Discrete Optimization - ADM2

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Version
git: 7f45928
compiled: Saturday 7$^{\text{th}}$ May, 2022 22:18

Abstract

The following lecture notes are my personal (and therefore unofficial) write-up for 'Discrete Optimization' aka 'ADM II', which took place in summer semester 2022 at Technische Universität Berlin. I do not guarantee correctness, completeness, or anything else.

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Summary of lectures

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Definitions for ILP. Binary LP. Disaggregation.

Lecture 2 (Di 19 Apr 2022) **7**

Intuition why IP is hard. Big-$\mathcal{O}$ notation

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Hardness. Decision problems.

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NP-complete. Problems in **NPC**. Reductions. Weakly vs. strongly.

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co$-$**NP**. Ellipsoid method. Separation vs. optimization.

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co$-$**NP**. Ellipsoid method. Separation vs. optimization.

Part I

Lecture notes

Lecture 1
Di 19 Apr 2022

1 Introduction

In ADM1 we often already worked with Integer Programming and just assumed everything is fine. In this course however, we want to find out how Integer Programming actually works, why it is generally "hard", and under which circumstances it is "easy".

Definition 1.1 (Flavors of IP)**.** First of all, we want to define different variants of **Integer Programming**:

- **Pure Integer Programming** assumes *all* variables are integer.
- **Mixed Integer Programming** also allows some variables to be real.
- **Binary Integer Programming**, also called 0-1-Integer-Programming, restricts the integer variables to $\mathbb{B} := \{0, 1\}$. Mixed variants are also possible.

Question 1.2. Why is IP harder than LP? Naively, one would assume this should *not* be the case, because our search space is smaller (at most countably infinite)!

Let's solve the IP in ADM1-style - suppose

$$\begin{aligned} \max_x \quad & w^T x \\ \text{s.t.} \quad & x \in Q = \{x \in \mathbb{Z}^n : Ax \leqslant b\} \end{aligned}$$

For simplicity, assume Q is bounded. Then the set of feasible points in Q is finite, and therefore we can consider the polytope

$$\operatorname{conv}(Q) = \{x \in \mathbb{R}^n \mid A'x \leqslant b'\}$$

for suitable A' and b'. Notice all vertices must be in Q and thus are integral. As a consequence, it is sufficient to solve the LP

$$\begin{aligned} \max_x \quad & w^T x \\ \text{s.t.} \quad & A'x \leqslant b'. \end{aligned}$$

Warning. Computing A', b' is non-trivial! In fact, computing the **integer hull** $\operatorname{conv}(Q)$ is what makes IP hard.

1.1 IP is "hard"

We can gather more evidence that IP must be hard.

Theorem 1.3. Every logical statement can be expressed with integer programming.

Proof. It suffices to show that for variables $x_1, x_2, y \in \mathbb{B}$ we can find IPs that model $\wedge, \vee, \neg$.

- $y = x_1 \wedge x_2$ can be modeled as

$$\begin{aligned} y &\leqslant x_1 \\ y &\leqslant x_2 \\ y &\geqslant x_1 + x_2 - 1 \end{aligned}$$

- $y \coloneqq x_1 \vee x_2$ can be modeled as

$$\begin{aligned} y &\geqslant x_1 \\ y &\geqslant x_2 \\ y &\leqslant x_1 + x_2 \end{aligned}$$

- $y \coloneqq \neg x$ can be modeled as

$$y = 1 - x$$

□

Theorem 1.4. IP can model (finite) unions of polyhedra, e.g. non-convex problems.

Proof. Consider

$$P \coloneqq \bigcup_{i=1}^{k} \underbrace{\{x \mid A_i x \leqslant b_i\}}_{P_i}$$

and introduce auxiliary binary variables

$$y_i \coloneqq \begin{cases} 1, & \text{if } x \in P_i, \\ 0, & \text{if we don't care.} \end{cases}$$

Now, assume $M \in \mathbb{R}$ large enough (**Big-M method**), such that

$$P_i \subseteq \overline{P} \coloneqq \{x \mid A_i x \leqslant b_i + M \cdot \mathbf{1}\}$$

for all i. Thus, we can construct following IP:

$$\begin{aligned}\min_x \quad & w^T x \\ \text{s.t.} \quad & A_i x \leqslant b_i + (1 - y_i) M \cdot \mathbf{1}, \quad i = 1, \dots, k, \\ & \sum_i y_i = 1\end{aligned}$$

This forces exactly one y_i to 1, resulting that $x \in P_i$ and $x \in \overline{P}$ is sufficient. We call this method **righthandside Big-M**, short RHS. Further information can be found in [Geo99, Ch. 1]. □

Note. It's also possible to handle M as a symbolic value, but this makes other things more complicated.

Problem. Finding M big enough can make LP hard to solve, because of matrix-inversions getting numerically unstable.

Alternative proof of Theorem 1.4. Assume that we can bound each $x \in P_i$ by u_i, i.e. $x \leqslant u_i$ (note that this is basically a hidden big-M!). Now we can disaggregate x for each P_i as its own private $x_i \in \mathbb{R}^k$, and analogously introduce $y_i \in \mathbb{B}$ to restrict ourselves to one polyhedron:

$$\begin{aligned}\min_x \quad & w^T x \\ \text{s.t.} \quad & A_i x_i \leqslant y_i b_i, \quad i = 1, \dots, k, \\ & x_i \leqslant y_i u_i, \quad i = 1, \dots, k, \\ & \sum_{i=1}^n y_i = 1, \\ & \sum_{i=1}^n x_i = x, \\ & x, x_i \geqslant 0\end{aligned}$$

Again, exactly one y_i is equal to 1, forcing the other x_j to be equal to 0, and thus setting x to x_i. We call this method **upper bound on x Big-M**, short UBX. □

Remark 1.5. In general, it cannot be said if RHS or UBX is better. Even though RHS only introduces n new variables as opposed to UBX's nk variables, UBX's disaggregated formulation often is *tighter* in the sense that the LP relaxation is closer to the integer hull.

Conclusion 1.6. Summarizing, following facts deliver an intuition why IP should be hard:

1. Consider a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b\}$ and its integer hull Q. Even

though P can be "smooth" (e.g. cube), its integer hull can look more like a "disco ball" with many facets.

2. In real life, many problems can be modeled with decision variables. IP can handle this, see Theorem 1.3.
3. Additionally, IP also can handle non-convex problems, see Theorem 1.4.

Lecture 2
Di 19 Apr 2022

2 Modelling using IP

$$
\begin{aligned}
\min_{\lambda} \quad & \sum_i \lambda_i f(a_i) \\
\text{s.t.} \quad & \sum_{i=1}^{k-1} y_i = 1, \\
& \lambda_i \leqslant y_{i-1} + y_i, \quad i = 2, \ldots, k, \\
& \lambda_1 \leqslant y_1, \\
& \lambda_k \leqslant y_{k-1}, \\
& y_i \in \mathbb{B}
\end{aligned}
$$

This allows λ_{i-1}, λ_i to be positive and rest negative. [1].

Now, given an IP Q could be formulated by *many* P's.

$$
\begin{aligned}
Q :& \{0000, 1000, 0100, 0010, 0110, 0101, 0011\} \\
P_1 =& \{x \in \mathbb{R}^4 \mid 93x_1 + 49x_2 + 37x_3 + 29x_4 \leqslant 111\} \\
P_2 =& \{x \in \mathbb{R}^4 \mid 2x_1 + x_2 + x_3 + x_4 \leqslant 2\} \\
P_3 =& \{x \in \mathbb{R}^4 \mid 2x_1 + x_2 + x_3 + x_4 \leqslant 2, \\
& x_1 + x_2 \leqslant 1, \\
& x_1 + x_3 \leqslant 1, \\
& x_1 + x_4 \leqslant 1\}
\end{aligned}
$$

Then, $P_3 \subsetneq P_2 \subsetneq P_1$.

[1] Ch 1: Nemhauser Wolsey

Facility location

Consider following boolean variables:

$$m: \qquad y_i = \begin{cases} 1, & \text{open warehouse } i, \\ 0, & \text{else} \end{cases}$$

$$n: \qquad x_{ij} = \begin{cases} 1, & \text{if serve store } j \text{ from warehouse } i, \\ 0, & \text{else} \end{cases}$$

Complexity

Question: Easy vs. hard?
(some recap of big-$\mathcal{O}$-notation and $\mathcal{P}$ vs. $\mathcal{NP}$)

Lecture 3
Di 26 Apr 2022

3 Hardness

We cheat and restrict hardness to *decision problems*, that is, problems that can be answered by "Yes" or "No" only.

Example 3.1. Possible decision problems could be:

1. Does there exist a Hamiltonian cycle?
2. Is the LP feasible?

Question 3.2. How do we model an optimization problem as an decision problem?

Answer. We can simply introduce a parameter z which we use as a bound for the value we want to optimize.

Example 3.3. Possible reformulations of optimization problems are therefore:

1. Does there exist a feasible x with $c^T x \leqslant z$?
2. Does there exist a spanning tree with cost less than z?
3. Is there a clique with size less than z?

Definition 3.4. A **clique** C is a subset of nodes V of a graph $G = (V, E)$ s.t. for all $i, j \in C$ it must hold true that $(i, j) \in E$.

Theorem 3.5. When we model an optimization problem as a decision problem, then there exist a oracle-polynomial way to solve the optimization problem using the decision problem as an oracle.

Algorithm 1: Oracle-polynomial algorithm for max-clique

```
Use binary search to find z* in O(log n)
G' ← G = (V, E)
for i = 1, ..., n do
    G'' ← G', but remove all edges incident to node i
    if Call of decision oracle on G'', z* is true then
        G' ← G''
    end
end
```

Theorem 3.6. Final $\overline{G} := G'$ is a max-clique.

Proof. The size of a max clique in G' never goes below z^*. Therefore, there exists a clique $C \subseteq \overline{G}$ with $|C| = z^*$. Suppose $\overline{G}$ has more than z^* nodes. Then $i \in \overline{G} \backslash C$. But then the algorithm would have deleted this node! □

Corollary 3.7. If we have an optimal oracle, then one can solve decision version in oracle-polynomial time using the optimal oracle.

Conclusion 3.8. Optimization and decision version differ only by a polynomial factor of complexity. Therefore, either both are easy or both are hard.

Definition 3.9 (Certificate)**.** Given an instance of any problem with size n, a **certificate** is a binary-encoded string that is generated by some algorithm specific to the problem, taking the instance as input. We say the certificate is a **succinct certificate**, if its *length* is polynomial in the input size n.

Definition 3.10 (**NP**)**.** We say a (decision) problem P lies in **NP**, if for all Yes-instances I there exists a succinct certificate C and a certificate checking algorithm A that confirms $A(I, C)$ in polynomial time.

Theorem 3.11. Max-clique lies in **NP**

Proof. We use our clique C directly as the certificate.

Algorithm 2: Certificate checking for max-clique

```
if |C| < z then
    return NO
end
else
    for i, j ∈ C do
        if (i, j) ∉ E then
            return NO
        end
    end
end
return YES
```

□

Remark 3.12. Note that we don't care for No-instances! In order to verify them we would need to list all $\binom{n}{z}$ subsets (for max-clique), which is *not* polynomial.

Theorem 3.13. **P** $\subseteq$ **NP**

Proof. Let $P \in \mathbf{P}$. Then there exists a polynomial algorithm A. Record the steps of A on an instance I and use this as a polynomial certificate. □

Theorem 3.14. LP $\in$ **NP**, using decision variant if there is any feasible x.

Proof. If feasible, there exists a basic feasible solution x^*. We verify by checking $Bx^* = b$. One can show that x^* has polynomial bits. □

Let's also have a look at the canonical **NP** problem:

Definition 3.15 (Satisfiability problem, SAT)**.** Consider n logical variables $v_1, ..., v_n$, allowing also the negated literals $\overline{v_i}$. Additionally, we have m clauses $C_1, ..., C_m$, which are subsets of the literals. Determining if there is an assignment such that the overall clause is true (i.e. each subclause has at least one true literal) is known as the **satisfiability problem**, for short SAT.

Example 3.16. A few examples:

1. $(v_1 \vee v_2 \vee v_3) \wedge (\overline{v_1} \vee \overline{v_2} \vee \overline{v_3})$
   This instance is true for $v = (110)$.

2. $(v_1 \vee v_2) \wedge (\overline{v_1} \vee v_2) \wedge (\overline{v_2} \vee v_3) \wedge (\overline{v_3} \vee \overline{v_4})$
   One can check that this instance is always false.

Theorem 3.17. SAT $\in$ **NP**

Proof. The satisfiability truth assignment is a succinct certificate. □

Theorem 3.18 (Cook)**.** If $P \in$ **NP**, then P has an oracle-polynomial algorithm with SAT as an oracle.

Proof. Suppose $P \in$ **NP**, then P has a non-deterministic Turing Machine with polynomial size. □

Lecture 4
Do 28 Apr 2022

This means SAT is the hardest problem in **NP**.

We remind ourselves that IP can formulate logic, and therefore can encode SAT formulas.

Example 3.19. Translating from Example 3.16:

1.

$$\begin{aligned} x_1 + x_2 + x_3 &\geqslant 1 \\ (1 - x_1) + (1 - x_2) + (1 - x_3) &\geqslant 1 \\ x &\in \mathbb{B}^3 \end{aligned}$$

2.

$$\begin{aligned} x_1 + x_2 &\geqslant 1 \\ (1 - x_1) + x_2 &\geqslant 1 \\ (1 - x_2) + x_3 &\geqslant 1 \\ (1 - x_2) + (1 - x_3) &\geqslant 1 \\ x &\in \mathbb{B}^3 \end{aligned}$$

Definition 3.20 (Reduction)**.** We say $P \propto Q$ ("P **reduces to** Q") if there exists a polynomial algorithm A such that

1. for all instances $I \in P$, $A(I)$ is element of Q,
2. I is **Yes-preserving**, e.g. I is Yes-instance of P iff $A(I)$ is Yes-instance of Q.

Definition 3.21 (**NP**-complete)**.** A problem P is **NP-complete**, if

1. $P \in \mathbf{NP}$, and
2. for all $Q \in \mathbf{NP}$ it holds that $Q \propto P$.

We call the set of all **NP**-complete problems **NPC**.

Proof Strategy. In order to show a problem P is **NP**-complete we first describe a way to construct a succinct certificate, and state an algorithm that describes how we use the certificate to verify a Yes-instance is indeed a Yes-instance.

After that, we find a suitable problem Q, which is known to be **NP**-complete, and try to proof $Q \propto P$. We do this by converting each instance of Q into an instance of P in polynomial time, and verify that the conversion is Yes-preserving.

Theorem 3.22. SAT is as hard as 0-1-IP

Proof. We know 0-1-IP $\in$ **NP**, and therefore 0-1-IP $\propto$ SAT. It remains to show SAT $\propto$ 0-1-IP: Let $I \in$ SAT with clauses $c_j = l_1, \ldots, l_k$. We convert each clause to the inequality $l_1 + \ldots + l_k \geqslant 1$ for binary l. It was previously shown this encodes exactly the logic formula. □

insert reference lec01

Definition 3.23 (3SAT)**.** We define 3SAT as a variant of SAT where we only allow clauses with exactly 3 literals, e.g. $|C_j| = 3$.

Theorem 3.24. 3SAT $\in$ **NPC**

Proof. 3SAT $\in$ **NP** follows directly from SAT $\in$ **NP**. It remains to show SAT $\propto$ 3SAT. Consider clause $C_j = (l_1 \vee \ldots \vee l_k)$ for $k > 3$. Add $k-3$ new variables $y_{2,j}, \ldots, y_{k-2,j}$ and replace C_j with

$$(l_1 \vee l_2 \vee y_{2,j}) \wedge (\overline{y}_{2,j} \vee l_3 \vee y_{3,j}) \wedge \ldots \wedge (\overline{y}_{k-2,j} \vee l_{k-1} \vee l_k)$$

One can figure out via proof tables and induction that this is indeed Yes-preserving. □

Definition 3.25 (Node cover, NC)**.** Given graph $G = (N, E)$, we say $C \subseteq N$ is a **node cover** if for every edge in E at least one of the nodes is in N. We define NC as the decision problem if there is a node cover of at most size z.

Theorem 3.26. NC $\in$ **NPC**

Proof. We can easily check if for a given C, it is indeed a node cover in polynomial time. Therefore NC $\in$ **NPC**. We want to reduce from 3SAT:

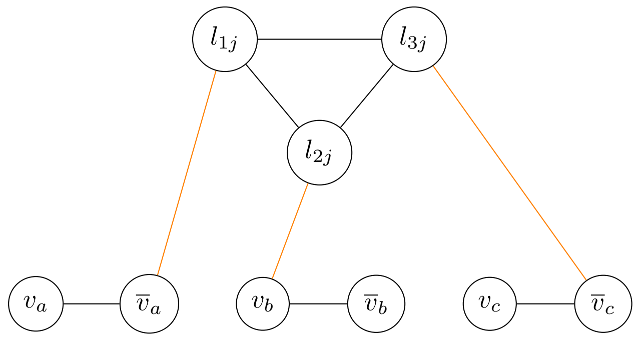


Figure 1: Schema of how to use triangle "gadgets" for a single clause C_j

Consider an instance of 3SAT and construct a graph as shown in *Figure* 1, e.g. for each variable v_i construct an edge between nodes v_i and $\overline{v}_i$, and for each clause C_j construct a triangle l_{1j}, l_{2j}, l_{3j}. Now, connect each node of the triangle with the corresponding literal in the clause (the orange edges). Using this construction, we want to proove that there is a node cover of size $n + 2m$ iff the 3SAT instance is valid.

Suppose the 3SAT instance is feasible. We use the n nodes of the feasible labeling corresponding to the literals. Now, because the labelling is valid, at least one orange edge per triangle must be covered, by construction. Therefore, we can choose 2 additional nodes per triangle that cover the triangle and the remaining orange edges.

On the other side, suppose there is a node cover of size (at most) $n+2m$. Analoguous, each triangle must have at least 2 chosen nodes to cover each edge, and each literal-pair at least 1 node, meaning our bounds must actually be exact to not overshoot $n + 2m$. Therefore, the node cover represents a valid truth assignment, which is also a valid labelling, because each clause has a remaining orange edge, which is covered by one of the literals.

Therefore, our reduction is Yes-preserving. □

Remark 3.27. NC in bipartite graphs is in **P**.

Definition 3.28 (Independent set, IS)**.** For a graph $G = (N, E)$ we call $S \subseteq N$ a **independent set** (or **stable set**) if no edge has both nodes in S. The decision problem, called IS, if there is a independent set of size at least z.

Theorem 3.29. IS $\in$ **NPC**

Proof. IS $\in$ **NP** trivial. We can also easily show that C is a node cover iff $N \backslash C$ is stable. □

Theorem 3.30. CLIQUE $\in$ **NPC**

Proof. CLIQUE $\in$ **NP** trivial. We can also easily show that C is a clique in $G = (N, E)$ iff C is stable in $(N, \overline{E})$. ☐

Definition 3.31 (Partition, PART)**.** Given $a_1, ..., a_n \in \mathbb{Z}^+$. The decision problem if there is a set $S \subseteq \{1, ..., n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i$$

is called **partition problem**, PART.

Theorem 3.32. PART $\in$ **NPC**

Proof Sketch. We can show [Ber18, Ch. 15.5]:

$$\text{SAT} \propto \text{3-dim match} \propto \text{subset sum} \propto \text{PART}$$

Remark 3.33. Still, PART has a pseudopolynomial algorithm using dynamic programming.

Definition 3.34. If a (numerical) problem is only **NP**-complete if it is dependent on the size of the numbers (e.g. exponentially in count of numbers), we call it **weakly NP-complete**. Otherwise, we call it **strongly NP-complete**.

Definition 3.35 (3-partition, 3PART)**.** Given the numbers $a_1, ..., a_{3k} \in \mathbb{Z}$. The problem, if we can partition these numbers in sets of 3 such that every set has the same value, is called **3-Partition**, or 3PART.

Theorem 3.36. 3SAT is strongly **NP**-complete.

Proof. ☐

proof 3part NPC

Remark 3.37. Only weakly **NP**-complete problems could have pseudopolynomial algorithms (except **P** = **NP**).

Lecture 5
Di 03 May 2022

Definition 3.38 (**NP**-hard)**.** Consider an optimization problem P. We can't have $P \in \mathbf{NP}$, but because of Theorem 3.5 we can introduce the notion to call P **NP-hard**, if its decision variant is in **NPC**.

Definition 3.39 ($\mathsf{co}-\mathbf{NP}$)**.** We say $P \in \mathsf{co}-\mathbf{NP}$, if we have a succinct certificate for verifying No-instances.

Example 3.40. Given a matrix A. We call it totally unimodular, if every square submatrix has determinant 0 or 1. Deciding if A is totally unimodular is in $\mathsf{co}-\mathbf{NP}$, because giving a failing submatrix as a succinct certificate is easy.

Theorem 3.41. The decision version of LP is in $\mathsf{co}-\mathbf{NP}$.

Proof. The answer to the decision problem is No iff

1. the system is infeasible, or
2. the system is feasible, but the optimal cost is larger than the z we want.

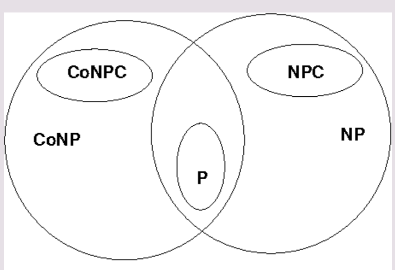
Because both can be decided with the tools we have in polynomial time, LP is indeed in $\mathsf{co}-\mathbf{NP}$. □

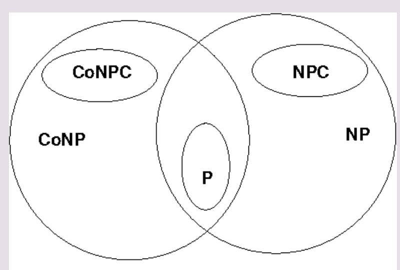
Definition 3.42 ($\mathsf{co}-\mathbf{NP}$-complete)**.** Analoguous to Definition 3.21, we can also define **$\mathsf{co}-\mathbf{NP}$-completeness**, or $\mathsf{co}-\mathbf{NPC}$, for the "most difficult" problem in $\mathsf{co}-\mathbf{NP}$.

Remark 3.43. It holds that $\mathsf{co}-\mathbf{NPC} \cap \mathbf{NPC} = \varnothing$, except $\mathbf{P} = \mathbf{NP}$.

images p np

Open Question 3.44. Is $\mathbf{P} = \mathbf{NP}$?





4 Complexity in linear programming

Recall (Ellipsoid Algorithm)**.** As discussed in ADM1, the **ellipsoid method** can be used to determine feasibility of LPs in polynomial time. One could also call the ellipsoid method a fancy "n-dimensional binary search". A rough draft how the algorithm worked:

1. Reduce optimization version to decision version and introduce bound $L = mn \cdot \log(\text{max abs. data})$
2. Volume-based argument: If the LP is feasible, there is a solution within the centered cube with length 2^L.
3. Volume is zero: Perturb the problem to $Ax \leqslant b + 2^{-L}$, which maintains feasibility, but now has positive volume.

For details, refer to the slides from ADM1.

Remark 4.1. The key step of the ellipsoid method is to find a hyperplane that separated the current x from the considered polyhedron. This is equivalent to solving the **separation problem** SEP. Especially, iff $\mathsf{SEP} \in \mathbf{P}$, then $\mathsf{OPT} \in \mathbf{P}$.

Definition 4.2. Let $\mathcal{Q}$ be the class of full-dimensional polytopes with 0 inside. We define the **polar** Q^* for $Q \in \mathcal{Q}$ as

$$Q^* := \{y \in \mathbb{R}^n \mid y^T x \leqslant 1 \forall x \in Q\}.$$

Theorem 4.3. Considering this class of polytopes, one can proove [Ber18, Ch. 4, Thm. 4.22]:

1. Q^* is also a full-dimensional polytope with 0.
2. $(Q^*)^* = Q$
3. v is a vertex of Q iff $v^T y \leqslant 1$ is a facet of Q^*

Theorem 4.4. Suppose we can solve OPT on $\mathcal{Q} \in \mathbf{P}$ with algorithm A. Then we can use A as an oracle to solve SEP on $\mathcal{Q}^*$ in polynomial time.

Proof. Suppose $Q^* \in \mathcal{Q}^*$, and we want to separate y^0. Use A to solve OPT on Q with objective function $\max (y^0)^T x$ to get $x^* \in Q$. This yields two cases:

- $(y^0)^T x^* \leqslant 1$: Then this holds for all $x \in Q$, and thus $y^0 \in Q^*$ by definition.
- $(y^0)^T x^* > 1$: Consider hyperplane $(x^*)^T y$. From $x^* \in Q$ it follows that for all $y \in Q^*$, that $(x^*)^T y \leqslant 1$, but $(x^*)^T y^0 > 1$. Thus, we found a separating hyperplane.

□

Theorem 4.5. SEP $\in$ **P** for $\mathcal{Q}$ iff OPT $\in$ **P** for $\mathcal{Q}$

Proof. Using what we proven so far:

$$\begin{aligned} & \mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q} && \overset{4.4}{\Longrightarrow} \mathsf{SEP} \in \mathbf{P} \text{ for } \mathcal{Q}^* \\ \overset{\text{Ellips.}}{\Longrightarrow} & \mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q}^* && \overset{4.4}{\Longrightarrow} \mathsf{SEP} \in \mathbf{P} \text{ for } (\mathcal{Q}^*)^* = Q \\ \overset{\text{Ellips.}}{\Longrightarrow} & \mathsf{OPT} \in \mathbf{P} \text{ for } \mathcal{Q} \end{aligned}$$

□

Lecture 6
Di 03 May 2022

Content lec06

Theorem 4.6 (Minkowski). For a polyhedron P it holds $x \in P$ iff there exist vertices $v_1, \ldots, v_k$ and rays $r_1, \ldots, r_l$, such that

$$\sum_i \lambda v_i + \sum_j \mu_j r_j = x$$

$$\sum_i \lambda_i = 1$$

$$\lambda, \mu \geqslant 0$$

Proof. See ADM1. □

add ref HW?

Conclusion 4.7. Depending on the representation we have, we have different ways to solve OPT and SEP:

	Hull repr.	Vertex-repr
OPT	LP Simplex/Ellipsoid	Brute Force
SEP	Brute Force	LP (Homework)

picture example

Example 4.8. Consider the n-cube $C^n := \{x \in \mathbb{R}^n \mid -1 \leqslant x_i \leqslant 1\}$. It has $2n$ facets, but 2^n vertices.

Now, consider the polar of C^n, which can be shown to be the n-octahedron O^n. Remember the intuition, that the polar exchanges vertices with facets. Indeed it holds that now, we have 2^n facets, but only $2n$ vertices.

hyperlink polymake

Information. Polymake is a tool for converting between H-representation and V-representation.

Question 4.9. Consider the problem of finding a solution x to $Ax = b$. How do we construct succinct certificates of feasibility and infeasibility?

Theorem 4.10. Exactly one of the following systems is feasible:

$$Ax = b \qquad \text{vs.} \qquad \begin{aligned} y^T A &= 0 \\ y^T b &= 1 \end{aligned}$$

Proof. Suppose both are feasible. Then we have solutions y^0, x^0, and can construct following contradiction:

$$\begin{aligned} & Ax^0 = b \\ \Leftrightarrow \qquad & \underbrace{(y^0)^T A}_{0} x^0 = (y^0)^T b = 1 \end{aligned}$$

It remains to proove at least one system is feasible. We can use **Gaussian Elimination** for that: Gaussian Elimination either yields a solution x^0 we can use as a succinct certificate for feasibility, or determine it is infeasible by yielding the row multiplier y^0 as a succinct certificate of infeasibility. □

Proof Strategy. The method we used in previous proof is called **Theorem of the Alternative**.

Question 4.11. Now consider the problem of finding a solution x to $Ax \leqslant b$. How do we construct succinct certificates of feasibility and infeasibility?

Theorem 4.12 (Farkas Lemma)**.** Exactly one of the following systems is feasible:

$$Ax \leqslant b \qquad \text{vs.} \qquad \begin{aligned} y^T A &= 0 \\ y^T b &< 0 \\ y &\geqslant 0 \end{aligned}$$

This is also known as **Farkas Lemma**.

Proof. Suppose both are feasible. Analoguous to previous proof we can see the contradiction:

$$\begin{aligned} & Ax^0 \leqslant b \\ \Leftrightarrow \qquad & \underbrace{(y^0)^T A}_{0} x^0 \leqslant (y^0)^T b < 0 \end{aligned}$$

At least one system is feasible, which we can see by using Phase 1 of the Simplex Algorithm, and the Ellipsoid Method, which can generate certificates x or else y. □

Definition 4.13 (Diophantine equations)**.** An equation of the form $Ax = b$, for $x \in \mathbb{Z}^n$, is called **diophantine equation**.

Theorem 4.14. Exactly one of following is feasible:

$$\begin{array}{ccc} Ax \leqslant b & \text{vs.} & y^T A \in \mathbb{Z}^n \\ x \in \mathbb{Z}^n & & y^T b \notin \mathbb{Z} \end{array}$$

Proof. Suppose both are feasible. Then

$$Ax^0 = b$$

$$\Leftrightarrow \qquad \underbrace{(y^0)^T A}_{\mathbb{Z}^n} \underbrace{x^0}_{\mathbb{Z}^n} = \underbrace{(y^0)^T b}_{\notin \mathbb{Z}}$$

We can use the **Hermite Normal Form** algorithm to show that at least one system is feasible. Note that HNF is a polynomial algorithm. ☐

Conclusion 4.15. Summing everything up for feasibility of linear systems:

	continuous	integer
$=$	G.E.	HNF
$\leqslant$	LP	not possible

The problem with integer inequality systems is missing duality, e.g. there is no way of generating succinct certificates for verifying infeasibility, making it impossible to use the Theorem of the Alternative.

Another usage of Theorem of the Alternative:

$$\begin{aligned} Ax &= b & \Leftrightarrow && Ax &\leqslant b \\ x &\geqslant b & && -Ax &\leqslant -b \\ & & && -x &\leqslant 0 \end{aligned}$$

$$\begin{aligned} &\overset{4.12}{\text{vs.}} & (y^1)^T A - (y^2)^T A - (y^3)^T &= 0 \\ & & (y^1)^T b - (y^2)^T b &< 0 \\ & & y^1, y^2, y^3 &\geqslant 0 \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow & y^T A &\geqslant 0 \\ & & y^T b &< 0 \\ & & y &\text{ free} \end{aligned}$$

Theorem 4.16 (Gourdan)**.** Consider $Ax < 0$.

write Gourdan

Consider an LP with lower and upper bounds:

$$\begin{aligned} \min \quad c^T x \\ Ax = b \\ l \leqslant x \leqslant u \end{aligned}$$

Decompose:

$$\begin{aligned} \min \quad c^T x \\ Ax = b \\ x \geqslant l \\ -x \geqslant u \\ x \text{ free} \end{aligned}$$

Dualize:

$$\begin{aligned} \max \quad b^T y + l^T \lambda - u^T \mu \\ y^T A + \lambda^T - \mu^T = c^T \\ \lambda, \mu \geqslant 0 \\ y \text{ free} \end{aligned}$$

Rewrite

Part II

Appendix

A Exercise sheets

1. exercise sheet

Exercise 1.1. Did on paper.

2. exercise sheet

Exercise 2.1. An correct ordering is given by:

$$
\begin{aligned}
\mathrm{O}\left(\varepsilon^{n}\right) &\subseteq \mathrm{O}\left(n^{\varepsilon-1}\right) \subseteq \mathrm{O}\left(n^{-\varepsilon}\right) \subseteq \mathrm{O}\left(\frac{\log n}{n^{\varepsilon}}\right) && (1)\\
&\subseteq \mathrm{O}\left(\frac{1}{\log n}\right) \subseteq \mathrm{O}\left(\frac{\log^2 n}{\log n}\right) \subseteq \mathrm{O}\left(\frac{1}{\log^2 n}\right) && (2)\\
&\subseteq \mathrm{O}\left(e^{\frac{1}{n}}\right) = \mathrm{O}\left(1\right) = \mathrm{O}\left(\left(1-\frac{1}{n}\right)^{n}\right) && (3)\\
&\subseteq \mathrm{O}\left(\log n\right) \subseteq \mathrm{O}\left(\frac{n^{\varepsilon}}{\log n}\right) \subseteq \mathrm{O}\left(n^{\varepsilon}\right) \subseteq \mathrm{O}\left(n^{\varepsilon}\log n\right) \subseteq \mathrm{O}\left(n^{1-\varepsilon}\right) && (4)\\
&\subseteq \mathrm{O}\left(\frac{n}{\log n}\right) \subseteq \mathrm{O}\left(n\log n\right) \subseteq \mathrm{O}\left(n^{2}\right) \subseteq \mathrm{O}\left(n^{2}\log n\right) \subseteq \mathrm{O}\left(n^{e}\right) && (5)\\
&\subseteq \mathrm{O}\left(n^{\log n}\right) \subseteq \mathrm{O}\left(e^{n}\right) \subseteq \mathrm{O}\left((\log n)^{n}\right) \subseteq \mathrm{O}\left(n!\right) && (6)
\end{aligned}
$$

These can mostly achieved by the fact that $n^x \in \mathrm{O}(n^y)$ if $x \leqslant y$, and $(\log n) \cdot n^x \in \mathrm{O}(n^y)$ if $y > x$, otherwise the other way around. Additionally, it is often useful to consider the logarithm of the functions we compare, because it maintains monotonocity.

Exercise 2.2. Analoguous to the lecture we can introduce constraints, such that $y_{ij} = x_i \wedge x_j$:

$$
\begin{aligned}
y_{ij} &\leqslant x_i\\
y_{ij} &\leqslant x_j\\
y_{ij} &\geqslant x_i + x_j - 1\\
y_{ij} &\in [0,1]
\end{aligned}
$$

Exercise 2.3. We can show that $f(x_1) = \max(c_1 x_1, c_1 p + c_2 x_1 - c_2 p)$ using a case distinction.

- $x_1 = p$: Trivial.
- $x_1 > p$: Consider $c_1 < c_2$. Multiplying by $x_1 - p$ (which is positive) and rearranging yields $c_1 x_1 < c_1 p + c_2 x_1 - c_2 p$.
- $x_1 < p$: Analoguous, but now $x_1 - p$ is negative, which reverses the inequality.

As shown in ADM1, the maximum of linear functions can be written as an LP by introducing a helper variable as follows:

$$\begin{aligned}
\min \quad & z + \sum_{i=2}^{n} c_i x_i \\
\text{s.t.} \quad & Ax = b \\
& l \leqslant x_1 \leqslant u \\
& x_2, ..., x_n \leqslant 0 \\
& z \geqslant c_1 x_1 \\
& z \geqslant c_1 p + c_2 x_1 - c_2 p
\end{aligned}$$

3. exercise sheet

Exercise 3.1.

1. We can show easily that SPATH $\in$ **P** by using the fact from ADM1, that breadth-first search started from s finds a shortest path to t in polynomial time. Therefore, if the shortest path has length $k^* \leqslant k$, we can return true, and false otherwise.
2. We first show LPATH $\in$ **NP**: Suppose an instance of LPATH is true, then there is a path of at least length k. Therefore, we can simply use this path as a succinct certificate and verify in polynomial time that the path is indeed valid.

   It remains to show that we can reduce a **NP**-complete problem to LPATH. It suffices to show UHAMPATH $\propto$ LPATH: Suppose we have an instance $((V, E), s, t)$ of UHAMPATH. We can simply reduce it to the problem of finding a path of at least length $|V| - 1$ starting in s and ending in t, because every such path is indeed a hamiltonian path, because every vertex needs to be visited exactly once. Therefore, if there is a hamiltonian path, it is already a path of at least length $|V| - 1$. For the other direction, if there is a path of at least length $|V| - 1$, then it must visit every node exactly once in order to be a valid path.
   This shows that the reduction is Yes-preserving.

Exercise 3.2. If DOUBLESAT is true, then we can choose any two valid assignments as a succinct certificate and easily verify their correctness in polynomial time.

It remains to show $\mathsf{SAT} \propto \mathsf{DOUBLESAT}$: Starting from our SAT-instance, we can simply introduce two new variables a, b and a new clause $a \vee b$. This construction is Yes-preserving, because if the original instance is infeasible, the new instance still has no assignments. On the other hand, if there is a valid assignment in the original, then we now have at least 3 valid instances for different assignments of a and b.

Exercise 3.3. We notice that $a \vee b$ is equivalent to $\neg a \implies b$, and $\neg b \implies a$. By doing this for all clauses, we can construct a graph with the literals as vertices, and the implications as directed edges. Now, checking for each literal pair $l, \neg l$ if both can reach one another by a directed path suffices to show feasibility:

If previous condition holds true, then by logic it must hold that a feasible assignment satisfies $l \Leftrightarrow \neg l$, which is impossible. On the other hand, if this is never the case, then there must be a feasible solution:

We can construct this solution by iteratively setting either l or $\neg l$ to true, depending if $l \implies \neg l$ holds, and then also set every implied variable to true. If we would encounter a variable r which is already false, then $\neg r$ must be true, and therefore all further implications would need to be true by construction. Because $l \implies r$, also $\neg r \implies \neg l$, meaning that l would be already false - contradiction!
Therefore, our construction always works.

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Literature

[Ber18] Jens Vygen Bernhard Korte. *Kombinatorische Optimierung*. 3rd ed. Springer, 2018. ISBN: 978-3-662-57690-8. URL: https://link.springer.com/book/10.1007/978-3-662-57691-5.

[Geo99] Laurence Wolsey George Nemhauser. *Integer and Combinatorial Optimization*. Wiley, 1999. ISBN: 978-0-471-82819-8. URL: https://onlinelibrary.wiley.com/doi/book/10.1002/9781118627372.