cap U_2 = \varnothing$.

**Example** (Examples of Connected Sets)**.**

$$\mathbb{R}, \mathbb{R}^n, [a, b]$$

**Theorem 63.** Given $(X, d_X), (Y, d_Y), f : X \to Y$ continuous, then if $X$ is connected, then $f(x)$ is also connected.

**Proof.** Suppose, for the sake of contradiction, $f(x)$ is disconnected, then $f(X) = U_1 \cup U_2$ such that $U_1 \cap U_2 = \varnothing$. Then $f^{-1}(U_1) \cup f^{-1}(U_2)$ disconnected which is a contradiction. Hence $X$ is connected. ∎

**Theorem 64** (Intermediate Value Theorem)**.** Let $(X, d)$ connected and $f : X \to \mathbb{R}$ continuous. If $a, b \in (X)$ and $a < r < b$, then $r \in f(X)$.

**Proof.** Suppose $r \in f(x)$. Let $A = (-\infty, r)$ and $B = (r, \infty)$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a disjoint union of open sets, hence a contradiction. ∎

## Lecture 51: Day 51

**Definition 83** (Directional Derivative)**.** Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$, a point $a \in U$ and $U \in \mathbb{R}^n$, then:

$$D_u(f(a)) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}.$$

**Remark.** Recall in single variable calculus:

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

Now a function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by $\lambda(t) = f'(a)t$, then:

$$\begin{aligned} \lim_{t \to 0} \frac{f(a+t) - f(a) - \lambda(t)}{t} &= \lim_{t \to 0} \frac{f(a+t) - f(a)}{t} - f'(a) \\ &= f'(a) - f'(a) \\ &= 0 \end{aligned}$$

Thus as $t \to \infty$, then $\lambda(t) \approx f(a+t) - f(a)$.

**Definition 84** (Differentiability)**.** Given $U \overset{op}{\subset} \mathbb{R}^n$ and a point $a \in U$, $f$ is differentiable at $a$ if there exists a linear mapping $B : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $h \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{f(a+h) - f(a) - Bh}{|h|} \to 0, \qquad h \to 0.$$

Thus $Bh \approx f(a+h) - f(a)$.

**Theorem 65.** If $f$ is differentiable at $a$, then for every $u$, the directional derivative of $f$, in the direction of $u$ at $u$ exists.

**Proof.** The function $f$ is differentiable at $a$, so

$$\frac{f(a+tu) - f(a) - B(tu)}{|tu|} \to 0, \qquad t \to 0.$$

Furthermore,

$$\frac{t}{|tu|} \frac{f(a+tu) - f(a) - B(tu)}{t} = \frac{t}{|t|} \frac{1}{u} \frac{f(a+tu) - f(a)}{t} - Bu \to 0, \qquad t \to 0.$$

Thus,

$$\frac{f(a+tu) - f(a)}{t} \to Bu, \qquad t \to 0.$$

■

**Remark.** We will use $Df(a)$ to denote the derivative.

## 8.2 Lebesgue's Criterion for Riemann Integrability

### Lecture 52: Day 52

**Observe.** The following is called Thomae's Function, which is continuous on the set of irrationals, and has discontinuities at every rational. However, this function is still integrable on $[0, 1]$ with $\int_0^1 t = 0$.

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This set may fail to be continuous at rational points, and is infinite cardinality. Generally countably infinite sets are the smallest type of infinite set. We have previously looked at the length of sets, which for example, a cantor set continues to take infinite interval subset construction resulting in some length of intervals. Cantor sets are zero length. However, we will look at measures which are more detailed and specific.

**Definition 85** (Measure Zero). A subset $A$ has measure zero if, for all $\varepsilon > 0$, there exists a countable collection of open intervals $O_n$ with the property that $A$ is contained in the union of all intervals $O_n$ and the sum of the lengths of all the intervals is less than or equal to $\varepsilon$. More precisely,

$$A \subseteq \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \sum_{n=1}^{\infty} |O_n| \leq \varepsilon.$$

**Theorem 66** (Lebesgue's Theorem). Let $f$ be a bounded function defined on the interval $[a, b]$. Then, $f$ is Riemann integrable if and only if the set of points where $f$ is not continous has measure zero.

**Proof.** Let $M > 0$ satisfy $|f(x)| \leq M$ for all $x \in [a, b]$, and let $D$ and $D^n$ be defined as in

$$D = \{x \in [a, b] : f \text{ is not continuous at } x\}.$$

$$D^\alpha = \{x \in [a, b] : f \text{ is not } \alpha\text{-continuous at } x.\}$$

Suppose that $D$ has measure zero. ($\Leftarrow$). Let $\varepsilon > 0$ and set

$$\alpha = \frac{\varepsilon}{2(b - a)}.$$

**Problem.** Show that there exists a finite collection of disjoint open intervals $\{G_1, G_2, \ldots, G_N\}$ whose union contains $D^\alpha$ and that satisfies

$$\sum_{n=1}^{N} |G_n| < \frac{\varepsilon}{4M}.$$

**Problem.** Let $K$ be what remains of the interval $[a, b]$ after the open intervals $G_n$ are all removed; that is, $K = [a, b] \setminus \bigcup_{n=1}^{N} G_n$. Argue that $f$ is uniformly $\alpha$-continuous on $K$.

**Problem.** Finish the proof in this direction by explaining how to construct a partition $P_\varepsilon$ of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \varepsilon$. It will be helpful to break the sum

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k$$

into two parts-one over those subintervals that contain points of $D^\alpha$ and the other over subintervals that do not. ($\Rightarrow$) For the other direction, assume $f$ is Riemann-integrable. We must argue that the set $D$ of discontinuities of $f$ has measure zero.

Let $\varepsilon > 0$ be arbitrary, and fix $\alpha > 0$. Because $f$ is Riemann-integrable, there exists a partition $P_\varepsilon$ of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \alpha\varepsilon$.

**Problem.** (a) Prove that $D^\alpha$ has measure zero. Point out that it is possible to choose a cover for $D^\alpha$ that consists of a finite number of open intervals.

(b) Show how this implies that $D$ has measure zero.

■

# Part III

# Problem Sets