Discrete Optimization - ADM2

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Abstract

The following lecture notes are my personal (and therefore unofficial) write-up for 'Discrete Optimization' aka 'ADM II', which took place in summer semester 2022 at Technische Universität Berlin. I do not guarantee correctness, completeness, or anything else.

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Summary of lectures

Lecture 1 (Di 19 Apr 2022) **4**

Definitions for ILP. Binary LP. Disaggregation.

Lecture 2 (Di 19 Apr 2022) **4**

Intuition why IP is hard. Big-$\mathcal{O}$ notation

Lecture 3 (Di 26 Apr 2022) **5**

Hardness. Decision problems.

Lecture 4 (Do 28 Apr 2022) **8**

NP-complete. Problems in **NPC**. Reductions. Weakly vs. strongly.

Lecture 5 (Di 03 May 2022) **11**

$\mathsf{co}-\mathbf{NP}$. Ellipsoid method. Separation vs. optimization.

Part I

Lecture notes

Lecture 1
Di 19 Apr 2022

1 Test

I didn't make any notes.

Lecture 2
Di 19 Apr 2022

2 Modelling using IP

$$
\begin{aligned}
\text{s.t.} \quad & \lambda_1 \leqslant y_1 \\
& \lambda_i \leqslant y_{i-1} + y_i, \qquad i > 1 \\
& \lambda_k \leqslant y_{k-1} \\
& \sum_{i=1}^{k-1} y_i = 1 \\
& y_i \in \mathbb{B}
\end{aligned}
$$

This allows λ_{i-1}, λ_i to be positive and rest negative. [1].

Now, given an IP Q could be formulated by *many* P's.

$$
\begin{aligned}
Q :& \{0000, 1000, 0100, 0010, 0110, 0101, 0011\} \\
P_1 =& \{x \in \mathbb{R}^4 \mid 93x_1 + 49x_2 + 37x_3 + 29x_4 \leqslant 111\} \\
P_2 =& \{x \in \mathbb{R}^4 \mid 2x_1 + x_2 + x_3 + x_4 \leqslant 2\} \\
P_3 =& \{x \in \mathbb{R}^4 \mid 2x_1 + x_2 + x_3 + x_4 \leqslant 2, \\
& x_1 + x_2 \leqslant 1, \\
& x_1 + x_3 \leqslant 1, \\
& x_1 + x_4 \leqslant 1\}
\end{aligned}
$$

Then, $P_3 \subsetneq P_2 \subsetneq P_1$.

Facility location

Consider following boolean variables:

$$
m : \qquad y_i = \begin{cases} 1, & \text{open warehouse } i, \\ 0, & \text{else} \end{cases}
$$

$$
n : \qquad x_{ij} = \begin{cases} 1, & \text{if serve store } j \text{ from warehouse } i, \\ 0, & \text{else} \end{cases}
$$

[1] Ch 1: Nemhauser Wolsey

Complexity

Question: Easy vs. hard?
(some recap of big-$\mathcal{O}$-notation and $\mathcal{P}$ vs. $\mathcal{NP}$)

Lecture 3
Di 26 Apr 2022

3 Hardness

We cheat and restrict hardness to *decision problems*, that is, problems that can be answered by "Yes" or "No" only.

Example 3.1. Possible decision problems could be:

1. Does there exist a Hamiltonian cycle?
2. Is the LP feasible?

Question 3.2. How do we model an optimization problem as an decision problem?

Answer. We can simply introduce a parameter z which we use as a bound for the value we want to optimize.

Example 3.3. Possible reformulations of optimization problems are therefore:

1. Does there exist a feasible x with $c^T x \leqslant z$?
2. Does there exist a spanning tree with cost less than z?
3. Is there a clique with size less than z?

Definition 3.4. A **clique** C is a subset of nodes V of a graph $G = (V, E)$ s.t. for all $i, j \in C$ it must hold true that $(i, j) \in E$.

Theorem 3.5. When we model an optimization problem as a decision problem, then there exist a oracle-polynomial way to solve the optimization problem using the decision problem as an oracle.

Algorithm 1: Oracle-polynomial algorithm for max-clique

Use binary search to find z^* in $\mathcal{O}(\log n)$
$G' \leftarrow G = (V, E)$
for $i = 1, \ldots, n$ **do**
    $G'' \leftarrow G'$, but remove all edges incident to node i
    if *Call of decision oracle on* G'', z^* *is true* **then**
        $G' \leftarrow G''$
    end
end

Theorem 3.6. Final $\overline{G} := G'$ is a max-clique.

Proof. The size of a max clique in G' never goes below z^*. Therefore, there exists a clique $C \subseteq \overline{G}$ with $|C| = z^*$. Suppose $\overline{G}$ has more than z^* nodes. Then $i \in \overline{G} \backslash C$. But then the algorithm would have deleted this node! □

Corollary 3.7. If we have an optimal oracle, then one can solve decision version in oracle-polynomial time using the optimal oracle.

Conclusion 3.8. Optimization and decision version differ only by a polynomial factor of complexity. Therefore, either both are easy or both are hard.

Definition 3.9 (Certificate)**.** Given an instance of any problem with size n, a **certificate** is a binary-encoded string that is generated by some algorithm specific to the problem, taking the instance as input. We say the certificate is a **succinct certificate**, if its *length* is polynomial in the input size n.

Definition 3.10 (**NP**)**.** We say a (decision) problem P lies in **NP**, if for all Yes-instances I there exists a succinct certificate C and a certificate checking algorithm A that confirms $A(I, C)$ in polynomial time.

Theorem 3.11. Max-clique lies in **NP**

Proof. We use our clique C directly as the certificate.

Algorithm 2: Certificate checking for max-clique

```
if |C| < z then
  | return NO
end
else
  for i, j ∈ C do
    if (i, j) ∉ E then
      | return NO
    end
  end
end
return YES
```

□

Remark 3.12. Note that we don't care for No-instances! In order to verify them we would need to list all $\binom{n}{z}$ subsets (for max-clique), which is *not* polynomial.

Theorem 3.13. $\mathbf{P} \subseteq \mathbf{NP}$

Proof. Let $P \in \mathbf{P}$. Then there exists a polynomial algorithm A. Record the steps of A on an instance I and use this as a polynomial certificate. □

Theorem 3.14. $\mathsf{LP} \in \mathbf{NP}$, using decision variant if there is any feasible x.

Proof. If feasible, there exists a basic feasible solution x^*. We verify by checking $Bx^* = b$. One can show that x^* has polynomial bits. □

Let's also have a look at the canonical **NP** problem:

Definition 3.15 (Satisfiability problem, SAT)**.** Consider n logical variables $v_1, ..., v_n$, allowing also the negated literals $\overline{v_i}$. Additionally, we have m clauses $C_1, ..., C_m$, which are subsets of the literals. Determining if there is an assignment such that the overall clause is true (i.e. each subclause has at least one true literal) is known as the **satisfiability problem**, for short SAT.

Example 3.16. A few examples:

1. $(v_1 \vee v_2 \vee v_3) \wedge (\overline{v_1} \vee \overline{v_2} \vee \overline{v_3})$
   This instance is true for $v = (110)$.
2. $(v_1 \vee v_2) \wedge (\overline{v_1} \vee v_2) \wedge (\overline{v_2} \vee v_3) \wedge (\overline{v_3} \vee \overline{v_4})$
   One can check that this instance is always false.

Theorem 3.17. SAT $\in$ **NP**

Proof. The satisfiability truth assignment is a succinct certificate. □

Theorem 3.18 (Cook)**.** If $P \in$ **NP**, then P has an oracle-polynomial algorithm with SAT as an oracle.

Proof. Suppose $P \in$ **NP**, then P has a non-deterministic Turing Machine with polynomial size. □

Lecture 4
Do 28 Apr 2022

This means SAT is the hardest problem in **NP**.

We remind ourselves that IP can formulate logic, and therefore can encode SAT formulas.

Example 3.19. Translating from Example 3.16:

1.

$$\begin{aligned} x_1 + x_2 + x_3 &\geqslant 1 \\ (1 - x_1) + (1 - x_2) + (1 - x_3) &\geqslant 1 \\ x &\in \mathbb{B}^3 \end{aligned}$$

2.

$$\begin{aligned} x_1 + x_2 &\geqslant 1 \\ (1 - x_1) + x_2 &\geqslant 1 \\ (1 - x_2) + x_3 &\geqslant 1 \\ (1 - x_2) + (1 - x_3) &\geqslant 1 \\ x &\in \mathbb{B}^3 \end{aligned}$$

Definition 3.20 (Reduction)**.** We say $P \propto Q$ ("P **reduces to** Q") if there exists a polynomial algorithm A such that

1. for all instances $I \in P$, $A(I)$ is element of Q,
2. I is **Yes-preserving**, e.g. I is Yes-instance of P iff $A(I)$ is Yes-instance of Q.

Definition 3.21 (**NP**-complete)**.** A problem P is **NP-complete**, if

1. $P \in$ **NP**, and
2. for all $Q \in$ **NP** it holds that $Q \propto P$.

We call the set of all **NP**-complete problems **NPC**.

Proof Strategy. In order to show a problem P is **NP**-complete we first describe a way to construct a succinct certificate, and state an algorithm that describes how we use the certificate to verify a Yes-instance is indeed a Yes-instance.

After that, we find a suitable problem Q, which is known to be **NP**-complete, and try to proof $Q \propto P$. We do this by converting each instance of Q into an instance of P in polynomial time, and verify that the conversion is Yes-preserving.

Theorem 3.22. SAT is as hard as 0-1-IP

Proof. We know 0-1-IP $\in$ **NP**, and therefore 0-1-IP $\propto$ SAT. It remains to show SAT $\propto$ 0-1-IP: Let $I \in$ SAT with clauses $c_j = l_1, \dots, l_k$. We convert each clause to the inequality $l_1 + \dots + l_k \geqslant 1$ for binary l. It was previously shown this encodes exactly the logic formula. ☐

insert reference lec01

Definition 3.23 (3SAT)**.** We define 3SAT as a variant of SAT where we only allow clauses with exactly 3 literals, e.g. $|C_j| = 3$.

Theorem 3.24. 3SAT $\in$ **NPC**

Proof. 3SAT $\in$ **NP** follows directly from SAT $\in$ **NP**. It remains to show SAT $\propto$ 3SAT. Consider clause $C_j = (l_1 \vee \dots \vee l_k)$ for $k > 3$. Add $k - 3$ new variables $y_{2,j}, \dots, y_{k-2,j}$ and replace C_j with

$$(l_1 \vee l_2 \vee y_{2,j}) \wedge (\overline{y}_{2,j} \vee l_3 \vee y_{3,j}) \wedge \dots \wedge (\overline{y}_{k-2,j} \vee l_{k-1} \vee l_k)$$

One can figure out via proof tables and induction that this is indeed Yes-preserving. ☐

Definition 3.25 (Node cover, NC)**.** Given graph $G = (N, E)$, we say $C \subseteq N$ is a **node cover** if for every edge in E at least one of the nodes is in N. We define NC as the decision problem if there is a node cover of at most size z.

Theorem 3.26. NC $\in$ **NPC**

Proof. We can easily check if for a given C, it is indeed a node cover in polynomial time. Therefore NC $\in$ **NPC**. We want to reduce from 3SAT:

Consider an instance of 3SAT and construct a graph as shown in *Figure* 1, e.g. for each variable v_i construct an edge between nodes v_i and $\overline{v}_i$, and for each clause C_j construct a triangle l_{1j}, l_{2j}, l_{3j}. Now, connect each node of the triangle with the corresponding literal in the clause (the orange edges). Using this construction, we want to proove that there is a node cover of size $n + 2m$ iff the 3SAT instance is valid.

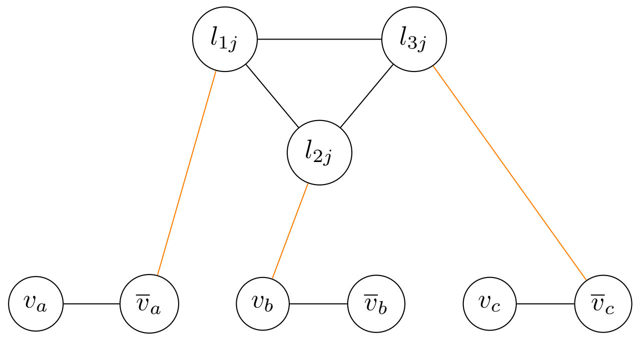


Figure 1: Schema of how to use triangle "gadgets" for a single clause C_j

Suppose the 3SAT instance is feasible. We use the n nodes of the feasible labeling corresponding to the literals. Now, because the labelling is valid, at least one orange edge per triangle must be covered, by construction. Therefore, we can choose 2 additional nodes per triangle that cover the triangle and the remaining orange edges.

On the other side, suppose there is a node cover of size (at most) $n+2m$. Analoguous, each triangle must have at least 2 chosen nodes to cover each edge, and each literal-pair at least 1 node, meaning our bounds must actually be exact to not overshoot $n+2m$. Therefore, the node cover represents a valid truth assignment, which is also a valid labelling, because each clause has a remaining orange edge, which is covered by one of the literals.

Therefore, our reduction is Yes-preserving. ☐

Remark 3.27. NC in bipartite graphs is in **P**.

Definition 3.28 (Independent set, IS)**.** For a graph $G = (N, E)$ we call $S \subseteq N$ a **independent set** (or **stable set**) if no edge has both nodes in S. The decision problem, called IS, if there is a independent set of size at least z.

Theorem 3.29. IS $\in$ **NPC**

Proof. IS $\in$ **NP** trivial. We can also easily show that C is a node cover iff $N \backslash C$ is stable. ☐

Theorem 3.30. CLIQUE $\in$ **NPC**

Proof. CLIQUE $\in$ **NP** trivial. We can also easily show that C is a clique in $G = (N, E)$ iff C is stable in $(N, \overline{E})$. ☐

Definition 3.31 (Partition, PART)**.** Given $a_1, ..., a_n \in \mathbb{Z}^+$. The decision problem if there is a set $S \subseteq \{1, ..., n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i$$

is called **partition problem**, PART.

Theorem 3.32. PART $\in$ **NPC**

Proof Sketch. We can show [Ber18, Ch. 15.5]:

$$\mathsf{SAT} \propto \text{3-dim match} \propto \text{subset sum} \propto \mathsf{PART}$$

Remark 3.33. Still, PART has a pseudopolynomial algorithm using dynamic programming.

Definition 3.34. If a (numerical) problem is only **NP**-complete if it is dependent on the size of the numbers (e.g. exponentially in count of numbers), we call it **weakly NP-complete**. Otherwise, we call it **strongly NP-complete**.

Definition 3.35 (3-partition, 3PART)**.** Given the numbers $a_1, ..., a_{3k} \in \mathbb{Z}$. The problem, if we can partition these numbers in sets of 3 such that every set has the same value, is called **3-Partition**, or 3PART.

Theorem 3.36. 3SAT is strongly **NP**-complete.

Proof. □

proof 3part NPC

Remark 3.37. Only weakly **NP**-complete problems could have pseudopolynomial algorithms (except **P** = **NP**).

Lecture 5
Di 03 May 2022

Definition 3.38 (**NP**-hard)**.** Consider an optimization problem P. We can't have $P \in$ **NP**, but because of Theorem 3.5 we can introduce the notion to call P **NP-hard**, if its decision variant is in **NPC**.

Definition 3.39 (co−**NP**)**.** We say $P \in$ co−**NP**, if we have a succinct certificate for verifying No-instances.

Example 3.40. Given a matrix A. We call it totally unimodular, if every square submatrix has determinant 0 or 1. Deciding if A is totally unimodular is in co−**NP**, because giving a failing submatrix as a succinct certificate is easy.

Theorem 3.41. The decision version of LP is in co−**NP**.

Proof. The answer to the decision problem is No iff

1. the system is infeasible, or
2. the system is feasible, but the optimal cost is larger than the z we want.

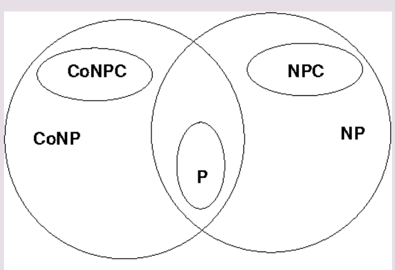
Because both can be decided with the tools we have in polynomial time, LP is indeed in co−**NP**. □

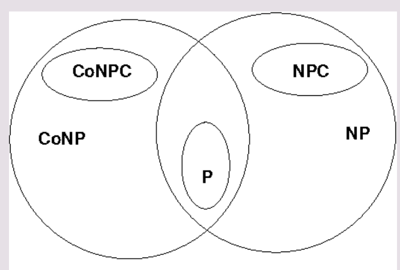
Definition 3.42 (co−**NP**-complete)**.** Analoguous to Definition 3.21, we can also define **co−NP-completeness**, or co−**NPC**, for the "most difficult" problem in co−**NP**.

Remark 3.43. It holds that co−**NPC** $\cap$ **NPC** $= \varnothing$, except **P** $=$ **NP**.

images p np

Open Question 3.44. Is **P** $=$ **NP**?





Recall (Ellipsoid Algorithm)**.** As discussed in ADM1, the **ellipsoid method** can be used to determine feasibility of LPs in polynomial time. One could also call the ellipsoid method a fancy "n-dimensional binary search". A rough draft how the algorithm worked:

1. Reduce optimization version to decision version and introduce bound $L = mn \cdot \log(\text{max abs. data})$
2. Volume-based argument: If the LP is feasible, there is a solution within the centered cube with length 2^L.
3. Volume is zero: Perturb the problem to $Ax \leqslant b + 2^{-L}$, which maintains feasibility, but now has positive volume.

For details, refer to the slides from ADM1.

Remark 3.45. The key step of the ellipsoid method is to find a hyperplane that separated the current x from the considered polyhedron. This is equivalent to solving the **separation problem** SEP. Especially, iff SEP $\in$ **P**, then OPT $\in$ **P**.

Definition 3.46. Let $\mathcal{Q}$ be the class of full-dimensional polytopes with 0 inside. We define the **polar** Q^* for $Q \in \mathcal{Q}$ as

$$Q^* := \{y \in \mathbb{R}^n \mid y^T x \leqslant 1 \forall x \in Q\}.$$

Theorem 3.47. Considering this class of polytopes, one can proove [Ber18, Ch. 4, Thm. 4.22]:

1. Q^* is also a full-dimensional polytope with 0.
2. $(Q^*)^* = Q$
3. v is a vertex of Q iff $v^T y \leqslant 1$ is a facet of Q^*

Theorem 3.48. Suppose we can solve OPT on $\mathcal{Q} \in$ **P** with algorithm A. Then we can use A as an oracle to solve SEP on $\mathcal{Q}^*$ in polynomial time.

Proof. Suppose $Q^* \in \mathcal{Q}^*$, and we want to separate y^0. Use A to solve OPT on Q with objective function $\max (y^0)^T x$ to get $x^* \in Q$. This yields two cases:

- $(y^0)^T x^* \leqslant 1$: Then this holds for all $x \in Q$, and thus $y^0 \in Q^*$ by definition.
- $(y^0)^T x^* > 1$: Consider hyperplane $(x^*)^T y$. From $x^* \in Q$ it follows that for all $y \in Q^*$, that $(x^*)^T y \leqslant 1$, but $(x^*)^T y^0 > 1$. Thus, we found a separating hyperplane.

$\square$

Theorem 3.49. SEP $\in$ **P** for $\mathcal{Q}$ iff OPT $\in$ **P** for $\mathcal{Q}$

Proof. Using what we proven so far:

$$\begin{aligned}
&\mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q} && \overset{3.48}{\Longrightarrow} \mathsf{SEP} \in \mathbf{P} \text{ for } \mathcal{Q}^* \\
\overset{\text{Ellips.}}{\Longrightarrow} &\mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q}^* && \overset{3.48}{\Longrightarrow} \mathsf{SEP} \in \mathbf{P} \text{ for } (\mathcal{Q}^*)^* = Q \\
\overset{\text{Ellips.}}{\Longrightarrow} &\mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q}
\end{aligned}$$

□

Part II

Appendix

A Exercise sheets

1. exercise sheet

Exercise 1.1. Did on paper.

2. exercise sheet

Exercise 2.1. An correct ordering is given by:

$$\begin{aligned}
\mathrm{O}\left(\varepsilon^{n}\right) &\subseteq \mathrm{O}\left(n^{\varepsilon-1}\right) \subseteq \mathrm{O}\left(n^{-\varepsilon}\right) \subseteq \mathrm{O}\left(\frac{\log n}{n^{\varepsilon}}\right) & (1)\\
&\subseteq \mathrm{O}\left(\frac{1}{\log n}\right) \subseteq \mathrm{O}\left(\frac{\log^2 n}{\log n}\right) \subseteq \mathrm{O}\left(\frac{1}{\log^2 n}\right) & (2)\\
&\subseteq \mathrm{O}\left(e^{\frac{1}{n}}\right) = \mathrm{O}\left(1\right) = \mathrm{O}\left(\left(1-\frac{1}{n}\right)^{n}\right) & (3)\\
&\subseteq \mathrm{O}\left(\log n\right) \subseteq \mathrm{O}\left(\frac{n^{\varepsilon}}{\log n}\right) \subseteq \mathrm{O}\left(n^{\varepsilon}\right) \subseteq \mathrm{O}\left(n^{\varepsilon}\log n\right) \subseteq \mathrm{O}\left(n^{1-\varepsilon}\right) & (4)\\
&\subseteq \mathrm{O}\left(\frac{n}{\log n}\right) \subseteq \mathrm{O}\left(n\log n\right) \subseteq \mathrm{O}\left(n^{2}\right) \subseteq \mathrm{O}\left(n^{2}\log n\right) \subseteq \mathrm{O}\left(n^{e}\right) & (5)\\
&\subseteq \mathrm{O}\left(n^{\log n}\right) \subseteq \mathrm{O}\left(e^{n}\right) \subseteq \mathrm{O}\left((\log n)^{n}\right) \subseteq \mathrm{O}\left(n!\right) & (6)
\end{aligned}$$

These can mostly achieved by the fact that $n^x \in \mathrm{O}(n^y)$ if $x \leqslant y$, and $(\log n) \cdot n^x \in \mathrm{O}(n^y)$ if $y > x$, otherwise the other way around. Additionally, it is often useful to consider the logarithm of the functions we compare, because it maintains monotonocity.

Exercise 2.2. Analoguous to the lecture we can introduce constraints, such that $y_{ij} = x_i \wedge x_j$:

$$\begin{aligned}
y_{ij} &\leqslant x_i\\
y_{ij} &\leqslant x_j\\
y_{ij} &\geqslant x_i + x_j - 1\\
y_{ij} &\in [0,1]
\end{aligned}$$

Exercise 2.3. We can show that $f(x_1) = \max(c_1 x_1, c_1 p + c_2 x_1 - c_2 p)$ using a case distinction.

- $x_1 = p$: Trivial.
- $x_1 > p$: Consider $c_1 < c_2$. Multiplying by $x_1 - p$ (which is positive) and rearranging yields $c_1 x_1 < c_1 p + c_2 x_1 - c_2 p$.
- $x_1 < p$: Analoguous, but now $x_1 - p$ is negative, which reverses the inequality.

As shown in ADM1, the maximum of linear functions can be written as an LP by introducing a helper variable as follows:

$$\begin{aligned}
\min \quad & z + \sum_{i=2}^{n} c_i x_i \\
\text{s.t.} \quad & Ax = b \\
& l \leqslant x_1 \leqslant u \\
& x_2, ..., x_n \leqslant 0 \\
& z \geqslant c_1 x_1 \\
& z \geqslant c_1 p + c_2 x_1 - c_2 p
\end{aligned}$$

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[Ber18] Jens Vygen Bernhard Korte. *Kombinatorische Optimierung*. 3rd ed. Springer, 2018. ISBN: 978-3-662-57690-8. URL: https://link.springer.com/book/10.1007/978-3-662-57691-5.