Modern Algebra and Geometry

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MATH 4000/6000 Notes

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Chapter 1

Introduction into Algebra

1.1 Well Ordering and Induction

The first thing we will look at is the well-ordering axiom. This takes into account that there is an order relation (¡) on all integers of Z, such that:

Definition 1.1.1: Well-Ordering

Every nonempty subset of $\mathbb{Z}^{\geqslant 0}$ contains a smallest element.

The direct consequence of this definition is Mathematical Induction. Mathematical Induction is a proof technique that uses recursive techniques to prove that a statement is true for all elements past its base case.

Theorem 1.1.1 Principle of Mathematical Induction

Assume that $n \in \mathbb{Z}^{\geqslant 0}$ and P(n) is given.

1. P(0) is a true statement.
2. When P(k) is true, then P(k + 1) is also true.

Then P(n) is true for all $n \in \mathbb{Z}^{\geqslant 0}$.

A remark on this theorem is that P(k) does not have to be true, but we assume so. This is called the induction hypothesis. In proofwriting, if we are given an "If... Then..." statement, we generally assume that the statement before the "Then" is true, and attempt to prove the rest. This is the same thing we have proved through Induction. It can be seen as a result of continued direct proofs compiled together and generalized to become the induction we know today. The following example is how we use Induction in today's world, and it's important to note how we use it compared to how one may have done it for a proofs course. In other words, a practical application of how a researcher would use induction.

Example 1.1.1

A set of n elements has 2^n subsets
$P(0) : 2^0 = 1$ subsets.
$P(1) : 2^1 = 2$ subsets.
$P(3) : 2^3 = 8$ subsets.
Assume P(k) is a set with k elements and has 2^k subsets. Now prove $P(k + 1) = 2^{k+1}$ subsets.
In a more standardized proofwriting, we can define a set

$$S := \{n \in \mathbb{Z}^{\geqslant 0} : \ P(n) \ is \ true\},$$

and show that $S = \mathbb{Z}^{\geqslant 0}$. Let our induction hypothesis be "P(n) is true". Since we have shown that our base case : P(0) is true, then we assume P(k) is true and attempt to prove P(k+1). Let's suppose that since P(n) is true, then $\#S = k$, which is the cardinal of set S. If we are to add a new element to set S and attempt to prove $k+1$, every subset has the option to choose between including $k+1$ or not including $k+1$. Therefore set S has $2 * 2^k = 2^{k+1}$ subsets. Thus proving $k+1 \in S$, therefore $S = \mathbb{Z}^{\geqslant 0}$. ■

1.1.1 A variation on Induction

Now with mathematical induction, also just referenced as induction, we can also show another type called Strong or Complete Induction.

Theorem 1.1.2 Principle of Complete Induction
Assume that $n \in \mathbb{Z}^{\geqslant 0}$, P(n) is given. If

1. P(0) is true, and
2. P(j) is true for all j such that $0 \leqslant j \leqslant t$, then P(t) is also true.

Proof: Let's prove this through induction. Let our induction hypothesis be if "P(j) is true for all j such that $0 \leqslant j \leqslant t$, then P(t) is also true" Suppose there is a set S, such that

$$S := \{n \in \mathbb{Z}^{\geqslant 0} : \; P(j) \; is \; true \; for \; all \; j \; such \; that \; 0 \leqslant j \leqslant n\}$$

For our base case, let's set n = 0, and suppose that $0 \in S$, thus P(0) is true.
Now assume P(k) is true, therefore P(k+1) is also true due to our induction hypothesis. Therefore $k \in S$ and $k+1 \in S$ is true. Therefore by induction, $S = \mathbb{Z}^{\geqslant 0}$, and we have proved Complete Induction. ■

Similar to how we used weak or regular induction to prove complete induction, we can do the same in reverse. In fact, we can prove all of these theorems and definitions using one another. We can use the well-ordering axiom to prove mathematical induction and use mathematical induction to prove complete induction. To complete the loop, prove well ordering through complete induction. On a harder note, we can prove regular induction through complete induction, but it is possible.

$Well-Ordering \implies Induction$

Proof: Let us define the set S as

$$S := \{n \in \mathbb{Z}^{\geqslant 0} : \; P(n) \; is \; false\} \subseteq \mathbb{Z}^{\geqslant 0}.$$

Our goal in this proof is to show that the set $S = \phi$.
Assume $S \neq \phi$. Then let $d \in S$ be the smallest element. Let P(0) be true, but this means that $d \neq 0$. So that means $d \geqslant 1$. So if $d-1 \geqslant 1$, then $d-1 \in \mathbb{Z}^{\geqslant 0}$. Since $d-1 < d, \; then \; d-1 \in S$, so P(d-1) is true. By assumption $P(d-1) \implies P(d)$ so P(d) is true, so $d \notin S$. So $S = \phi$, therefore P(k) is true for all $k \in \mathbb{Z}^{\geqslant 0}$. ■

Now that we have jump-started the proofwriting structure in our heads, let's go ahead and start this course with our next topic: Fundamentals of Arithmetic and Divisibility.

Chapter 2

Fundamentals of Arithmetic and Divisibility

2.1 Axioms

Axioms are trivial definitions used in everyday life — or even mathematics — that we take for granted. They are definitions that are inarguable and are the core of math today. I never quite understood the hierarchy of math statements, but this is a way to look at it: Axioms are a specific type of definition that is just taken as a fact or true. Definitions are similar to axioms in which they build the premise of future statements, these may or may not include proofs to explain why this may be true. Lemmas are true statements that are not important in the long run, but are trivial to understand to understand future statements, generally are associated with proof. Propositions are important statements that must be associated with proof and are vital research building blocks. Theorems are big conclusion that wraps each concept mentioned in a paper into one central idea and are even more important than propositions, these also require proofs to be stated alongside the statement. Now the following axioms or properties are what we accept without another thought, but they are important to mention to understand future content when they are brought up again.

Definition 2.1.1: Additive Properties

1. Addition is well-defined. Given a,b Z, a + b is a uniquely defined integer.
2. Substitution Law: Since addition is well-defined, if a = b, and c = d, then a + c = b + d.
3. Commutative Law: For all a,b Z, a + b = b + a.
4. Associative Law: For all a,b,c Z, (a + b) + c = a + (b + c).
5. There exists a zero element 0 Z, called the additive identity, satisfying 0 + a = a for any a Z.
6. For all a Z, there exists a unique additive inverse, -a Z, satisfying a + (-a) = 0

Definition 2.1.2: Multiplicative Properties

Multiplication is well-defined. Given a,b Z, a · b is a uniquely defined integer.

Substitution Law: If a = b and c = d, then ac = bd.

Z is closed under multiplication, for all a,b Z, a · b Z.

Commutative Law: For all a,b Z, ab = ba.

Associative Law: For all a,b,c Z, (ab)c = a(bc)

1 Z is the multiplicative identity, satisfying 1 · a

Definition 2.1.3: Distributive Property

For all a,b,c Z, a(b + c) = ab + ac.

Definition 2.1.4: Trichotomy Principle

Z can be split into three distinct sets.

$$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$$

Definition 2.1.5: Positivity Axiom

The sum or product of positive integers is positive.

Definition 2.1.6: Discrete Property

We have learned these already but they are the Well-Ordering Principle of N, and the Principle of Induction.

2.2 Division

Now that we have learned the axioms of arithmetic, let us learn about the division algorithm.
We have all (hopefully) learned how to divide in grade school. As a revision, you can divide a number evenly by some other number and whatever is left over will result as the remainder. This can be written more formally as:

$$dividend = (divisor)(quotient) + (remainder)$$

Now there is an important understanding I wanted to show to the audience. Every basic arithmetic operation can be written in terms of addition and multiplication. We will later see with rings that we make our lives easier by doing subtraction which shows both an inverse and additive property. But for now, that's all mumbo jumbo.

Theorem 2.2.1 Division Algorithm
Suppose $a, b \in \mathbb{Z}, b \geqslant 0, a = qb + r$ such that $\exists q, r \in \mathbb{Z}$, with $0 \leqslant r < b$.

Proof: Let there be set S such that

$$S := \{a - xb : a - xb \geqslant 0, x \in \mathbb{Z}\}$$

Check $S \neq \phi$

Given a and b, find x, such that $a - xb$. If $a \geqslant 0$, let $x = 0$, then $a - xb \implies a \geqslant 0$.
If $a < 0$ and let $x = a$, then $a - ab = a(1 - b)$, and since $b > 0, b \geqslant 1$, therefore $1 - b \leqslant 0$.
Since $S \neq \phi$ then S is well-ordered. $\exists r \in S$, such that r is the smallest element of S.

Claim: $r \geqslant 0$ and $r < b$. Since $r \in S, \exists q \in \mathbb{Z}$ such that $r \geqslant 0$ and $r = a - qb$. Prove that $r < b$.
Suppose $r \geqslant b$, then we can let

$$\begin{aligned} d &= a - (q+1)b \\ &= a - qb - b \\ &= r - b \\ r - b &\geqslant 0 \end{aligned}$$

So $0 \leqslant b < r, d = a - (q+1)b$, therefore $d \in S$, but $d < r$. Therefore we have a contradiction that r is the smallest element of S, therefore $r < b$. ■

There is a lot to dissect here. I want to dedicate special focus to this theorem. This will lay the foundation so glance your eyes on this beauty and take it in its glory. But in all seriousness, this is a really important topic to take in so let's explain it thoroughly. Similar to what we have in Figure 2.1 with the dividend equation, we just broke it down and generalized it using proof notation. So given that "a" is some dividend, we have divisor "b", and quotient "q" that are multiplied then added with remainder "r". There is also a reason why the division algorithm requires that r be less than b but at minimum 0. This may be trivial, but if r is greater than b, we can subtract r-b, and get the new remainder. It has the most optimized equation. Now that we understand what we are doing in more understandable terms, let us look at our proof itself and implement it as a core memory as how a child may remember their guardian.

Example 2.2.1

Let S be a set of remainders. We can do this through example. If

$$a = 81$$

$$b = 8$$

x is a variable

$$r = a - bx$$

If we let x = 1 for example, then r = 73.
If we let x = 4 for example, then r = 49.
If we let x = 10 for example, then r = 1.
If we let x = 11 for example, then r = -7.
However, r can only be at minimum 0, therefore r cannot be -7.
Therefore our most optimized r is when x = 10.
Of course, x can go in the opposite direction, since we did not bound Z only to non-negative integers.

Thus we have shown an example of the division algorithm. Now that we understand the values that set S can contain, even though we have provided proof, we must still prove this through math and generalize it. And that's exactly what we spend the rest of the proof doing. We answer questions in this proof such as, what if a is greater than 0 or less than 0? And what happens if r is greater than b, which we show that r is not the smallest integer which means we can technically have a solution of

Example 2.2.2

$$a = 81$$

$$b = 8$$

$$x is a variable$$

$$r = a - bx$$

If we let x = 1 for example, then r = 73.
If we let x = 4 for example, then r = 49.
If we let x = 10 for example, then r = 1.
If we let x = 11 for example, then r = -7.
However, r can only be at minimum 0, therefore r cannot be -7.
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Thus we have shown an example of the division algorithm. Now that we understand the values that set S can contain, even though we have provided proof, we must still prove this through math and generalize it. And that's exactly what we spend the rest of the proof doing. We answer questions in this proof such as, what if a is greater than 0 or less than 0? And what happens if r is greater than b, which we show that r is not the smallest

integer which means we can technically have a solution of

$$a = 200$$

$$b = 2$$

$$x = 10$$

$$r = 180,$$

and this is a valid solution by the division algorithm if we did allow r to not be the smallest, even though we know it's not exactly true.

Proposition 2.2.1 Uniqueness in the Division Algorithm

The integers $q, r \in \mathbb{Z}$, in the division algorithm are unique.

Proof: Given $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^{\geqslant 0}$, Suppose $a = q_1 b + r_1$, such that $q_1, r_1 \in \mathbb{Z}$ and $0 \leqslant r_1 < b$. Also suppose that $a = q_2 b + r_2$, such that $q_2, r_2 \in \mathbb{Z}$ and $0 \leqslant r_2 < b$.
Claim: $q_1 = q_2$ and $r_1 = r_2$.

$$\begin{aligned} a &= q_1 b + r_1 \\ -a &= q_2 b + r_2 \\ 0 &= (q_1 - q_2)b + r_1 - r_2 \\ r_2 - r_1 &= (q_1 - q_2)b. \end{aligned}$$

Thus, $-b < r_2 - r_1 < b$. Therefore $-b < (q_1 - q_2)b < b$, then $-1 < q_1 - q_2 < 1$. Since $q_1, q_2 \in \mathbb{Z}$, and the only integer that is greater than -1 and less than 1 is 0, then $q_1 - q_2 = 0$. Therefore $q_1 = q_2$. Then

$$\begin{aligned} 0 &= (q_1 - q_2)b + r_1 - r_2 \\ 0 &= (0)b + r_1 - r_2 \\ 0 &= r_1 - r_2. \end{aligned}$$

Thus $r_1 = r_2$. ■

This proposition demonstrates that q and r are unique, and this is really important to show in math when we are proving an algorithm. Regardless of what q and r are, if they exist, then they are unique, sounding trivial but as we see the proof is rather... less trivial. This one is a bit more straightforward therefore there won't be a conceptualizing analysis on this proof. This is also just further building the proof techniques we have at our arsenal and allowing one to understand the algorithm through and through.

Definition 2.2.1: Logical Divide

Suppose $a, b \in \mathbb{Z}$. Let us define the logical divide of b divides a as $b \mid a$.
If $\exists q \in \mathbb{Z}$ in this logic, then $a = bq$. If $b = 0, a \neq 0$, then $b \nmid a$, because $0q = 0$, and $a \neq 0$.

There isn't a strict name for this definition as far as I know, therefore I created a name for it. Logical Divide. It is the logical notation for the phrase "x divides y", and it is trivial to Abstract Algebra. It is slightly different than say previous computationally algebraic courses, where one just computes some division and may even end up with a completed or incomplete (rational or not) answer. Note that up to now we are only sticking with the integers, and this is a really important fact to keep in mind. Therefore when we say that 2—4, then we really mean that 4 is evenly divisible by 2, but 3 does not divide 4, even if we can write it in terms of a decimal. Another way we can explain this topic is through the division algorithm. If it doesn't look similar, we can write b divides a as, a = bq + r, where r = 0. Now does this mean that if a = 0, does 0—0? Honestly, it's a debated topic in algebra and number theory, some may state yes, others may state no. But what's important, is that the majority say no, the same reason why your calculator cannot divide 0 from 0.
Now if a = 0 and b—a, there is an integer q in Z, such that a = bq and q is unique. This is proof we will not get into it for the sake of saving time and space, but it is a nice practice exercise.
One proof we will be looking at is:

Lemma 2.2.1
Assume $b \mid a, b \neq 0$, so $a = bq$ for $q \in \mathbb{Z}$, then $-b \mid a$.

Proof: $a = (-b)(-q)$, so $-b \mid a$, for $q \in \mathbb{Z}$. Similarly $b \mid -a$. ■

This is just a fun fact to rationalize that these four results are possible: b—a, b—-a, -b—a, -b—-a. Now on a larger note, we must prove transitivity through logical divides.

Lemma 2.2.2
Suppose $a, b, c \in \mathbb{Z}$. If $c \mid b$ and $b \mid a$¡ then $c \mid a$.

Proof: $\exists q_1, q_2 \in \mathbb{Z}$, such that $a = bq_1$ and $b = cq_2$. So $a = (cq_2)q_1 = c(q_2 q_1)$. So $c \mid a$. ■

One thing to note is that divisibility is anti-reflexive, which means if b—a and a do not equal b or -b, then a does not divide b. There is a statement that could be said about linear combinations of a and b. If there is an integer c that divides both a and b, then there exists integers x and y, such that c—xa+yb. Therefore, c divides any linear combination of a and b. The proof of this is similar to the previous proof before. The idea is if you can write a and b in terms of c, then the linear combination could also be written in terms of c. Thus showing divisibility. Try to implement this on your own. If it hasn't been noticeable, there is nothing more to learning a course outside of learning the definitions and theorems.

Definition 2.2.2: Greatest Common Divisor

The GCD of a and b, written as $gcd(a, b) = d$, and $d > 0$ such that $d \mid a$ and $d \mid b$ and if $c \mid a$ and $c \mid b$, then $c \mid d$, and $c \leqslant d$.

The greatest common divisor is a concept that we have learned in grade school. If we recall, we can write the gcd(4,6) = 2, since 2—4 and 2—6.

Theorem 2.2.2 Linear Combinations of GCD
Let $a, b \in \mathbb{Z}$, not both 0. Let there be set S such that

$$S := \{xa + yb : x, y \in \mathbb{Z}, xa + yb > 0$$

Then $S \neq \phi$ and $S \subseteq \mathbb{Z}^{\geqslant 0}$, then by the well-ordering principle, S has the smallest element called d. Then $d = gcd(a, b)$.
The key statement is if $d = gcd(a, b)$, then $\exists x, y \in \mathbb{Z}$ such that $d = xa + yb$.

Proof: Let $S \neq \phi$. $\exists d \in S$ such that $\forall t \in S, d \leqslant t$ since $d \in S$. Then $\exists x, y \in \mathbb{Z}$ such that $d = xa + yb$. Now our goal is to prove that $d \mid a$.
If $d \in S$, then $d > 0$, so $\exists q, r \in \mathbb{Z}$ and $a = qd + r$ when $0 \leqslant r \leqslant d$.
Suppose $r > 0$, then

$$\begin{aligned} r &= a - qd \\ &= a - q(xa + yb) \\ &= a - qxa - qyb \\ &= (1 - qx)a - (qy)b. \end{aligned}$$

So r is a linear combination of a and b. Since $r > 0$ and $r < d$, then $r \in S$, contradicting the assumption that d is the smallest element of S.
If $r = 0$, then $a = qd$, therefore $d \mid a$.
Similarly we can show $d \mid b$.
Now suppose $c \mid a$ and $c \mid b$, then $c \mid xa + yb$, which is a linear combination of a and b, which equals d. Therefore d is unique.
Suppose $t > 0$ has the property that if $c \mid a$, $c \mid b$, then $c \mid t$, and $t \mid a$, $t \mid b$, then $t \mid d$ and $d \mid t$.
Therefore $d = t$. ■

If the gcd of any two integers ever equals 1, then we say that a and b are relatively prime. If they are relatively prime, then by the previous theorem, the linear combination will also equal 1.

Theorem 2.2.3

Suppose $gcd(a, b) = 1$ and $c \in \mathbb{Z}$ such that $a \mid bc$, then $a \mid c$.

Proof: Since the $gcd(a, b) = 1$, then $\exists x, y \in \mathbb{Z}$ such that $xa + yb = 1$. Therefore,

$$xa + yb = 1$$
$$cxa + cyb = c$$
$$(cx)a + y(bc) = c,$$

then $a \mid a$ and $a \mid bc$, therefore $a \mid c$. ■

2.3 Algorithms

The last thing we will be looking at in this chapter is the extended gcd algorithm. The idea behind this is to use the gcd algorithm and then reverse the process in order to find the factors of the linear combination. This is more of a computational math. The GCD algorithm can be written in terms of the Division Algorithm and continuing to find the terms that make up the two factors. An idea of this is using the gcd(109, 26).

$$109 = 26(4) + 5$$

Because 109 can be split up by 26 and have a remainder of 5, this is no different than having a gcd of (26,5).

$$26 = 5(5) + 1$$

Now because we are left with a remainder of one, and one can go into any number, then 1 is our final answer for the gcd of (109, 26). This is a way to do the gcd algorithm through division. But what if we are to set this the other way around?

$$1 = 26–5(5)$$

Similar to what we did before, we are shifting all elements in the equation to create the one above.

$$1 = 26–5(109–26(4))$$

$$1 = (-5)(109) + (21)(26)$$

Thus we have found the linear combination factors of the equation.

2.4 Primes

In the realm of mathematics, prime numbers are the true VIPs. The Fundamental Theorem of Arithmetic serves as a bouncer taking off the cheap costumes that all the composites wear making sure only primes get through. In this post, we will look at how the FTA classifies numbers and how the primes are the real deal when it comes to these costumes. I'm excited about this topic because it practically is my field of interest!

Definition 2.4.1: Prime Integer

Let $p \in \mathbb{Z}$. p is prime if the only divisors of p are $-1, 1, -p, p$ and $p \neq -1, 0, 1$.

This definition has two criteria, 1) the divisors of p are restricted; 2) p is not equal to restricted values. We use the term restricted to denote more so a finite set of values, but this sounds like a stronger claim.
(1) By the only divisors of p, we mean that if you are to divide p by any other integer, using the division algorithm, we will get a remainder. Using the previous content learned, we will learn that the GCD of p and any relatively prime, or co-prime, is 1.
(2) When we have p not equal to a select few values, then this ensures that the prime number does not contradict the first criterion. This definition helps identify and distinguish prime numbers from other integers.

Theorem 2.4.1 Euclid's Lemma

Suppose p is prime, and $b, c \in \mathbb{Z}$ with $p|bc$, then $p|b$ or $p|c$. Proof. Suppose $p \nmid b$. We claim that the $gcd(p, b) = 1$.‘

Proof: Suppose $d = gcd(p, b)$. Then $d > 0, d|p, d|b$, and since p is prime, then $d = 1$ or $d = p$. But $d \neq p$ since $p \nmid b$, so $d = 1$. Let's assume that p is prime. Then p would have some divisors $d, t \in \mathbb{Z}$, such that $p = dt$. Then according to our assumption, if p is prime, then the only divisors are $-1, 1 and -p, p$. Therefore, when $p|d$, then $d = -p, p$ and $t = -1, 1$. Or when $p|t$, then $t = -p, p$, and $d = -1, 1$. Thus p is prime. ■

This "lemma" is something Euclid used to prove something bigger. The name stuck as "Euclid's Lemma", however, it is the foundation for fields such as Number Theory. Its more appropriate name is the Fundamental Property of Prime Numbers. It sounds like a really basic lemma, but it does undermine its true essence. It shows that prime numbers are the building blocks of all integers and that a number divisible by a prime must be divisible by that prime individually or by another prime factor.

Theorem 2.4.2 Fundamental Theorem of Arithmetic (FTA)

If $n \in \mathbb{Z}$ and $n \neq -1, 0, 1$, then n can be written uniquely as a product of primes up to order and sign.

In other words, the theorem tells us that every composite number can be broken down into a unique set of prime factors. These prime factors are the building blocks of all positive integers. The uniqueness of the factorization means that no matter how you break down a composite number into its prime factors, the set of primes you obtain will always be the same, even if the order and sign of the primes might differ.

Lemma 2.4.1

Suppose p is prime. $a_1, a_2, a_3, \ldots, a_n \in \mathbb{Z}$ such that $p|a_1a_2 \ldots a_n$. Then $p|a_1$ for some $i \in \mathbb{N}$.

Proof of Lemma: By induction on i, let $n = 1$. If $p|a_1$, then $p|a_1$ is true. Let $n = 2$. If $p|a_1a_2$, then $p|a_1$ or $p|a_2$ is true. Suppose it's true for given $n = k$. Now let $n = k + 1$. If $p|a_1a_2 \ldots a_k a_{k+1} = p|(a_1a_2 \ldots a_k)a_{k+1}$, then $p|(a_1a_2 \ldots a_k)$ or $p|a_{k+1}$. By induction, $p|a_i$ for some $i, 1 \leqslant i \leqslant k$, or $p|a_{k+1}$. ■

Now note that we haven't quite proved the Fundamental Theorem of Arithmetic, but something we have shown is a corollary of Euclid's Lemma, which relates to FTA a little bit. So let's go ahead and link these two out.

Proof of FTA Theorem:

Claim 1. Existence of Factorization:

Suppose $n \in \mathbb{Z}$ and $n \neq -1, 0, 1$. Then there exists primes $p_1 \ldots p_k$ such that $n = p_1 \ldots p_k$. If $n \in \mathbb{Z}$ is a negative integer, then $n = -m$ is a positive integer. If $n = p_1 \ldots p_k$, p prime, then $n = (-p_1)p_2p_3 \ldots p_k$ is also a product of primes. So $n \in \mathbb{N}$ is true for all $n \in \mathbb{Z}$.

Proof of Claim 1: Suppose n is not prime, then $\exists a > 1$. So $n = ab$, given $b \in \mathbb{Z}$ and $b > 1$. Now apply strong induction on a and b.
$a = p_1 \ldots p_r$, p_i prime
$b = q_1 \ldots q_s, q_i$ prime
$n = ab = p_1q_1 \ldots p_rq_s$ as a product of primes, i.e. if $n = p_1 \ldots p_r = q_1 \ldots q_s$, then $r = s$ and after rearranging $r_i = -q_i, q_i$, for each i. ■

Claim 2. Uniqueness of Factorization:

From existence, we can show uniqueness. By induction on the $min\{r, s\}$. Let our base case be $k = 1$. We can assume $r = 1$, so $p_1 = q_1 \ldots q_k, p_1$ prime, all q_i prime. So $s = 1, p_1 = q_1$.

Proof of Uniqueness: Assume uniqueness for k, prove for $k+1$. Suppose $n = p_1 \ldots p_r = q_1 \ldots q_s$, $min\{r, s\} = k + 1$, as we can assume that $r = k + 1$. Then $p_1 \ldots p_k p_{k+1} = q_1 \ldots q_s$. So $p_1 q_1 \ldots q_s$, assume $p_1 = q_1$. So $p_1 = -q_1, q_1$. Then let's replace this singular prime, $p_1 \ldots p_{k+1} = (p_1)q_2 \ldots q_s$. Then $p_1(p_2 \ldots p_{k+1}) = p_1(q_2 \ldots q_s)$. By cancellation law, then we can cross out the p_1's. Then the minimum number of terms is k. By the induction hypothesis, $s = k + 1$, and after rearranging $q_i = p_i$ all $i > 1$. So uniqueness is trying for k + 1. So true for all $k \in \mathbb{N}$. ■

The uniqueness of the factorization means that no matter how you break down a composite number into its prime factors, the set of primes you obtain will always be the same, even if the order of the primes might differ. For example, consider the number 60. The Fundamental Theorem of Arithmetic tells us that 60 can be expressed as a product of prime factors uniquely:

$$60 = 2 * 2 * 3 * 5$$

This factorization is unique for the number 60. You can change the order of the factors, but the set of primes (2, 3, and 5) will remain the same.

In conclusion, we've explored some fascinating concepts in the world of mathematics, all centered around the remarkable prime numbers and the bedrock principle of the Fundamental Theorem of Arithmetic.

The Fundamental Theorem of Arithmetic, our mathematical Rosetta Stone, reveals the secret language of integers. It tells us that every positive integer greater than 1 can be expressed uniquely as a product of prime factors. This unique factorization into prime numbers underpins countless mathematical discoveries, making it a cornerstone of number theory.

Chapter 3

Congruence Classes in $\mathbb{Z}$

3.1 Congruences

When we talk about congruence classes mod n, we're essentially grouping integers based on the remainder they leave when divided by n. This creates a fascinating classification system, where numbers that share the same remainder form a class. It's like organizing a grand masquerade ball, where every guest wears a mask that matches their remainder modulo n, allowing them to join a specific group similar to classes/grades in school.

Definition 3.1.1: Congruence

Suppose $n \in \mathbb{N}$. If $a, b \in \mathbb{Z}$, we define $a \equiv b \mod n$ as a congruence. We say "a is congruent to b modulo n" if and only if $n|(b-a)$.

Lemma 3.1.1

$a \equiv b \mod n$ then $n|a-b$ if and only if there exists $q \in \mathbb{Z}$ such that $b = qn + a$. Prove this exercise on your own.

Definition 3.1.2: Equivalence Relation

Given S is a set and $\sim$ is a relation on S. $\sim$ is an equivalence relation if for all $a, b, c \in S$

1. $a \sim a$ (reflexive);
2. If $a \sim b$, then $b \sim a$ (symmetric);
3. If $a \sim b$ and $b \sim c$, then $a \sim c$ (transitive).

We will learn and find uses for the equivalence relation, but to connect it to the topics at hand, $a \equiv b$ is an equivalence relation which envelopes congruences and what we will learn about congruence classes. Essentially, $a \equiv b$ is the same as $a \sim b$.

Lemma 3.1.2

Congruence mod n is an equivalence relation.

Proof: **Case 1.**
Let $a \in \mathbb{Z}, a \equiv a \mod n$, because $a - a = 0$ and $n|0$.
Case 2.
Suppose $a \equiv b \mod n$ then $n|a-b$ and due to properties of logical divide, $n|b-a$. Thus $b \equiv a \mod n$. **Case 3.**
Suppose $a, b, c \in \mathbb{Z}$, $a \equiv b \mod n$ and $b \equiv c \mod n$, so $n|b-a$ and $n|c-b$, so $n|(a-b)+(b-c) = n|a-c$. Thus $a \equiv c \mod n$. ■

Definition 3.1.3: Equivalence Classes

Suppose $\sim$ is a equivalence relation on S if $a \in S$. The equivalence class of a is $[a] := \{b \in S : b \sim a$.

Let's consider an equivalence relation $\sim$ on the set of integers $\mathbb{Z}$, where $a \sim b$ if and only if $a \equiv b \mod 5$ (congruence modulo 5). In this case:

$$[2]_5 = \{\dots, 8, 3, 2, 7, 12, \dots\}$$

This is the equivalence class of 2, consisting of all integers that are congruent to 2 modulo 5. Equivalence classes provide a systematic way of grouping elements in a set based on their relationships under an equivalence relation.

Definition 3.1.4: Congruence Classes

For a congruence mod n, if $a \in \mathbb{Z}$, $[a] := \{b \in \mathbb{Z} : b \equiv a \mod n\}$.

Congruence classes provide a systematic way of grouping integers based on their remainders when divided by n under a congruence relation. They are essential in modular arithmetic, number theory, and algebraic structures, contributing to a deeper understanding of mathematical relationships and structures. Two equivalence classes are the same if they include each other, for example, if $[a] = [b]$, then $a \in [b]$ and $b \in [a]$. The set S is the distinct union of its distinct equivalence classes. I.e. every element of S is in some equivalence class.

Proposition 3.1.1

If $a, b \in S$, then either $[a] = [b]$ or $[a] \bigcap [b] = \phi$.

Proposition 3.1.2

$[a] = [b] \iff a \equiv b \mod n$.

Proof.: ($\implies$). $[a] := \{x : x \equiv a \mod n\}$. so $a \in [a]$ since $a \equiv a \mod n$. So $a \in [b], [b] := \{x \in \mathbb{Z} : x \equiv b \mod n\}$, so $a \equiv b \mod n$.
($\impliedby$). **Case 1.** $[a] \subseteq [b]$**.**
Let $c \in [a]$, then $c \equiv a \mod n$. By transitivity, $c \equiv b \mod n$ so $c \in [b]$ so $[a] \subseteq [b]$.
Similarly, we can show $[b] \subseteq [a]$. Thus $[a] = [b]$. ■

This relationship provides a clear connection between the equality of equivalence classes and the congruence of integers modulo nn.

Proof of Proposition 3.0.1: We need to prove if $[a] \bigcap [b] \neq \phi$ then $[a] = [b]$. Let $c \in [a] \bigcap [b]$, then $c \equiv a \mod n, c \equiv b \mod n$. So by the previous proposition, $[c] = [a] = [b]$, so $[a] = [b]$. ■

Proposition 3.1.3

Fix $n \geqslant 2$. The distinct congruence classes modulo n are $[0], [1], \dots, [n-1]$. In fact, if $a \in \mathbb{Z}$, then $[a] = [r]$ where r is the remainder when a is divided by r.

Proof: If $a = qn + r, 0 \leqslant r \leqslant n - 1$, then $a \equiv r \mod n$ so $[a] = [r]$. By the division algorithm, $[a]$ must be one of these classes. By uniqueness, these classes are unique. ■

3.2 Modular Arithmetic

Definition 3.2.1: Modular Arithmetic

Fix $n \in \mathbb{Z}^{\geqslant 2}$. Define addition and multiplication on congruence classes mod n.

$$[a] + [b] = [a + b]$$

$$[a] \cdot [b] = [ab]$$

Theorem 3.2.1

If $a, b, c, d \in \mathbb{Z}$, then $a + b \equiv c + d \mod n$ and $ab \equiv cd \mod n$.

Proof: Given $n|c - a$, $n|d - b$, then $n|(c - d) - (a + b)$, so $n|(c - a) + (d - b)$. Thus $n|c - a, n|d - b, n|d(c - a) + a(d - b), n|cd - ab + cd - ab$, therefore $n|cd - ab$. ■

Theorem 3.2.2 Well-Defined Modular Arithmetic

Modular arithmetic is well defined.

Proof: Suppose $[a] = [c]$ and $[b] = [d]$. Then by the previous theorem, $[a + b] = [c + d]$ and $[ab] = [cd]$. ■

Definition 3.2.2: $\mathbb{Z}_n$

The set of congruence classes mod n with addition is defined by:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11
3	3	10	11	12

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Commutative, associative, and distributive hold for $\mathbb{Z}_n$.
The additive identity is $[0] + [a] = [a]$.
The multiplicative identity is $[1a] = [a]$.

$\mathbb{Z}_n := \{[a] : a \in \mathbb{Z}\} = \{[0], [1], \ldots, [n - 1]\}$.

Axiom 3.2.1 Additive inverses in $\mathbb{Z}_n$

Every element in $\mathbb{Z}_n$ has an additive inverse.

$$[a] + [-a] = 0$$

3.3 Units and Divisors

Definition 3.3.1: Units in Congruence Classes

$[a]$ is a unit if $[a]$ has a multiplicative inverse.

Theorem 3.3.1

$[a]$ is a unit if and only if $gcd(a, n) = 1$.

Proposition 3.3.1

All classes in $\mathbb{Z}_p$ are units.

Proof of Proposition 3.2.1: Suppose $[a] \in \mathbb{Z}_p$ is a unit so $\exists x \in \mathbb{Z}$ such that $[xa] = 1$. Then

$$\begin{aligned} xa \equiv 1 \mod p &\iff xa = 1 + qp \\ &\iff xa - qp = 1 \\ &\iff gcd(a, p) = 1 \end{aligned}$$

■

Easy to show the opposite by showing a multiplicative inverse in the $\mathbb{Z}_p$. So $[a]$ has a multiplicative inverse. This proposition, using $n \neq p$ will show it is true for the **Theorem 3.2.3**.
To show that $[a]$ is a unit in $\mathbb{Z}_{32}$ and find $[a]^{-1}$ in $\mathbb{Z}_{32}$. Let $a = 4$ and find an $x \in \mathbb{Z}$ such that $x4 + q32 = 1$, and use the Extended Euclidean Algorithm to find this inverse. We find that $x = -7$, which means $[x] = [-7] = [25]$.

Definition 3.3.2: Zero-Divisors

$[a]$ is a zero-divisor in $\mathbb{Z}_n$ if $\exists [x] \neq [0]$, with $[ax] = [0]$.

Theorem 3.3.2

$[a]$ is a zero-divisor in $\mathbb{Z}_n$ if and only if the $gcd(a, n) \neq 1$.

Proof: Let's prove the contrapositive. Suppose $gcd(a, n) = 1$, then $[a]$ is not a zero-divisor. Assume $gcd(a, n) = 1$. Suppose $b \in \mathbb{Z}$ with $[ab] = [0]$. $[a]$ is not a zero-divisor if and only iff $[ab] = [0]$ implies $[b] = [0]$. By **Theorem 3.2.3**, $[a]$ is a unit in $\mathbb{Z}_n$, so there exists $x \in \mathbb{Z}$, such that $[xa] = 1$. So $[x]([ab]) = [x0] = [0]$ or $[xa][b] = [1][b] = [b] = [0]$. Conversely, suppose $gcd(a, n) = d > 1$. ■

For example, if we take $\mathbb{Z}_{12}$, then since $gcd(4, 12) = 4$, then $[4]$ is a zero-divisor.

Chapter 4

Rings

4.1 Rings

Definition 4.1.1: Ring

A set with $+, \times$, called R. Addition has the properties of being commutative and associative. Multiplication is at minimum associative, and together distributive. There is an additive identity, usually denoted by 0_R. But there is also a multiplicative identity, denoted by 1_R. There exists an additive inverse in R, b, such that $a + b = 0$, and are unique.

Definition 4.1.2: Subrings

S is a subring of R if for all $a, b \in S$, has closure under addition and multiplication. It must also have the additive identity and additive inverses per each element.

For example, in an introduction to proofs class we may have seen that

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

We learned them as sets, but looking at properties of rings and subrings, consider them all rings and subrings of the order. However, if we wanted to look outside of these number systems, let's look at matrices:

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}$$

Note that this is a subring of $Mat_2(\mathbb{R})$.

Definition 4.1.3: Field

A commutative ring, $\mathbb{F}$, $1 \in \mathbb{F}$. if $a \in \mathbb{F}$, such that a is a unit. $\mathbb{F}$ is called a field.

Definition 4.1.4: Subfield

If S is a subring of field $\mathbb{F}$, and also closed under multiplicative inverses, then it also a subfield.

We have previously learned that

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}$$

is a subring of $Mat_2(\mathbb{R})$. But I also claim it is a field itself.

Proof of Claim: Suppose $Mat_2(0) \notin M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, which means a and b not both 0.

$$\det M = a^2 + b^2$$

Let $M^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Thus we have shown that the subring also is closed under multiplicative inverses. Thus is a field. ■

Lemma 4.1.1

For n composite, $\mathbb{Z}_n$ is not a field if it has zero-divisors.

From now on $\mathcal{R}, \mathcal{S}$ is a ring and $\mathbb{F}$ is a field.

Now that we have delved into the intricacies of zero-divisors and units within rings, and units specifically within fields, we gain profound insights into the fundamental structure and classification of algebraic systems. Recognizing the presence of zero-divisors, elements that, when multiplied, yield the additive identity, and units, elements with multiplicative inverses, becomes paramount in distinguishing and categorizing rings and fields. These concepts serve as a foundational framework for characterizing the algebraic properties and behaviors of mathematical structures.

Definition 4.1.5: Integral Domain

Suppose n is a commutative ring with $1 \in \mathcal{R}$. We say $\mathcal{R}$ is an integral domain if $a \neq 0$ and $a \in \mathcal{R}$ and a is not a zero-divisor.

We can think of these integral domain rings as being almost a field but the only thing discerning them from being a field is the fact the only zero-divisor is $0_{\mathcal{R}} \in \mathcal{R}$. Remember that if there is $0 \in \mathcal{R}$ then it is no way it can be a field, since all fields have $0 \notin \mathbb{F}$, since all elements must have an inverse a.k.a a unit.

Corollary 4.1.1

$\mathbb{F}$ is an integral domain.

Proof: Suppose $a, b \in \mathbb{F}$ with $ab = 0$. Suppose $a \neq 0$, then a is a unit with inverse a^{-1}. then

$$\begin{aligned} a^{-1}(ab) &= a^{-1} \cdot 0 = 0 \\ &= 0 \\ (a^{-1}a)b &= 1b = 0 \\ &= b = 0 \end{aligned}$$

■

Let's look into something called extensions.

Definition 4.1.6: Field Adjoins

We call something an adjoin given that suppose we have $\mathbb{F} = \mathbb{Q}$. Note this field is a subfield of $\mathbb{R}$. Then an extension of $\mathbb{Q}$ is taking an element of $\mathbb{R} \setminus \mathbb{Q}$, and adding it to $\mathbb{Q}$. An example of this is,

$$\mathbb{Q}[\sqrt{7}] := \{a + b\sqrt{7} : a, b \in \mathbb{Q}\}$$

In fact an exercise to do is to show that $\mathbb{Q}[\sqrt{7}]$ is infact a subfield. Based on everything we have observed, we can definitely say that $\mathbb{Z}_p$ is a field and $\mathbb{Z}_n$ is not even an integral domain.

Axiom 4.1.1 Pigeonhole Principle

If you have $n + 1$ objects in n slots, one slot will have more than 1 element.

Theorem 4.1.1

Finite integral domain is a field.

Proof: Let F be a finite integral domain. We need to show that if $0 \neq u \in F$, then u has a multiplicative inverse. Consider the set $\{u, u^2, u^3, \ldots\}$. Suppose F has n elements, then there must be repetition. So $u^k = u^m$ for $m > k$.

$$u^m - u^k = 0$$
$$u^k(u^{m-k} - 1) = 0$$

Since F is an integral domain, then $u^k = 0$ or $u^{m-k} - 1 = 0$. Since $u \neq 0$, the $u^k \neq 0$. Then

$$u^{m-k} - 1 = 0$$
$$u^{m-k} = 1$$
$$u(u^{m-k-1}) = 1$$
$$u^{-1} = u^{m-k-1}.$$

Thus F is a field. ■

4.2 Homomorphisms and Isomorphisms

Definition 4.2.1: Homomorphism

Let $\mathcal{R}$ and $\mathcal{S}$ be rings. Suppose a function $f : \mathcal{R} \mapsto si$, with given that $a, b \in \mathcal{R}$, $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.

Definition 4.2.2: Isomorphism

Suppose f is a homomorphism which is bijective, then f is a isomorphism.

If f is an isomorphism, then $\mathcal{R}$ and $\mathcal{S}$ are isomorphic to each other. Suppose $\mathcal{R} = \mathbb{Z}$ and $\mathcal{S} = 2\mathbb{Z}$ and let $f : \mathcal{R} \mapsto \mathcal{S}$ defined by $f(m) = 2m$. Is an isomorphism?

Disproof:

$$f(m + n) = 2(m + n) = 2n + 2m$$
$$f(mn) = 2mn \neq f(m)f(n)$$

Not isomorphic. ■

Proposition 4.2.1

A bijection exists if and only if it has an inverse.

Proof: Let $g : \mathcal{S} \mapsto \mathcal{R}$, thus $f \circ g$ is the identity of $\mathcal{S}$. And $g \circ f$ is the identity of $\mathcal{R}$. Define $g(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Let $f(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = a + bi$. This is closed under addition and multiplication. It also has an inverse due to the determinate law for inverses. ■

Axiom 4.2.1 Isomorphism Properties

We can check properties of a homomorphism to check if it is isomorphic.

1. # of elements in $\mathcal{R} = \mathcal{S}$
2. # of units in $\mathcal{R} = \mathcal{S}$ (Check how many coprimes in both sets)
3. # of 0-divisors for both are the same.

For example we can state that $\mathbb{Z} \ncong \mathbb{Q}$. The reason being is that every element in $\mathbb{Q}$ is a unit where as the only unit in $\mathbb{Z}$ is 1. Similarly, $\mathbb{Z}_4 \ncong \mathbb{Z}_6$, due to the number of elements.
Perhaps in previous courses, such as Calculus III, you have looked at $\mathbb{R}^3$, which means a 3-tuple ordered pair which represents (x,y,z) in a space. However, this is a generalized fact. What if I wanted to have two points from different sets, but still create an ordered pair or tuple.

Axiom 4.2.2 Cartesian Product
If $\mathcal{R}$and $\mathcal{S}$ are rings, then $\mathcal{R} \times \mathcal{S} := \{(r, s) : r \in \mathcal{R}, s \in \mathcal{S}\}$ is also a ring under addition and multiplication.

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$$
$$(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2).$$

It will be a fun exercise to prove the following lemma or at least a couple of examples.

Lemma 4.2.1
If the $gcd(m, n) = 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$.

Note that in $\mathbb{Z} \times \mathbb{Z}$, the zero-divisors are $(0, 1), (1, 0)$.
Let $\mathcal{R} = \mathcal{S} = \mathbb{Z}$ in $\mathbb{Z} \times \mathbb{Z}$. Let $\pi : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$. Then we have that $\pi[(1, 0)] = 1$, which is a unit. So a homomorphism need not preserve zero-divisors.

Chapter 5

Polynomials

5.1 Polynomials

Definition 5.1.1: Polynomial

A polynomial with coefficients in a $\mathcal{R}$ is denoted by $\mathcal{R}[x]$, which is an extension field of x expanding the set to include

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n.$$

We can think of a_n as coefficients.

Proposition 5.1.1

We do addition and multiplication component-wise, which means given an i large enough, a will eventually be 0. To understand what I mean, let

$$\begin{aligned} f(x) &= a_0 + \ldots + a_nx^n \\ g(x) &= b_0 + \ldots + b_mx^m, \end{aligned}$$

given that $m \geqslant n$. Therefore

$$f(x) + g(x)$$

This informal definition raises several questions: What is x? Is it an element of R? If not, what does it mean to multiply x by a ring element? In order to answer these questions, note that an expression of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ makes sense, provided that the a_1 and x are all elements of some larger ring. An analogy might be helpful here. The number π is not in the ring of integers ($\mathbb{Z}$), but expressions such as $3 - 4\pi + 12\pi^2 + \pi^3$ and $8 - \pi^2 + 6\pi^5$ make sense in the real numbers ($\mathbb{R}$). Furthermore, it is not difficult to verify that the set of all numbers of the form $\sum_{i=0}^{n} a_i\pi^i$, with $n \geqslant 0$ and $a_1 \in \mathbb{Z}$, is a subring of $\mathbb{R}$ that contains both $\mathbb{Z}$ and π (Exercise 2). For the present, we shall think of polynomials with coefficients in a ring R in much the same way, as elements of a larger ring that contains both R and a special element x that is not in R. This is analogous to the situation in the preceding paragraph with R in place of $\mathbb{Z}$ and x in place of π, except that here we don't know anything about the element x or even if such a larger ring exists.
Feel free to check if $R[x]$ is a ring, but we will be concentrating on $\mathbb{Z}[x], \mathbb{Q}[x]\mathbb{R}[x], \mathbb{C}[x], \mathbb{Z}_p[x]$, and have their elements denoted by $f(x)$ or $P(x)$.

Definition 5.1.2: Degree of a Polynomial

If $f(x) \in \mathcal{R}[x]$, the degree of $f(x)$, denoted by $\deg f(x)$, is the largest n for which the coefficient of x^n is not 0. a_n is also called the leading term.

Definition 5.1.3: Additive Identity of $\mathcal{R}[x]$

a_n is 0.

If the $\deg f(x) = 0$, then the degree is undefined, which means the leading term is undefined.

Proposition 5.1.2 Degree Arithmetic

Suppose $\deg f(x) = m, \deg g(x) = n$,

$$\deg f(x) + g(x) \leqslant \max\{\deg f(x), \deg g(x)\}$$

if $m \neq n$

$$\deg f(x) + \deg g(x) = \max\{m, n\}$$

if $m = n$

$$\deg f(x) + \deg g(x) \leqslant \max\{m, n\}$$

If $f(x)g(x) = a_0b_0 + \ldots + a_nb_mx^{n+m}$, so $\deg f(x)g(x) \leqslant \deg f(x) + \deg g(x)$.
However if $\mathcal{R}$ is an integral domain, then

$$\deg f(x)g(x) = \deg f(x) + \deg g(x)$$

Let $f(x), g(x) \in \mathbb{Z}_4[x]$, $f(x) = 2x, g(x) = 2x^2$, then $f(x)g(x) = 4x^3 = 0$.

From now on $\mathcal{R}$ is an Integral Domain.

Given that $\mathcal{R}$ is an integral domain, one may naturally ask, what are the units of $\mathcal{R}$.

Lemma 5.1.1

Suppose $u(x)$ is a unit with a multiplicative inverse $v(x)$. Then

$$u(x)v(x) = 1 = 1 + 0x + 0x^2 + \ldots$$

5.2 Division

Theorem 5.2.1 Division Algorithm in Polynomial Fields

Suppose $\mathbb{F}$ is a field and $a(x), b(x) \in \mathbb{F}[x], b(x) \neq 0$. Then there exists a unique $r(x) \in \mathbb{F}[x]$ with

$$a(x) = q(x)b(x) + r(x)$$

with $\deg(r(x)) > \deg(b(x))$ or $r(x) = 0$.

Proof: **Case 1:** If $a(x) = 0$ or $\deg(a(x)) < \deg(b(x))$, then $q(x) = 0$ and $r(x) = a(x)$ because $a(x) = b(x)0 + a(x)$.
Case 2: If $a(x) \neq 0$ and $\deg(a(x)) > \deg(b(x))$, and $a(x)/b(x) = h(x)$, then $\deg(h(x)) < \deg(a(x))$. If $\deg(a(x)) = 0$, then $a(x) = a$, a constant in $\mathbb{F}$. $\deg(b(x)) < \deg(a(x))$ implies $b(x)$ equals a constant.

$$\begin{aligned} a(x) &= b(x)(b(x)^{-1}a(x)) + 0 \\ q(x) &= b(x)^{-1}a(x) \\ r(x0 &= 0. \end{aligned}$$

Assume division is using strong induction. For all polynomials of $\deg(a(x)) < \deg(b(x))$ assume $b(x), a(x)$. Then $a(x) = a_nx^{n-m}b(x) + h(x)$ such that $\deg(h(x)) < \deg(a(x))$. $h(x) = q_1(x)b(x) + r(x)$ such that $\deg(r(x)) < \deg(b(x))$

or $r(x) = 0$.
Proof of Uniqueness: Suppose

$$\begin{aligned} a(x) &= q_1(x)b(x) + r_1(x) \\ &= q_2(x)b(x) + r_2(x) \end{aligned}$$

where $\deg(r_1(x)), \deg(r_2(x)) < \deg(b(x))$ or $r_1(x), r_2(x) = 0$. So

$$\begin{aligned} [q_1(x) - q_2(x)]b(x) + [r_1(x) - r_2(x)] &= 0 \\ [q_1(x) - q_2(x)]b(x) &= [r_2(x) - r_1(x)]. \end{aligned}$$

So either $r_2(x) = r_1(x) = 0$ or $\deg(r_2(x) - r_1(x)) \leqslant \deg(r_1(x)) \leqslant \deg(r_2(x))$.
In any case $\deg(r_2(x) - r_1(x)) < \deg(b(x))$ or $r_2(x) = r_1(x) = 0$. Let's state $a(x) \neq 0$. Then $a(x)b(x) = a_n b_m x^{n+m} + \ldots a_0 b_0$, and $a_n b_m \neq 0$.
Suppose $(q_1(x) - q_2(x))b(x) \neq 0$. Then $\deg((q_1(x) - q_2(x))b(x)) = deg(q_1(x)q_2(x)) + deg(b(x)) \geqslant deg(b(x))$.
Conclusion: $(q_1(x) - q_2(x))b(x) = 0$ thus $q_1(x) = q_2(x)$. And since $r_2(x) - r_1(x) = 0$, then $r_2(x) = r_1(x)$. ■

The Division Algorithm for polynomial fields is a fundamental concept that allows you to divide one polynomial by another, similar to the division algorithm with integers. This algorithm helps you express one polynomial as a quotient of another polynomial plus a remainder.

Definition 5.2.1: Logical Divide of Polynomial Fields

Let $a(x), b(x) \in \mathbb{F}$ and $b(x) \neq 0$. We say $b(x)|a(x)$ if there exists a $q(x) \in \mathbb{F}$ such that $a(x) = q(x)b(x)$.

Definition 5.2.2: GCD of Polynomial Fields

Suppose $a(x), b(x) \in \mathbb{F}[x]$ not both 0. $d(x) = gcd(a(x), b(x))$ means $d(x)a(x), d(x)b(x)$, and if there exists a $c(x) \in \mathbb{F}[x]$ with $c(x)a(x), c(x)b(x)$, then $c(x)d(x)$ so $deg(c(x)) \leqslant deg(d(x))$.

Suppose we are in $\mathbb{Q}[x]$. Let

$$a(x) = (x-1)^2$$

and

$$b(x) = (x-1)(x-2).$$

Then the $gcd(a(x), b(x)) = x - 1$. But wait, doesn't $2x - 2|a(x)$ and $b(x)$.
We got a problem on our hands... We have to figure out how to circumvent this solution and before we can do that, let's go ahead and introduce a new term.

Definition 5.2.3: Monic

If $d(x) \in \mathbb{F}[x]$ has a leading coefficient of 1, then $d(x)$ is monic.

In algebra, monic polynomials are commonly used in the context of irreducible polynomials (polynomials that cannot be factored further). Monic irreducible polynomials have a leading coefficient of 1, and this condition simplifies discussions of unique factorization.

Definition 5.2.4: Polynomial Associates

If $c(x), d(x) \in \mathbb{F}[x]$ and $c(x) = \beta d(x)$ and $\beta \in \mathbb{F}$ and $\beta \neq 0$, we say $c(x)$ and $d(x)$ are associates.

We can think of associates as polynomial constant multiples.

Theorem 5.2.2 GCD Theorem
Suppose $a(x), b(x) \in \mathbb{F}[x]$ not both 0. Let

$$S := \{u(x)a(x) + v(x)b(x) \neq 0 : u(x), v(x) \in \mathbb{F}[x]\}$$

, then there exists $u(x), v(x) \in \mathbb{F}[x]$, such that $d(x) = u(x)a(x) + v(x)b(x)$ and $d(x) = gcd(a(x), b(x))$. S has a unique monic polynomial of smallest degree which is the $gcd(a(x), b(x))$.

This theorem also answers the question to our gcd question, which shows that we want to have a monic polynomial of smallest degree as our $gcd(a(x), b(x))$. The set of degrees is a subset of $\mathbb{Z}^+$, and let $d(x)$ be a monic polynomial of minimal degree in S, so the theorem exists. The GCD (Greatest Common Divisor) Theorem for Polynomial Fields is a fundamental result in abstract algebra that addresses the existence and uniqueness of the greatest common divisor of two polynomials in a polynomial ring over a field. The theorem establishes a clear and precise method for finding the GCD of polynomials and its properties.

Proof: Let $d(x)$ be a monic polynomial such that $d(x) \in S$. If $c(x)$ is any polynomial in S, then $\deg(d(x)) \leqslant \deg(c(x))$. We need to show that $d(x)|a(x)$.

Let's use the division algorithm. Suppose $d(x) \neq 0$. We write $a(x) = q(x)d(x) + r(x)$, so $r(x) = 0$. We show this by saying $r(x)$ is a non-zero and $r(x) \in S$ and $r(x) = a(x) - q(x)d(x)$ where $d(x) = a(x)u(x) + b(x)v(x)$ such that

$$\begin{aligned} r(x) &= a(x) - q(x)(a(x)u(x) + b(x)v(x)) \\ &= 1 - q(x)u(x)a(x) - q(x)v(x)b(x)S. \end{aligned}$$

Contradicting $d(x)$ as being a polynomial with least degree. We conclude $r(x) = 0$, so $d(x)|a(x)$. Similarly $d(x)|b(x)$. Suppose $c(x)|a(x), c(x)b(x)$, then $c(x)|u(x)a(x) + v(x)b(x) = d(x)$. ■

Definition 5.2.5: Relatively Prime

$a(x), b(x) \in \mathbb{F}$, not both 0. a(x) and b(x) are relatively prime if $gcd(a(x), b(x)) = 1$.

Corollary 5.2.1 Consequence of GCD Theorem

Suppose $a(x), b(x) \in \mathbb{F}$ are relatively prime and $c(x) \in \mathbb{F}$. If $a(x)|b(x)c(x)$, then $a(x)|c(x)$.

Proof: By the gcd theorem, we have $1 = u(x)a(x) + v(x)b(x)$, so $c(x) = c(x)u(x)a(x) + c(x)v(x)b(x)$. Since $a(x)|c(x)u(x)a(x)$ and $a(x)|c(x)v(x)b(x)$, then $a(x)|c(x)$. ■

If we let $\mathcal{R} = \mathbb{F}[x]$, we notice that it has very similar properties to $\mathbb{Z}$, such that it has the division and gcd algorithm. In fact, it also will have relatively prime and an equivalence to primes but for polynomials. Let's look at this equivalence.

Definition 5.2.6: Irreducible

A polynomial $p(x) \in \mathbb{F}[x]$ is irreducible if $p(x) = a(x)b(x)$ for $a(x), b(x) \in \mathbb{F}[x]$ then $a(x)$ is an associate of $p(x)$ or $a(x)$ is a unit.

5.3 Irreducibility

Proposition 5.3.1 Polynomial Euclid's Lemma

Suppose $p(x) \in \mathbb{F}[x]$ which is irreducible and $b(x) \in \mathbb{F}[x]$ such that $p(x) \nmid b(x)$, then $gcd(p(x), b(x)) = 1$.

Proof: Let $d(x) = gcd(p(x), b(x))$. $d(x)|p(x), d(x)|b(x)$, and since $p(x)$ is irreducible, then $d(x)$ is monic, $d(x) = 1$, $d = cp(x)$ given that $c \in \mathbb{F}$.
If $d(x) = cp(x)$ and $d(x)|b(x)$, then $p(x)|b(x)$, a contradiction arose. Therefore $p(x)|b(x)$ or $p(x)|c(x)$. ■

Corollary 5.3.1

If $p(x)|a_1(x)\ldots a_n(x)$, given $a_i(x) \in \mathbb{F}[x]$, $p(x)$ is irreducible, then $p(x)|a_i(x)$ for some i. Then show the answer by induction on n.

Theorem 5.3.1

Suppose you have any polynomial $a(x) \in \mathbb{F}[x]$, then $a(x)$ has a factorization into irreducible polynomials. This factorization is unique up to order and associates.

Proof: Use strong induction on degree of $a(x)$.
Uniqueness. If

$$\begin{aligned} a(x) &= p_1(x) \dots p_r(x) \\ &= q_1(x) \dots q_s(x), \end{aligned}$$

where $p_i(x), q_i(x)$ are irreducible. Then let $r = s$ and after rearranging $q_i(x)$, $p_i(x)$ is an associate of $q_i(x)$ each.
Proof of Uniqueness. $p_1(x)|q_1(x)\dots q_s(x)$, $p_i(x)|q_i(x)$ for some i. Without loss of generality, since $q_1(x)$ is irreducible, then $p_1(x), q_1(x)$ are associates. Proceed to show this by induction on $\min\{r, s\}$. ■

Lemma 5.3.1 Irreducible degrees

Degree 1 polynomials are irreducible.
If a degree 2 polynomial is reducible, then it is made of linear polynomials.

Lemma 5.3.2 Freshman's Dream

In $\mathbb{Z}_2$, $(x+1)^2 = x^2 + 1$.

Proposition 5.3.2

If $f(x)$ is irreducible $\mathbb{F}[x]$, so are all associates $f(x)$

Take note of that for the equation $x^2 + ax + b$, there are 3 choices for each a, b which means 9 total choices for this polynomial. The number of monic polynomials of $\deg n$ in $\mathbb{Z}_p[x]$ is p^n. Total number of polynomials of $\deg n$ is $(p-1)p^n$.

Example 5.3.1

Prove that $x^2 + 2$ is irreducible in $\mathbb{Q}[x]$.

Proof:

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2}) \in \mathbb{R}$$

This factorization is unique. Since factorization $\mathbb{Q}[x]$ is also unique if $x^2 - 2$ had a factorization by linears. It would have include $(x - \sqrt{2})(x + \sqrt{2})$, but $\sqrt{2} \notin \mathbb{Q}$. ■

Let $\mathbb{F}$ be a field. Take $f(x) \in \mathbb{F}[x]$, there is a corresponding polynomial function, $\mathbb{F} \mapsto \mathbb{F}$ denoted by $f(x)$.

Theorem 5.3.2 Factor Theorem

Let $f(x) \in \mathbb{F}[x]$ and $a \in \mathbb{F}$ if $f(a) = 0$, then $(x-a)$ is a factor of the polynomial $f(x)$. i.e. $f(x) = g(x)(x-a)$.

Example 5.3.2

(a). Show that $x^2 + 2$ is irreducible in $\mathbb{Z}_5[x]$.

Proof of Example 5.2.2: We will do a proof by contradiction. Suppose x^2+2 is not irreducible. Then x^2+2 is made up of linear polynomials such that $(x+a)(x+b) = x^2+2$. But note that $(x+a)(x+b) = x^2+xa+xb+ab$, and we don't have a degree 1 in our polynomial. Therefore, $a = -b$, thus $(x+a)(x-a)$ will result in x^2+a^2, but note that $a^2 = 2$ or $a^2 = 3$, and $a = \pm\sqrt{2}$ or $a = \pm\sqrt{3}$, but $\sqrt{2}, \sqrt{3} \notin \mathbb{Z}_5$. Therefore, this polynomial, x^2+2 is irreducible. ■ ■

(b). Factor $x^4 - 4$ as a product of irreducibles in $\mathbb{Z}_5[x]$.

$$(x^2+2)(x^2-2)$$

However, (x^2-2) is not further reducible, since we will deal with an irrational $\sqrt{2}$, which is not in $\mathbb{Z}_5$.

Theorem 5.3.3 Remainder Theorem

Let $f(x) \in \mathbb{F}[x], a \in \mathbb{F}[x]$. Then $f(x) = g(x)(x-a)+r(x)$, given there exists $g(x) \in \mathbb{F}[x]$. $r(x)$ is a constant.

Proof of Remainder Theorem: By division algorithm, $f(x) = g(x)(x-a)+r(x)$ where $r(x) = 0$ or $\deg(r(x)) < \deg(x-a)$ or $r(x) = 0$.
If $\deg(r(x)) < \deg(x-1)$, then $\deg(r(x)) < 1$ implying that $\deg(r(x)) = 0$. So $r(x)$ is some constant or 0. ■

Proof of Factor Theorem: We know by remainder theorem $f(x) = g(x)(x-a)+r(x)$ where $r(x)$ is a constant, indexed function and by the previous example we now have that

$$\begin{aligned} f(a) &= g(x)(a-a)+r(x) \\ &= r(x). \end{aligned}$$

So $f(x) = g(x)(x-a)$. ■

Definition 5.3.1: Roots

a is a root of $f(x)$ if $f(a) = 0$.

Corollary 5.3.2 of Factor Theorem

Suppose $f(x) \in \mathbb{F}[x]$ has $\deg f(x) = n$, then $f(x)$ has at most n different roots.

Proof: By induction on $\deg f(x)$; Suppose $\deg f(x) = 0$, f is a non-zero constant with no roots.
$\deg f(x) = 1$, then $f(x) = a_1x + a_2$, $a \neq 0$. Only one root at $x = \frac{-a_2}{a_1}$. Assume true for polynomials of $\deg f(x) = n-1$. If $b \neq a$, then b is a root of f(x). $0 = f(b) = (b-a)g(b)$, $b-a \neq 0 \implies g(b) = 0$. By the induction hypothesis, there exists at most $n-1$ such b. So the number of roots of $f(x)$ is at most $1+(n-1) = n$.
■

If $f(x) \in \mathbb{Q}[x]$, then the rational root test tells us if $f(x)$ has a linear factor.

Definition 5.3.2: Rational Root Test

If $r|a_0$ and $s|a_n$ and $gcd(r,s) = 1$ then $\frac{r}{s}$ is a possible root given that $f(\frac{r}{s}) = 0$. Since a_0 and a_n have finitely many factors, then there are only finitely many factors to check.

For example, $2x^3 - x^2 + 1$ is irreducible due to the Rational Root Test, as we find the $r/s = 1/2, 1$ and their additive inverses. After checking all possibilities plugged into $f(x)$, we see none of them are 0.
Suppose $f(x) \in \mathbb{Z}[x]$ and $g(x), h(x) \in \mathbb{Q}[x]$ then $\exists \alpha, \beta \in \mathbb{Q}$ such that

$$f(x) = (\alpha g(x))(\beta h(x)) \in \mathbb{Z}[x].$$

Suppose also that if $f(x) \in \mathbb{Q}[x]$, $f(x)$ is only irreducible if and only if there is a $c \in \mathbb{Q}$ such that $cf(x)$ can let us assume that $cf(x) \in \mathbb{Z}[x]$. The rational root test tells us if they are linear which suffices to show there is irreducibility for degrees 2 and 3 but not higher. This builds the foundation for the following theorem.

Theorem 5.3.4 Gauss's Lemma of Irreducibility

Suppose $f(x) \in \mathbb{Z}[x]$, if $f(x)$ is irreducible in $\mathbb{Z}[x]$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

One may ask, is the converse possible given these assumptions? I claim not always.
It is possible when given that $g(x)h(x) \in \mathbb{Q}[x]$, and the $\deg g(x), h(x) < \deg f(x)$, therefore $f(x)$ is irreducible in $\mathbb{Q}[x]$. But what if we considered that $f(x)$ cannot even be written as a product of integer coefficients? This is a more simplified version of Gauss's lemma, but the actual lemma looks into something called primitivity, which is not looked into in this course.

Definition 5.3.3: Primitivity

$p(x)$ has integer coefficients and is called primitive if and only if the gcd of all the coefficients is 1.

If this is also true, then and only then will it be a bi-conditional statement.
This was a whole block of assumptions to unfold before displaying the if-then statement of (our) Gauss's lemma of irreducibility. But let's look at an example of how to apply this. Let $f(x) \in \mathbb{Q}[x], f(x) = 6x^2 - 5x + 1$, therefore it can be reduced into $f(x) = (x - \frac{1}{2})(6x - 2)$, therefore $f(\frac{1}{2}) = 0$. Thus we have shown a root in $\mathbb{Q}[x]$ which demonstrates that it is reducible. But we can also write this in the form of integer factors, as $f(x) = (2x - 1)(3x - 1) \in \mathbb{Z}[x]$ and you can verify this.

Lemma 5.3.3 Introductory Lemma

Suppose $f(x), g(x), h(x) \in \mathbb{Z}[x]$ where $f(x) = g(x)h(x)$. Let p be prime such that p divides every coefficient of $f(x)$, then either p divides every coefficient of $g(x)$ or $h(x)$.

Sketch of Proof: Suppose $f(x), g(x), h(x) \in \mathbb{Z}[x]$ where $f(x) = g(x)h(x)$. Then $a_0 = b_0c_0$, therefore $p|a_0$ which implies $p|b_0c_0$, and due to Euclid's lemma, then $p|b_0$ or $p|c_0$. Suppose $gcd(p, c_0) = 1$, then and $p|a_1 = b_0c_1 + b_1c_0$, then we know $p \nmid c_0$ implying $p|b_1$. Let there exist α, β such that $\alpha g(x), \beta h(x) \in \mathbb{Z}[x]$, then $\alpha\beta f(x) = (\alpha g(x))(\beta h(x))$. By canceling primes, dividing $\alpha\beta$, and using the introductory Lemma we get $f(x)$ being a product of polynomials of integer coefficients. ■

Theorem 5.3.5 Eisenstein's Theorem of Irreducibility

Suppose $f(x) \in \mathbb{Z}[x]$. Let $\deg f(x) = n$. Suppose $p \nmid a_n$ $p|a_i$ for $i < n$, $p^2 \nmid a_0$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Proof: Suppose $f(x)$ is reducible in $\mathbb{Q}[x]$, then $f(x)$ is reducible in $\mathbb{Z}[x]$ by Gauss's Lemma. So $f(x) = g(x)h(x)$, $g(x), h(x) \in \mathbb{Z}[x]$, $\deg g(x), h(x) < \deg f(x) = n$. So $p|a_0 = b_0c_0$ and so forth following Introductory Lemma. ■

Lemma 5.3.4

Linear Polynomials are not reducible

Sketch of Proof: Following Eisenstein's proof, we find that if linear polynomials are reducible then this contradicts Eisenstein's. ■

Let $f(x) = 2x^4 + 15x^3 + 30x^2 + 60x - 21$. $3 \nmid 2, 3|15, 30, 60, 21$, but $9 \nmid 21$. So $f(x)$ is irreducible by Eisenstein.

Theorem 5.3.6 Reduction mod P

Let $f(x) \in \mathbb{Z}[x]$. Let $p \nmid a_n$. Consider $\overline{f(x)} = \overline{a_n}x^n + \ldots + \overline{a_0}$ where $\overline{a_i}$ Is congruence class $a_i \mod p$. If $\overline{f(x)}$ is irreducible in $\mathbb{Z}_p[x]$ then $\overline{f(x)}$ is irreducible in $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$. Converse is not true.

Let $f(x) = x^4 + 3x^3 + 6x^2 + 1 \in \mathbb{Q}[x]$. Try $p = 2$, $\overline{f(x)} = x^4 + x^3 + 1$ has no factors and roots, so it is not linear and irreducible.

Proof: Suppose $f(x)$ is irreducible in $\mathbb{Q}[x]$, then $f(x) = g(x)h(x) \in \mathbb{Z}[x]$. $\underline{\deg g(x), h(x) < n}$. Let $\overline{f(x)} = f(x)$ mod p, since $p \nmid a_n$, and $a_n \neq 0$, so $\deg \overline{f(x)} = n$ implying that $k, m < n$. $\overline{f(x)}$ is a product of polynomials of smaller degree which is a contradiction, so $f(x)$ must be irreducible in $\mathbb{Q}[x]$. ■

Theorem 5.3.7 Fundamental Theorem of Algebra

If $f(x) \in \mathbb{C}[x]$, then $f(x)$ is irreducible, if and only if $f(x)$ is linear, if and only if every non-constant of $f(x) \in \mathbb{C}[x]$ can be factored as a product of linear factors, if and only if every non-constant $f(x) \in \mathbb{C}[x]$ has a root.

For example $f(x) = x^2 + 1 \in \mathbb{C}[x]$ has complex roots $\pm i$. $f(X) = (x+i)(x-i)$. Let $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

$$\begin{aligned} e^{\theta i} &= \cos\theta + i\sin\theta \\ \left(e^{\theta i}\right)^3 &= \cos 3\theta + i\sin 3\theta \\ &= \cos 2\pi + i\sin 2\pi \\ &= \cos 4\pi + i\sin 4\pi \\ &= 1 \end{aligned}$$

Roots of $x^n - 1$ are $e^{\theta i}, e^{2\theta i}, e^{3\theta i}, \ldots, e^{(n-1)\theta i}$

Proposition 5.3.3

Suppose $f(x) \in \mathbb{R}[x]$, every irreducible $f(x)$ has degree 1 and 2.

Example 5.3.3

Suppose $f(x) \in \mathbb{R}[x]$ and has degree 3. By IVT, there exists a $c \in \mathbb{R}, f(c) = 0$, so by the factor theorem, $(x - c)$ is a factor of $f(x) \in \mathbb{R}[x]$.

Proof Part 1.: Consider $f(x) \in \mathbb{R}[x]$ as a polynomial of $\mathbb{C}[x]$. By FTA, $f(x)$ has a root in $\mathbb{C}$. If this root is real, then $f(x)$ has a linear factor. So we can assume that $\omega = a + bi$ is a root of $f(x)$.
■

Claim. So is $\overline{\omega} = a - bi$: Suppose $f(x)$, $a_i \in \mathbb{R}$. We assume $f(\omega) = 0$. In fact we can suppose ϕ is a homomorphism of $\mathbb{C}$, which leaves $\mathbb{R}$ fixed. i.e. if $a \in \mathbb{R}$, $\phi(a) = a$, then ω and $f(\omega) = 0$, then $f(\phi(\omega)) = 0$.
■

Lemma 5.3.5

Now let $\phi(a + bi) = a - bi$, therefore $\phi : \mathbb{C} \mapsto \mathbb{C}$, therefore ϕ is a isomorphism.

Proof. ctd: Since complex conjugation is an isomorphism $\mathbb{C} \mapsto \mathbb{C}$. Therefore $f(\overline{\omega}) = 0$ also. Now suppose $f(\omega) = 0$, $\omega \notin \mathbb{R}$ and $f(\overline{\omega}) = 0$, $(x-\omega), (x-\overline{\omega})$ are factors of $\mathbb{C}[x]$ of $f(x)$. But $(x-\omega), (x-\overline{\omega}) = x^2 - (\omega + \overline{\omega})x + \omega\overline{\omega}$, which $(\omega + \overline{\omega}) \in \mathbb{R}$, $\omega\overline{\omega} \in \mathbb{R}$. Therefore all factors are in $\mathbb{R}[x]$ hence, degree 1 or 2. ■

5.4 Congruences

Theorem 5.4.1

Let $m(x) \in \mathbb{F}[x]$. If $a(x), b(x) \in \mathbb{F}[x]$. Let's define $a(x) \equiv b(x) \mod m(x)$

Definition 5.4.1: Congruences

Let $m(x) \in \mathbb{F}[x]$. If $a(x), b(x) \in \mathbb{F}[x]$. Let's define $a(x) \equiv b(x) \mod m(x)$ if $m(x) | a(x) - b(x)$ if and only if there exists $q(x) \in \mathbb{F}[x]$, $a(x) - b(x) = q(x)m(x)$. If and only if $a(x) = b(x) + q(x)m(x)$

Definition 5.4.2: Congruence Class

Congruence of $a(x) \in \mathbb{F}[x]$ is denoted by $[a(x)]$. It consists of

$$[a(x)] := \{b(x) \in \mathbb{F}[x] : b(x) \equiv a(x) \mod m(x)\}$$

Definition 5.4.3: Polynomial Division Algorithm

Suppose $g(x) \in \mathbb{F}[x]$. $g(x) = q(x)m(x) + r(x)$, $\deg r(x) < \deg m(x)$ or $r(x) = 0$. If $r(x) \equiv g(x) \mod m(x)$, so $g(x) \in [r(x)]$. So every $g(x)$ is in exactly one of these congruence classes.

Lemma 5.4.1

In $\mathbb{Z}_p[x]$ if $\deg m(x) = n$, there are exactly p^n different congruence classes.

Similar to congruence classes in the integers, we also have similar ideas for addition and multiplication for polynomial congruences.

Definition 5.4.4: Modular Operations

Addition:

$$[a(x)] + [b(x)] = [a(x) + b(x)]$$

Multiplication:

$$[a(x)][b(x)] = [a(x)b(x)]$$

We can use this to check if it is well-defined.

Lemma 5.4.2 Well-Defined

Suppose $[a(x)] = [c(x)], [b(x)] = [d(x)]$.

1. $[a(x) + b(x)] = [c(x) + d(x)]$
2. $[a(x)b(x)] = [c(x)d(x)]$

Chapter 6

Ideals and Quotient Rings

Definition 6.0.1: Quotient Rings

Congruence classes mod $f(x)$ are noted by $\mathbb{F}[x]/(f(x))$ which is a ring. In fact the additive identity of this ring is $[0] = [f(x)]$. This together is called a quotient ring closed under addition.

Theorem 6.0.1 Class of g(x)

$[g(x) \in \mathbb{F}[x]$ is a unit if and only if $gcd(f(x), g(x)) = 1$, then $g(x), f(x)$ are relative prime.

Proof: ($\Longleftarrow$). Suppose $gcd(f(x), g(x)) = 1$, then there exists $w(x), v(x)$ such that $w(x)g(x) + v(x)f(x) = 1$, so $[w(x)g(X)] = [1]$ so $[w(x)] = [g(x)]^{-1}$.
($\Longrightarrow$). Suppose $w(x)g(x) \equiv 1 \mod f(x)$, so $f(x)|w(x)g(x) - 1$ therefore there exists a $v(x) \in \mathbb{F}[x]$.

$$w(x)g(x) - 1 = v(x)f(x)$$
$$w(x)g(x) - v(x)f(x) = 1,$$

Therefore, $gcd(f(x), g(x)) = 1$. ■

Corollary 6.0.1

If $f(x)$ is irreducible in $\mathbb{F}[x]$, then $\mathbb{F}[x]/(f(x))$ is a field.

Proof: If $g(x) \in \mathbb{F}[x]$, $[g(x)] \neq [0]$, $f(x) \nmid g(X)$, then $gcd(f(x), g(x)) = 1$, so $[g(x)]$ is a unit in $\mathbb{F}[x]/(f(x))$. ■

Let $\mathbb{E} = \mathbb{F}[x]/(f(x))$, such that we have an injection from $\mathbb{F} \mapsto \mathbb{E}$ where $a \mapsto [a]$. We can consider $\mathbb{F}$ now a subfield of $\mathbb{E}$.

Definition 6.0.2: Roots in Quotient Rings

Suppose $\mathbb{F} \subseteq \mathbb{E}$, let $\alpha = [x]$, such that $f(x) \in \mathbb{E}[x]$, then $f(\alpha) = [0]$.

Axiom 6.0.1

$\mathbb{F} \cong \mathbb{E}$.

Definition 6.0.3: Ideal

Let $\mathcal{R}$ be a commutative ring. Given that $I \subseteq \mathcal{R}$. We call I an ideal if an only if I is a subring of $\mathcal{R}$ and if $r \in \mathcal{R}$, $a \in I$, then $ra \in I$.

Definition 6.0.4: Congruence mod I

Suppose $r, s \in \mathcal{R}$, $r \equiv s \mod I$ if $r - s \in I$.

Theorem 6.0.2 Congruence mod I is an Equivalence Relation

Given $a, b, c \in \mathcal{R}$ we have the following properties.
Reflexive. $a \equiv a \mod I$ because $a - a = 0 \in I$.
Symmetric. $a \equiv b \mod I, b \equiv a \mod I$ because $b - a, a - b \in I$.
Transitive. If $a \equiv b \mod I, b \equiv c \mod I$, then $a \equiv c \mod I$ because $a - b, b - c, a - c \in I$.

Definition 6.0.5: Coset

Instead of $[a]$ for $a \mod I$, we have the notation $a + I := \{a + i : i \in I\}$ called a coset.

For example $\mathbb{Z}_m[a] = a + m\mathbb{Z}$

Definition 6.0.6: Quotient Ring

$\mathcal{R}/I$ is called a quotient ring.

Theorem 6.0.3 Addition on Ideals

$$(a + I) + (b + I) = a + b + I$$
$$(a + I)(b + I) = (ab) + I$$

Proof: Suppose $a + I = c + I$ and $b + I = d + I$. Since $c - a, d - b \in I$, then $(c - a) + (d - b) \in I$ implies $(c + d) - (a + b) \in I$ which implies $c + d + I = a + b + I$.
To prove multiplication, since $c-a, d-b \in I$, then $c(d-b), b(c-a) \in I$ due to absorption property. $c(d-b)+b(c-a) \in I \implies cd - cb + cb - ba \in I$. Then $ab + I = cd + I$. ■

Quotient Rings are independently associated with homomorphism $\phi : \mathcal{R} \mapsto \mathcal{S}$.

Definition 6.0.7: Generators

If $\mathcal{R}$ is any commutative ring, let $a \in \mathcal{R}$, the ideal generated by a is $\{ra : r \in \mathcal{R}\} =: (a)$.

Lemma 6.0.1

(a) is an ideal of $\mathcal{R}$.

Proof: **Case 1.** if$r_1 a, r_2 a \in (a)$, then $r_1 a + r_2 a = (r_1 + r_2)a \in (a)$.
Case 2. if $ra \in (a), s \in \mathcal{R}$, then $s(ra) = (rs)a \in (a)$. ■

These generators are called the principle ideal generated by a.

Theorem 6.0.4

If $p(x)$ is irreducible in $\mathbb{F}[x]$ if and only if $\mathbb{F}[x]/(p(x))$ is a field if and only if $\mathbb{F}[x]/(p(x))$ is an integral domain.

Let $\mathcal{R}$ be a commutative ring with $1 \in \mathcal{R}$. Let A be any subset of the ideal generated by A which is the set of all finite linear combinations of elements.

$$(A) := \{r_1 a_2 + \ldots + r_n a_n : r_i \in \mathcal{R}, a_i \in A\}$$

Then (A) is the intersection of all ideals in $a \in A$.
Suppose $\mathcal{R} \in \mathbb{Z}, a, b \in \mathbb{Z}$ ideal generated by $(a, b) := \{xa + by : y, x \in \mathbb{Z}\} = \{r \cdot gcd(a, b) : r \in \mathbb{Z}\}$.
$\mathbb{Z}$ and $\mathbb{F}[x]$ are called principle ideal domains while $\mathbb{Z}[x], \mathbb{Q}[x, y]$ are not principle ideal domains.
$\phi : \mathbb{Z} \mapsto \mathbb{Z}/10\mathbb{Z}$, therefore $\phi(a) = [a]_{10} = a + 10\mathbb{Z}$.

Definition 6.0.8: Kernel

Let $K := \{x \in \mathbb{Z} : \phi(x) = 0\}$ which we learn is called the kernel of ϕ, $\ker \phi$.

Theorem 6.0.5

K is an ideal in $\mathcal{R}$.

From the previous example, $\ker \phi = (10) = 10\mathbb{Z}$. What we learned prior is that $\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}_{10}$.

Proof of Theorem: **(1).** Suppose $x, y \in \ker \phi$, then $\phi(x) = \phi(y) = 0, \phi(x + y) = \phi(x) + \phi(y) = 0$.
(2). Suppose $x, y \in \ker \phi, r \in \mathcal{R}$, then $(rx) = \phi(r)\phi(x) = \phi(r)0 = 0$.
So $\ker \phi$ is a ideal in $\mathcal{R}$. ■

Definition 6.0.9: Image

$$\operatorname{Im} \phi := \{s \in S : \exists r \in \mathcal{R}, \phi(r) = S\}$$

Theorem 6.0.6 First Isomorphism Theorem

Suppose $\phi : \mathcal{R} \mapsto \mathcal{S}$ is a homomorphism. Let $K = \ker \phi$. We can define $\overline{\phi} : \mathcal{R}/K \mapsto \operatorname{Im} \phi$ such that $\overline{\phi}(r + K) = \phi(r)$. Then $\overline{\phi}$ is an isomorphism from $\mathcal{R}/K$ to $\operatorname{Im} \phi$, so $\mathcal{R}/K \cong \operatorname{Im} \phi$.

Proposition 6.0.1

Suppose $\phi : \mathcal{R} \mapsto \mathcal{S}$ is a ring homomorphism, then ϕ is injective if and only if $\ker \phi = \{0\}$.

Proof of Proposition: $(\implies)$. Suppose ϕ is injective. Let $r \in \ker \phi$, so $\phi(r) = 0$, but $\phi(0) = 0$, so ϕ is injective $r = 0$.
$(\impliedby)$. Suppose $\ker \phi = \{0\}$. Let $r, s \in \mathcal{R}$ with $\phi(r) = \phi(s)$.

$$\begin{aligned}\phi(r) - \phi(s) &= 0\\ \phi(r - s) &= 0.\end{aligned}$$

So $r - s \in \ker \phi$, so $r - s = 0$, therefore $r = s$. Therefore ϕ is injective. ■

Proof of First Isomorphism Theorem: Assume ϕ is a homomorphism. Suppose $r, s \in \mathcal{R}$, $\overline{\phi}(r + s) = \overline{\phi}(r) + \overline{\phi}(s)$, $\overline{\phi}(rs) = \overline{\phi}(r)\overline{\phi}(s)$
$\overline{\phi}$ is surjective. Suppose $s \in \operatorname{Im} \phi$, then $\exists r \in \mathcal{R}$ such that $\phi(r) = s$, so $\overline{\phi}(r) = s$. ■

Example 6.0.1

Prove $\mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})$.

Proof: Define $\phi : \mathbb{Q}[x] \mapsto \mathbb{C}$ so $\phi(f(x)) = f(\sqrt{2})$. Let $\ker \phi := \{f(x) \in \mathbb{Q}[x] : f(\sqrt{2}) = 0\}$.

$$x^2 - 2 \in \ker \phi$$

Claim. $\ker \phi$ is the ideal generated by $x^2 - 2$.
By the first isomorphism theorem, we find that $\mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})$ ■

Chapter 7

Field Extensions

Definition 7.0.1: Vector Space

A vector space over $\mathbb{F}$ is an additive abelian (commutative) group V equipped with scalar multiplication such that $a, a_1, a_2 \in \mathbb{F}$ and $v, v_1, v_2 \in V$.

1. $a(v_1 + v_2) = av_1 + av_2$.
2. $(a_1 + a_2)v = a_1 v + a_2 v$.
3. $a_1(a_2 v) = (a_1 a_2)v$.
4. $1_= v$.

Definition 7.0.2: Span

If every element of a vector space $V/\mathbb{F}$ is in a linear combination, we say set $\{v_1, v_2, \ldots, v_n\}$ span $V/\mathbb{F}$.

Definition 7.0.3: Linearly Independent

A subset of a vector space $V/\mathbb{F}$ is linearly independent over $\mathbb{F}$ when there is a linear combination with $c_i \in \mathbb{F}$, then $c_i = 0_{\mathbb{F}}$ for all i. else is dependent.

Definition 7.0.4: Basis

The subset is linearly independent and spans $V/\mathbb{F}$.

Definition 7.0.5: Dimension

If $p(x) \in \mathbb{F}[x]$ is irreducible, then $\mathbb{E}$ is an extension field of $\mathbb{F}$. Infact this is called a vector space over $\mathbb{F}$. Denoted by $[\mathbb{E} : \mathbb{F}]$.

Theorem 7.0.1

Suppose K is an extension field of dimension $[K : \mathbb{E}]$, then

$$[K : \mathbb{F}] = [K : \mathbb{E}][\mathbb{E} : \mathbb{F}].$$

Proof: Suppose $[\mathbb{E} : \mathbb{F}] = n$. Suppose $v_1, \ldots, v_n \in \mathbb{E}$ which are basis for $\mathbb{E}/\mathbb{F}$. Suppose $[K : \mathbb{E}] = m$. Suppose $w_1, \ldots, w_m \in K$, basis for $K/\mathbb{E}$. Our claim is that $\{w_i v_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is the basis for $K/\mathbb{F}$. Which can also be stated as $\{w_i v_j\}$ span K. Let $u \in K$, $\{w_i\}$ span $K/\mathbb{E}$. So $u = \sum \alpha_i w_i, \alpha_i \in \mathbb{E}$. Each $\alpha_i = \sum \beta_{ij} v_j, \beta_{ij} \in \mathbb{F}$, so $u = \sum \beta_{ij} w_i v_j$. So $\{w_i v_j\}$ span K.
Suppose $\sum \beta_{ij} v_j w_i = 0$, $\forall i, \sum \beta_{ij} v_j \in \mathbb{E}$ since $\{w_i\}$ are linearly independent $/\mathbb{E}$.

$\sum \beta_{ij} v_j = 0$ for each i.
Since $\{v_j\}$ are a basis for $\mathbb{E}/\mathbb{F}$, $\beta_{ij} = 0$ for each j, i. Suppose $\mathbb{E}$ is an extension field of $\mathbb{F}$ and $u \in \mathbb{E}$. ■

Definition 7.0.6: Algebraic and Transcendental Functions

Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{E} = \mathbb{R}$, $u = \pi$. There is no polynomial $p(u) = 0$, $p(x) \in \mathbb{Q}$. If there is no such polynomial, we say u is transcendental $/\mathbb{F}$.
If there is such polynomial, we say u is algebraic $/\mathbb{F}$.

To understand two versions of field extensions, let's look at when $\mathbb{Q}[\alpha] = \mathbb{Q}(\alpha)$. We can use the first isomorphism theorem.

Lemma 7.0.1 $\mathbb{Q}[\alpha] = \mathbb{Q}(\alpha)$

$$\begin{aligned} \text{A function } \phi : \mathbb{Q}[x] \mapsto \mathbb{Q}[\alpha] \text{ is injective } &\iff \ker\phi = \{0\}; \\ &\iff \nexists f(x) \in \mathbb{Q}[x] : f(x) = 0; \\ &\iff \alpha \text{ is transcendental of } \mathbb{Q}. \end{aligned}$$

Proof Part One: Using the first isomorphism theorem, we can let ϕ be a homomorphism,

$$\operatorname{Im}\phi = \{f(\alpha) : f(x) \in \mathbb{Q}[\alpha]\} = \mathbb{Q}[\alpha].$$

So it is a surjective function. In fact

$$\ker\phi = \{f(x) \in \mathbb{Q}[x] : f(x) = 0\},$$

This ϕ is injective. Suppose α is transcendental/$\mathbb{Q}$, then $\mathbb{Q}[\alpha] \cong \mathbb{Q}[x]$. Therefore $\mathbb{Q}[\alpha]$ is a ring not a field. ■

Definition 7.0.7: Minimal Polynomial

Suppose $p(x)$ is a monic polynomial of smallest degree, this is called the minimal polynomial of $\alpha/\mathbb{Q}$.

Lemma 7.0.2
$p(x)$ is irreducible.

Proof: Suppose

$$\begin{aligned} p(\alpha) &= q(x)g(x) \\ &= q(x)g(x) = 0. \end{aligned}$$

So either $q(x) = 0$ or $g(x) = 0$. Since $p(x)$ is smallest degree, either $q(x)$ or $g(x)$ is a unit in $\mathbb{Q}[x]$. ■

Continuation of Proof Sketch of Lemma 7.0.1: Using the first isomorphism, suppose α is algebraic. Let $\mathbb{Q}[x] \mapsto \mathbb{Q}[\alpha]$ and this map has $\ker\phi = (p(x))$. $p(x)$ is an irreducible minimal polynomial of α. Therefore $\mathbb{Q}[x]/(p(x)) \cong \mathbb{Q}[x]$. Since $p(x)$ is irreducible, then $\mathbb{Q}[x]/(p(x))$ so $\mathbb{Q}(x)$ is a field and $\mathbb{Q}[\alpha] = \mathbb{Q}(\alpha)$. ■

We have previously learned that $\mathbb{Q}(r)$ is a vector space.

$\mathbb{Q}(r) = \mathbb{Q}[r]$

What is the $\dim[\mathbb{Q}[r] : \mathbb{Q}]$? We have to find the basis. So what is the basis of $\mathbb{Q}[r]/\mathbb{Q}$.

Lemma 7.0.3
A basis is $1, r, r^2, \ldots, r^{n-1}$ where $n = \deg f(x)$.

Proof: $\mathbb{Q}[r] = \{f(x) : f(x) \in \mathbb{Q}[x]\}$. $|f(r) = 0, \deg f(x) = n|$. ■

Lemma 7.0.4

Basis when we mod out $f(x)$, therefore $f(x)$ is the minimum polynomial or $r/\mathbb{Q}$.

Proof: Suppose $g(x) \in \mathbb{Q}[r]$. By the division algorithm, $g(x) = f(x)q(x) + s(x)$. Plug in r:

$$g(r) = f(r)q(r) + s(r),$$

so $g(r) = s(r)$ since $f(r) = 0$. Therefore $s(r) = 0$ or $\deg s(r) < \deg f(x)$ or $s(r)$ is some polynomial. So $g(r)$ is the linear combination of $1, r, \ldots, r^{n-1}$. So $1, r, r^2, \ldots, r^{n-1}$ span $\mathbb{Q}[r] = \mathbb{Q}(r)$. ■

Chapter 8

Geometric Constructions

Which regular n-gons can be constructed?

Definition 8.0.1: Construct

a is constructable if you can construct a line of length a.

Definition 8.0.2: Constructable Point

A point in $\mathbb{R}^2$ is constructable if its coordinates are constructable.

Definition 8.0.3: Constructable Line

A constructable line is made of constructable points.

Theorem 8.0.1

Constructable numbers are in extension field $\mathbb{Q}$.

Proof: Suppose a, b are constructable, they are closed under subtraction. ■

Theorem 8.0.2

$\mathbb{F}$ is constructable so is $\sqrt{a}$.

Proof: Suppose a triangle enclosed in a semicircle with triangle length 1 and radius $\frac{a+1}{2}$. The distance, x, is $\frac{a+1}{2}$.

$$\begin{aligned} x^2 &= \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 \\ &= a \end{aligned}$$

Which shows distance $x = \sqrt{a}$ ■

Suppose we have constructable points, how do we get new points intersecting lines, circles, and lines on circles?

Definition 8.0.4: New Constructable Points

$\mathbb{F}[\alpha]$ where $[\mathbb{F}[\alpha : \mathbb{F}] = 2$. Which means any constructable point lies in a field:

$$\mathbb{Q} \subseteq \mathbb{Q}[a_1] \subseteq \mathbb{Q}[a_1, a_2] \subseteq \ldots \subseteq \mathbb{F}$$

Therefore $\mathbb{F}_k = \mathbb{F}_{k-1}[a_k]$, thus $[\mathbb{F}_k : \mathbb{F}_{k-1}] = 2$.

Let α be the root of a quadratic polynomial. So no constructable numbers must lie in field $\mathbb{F}$ where the $[\mathbb{F} : \mathbb{Q}] = 2^n$ for some n. Therefore $\sqrt[3]{2}$ is not constructable.

Lemma 8.0.1 Constructable Points

Let $r \in \mathbb{R}$ be a constructable with a straightedge and compass $\iff$ r lies in a field extension, $\mathbb{E}$ with $[\mathbb{E} : \mathbb{Q}] = 2^n$ (power of 2).

π is not constructable and neither is it algebraic. Therefore constructable points are also only possible iff $[\mathbb{Q}(r) : \mathbb{Q}] = 2^k$ for some k. We will show that we cannot trisect $60°$ since we can construct $60°$, implying that not every angle can be trisected.
Because $20° = \theta = \frac{\pi}{4}$ can be constructed the $\cos\theta$ can be constructed.

$$\begin{aligned} cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2cos^2\theta - 1 \end{aligned}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

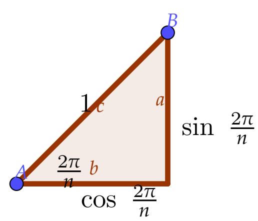
If $\theta = \frac{\pi}{4}$ then $\cos 3\theta = \frac{1}{2}$ and let $x = \cos 20$. Then

$$\begin{aligned} \frac{1}{2} &= 4x^3 - 3x \\ 0 &= 4x^3 - 3x\frac{1}{2} \\ 0 &= 8x^3 - 6x - 1 \end{aligned}$$

Our claim is that $8x^3 - 6x - 1$ is irreducible/$\mathbb{Q}$, which we can use the root test to check that it is indeed irreducible.

Question: Which regular n-gons can be constructed. i.e. for which n can angle $\frac{2\pi}{n}$ be constructed.

Such an angle can be constructed if and only iff $\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}$ can be constructed



if and only iff $(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n})$ is a constructable point if and only if $\rho = \cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}$, $[\mathbb{Q}(\rho) : \mathbb{Q}] =$ power of two, $\rho^n = 1$, ρ is an n^{th} root of 1 satisfying $x^n - 1 = 0$. Suppose $n = 2^{a_1}p_2^{a_2}\dots\rho_k^{a_k}$ is a factorization of regular n-gons is $p_1 = 2, p$ odd for $j \geqslant 2$ constructable if and only if $a_j = 1$ for $j \geqslant 2$ and each $p_i - gon$ is constructable.

Definition 8.0.5: Fermat Prime

If $2^{2^k} + 1 = p$ is prime, then p is a fermat prime.

Corollary 8.0.1

Let $\phi : \mathbb{Q}[x] \mapsto \mathbb{Q}[x] := \{f(x) : f(x) \in \mathbb{Q}[x]\}$. By $\phi(p(x)) = p(\alpha)$, this shows surjectivity. Proof. Suppose

$\beta \in \mathbb{Q}[x]$, then $\exists f(x) \in \mathbb{Q}[x]$ such that $\beta = f(x)$, so $\phi(f(x)) = f(\alpha) = \beta$. Q.E.D ϕ is injective if and only if $\ker \phi = \{0\}$. This is a consequence of the first isomorphism theorem.

Proof: $\ker \phi = \{0\} \iff (f(x) = 0 \implies f(x) = 0) \iff \alpha/\mathbb{Q}$ is transendental ■