

# Probability and Computing

Lecturers

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Notes

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## Abstract

The following lecture notes are my personal (and therefore unofficial) write-up for 'Probability and Computing' aka 'ProbComp', which took place in summer semester 2024 at Hasso-Plattner-Institut. I do not guarantee correctness, completeness, or anything else. Importantly, note that I willfully changed some specific notations, reordered some material, and left out parts that I didn't find worth typing down.

If you miss something, feel free to contribute in the repository!

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## Summary of lectures

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## Part I

## Lecture notes

Lecture 1  
Mo 15 Apr 2024

## 1 Probabilistic method

This section will introduce how probability can be used to solve problems that at first glance do not have anything to do with probability. Consider following combinatorial problem.

**Definition 1.1** (Ramsay numbers). We define  $R_k$  as the smallest integer  $n$  such that any graph with  $n$  vertices must contain either a clique or an independent set of size  $k$ . We call  $R_k$  the (symmetric) **Ramsey numbers**.

Let us first look at some examples to get a feel for what this section is about.

**Example 1.2.** The first few Ramsay numbers are given as

- $R_2 = 2$ : Obviously,  $R_k \geq k$ . Furthermore, there are only two different graphs with two nodes, i.e. with or without an edge. In the former case, both nodes form a clique, in the latter they form an independent set of size 2.
- $R_3 = 6$ : First, let us show  $R_3 \geq 6$ . Consider a cycle of 5 nodes. Then, there is no clique of size 3 (since there is no 3-cycle). Also, among any three nodes two nodes are connected by an edge, so there is no independent set. Therefore, this is a counterexample. Now, consider a graph with 6 nodes. Suppose there is no clique of size 3. Then it suffices to show that there is an independent set of 3 nodes. Indeed, with an ugly case distinction this is possible: If there are no cycles, the graph is bipartite, so there is an independent set of at least 3. Otherwise, there exists at least a 4-cycle, but no 3-cycle (i.e. chord-free). We can then select at least two independent nodes from the cycle, and if needed the missing third node from the non-cycle nodes such that they form an independent set.
- $R_4 = 18$ . Trust me, we do not want the proof here.
- $R_5 \in [43, 48]$ . The exact value is indeed still unknown!

As we can see, even for small  $k$ , it is not trivial to determine their Ramsay number. Instead, let us try to at least find some bound for their value.

**Theorem 1.3.** For every  $k \geq 1$  holds  $R_k > 2^{k/2}$ .

*Proof.* Consider a uniform distribution over all graphs with  $n$  vertices (i.e. the Erdős–Rényi random graph  $\mathcal{G}(n, \frac{1}{2})$ ). Each edge in particular exists with probability  $\frac{1}{2}$ . Now, have

a look at  $p := P(G \sim \mathcal{G}(n, \frac{1}{2}) \text{ has a } k\text{-clique or } k \text{ independent set})$ . If we can show that this probability  $p$  is less than 1, then this means there is a graph with  $n$  vertices such that the property is *not* satisfied, and therefore  $R_k > n$ .

Let  $S$  be a  $k$ -tuple of vertices.  $S$  is per definition a  $k$ -clique if *all* or *none* of its edges is existent. Therefore, its probability of being either one is given as

$$P(S \text{ is } k\text{-clique or } k\text{-independent set}) = 2 \cdot \frac{1}{2^{\binom{k}{2}}}. \quad (1)$$

The total number of  $k$ -subsets given  $n$  vertices is given as  $\binom{n}{k}$ , so by basic properties of probability and binomial coefficients we see

$$p \leq \binom{n}{k} \cdot \frac{1}{2^{\binom{k}{2}}} \leq \frac{n^k}{k!} 2^{1 - \frac{k^2 - k}{2}} \quad (2)$$

For  $n = 2^{k/2}$  the right-hand side reduces to  $\frac{2^{k+2}}{k!}$ , which can be shown easily to be smaller than 1 for  $k \geq 3$ .  $\square$

Notice how we suddenly imposed a probabilistic view on this presumably deterministic problem! This technique of using a suitable random model to demonstrate the existence and/or non-existence of certain properties is known as **Probabilistic Method**.

Before we have a look at another example, we need following lemma.

**Lemma 1.4.** Let  $X$  be a discrete random variable over a set  $\Omega$  with  $\mathbb{E}[X] = \mu$ . Then,  $P(X \geq \mu) > 0$  and  $P(X \leq \mu) > 0$ .

*Proof.* Assume  $P(X \geq \mu) = 0$ . Then  $P(X = x) = 0$  for  $x \geq \mu$ . By definition and assumption therefore

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} xP(X = x) = \sum_{x < \mu} xP(X = x) \quad (3)$$

$$< \sum_{x < X(\Omega)} \mu P(X = x) = \mu \sum_{x \in X(\Omega)} P(X = x) = \mu. \quad (4)$$

This is a contradiction!  $\square$

Consider following problem.

**Definition 1.5** (Max-Cut). Given a graph  $G = (V, E)$ , find a set  $A$  maximizing the number of edges between  $A$  and  $V \setminus A$ . We call this the **Maximum Cut Problem**.

As it is usual in lectures of this kind, its canonical decision variant is indeed a NP-complete problem. Again, let us try instead to find a "good" cut.

**Theorem 1.6** (Minimal Max-Cut). Given a graph  $G = (V, E)$  with  $|E| = m$ . There exists  $A \subseteq V$  with at least  $\frac{m}{2}$  cut size.

*Proof.* Choose  $A$  uniformly over  $\mathcal{P}(V)$ , i.e. every node is chosen with probability  $\frac{1}{2}$ . Then, every edge is with probability  $\frac{1}{2}$  included in the cut, which happens iff exactly one of the nodes of the edge is in  $A$ . Let  $X$  be the number of cut edges, and  $X_e$  be the indicator variable for  $e$  being in the cut. By linearity of expectancy,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{e \in E} X_e\right] = \sum_{e \in E} \mathbb{E}[X_e] = m \cdot \frac{1}{2}. \quad (5)$$

Using [Lemma 1.4](#) there is positive probability for a choice of  $A$  having cut size at least  $m/2$ .  $\square$

You might have noticed that these are non-constructive results, so naturally following question emerges: Can we use these results to construct a solution? Indeed, there is a simple answer!

**Definition 1.7** (Las-Vegas Algorithm). Let  $T$  be the run-time of an algorithm and  $n$  its input size. We call a **Las-Vegas Algorithm** an algorithm which

1. always returns the correct answer with  $\mathbb{E}[T] \in \mathcal{O}(n^k)$  for some  $k$ , or
2. has runtime always in  $\mathcal{O}(n^k)$  for some  $k$ , and returns a correct answer with probability at least  $\delta > 0$  (and otherwise returns no answer).

**Remark 1.8.** Both definitions are equivalent, which stems from the fact the second variant can be seen as a geometric process: Consider the number  $T$  of attempts until the algorithm returns a correct result. Then,  $\mathbb{E}[T] = \frac{1}{1-\delta}$ , so the total runtime is still  $\mathcal{O}(n^k)$  in expectancy.

Turning back to the Max-Cut problem, let us transform our result of [Theorem 1.6](#) into a Las-Vegas algorithm. Simply generate  $A$  randomly as constructed in the proof, and check if it has enough cut edges in polynomial time. The probability that this works is  $P(X_A \geq \frac{m}{2}) = p$ . However, this is just an abstract value - let us try to find a more meaningful way of expressing  $p$ :

$$\frac{m}{2} = \mathbb{E}[X_A] = \sum_{i < \frac{m}{2}} i \cdot P(X_A = i) + \sum_{i \geq \frac{m}{2}} i \cdot P(X_A = i) \quad (6)$$

$$\leq \left(\frac{m}{2} - 1\right) \sum_{i < \frac{m}{2}} P(X_A = i) + m \sum_{i \geq \frac{m}{2}} P(X_A = i) = \left(\frac{m}{2} - 1\right) (1 - p) + mp \quad (7)$$

Here we just upper-bound the corresponding first factors in each sum, and then use  $P(X_A \geq \frac{m}{2}) = p$ . Using some easy algebra, we deduce  $p \geq \frac{1}{\frac{m}{2} + 1}$ , so our Las-Vegas approach seems to get gradually worse the more edges our graph has.

Interestingly, we do not even need a randomized algorithm to find a big cut. Instead, using probabilistic arguments we can construct a deterministic algorithm still running in polynomial time. This technique is known as **Derandomization**.

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**Algorithm 1:** Find Big-Cut of  $G = (V, E)$

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 $A \leftarrow \emptyset, B \leftarrow \emptyset$ 
for  $k = 1, \dots, n$  do
    if  $|\{(v_k, u) \in E \mid u \in A\}| \leq |\{(v_k, u) \in E \mid u \in B\}|$  then
         $A \leftarrow A + v_k$ 
    end
    else
         $B \leftarrow B + v_k$ 
    end
end
return  $A$ 

```

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The idea is to greedily decide for each vertex if we want it in our cut set or not based on a case distinction using conditional expectation.

**Theorem 1.9.** Given a graph  $G = (V, E)$ . Let  $A \sim \mathcal{U}_{\mathcal{P}(V)}$ ,  $C_A$  the amount of cuts, and  $x_1, \dots, x_n$  indicate if the vertices  $v_1, \dots, v_n \in A$ . Then **Algorithm 1** satisfies following statements:

1.  $\mathbb{E}[C_A \mid x_1, \dots, x_k] \leq \mathbb{E}[C_A \mid x_1, \dots, x_k, x_{k+1}]$  after iteration  $k + 1$  of the for-loop (such that  $x_i$  is fixed by the algorithm).
2. The algorithm runs in  $\mathcal{O}(n + m)$  and returns a big cut of size at least  $m/2$ .

*Proof.* Notice for  $1 \leq k < n$  by definition of conditional expectancy

$$\mathbb{E}[C_A \mid x_1, \dots, x_k] = \frac{1}{2} \mathbb{E}[C_A \mid x_1, \dots, x_k, x_{k+1} = 1] + \frac{1}{2} \mathbb{E}[C_A \mid x_1, \dots, x_k, x_{k+1} = 0].$$

So, at least one of the choices for  $x_{k+1}$  satisfy the required lower bound for the conditional expectancy. It remains to prove that our algorithm also chooses this value, i.e. the choice with larger conditional expectancy. Let us observe how the expected number of cut edges can change if we fix the position of vertex  $v_{k+1}$ . Consider following cases for an edge  $e \in E$ :

- $e$  does not contain  $v_{k+1}$ . Then, by independence, the expected value of its Is-Cut-Edge indicator  $X_e$  conditioned on all fixed vertices does not change by fixing  $v_{k+1}$  (i.e 1 or 0 if both vertices are determined, else  $\frac{1}{2}$ ).
- $e$  contains  $v_{k+1}$ , but the other vertex is not fixed. Again, aforementioned expected value does not change since there is still a  $\frac{1}{2}$  probability for the other vertex being in  $A$ .
- $e$  contains  $v_{k+1}$ , but the other vertex *is* fixed. Then we suddenly fix the value of  $X_e$  depending on the choice of  $x_{k+1}$ . In particular, this changes the conditional expectancy of  $X_e$ , increasing it to 1 or decreasing it to 0, and therefore the conditional expectancy on the number of cut edges changes in the same way.

In summary, the change in conditional expectancy by fixing  $x_{k+1}$  only depends on the neighborhood of fixed vertices of  $v_{k+1}$ . If there are more neighbors that are fixed to be  $A$  than not in  $A$ , then

$$\mathbb{E}[C_A \mid x_1, \dots, x_k, x_{k+1} = 0] \geq \mathbb{E}[C_A \mid x_1, \dots, x_k, x_{k+1} = 1],$$

otherwise the other way around, adhering to our algorithm design.

In fact, by induction it immediately follows that  $\frac{m}{2} \leq \mathbb{E}[X_A] \leq \mathbb{E}[X_A \mid x_1, \dots, x_n]$  for  $x_1, \dots, x_n$  being chosen according to the algorithm. So, our algorithm works, and has mentioned runtime.  $\square$

Lecture 2  
Mi 17 Apr 2024

**Definition 1.10** (SAT problem). Given a boolean formula in  $k$ -CNF (conjunctive normal form). The decision problem if there is an assignment that the formula is true is called  $k$ -SAT.

Assume we have a  $k$ -SAT instance  $\varphi$ , and assume each variable appears in exactly one clause. Then it is possible to show  $\varphi \in \text{SAT}$ .

**Example 1.11.** For 3-SAT consider  $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_4 \vee \neg x_5 \vee x_6)$ . Then, a valid assignment is  $x_1 = x_2 = x_3 = 1, x_4 = x_5 = x_6 = 0$ .

It is rather intuitive this statement must hold, since we enforce some form of indepenence. This makes it interesting to look at from the lense of the probability, and its notion of indepenence.

So, let us assume a uniform random assignment  $\alpha$  for such a  $\varphi$  (with  $n$  clauses). Let  $E_i$  be the event that clause  $i$  is not satisfied by  $\alpha$ . Then,  $\bigcup E_i$  is the event that  $\varphi$  is not satisfied by  $\alpha$ , and thus  $\bigcap \overline{E_i}$  denotes  $\varphi$  being satisfied by  $\alpha$ . Similarly to previous applications of the probabilistic method we see that

$$P\left(\bigcap_{i=1}^n \overline{E_i}\right) = \prod_{i=1}^n P(\overline{E_i}) = \prod_{i=1}^n (1 - 2^{-k}) > 0 \quad (8)$$

shows that there *are* assignments that satisfy  $\alpha$ . However, we were now able to use indepenence of  $E_i$  in this case.

While this example does not seem too spectacular at first, we can now try to leviate our conditions and only assume limited indepenence. Let us introduce a new construct.

**Definition 1.12** (Dependency graph). Consider a set of events  $E_1, \dots, E_n$ . We call  $G = (V, E)$  with  $V = \{1, \dots, n\}$  the **Dependency Graph** if  $E$  satisfies that every  $E_i$  is mutually independent of the set  $\{E_j \mid (i, j) \neq E_i\}$ .



**Lemma 1.13.** If events  $E_1, \dots, E_n$  are mutually independent, then its counterparts  $\overline{E}_1, \dots, \overline{E}_n$  are mutually independent.

Using these notions, we can now introduce a rather strong tool!

**Theorem 1.14** (Symmetric **Lovász Local Lemma**). Let  $E_1, \dots, E_n$  be a set of events and assume

1.  $P(E_i) \leq p$  for some fixed  $p$ ,
2. the maximum degree of the dependency graph of these events is  $d$ , and
3.  $4dp \leq 1$ .

Then  $P(\bigcap_{i=1}^n \overline{E}_i) > 0$ .

The proof is rather technical, but the main idea is to use two nested inductions, and rewrite our result as a fancy product using conditional probabilities.

*Proof.* Let  $S \subseteq \{1, \dots, n\}$ . We will show by induction on the size  $s$  of  $S$  that for all  $k \notin S$  it holds

$$P(E_k \mid \bigcup_{j \in S} \overline{E}_j) \leq 2p. \quad (9)$$

For the base case  $s = 0$  by assumption  $P(E_k) \leq p \leq 2p$ .

For the induction step we first need to show  $P(\bigcap_{j \in S} \overline{E}_j) > 0$ . Again, we use another induction. For  $s = 1$  this is clear by assumption. (Notice  $p < 1$ ). Now, w.l.o.g. let  $S = \{1, \dots, s\}$ . By definition of conditional expectancy,

$$\begin{aligned} P\left(\bigcap_{j \in S} \overline{E}_j\right) &= P(\overline{E}_s \mid \bigcap_{j=1}^{s-1} \overline{E}_j) \cdot P\left(\bigcap_{i=1}^{s-1} \overline{E}_i\right) \\ &= \underbrace{\left(1 - P(E_s \mid \bigcap_{j=1}^{s-1} \overline{E}_j)\right)}_{>0 \text{ by outer induction}} \cdot \underbrace{P\left(\bigcap_{i=1}^{s-1} \overline{E}_i\right)}_{>0 \text{ by inner induction}} > 0 \end{aligned}$$

This concludes the inner induction, so let us continue with the outer induction step. Let us split  $S$  into  $S_1 = \{j \in S \mid (k, j) \in E\}$ ,  $S_2 = S \setminus S_1$ . Denote with  $F_k = \bigcup_{i=1}^k \overline{E}_i$ . Then

$$P(E_k \mid F_s) = \frac{P(E_k \cap F_s)}{P(F_s)} = \frac{P(E_k \cap F_{S_1} \cap F_{S_2})}{P(F_{S_1} \cap F_{S_2})} = \frac{P(E_k \cap F_{S_1} \mid F_{S_2})}{P(F_{S_1} \mid F_{S_2})}$$

Consider both parts of the final fraction, and let us bound them:

$$\begin{aligned}
 P(E_k \cap F_{S_1} \mid F_{S_2}) &\leq P(E_k \mid F_{S_2}) \leq p \\
 P(F_{S_1} \mid F_{S_2}) &= 1 - P(\overline{F_{S_1}} \mid F_{S_2}) = 1 - P\left(\bigcup_{i \in S_1} E_i \mid F_{S_2}\right) \\
 &\geq 1 - \sum_{i \in S_1} P(E_i \mid \bigcap_{j \in S_2} \overline{E_j}) \geq 1 - |S_1| \cdot 2p \geq 1 - 2pd \geq \frac{1}{2}
 \end{aligned}$$

Applying these results yields  $2p$  as an upper bound for  $P(E_k \mid F_s)$ , which also concludes the outer induction.

Continuing with the actual statement, we get

$$P\left(\bigcap_{i=1}^n \overline{E_i}\right) = P(\overline{E_n} \mid \bigcap_{i=1}^{n-1} \overline{E_i}) \cdot P\left(\bigcap_{i=1}^{n-1} \overline{E_i}\right) = \prod_{i=1}^n P(\overline{E_i} \mid \bigcap_{j=1}^{i-1} \overline{E_j}) \geq (1 - 2p)^n > 0.$$

□

Turning back to our  $k$ -SAT problem, we are now able to show a stronger version.

**Theorem 1.15.** Given a  $k$ -SAT instance  $\varphi$  (with  $n$  clauses), and assume that no variable appears in more than  $\frac{2^k}{4k}$  clauses. Then,  $\varphi \in \text{SAT}$ .

*Proof.* We motivate Theorem 1.14. Firstly, we see  $P(E_i) \leq 2^{-k}$ . For each vertex of the dependency graph its degree is at most  $\frac{2^k}{4k} \cdot k$ . Therefore,  $4 \cdot 2^{-k} \cdot \frac{2^k}{4k} k \leq 1$ , and  $P(\bigcap_{i=1}^n \overline{E_i}) > 0$  implies the existence of a valid assignment. □

Some more applications.

**Theorem 1.16.** Assume  $n$  pairs of vertices need to be connected using  $n$  disjoint paths on a given network  $E$ . Each pair  $i$  can choose from a collection  $F_i$  of  $m$  paths. If any path in  $F_i$  shares edges by at most  $k$  paths in  $F_j$  and  $\frac{8nk}{m} \leq 1$ , then we can always choose an edge-disjoint collection of paths.

*Proof.* Consider a probability space where every pair  $i$  chooses a path in  $F_i$  uniformly distributed, i.e. with probability  $1/m$ . We define  $E_{i,j}$  as the "bad" event that paths  $i, j$  share any edges, which occurs with probability  $k/m$ . The degree of the corresponding dependency graph then is  $2n$ . By Theorem 1.14, we are done.

□ Lecture 3  
Mo 22 Apr 2024

## 2 Random graphs

In this section we want to try answer following question.

**Question 2.1.** What does the "average" graph look like?

We can introduce two notions for distributions over graphs.

**Definition 2.2.** 1. The  $\mathcal{G}_{n,m}$  model is a probability distribution over  $\mathcal{G}$  given by a uniform distribution over all graphs with  $n$  nodes and  $m$  edges.  
2. The  $\mathcal{G}_{n,p}$  model is a probability distribution over  $\mathcal{G}$  for graphs with  $n$  nodes such that the existence of every edge is drawn with probability  $p$ .

However, for our goals it is easier to work with the second notion. If  $G \sim \mathcal{G}_{n,p}$ , then  $\mathbb{E}[|E(G)|] = \binom{n}{2}p$ .

**Lemma 2.3.** For all  $G \sim \mathcal{G}_{n,p}, G' \sim \mathcal{G}_{n,m}$  it holds that for any graph  $H$

$$P(H = G \mid |E(H)| = m) = P(H = G') \quad (10)$$

*Proof.* Using some simple transformations and thinking about the probabilities of our graphs we see

$$\begin{aligned} P(H = G \mid |E(H)| = m) &= \frac{P(H = G \cap |E(H)| = m)}{P(|E(H)| = m)} = \frac{P(H = G)}{P(|E(H)| = m)} \\ &= \frac{p^m (1-p)^{\binom{n}{2}-m}}{\binom{\binom{n}{2}}{m}} p^m (1-p)^{\binom{n}{2}-m} \\ &= \frac{1}{\binom{\binom{n}{2}}{m}} = P(H = G') \end{aligned}$$

□

This gives us the ability to try prove well-known graph problems on random graphs. For example, we can ask ourselves if  $G \sim \mathcal{G}_{n,p}$  contains  $K_4$ . Consider  $C \subseteq V(G)$  such that  $|C| = 4$ , and let  $X_C$  be the indicator variable if  $G[C] = K_4$ . Furthermore, let  $X$  be the number of 4-cliques in  $G$ .

We easily see that  $P(X_C = 1) = p^6 = \mathbb{E}[X_C]$ , and  $\mathbb{E}[X] = \binom{n}{4}p^6 \in \theta(n^4 p^6)$ . Therefore

- $p \ll n^{-2/3}$  implies  $\mathbb{E}[X] \rightarrow_{n \rightarrow \infty} 0$ , and
- $p \gg n^{-2/3}$  implies  $\mathbb{E}[X] \rightarrow_{n \rightarrow \infty} \infty$ .

What happens though in the case of equality, i.e  $p(n) = n^{-2/3}$ ?

**Definition 2.4.** We call  $f(n)$  a **Threshold** for a property  $Q$  in  $\mathcal{G}_{n,p}$  if

- $p \gg f(n)$  implies  $P(G \sim \mathcal{G}_{n,p} \text{ has } Q) \rightarrow_{n \rightarrow \infty} 1$ ,
- $p \ll f(n)$  implies  $P(G \sim \mathcal{G}_{n,p} \text{ has } Q) \rightarrow_{n \rightarrow \infty} 0$ .

Let us show  $p(n) = n^{-2/3}$  is a threshold for the existence of a 4-clique.

1. Case  $p \ll n^{-2/3}$ : Then using Markov's inequality we immediately see

$$P(X > 0) = P(X \geq 1) \leq \frac{\mathbb{E}[X]}{1} \rightarrow_{n \rightarrow \infty} 0. \quad (11)$$

2. Case  $p \gg n^{-2/3}$ : Then using Tschebychev's inequality we see

$$P(X = 0) \leq P(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \frac{\mathbb{V}[X]}{\mathbb{E}[X]^2}. \quad (12)$$

Having a closer at the variance by smart reordering, we conclude

$$\begin{aligned} \mathbb{V}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}\left[\left(\sum_C X_C\right)^2\right] - \mathbb{E}\left[\sum_C X_C\right]^2 \\ &= \sum_C \left(\mathbb{E}[X_C^2] - \mathbb{E}[X_C]^2\right) + \sum_{C \neq D} (\mathbb{E}[X_C X_D] - \mathbb{E}[X_C] \mathbb{E}[X_D]) \\ &= \sum_C \mathbb{V}[X_C] + \sum_{C \neq D} \mathbf{Cov}[X_C, X_D] \end{aligned}$$

It suffices to show that both sums independently tend to 0 for  $n \rightarrow \infty$  if divided by  $\mathbb{E}[X]^2$  as seen in (12).

Notice  $\mathbb{V}[X_C] = \mathbb{E}[X_C^2] - \mathbb{E}[X_C]^2 \leq \mathbb{E}[X_C^2] = p^6$ , so taken over all  $\binom{n}{4}$  instances of  $C$  the first sum has its upper bound in  $\Theta(n^4 p^6) \subseteq \Theta(n^8)$ . Since  $\mathbb{E}[X]^2 = p^{12} \gg n^{-8}$ , the first sum converges indeed to 0 for  $n \rightarrow \infty$ .

For the second sum, we need a case distinction over the overlap of  $C$  and  $D$ . Notice that we only need to consider  $\mathbb{E}[X_C X_D] \geq \mathbf{Cov}[X_C, X_D]$ .

- $|C \cap D| \leq 1$ : Then  $\mathbf{Cov}[X_C, X_D] = 0$ .
- $|C \cap D| = 2$ : Then  $\mathbb{E}[X_C X_D] = P(X_C X_D = 1) = p^{11}$ . This happens  $\binom{n}{6} \binom{6}{4} \binom{4}{2} \in \Theta(n^6)$ -times.
- $|C \cap D| = 3$ : Then  $\mathbb{E}[X_C X_D] = p^9$ . This happens  $\binom{n}{5} \binom{5}{4} \binom{4}{3} \in \Theta(n^5)$ -times.

Analogously, this concludes the convergence.

Another interesting property is the largest connected component of a random graph.

**Theorem 2.5.** For  $G \sim \mathcal{G}_{n,p}$  it holds that  $f(n) := \frac{1}{n}$  is a threshold for  $G$  having a connected component of size  $\Theta(n)$ . Furthermore, for  $p = \frac{c}{n}$ , it holds that

$$\text{Largest connected component is } \begin{cases} \Theta(\log n), & c < 1 \\ \Theta(n^{\frac{2}{3}}), & c = 1 \\ \Theta(n), & c > 1 \end{cases}$$

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**Theorem 2.6.** In almost every  $G \sim \mathcal{G}_{n, \frac{1}{2}}$ , the largest clique has size approximately  $2 \log_2(n)$ .

*Proof sketch.* Let  $X_k$  be the number of  $k$ -cliques in  $G \sim \mathcal{G}_{n, \frac{1}{2}}$ . As previously shown for 4-cliques, we easily generalize

$$g(k) := \mathbb{E}[X_k] = \binom{n}{k} 2^{-\binom{k}{2}}.$$

Let  $K_0(n)$  be the largest  $k$  such that  $g(k) \geq 1$ . If we show  $K_0(n) \approx 2 \log n$ , then  $g(k) \approx 2^{k \log n - k^2/2}$ . Furthermore, let  $c$  be a constant (to be determined later), and

$$K_1(n) := K_0(n) - c, \quad K_2(n) := K_0(n) + c. \quad (13)$$

We will show that

$$P(X_{K_1(n)} > 0) \xrightarrow{n \rightarrow \infty} 1, \quad (14)$$

$$P(X_{K_2(n)} > 0) \xrightarrow{n \rightarrow \infty} 0. \quad (15)$$

One can show (we just believe it) that

$$\mathbb{E}[X_{K_1}] \xrightarrow{n \rightarrow \infty} \infty, \quad (16)$$

$$\mathbb{E}[X_{K_2}] \xrightarrow{n \rightarrow \infty} 0. \quad (17)$$

Apparently following part gives us some intuition for that?

$$\frac{g(k+1)}{g(k)} = \frac{\binom{n}{k+1} 2^{-\binom{k+1}{2}}}{\binom{n}{k} 2^{-\binom{k}{2}}} = \frac{n-k}{k+1} \cdot 2^{-k}, \quad (18)$$

so for  $k \approx 2 \log n$  this is approximately

$$\frac{g(k+1)}{g(k)} \approx \frac{n}{2 \log n} n^{-2} \xrightarrow{n \rightarrow \infty} 0. \quad (19)$$

Using Markov's inequality (**First Moment Method**) we conclude

$$P(X_{K_2(n)} \geq 1) \leq \frac{\mathbb{E}[X_{K_2}]}{1} \xrightarrow{n \rightarrow \infty} 0. \quad (20)$$

which shows (15). Using Tschebychev's inequality ([Second Moment Method](#)) we can show

$$P(X_{K_1(n)} = 0) \leq P(|X_{K_1} - \mathbb{E}[X_{K_1}]| \geq \mathbb{E}[X_{K_1}]) \leq \frac{\mathbb{V}[X_{K_1}]}{\mathbb{E}[X_{K_1}]^2} \xrightarrow{n \rightarrow \infty} 1. \quad (21)$$

To show the limit actually holds, we use the same decomposition trick as in :

ref

$$\mathbb{V}[X_{K_1}] = \mathbb{V}\left[\sum_S X_S\right] = \sum_S \mathbb{V}[X_S] + \sum_{S \neq D} \mathbf{Cov}[X_S, X_D] \quad (22)$$

Let us define  $S \models D$  if  $|S \cap D| \geq 2$ . We simplify further

$$\begin{aligned} \sum_S \mathbb{V}[X_S] + \sum_{S \neq D} \mathbf{Cov}[X_S, X_D] &= \sum_S \mathbb{V}[X_S] + \sum_{S \models D} \mathbf{Cov}[X_S, X_D] \\ &\leq \sum_S \mathbb{E}[X_S^2] + \sum_{S \models D} \mathbb{E}[X_S X_D] \\ &= \sum_S \mathbb{E}[X_S] + \sum_{S \models D} \mathbb{E}[X_S X_D] \\ &= \mathbb{E}[X_{K_1}] + \sum_{S \models D} \mathbb{E}[X_S X_D]. \end{aligned}$$

Notice that we divide this term by  $\mathbb{E}[X_{K_1}]^2$  in (21), so the first summand of previous result reduces to  $1/\mathbb{E}[X_{K_1}]$  which tends to 0 by (16).

For the second sum, using even more reordering and conditional probabilities

$$\begin{aligned} \sum_{S \models D} \mathbb{E}[X_S X_D] &= \sum_{S \models D} P(X_S = 1 \cap X_D = 1) = \sum_{S \models D} P(X_S = 1 \mid X_D = 1) P(X_D = 1) \\ &= \sum_D P(X_D = 1) \left( \sum_{S: S \models D} P(X_S = 1 \mid X_D = 1) \right) \\ &= \sum_D P(X_D = 1) \left( \sum_{S: S \models D_0} P(X_S = 1 \mid X_{D_0} = 1) \right) \text{ for fixed } D_0 \\ &= \sum_{S: S \models D_0} P(X_S = 1 \mid X_{D_0} = 1) \cdot \sum_D P(X_D = 1) \\ &= \sum_{S: S \models D_0} P(X_S = 1 \mid X_{D_0} = 1) \cdot \mathbb{E}[X_{K_1}]. \end{aligned}$$

The factor vanishes after dividing by  $\mathbb{E}[X_{K_1}]$ , so we only remain with one last sum.

$$\sum_{S: S \models D_0} P(X_S = 1 \mid X_{D_0} = 1) = \sum_{i=2}^{K_1-1} 2^{-\binom{K_1}{2} + \binom{i}{2}} \binom{n-K_1}{K_1-i} \binom{K_1}{i}$$

Dividing by the second  $\mathbb{E}[X_{K_1}]$  using some binomial magic we conclude

$$\frac{\sum_{S: S \models D_0} P(X_S = 1 \mid X_{D_0} = 1)}{\mathbb{E}[X_{K_1}]} = \sum_{i=2}^{K_1-1} \underbrace{2^{\binom{i}{2}} \frac{\binom{K_1}{i} \binom{n-K_1}{K_1-i}}{\binom{n}{K_1}}}_{:=f(i)} \leq K_1 \max_{2 \leq i \leq K_1-1} f(i) \quad (23)$$

For our previously introduced  $c$ , if chosen large enough, then the maximum is reached for  $i = 2$ , thus

$$f(2) = \frac{K_1!}{(K_1-2)!} \cdot \frac{K_1!}{(K_1-2)!} \cdot \frac{(n-K_1)!(n-K_1)!}{n!(n-2K_1-2)!} \approx \frac{K_1^2 K_1^2}{n^2}$$

and we upper bound (23) by

$$\frac{K_1^5}{n^2} \approx \frac{\log n^5}{n^2} \xrightarrow{n \rightarrow \infty} 0$$

showing (14) and concluding this wonderful proof.  $\square$

Let us find a notion for randomizing  $k$ -SAT.

**Conjecture 2.7.** For all  $k$  there exists a threshold value  $r_k^* \in \mathbb{R}$  such that

$$P(\varphi_k(n, m) \in \text{SAT}) \xrightarrow{n \rightarrow \infty} \begin{cases} 0, & r > r_k^* \\ 1, & r < r_k^* \end{cases}. \quad (24)$$

**Note.** We know that  $3.52 \leq r_3^* \leq 4.51$ .

Lecture 5  
Mo 29 Apr 2024

### 3 Martingales and Concentration

In this section we study "bounds" of random variables.

**Definition 3.1 (Bounds).** We introduce several notions.

- A **Tail Bound** is an upper bound for  $P(X \geq a)$  given that  $X$  is non-negative.
- A **Deviation Inequality** is an upper bound on  $P(|X - a| \geq \varepsilon)$  where  $\varepsilon$  usually denotes  $\mathbb{E}[X]$ .

Let us introduce an important inequality.

**Theorem 3.2 (Azuma's Inequality).** Let  $X$  be a martingale with  $X_0 = 0$ . Suppose that  $|X_{n+1} - X_n| \leq 1$  for all  $n$ . Then, for any  $\lambda > 0$ , it holds that

$$P(X_n \geq \lambda) \leq \exp \frac{-\lambda^2}{2n}, \quad (25)$$

and equivalently

$$P(X_n \geq \lambda\sqrt{n}) \leq \exp \frac{-\lambda^2}{2}, \quad (26)$$

*Proof sketch.* By properties of a Martingale,

$$P(X_n \geq \lambda) \leq P(\exp X_n t \geq \exp t\lambda) \leq \mathbb{E}[\exp tX_n] \exp -t\lambda. \quad (27)$$

Consider the telescopic augmentation  $X_n = \sum_{i=1}^n (X_i - X_{i-1})$  and let us apply it to the previous expected value,

$$\mathbb{E}[\exp tX_n] = \mathbb{E}\left[\exp t \underbrace{\sum_{i=1}^n (X_i - X_{i-1})}_{:=F_n}\right] = \mathbb{E}\left[\prod_{i=1}^n \exp t(X_i - X_{i-1})\right] \quad (28)$$

$$= \mathbb{E}\left[\prod_{i=1}^{n-1} F_{i-1}\right] \mathbb{E}[\exp t(X_n - X_{n-1}) \mid F_{n-1}] \quad (29)$$

Notice that  $0 \leq \mathbb{E}[\exp t(X_i - X_{i-1}) \mid F_{i-1}] \leq \exp \frac{t^2}{2}$ , so using [Lemma 3.3](#) on this conditional expected value yields

$$\mathbb{E}[\exp t(X_i - X_{i-1}) \mid F_{i-1}] \leq \exp \left( t \mathbb{E} \left[ X_n - X_{n-1} \mid F_{n-1} + \frac{t^2}{2} \right] \right) = \exp \left( \frac{t^2}{2} \right). \quad (30)$$

Notice that we used the Martingale properties in the last step.  $\square$

**Lemma 3.3 (Hoeffding's Lemma).** If  $Y$  is a random variable bounded with  $a \leq Y \leq b$ , then

$$\mathbb{E}[\exp tY] \leq \exp t\mathbb{E}[Y] + \frac{t^2(b-a)^2}{8}. \quad (31)$$

We omit the proof for brevity. Roughly, the idea is to work with hyperbolic functions.

Notice that this version of Azuma's inequality uses quite restrictive requirements, which we in fact do not need. Instead, let us formulate following generalization which now allows for more dynamic step sizes and arbitrary start values.

**Theorem 3.4.** If  $X$  is a martingale and  $|X_{n+1} - X_n| \leq a_n$  for all  $n$ , then

$$P(|X_n - X_0| \geq \lambda) \leq 2 \exp \left( \frac{-\lambda^2}{2 \sum_{i=1}^n a_i^2} \right) \quad (32)$$

We omit the proof, but it is quite analogous.

Let us introduce following notion for continuity.



**Definition 3.5** (Lipschitz condition). Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . If there exists a  $C \in \mathbb{R}$  such that for any  $x, y \in \mathbb{R}^n$  whose values only differ in one place holds that

$$|f(x) - f(y)| \leq C \quad (33)$$

we say  $f$  satisfies the **Lipschitz Condition**.

The Lipschitz condition intuitively tries to limit the derivative in every point of  $f$ . In fact, one can show that this is an even stronger condition than "regular" continuity.

**Theorem 3.6** (**McDiarmid's Inequality**). Let  $X$  be a stochastic process, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a function satisfying the Lipschitz condition with Lipschitz constant  $c$ . Then, for the **Doob Martingale**<sup>a</sup> defined as

$$Z_k = \mathbb{E} \left[ f(X_1, \dots, X_n) \mid \bigotimes_{i=1}^k X_i \right] \quad (34)$$

we have

$$P(|Z_k - Z_0| \geq \lambda) \leq 2 \exp \left( \frac{-2\lambda^2}{kc^2} \right) \quad (35)$$

<sup>a</sup>The fact that  $Z$  is indeed a martingale is a classical result.

Again, we omit the proof, since it does not contain any exciting ideas.

**Remark 3.7.** Most of the time, only the special case  $k = n$  is interesting. (Conditional expectancies are hard to calculate!) In this case, the statement (35) simplifies to

$$P(f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \geq \lambda) \leq \exp \left( \frac{-2\lambda^2}{nc^2} \right). \quad (36)$$

Notice that the absence of the absolute value gets rid of the factor 2 (if needed).

We now turn to some applications of these results. In the context of bioinformatics, DNA sequences are the most important research topic. It might be useful to treat such sequences as random strings, and find interesting patterns in it.

**Problem** (Pattern matching). Find *interesting* substrings of some given random string following an arbitrary unknown distribution. We say a substring is *interesting* if it occurs more often than it would if the random string followed a uniform distribution.

Let  $N$  be the number of occurrences of a fixed substring  $B = (B_1, \dots, B_k)$  in the random string  $X = (X_1, \dots, X_n)$  given the underlying alphabet  $\eta$  with  $s := |\eta|$ . If  $X$  is drawn

uniformly, we can easily calculate the expected number of occurrences of  $B$  as

$$\mathbb{E}[N] = \mathbb{E} \left[ \sum_{i=1}^{n-k+1} \mathbb{1}(X_i = B_i, \dots, X_{i+k} = B_k) \right] \quad (37)$$

$$= P(X_i = B_i, \dots, X_{i+k} = B_k) = (n - k + 1) \frac{1}{s^k}. \quad (38)$$

Let us now treat  $N : \mathbb{R}^n \rightarrow \mathbb{R}$  as a function. Then  $|N(x) - N(y)| \leq k$ , and define the Doob martingale  $Z_k := \mathbb{E}[N \mid \bigotimes_{i=1}^k X_i]$ . This allows us to apply [Theorem 3.6](#), and deduce

$$P(|Z_n - Z_0| \geq \lambda) = P(|N - \mathbb{E}[N]| \geq \lambda) \leq 2 \exp \left( \frac{-2\lambda}{nk^2} \right). \quad (39)$$

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**Problem** (Balls-into-bins). Suppose we throw  $m$  balls into  $n$  bins. What is the (expected) number of empty bins?

Let  $X_i$  be the bin into which the  $i$ th ball falls. Then, for  $f : [n]^m \rightarrow [n]$  such that  $f$  counts the number of empty bins, we see  $|f(x) - f(y)| \leq 1$ , and thus

$$P(|f(X) - \mathbb{E}[f(X)]| \geq \lambda) \leq 2 \exp \left( \frac{-2\lambda^2}{m} \right).$$

**Problem** (Bin packing). Let  $X_1, \dots, X_n$  be random variables with values in  $[0, 1]$ . What is the (expected) number of bins of capacity 1 needed to pack all  $X_i$ ?

Define  $f : [0, 1]^n \rightarrow [n]$  such that  $f$  maps into the number of bins needed. We see  $|f(x) - f(y)| \leq 1$ , and thus

$$P(|f(X) - \mathbb{E}[f(X)]| \geq \lambda) \leq 2 \exp \left( \frac{-2\lambda^2}{m} \right).$$

**Problem** (Random knapsack). Let  $X_1, \dots, X_n$  be random variables with values in  $[0, 1]$ . Consider a knapsack of capacity  $b$ . A packing of  $X$  is a subset of some  $X_i$  such that their sum is at most  $b$ . We are interested in the *maximal* packing  $K : [0, 1]^n \rightarrow \mathbb{R}$ , i.e. the largest sum of  $X_i$ 's that is still at most  $b$ .

McDiarmid applies as above.

**Lemma 3.8.** Let  $K_n$  be the complete graph on  $n$  vertices. Let  $(A_1, \dots, A_m)$  be a partition of the edges in  $K_n$ . Furthermore, let  $f : \mathcal{G}_n \rightarrow \mathbb{R}$  a function defined on the space of all graphs with  $n$  vertices, and assume  $|f(G) - f(H)| \leq C$  if the symmetric difference  $E_G \triangle E_H$  is contained in *one*  $A_i$ . Then, for a random graph  $G$  on  $n$  vertices

$$P(f(G) - \mathbb{E}[f(G)] \geq \lambda) \leq \exp \left( \frac{-2\lambda^2}{mC^2} \right).$$

*Proof.* Trivial by [Theorem 3.6](#).  $\square$

**Corollary 3.9.** If  $|f(G) - f(H)| \leq C$  when  $G, H$  only differ by one edge, then

$$P(f(G) - \mathbb{E}[f(G)] \geq \lambda) \leq \exp\left(\frac{-4\lambda^2}{n(n-1)C^2}\right).$$

*Proof.* Trivial.  $\square$

**Corollary 3.10.** If  $|f(G) - f(H)| \leq C$  when  $G, H$  only differ at edges connected to one vertex, then we can take  $m = n$ .

*Proof.* Use  $A_i = \{\{j, i\} \in E \mid j < i\}$ .  $\square$

**Problem.** Let  $G$  be a graph on  $n$  vertices. A coloring of  $G$  is a map  $c : V_G \rightarrow [n]$  such that neighbors are not colored the same. The chromatic number  $\chi(G)$  of  $G$  is the minimal amount of different colors needed for a valid coloring of  $G$ . We are interested in the concentration of the chromatic number.

[Corollary 3.10](#) tells us that

$$P(|\chi(G) - \mathbb{E}[\chi(G)]| \geq \lambda) \leq 2 \exp\left(\frac{-2\lambda^2}{n}\right).$$

**Theorem 3.11.** Let  $G \sim \mathcal{G}_{n, \frac{1}{2}}$ . Then, following property for the probability of the concentration for the chromatic number holds for every  $\varepsilon > 0$ :

$$P(2 \log_2(n)(1 - \varepsilon) \leq \chi(G) \leq 2 \log_2(n)(1 + \varepsilon)) \xrightarrow{n \rightarrow \infty} 1.$$

*Proof idea.* Suppose we can find some  $s(n)$  such that the probability that  $G_{n,p}$  has no independent set of size  $s(n)$  is small. Then, we will iteratively color this set, remove it, and apply again  $s(n - s(n))$ . In fact, we will upper bound the mentioned probability by  $q(n) := \exp(-n^{4/3})$ .

More detailed, let  $\tilde{n} := \frac{n}{\log(n)^2}$ . We call a set  $W \subseteq V_G$  with  $|W| \geq \tilde{n}$  *bad*, if it contains no independent set of size  $s(\tilde{n})$ . We show that the probability that there exists a bad set in  $\mathcal{G}_{n,p}$  is upper-bounded by  $2^n q(\tilde{n})$ , which tends to 0 for  $n \rightarrow \infty$ . The procedure above then lets us iteratively

- color the set of size  $s(\tilde{n})$ ,
- remove those  $s(\tilde{n})$  vertices,

until we are left at most  $\tilde{n}$  vertices and color them uniquely. This process uses  $\frac{n}{s(\tilde{n})} + \tilde{n}$  colors.  $\square$

## Part II

# Appendix

## A Exercise sheets

## 1. Exercise Sheet

**Exercise 1.1.** Given a balanced graph  $H$  with  $v := |V(H)|, e := |E(H)|$ . Show that  $n^{v/e}$  is a threshold for the property if  $H$  is contained in a random graph  $G \sim \mathcal{G}_{n,p}$ .

*Proof.* For every subset  $S \subseteq V(G)$  with  $|S| = v$  let  $X_S$  be the number of containments of  $H$  in  $S$ , and  $X$  be the total number of containments. The probability for a specific containment is given as  $p^e$ . However, for every  $H$  independent of  $n, p$  there is a specific number of times  $c_H$  it can appear in such a subset  $S$ . Therefore,

$$\mathbb{E}[X] = \sum_{S \subseteq V(G)} \mathbb{E}[X_S] = \binom{n}{v} c_H p^e.$$

By Markov's inequality,

$$P(X > 0) = P(X \geq 1) \leq \frac{\mathbb{E}[X]}{1} = \binom{n}{v} c_H p^e \in O(n^v p^e)$$

If  $p \ll n^{-v/e}$ , then  $\binom{n}{v} c_H p^e \xrightarrow{n \rightarrow \infty} 0$ , proving one part of the threshold property.

For the other direction, consider the variance of  $X$ . Let  $H_i$  be the indicator for every possible containment of  $H$  in  $G$ . Then, the sum over all  $H_i$  is also  $X$ , i.e. it suffices to consider all pairs  $H_i$  and  $H_j$  for their covariance. Notice the covariance only depends on the number of overlapping (existing) edges. Let  $\hat{H} = H_i \cap H_j$  be the cross-sectional graph with edges only existing if they are in both graphs. Then,

$$\begin{aligned} \mathbf{Cov}[H_i, H_j] &= \mathbb{E}[H_i H_j] - \mathbb{E}[H_i] \mathbb{E}[H_j] \\ &= p^{e+e-|E(\hat{H})|} - p^{2e}. \end{aligned}$$

Also, consider how often each specific overlap  $\hat{H}$  could occur. The number of involved nodes is by inclusion-exclusion principle  $v+v-|V(\hat{H})|$ , therefore the pattern resulting in  $\hat{H}$  could occur  $\binom{n}{2v-|V(\hat{H})|}$ -times. Finally, we again introduce a constant  $c_{H, \hat{H}}$  denoting the number of ways two copies of  $H$  intersect in  $\hat{H}$ . Thus, combining everything we conclude

$$\mathbb{V}[X] = \sum_i \mathbb{V}[H_i] + \sum_{i,j} \mathbf{Cov}[H_i, H_j] \quad (40)$$

$$= \sum_{\hat{H} \subseteq H} c_{H, \hat{H}} n^{2v-|V(\hat{H})|} p^{2e-|E(\hat{H})|}. \quad (41)$$

Notice that variance is just a special form of covariance. Applying Chebyshev's

inequality we get

$$\begin{aligned} P(X = 0) &= P(|X - E(X)| = E(X)) \leq \frac{\mathbb{V}[X]}{\mathbb{E}[X]^2} \\ &= \sum_{\hat{H} \subseteq H} c_{H, \hat{H}} n^{-|V(\hat{H})|} p^{-|E(\hat{H})|}. \end{aligned}$$

Since  $H$  is balanced, every subgraph  $\hat{H}$  must satisfy  $e/v \geq |E(\hat{H})|/|V(\hat{H})|$  by definition. Therefore, if  $p \gg n^{-v/e}$ , then also  $p \gg n^{-|V(\hat{H})|/|E(\hat{H})|}$ , and  $p^{-|E(\hat{H})|} \ll n^{|V(\hat{H})|}$ , and the previous upper bound tends to 0 for  $n \rightarrow \infty$ . This proves the other condition for the threshold.  $\square$

**Exercise 1.2.** Given an undirected graph  $G = (V, E)$ . Let  $d \in \mathbb{N}$  and  $U \subseteq V$  the set of all vertices with degree at least  $d$ . Then there exists a dominating set for  $U$  of size at most

$$\left\lceil n \frac{\log(d+1) + 1}{d+1} \right\rceil.$$

*Proof.* Define  $p := \frac{\log(d+1)}{d+1}$  and randomly choose a subset  $S \subseteq V$  such that every node is chosen independently with probability  $p$ . Then, add a set  $|T|$  that contains every node  $u \in U$  if  $u$  is not already dominated by  $S$ . Clearly,  $S \cup T$  is a dominating set then. Let  $T_u$  be the indicator variable if  $u \in T$ . Consider the expected number of nodes in our final set

$$\mathbb{E}[|S \cup T|] = \mathbb{E}[|S|] + \mathbb{E}\left[\sum_u T_u\right] = \mathbb{E}[|S|] + |U|P(T_u = 1).$$

Notice  $|S|$  follows a binomial distribution  $\mathcal{B}_{n,p}$ . Also,  $P(T_u = 0)$  denotes the probability that  $u$  is not dominated by  $S$ , i.e. none of its  $\deg(u) \geq d$  neighbors or itself was chosen, resulting in the probability  $P(T_u = 0) = (1-p)^{d+1}$ . In summary,

$$\begin{aligned} \mathbb{E}[|S|] + |U|P(T_u = 1) &= np + |U|(1-p)^{d+1} \\ &\leq n(p + (1-p)^{d+1}) = n \left( \frac{\log(d+1)}{d+1} + \left(1 - \frac{\log(d+1)}{d+1}\right)^{d+1} \right) \\ &\leq n \left( \frac{\log(d+1)}{d+1} + e^{-\log(d+1)} \right) = n \frac{\log(d+1) + 1}{d+1}. \end{aligned}$$

Notice that we utilized Euler's inequality  $e^x > (1 + \frac{x}{n})^n$  in the last line using  $x = -\log(d+1)$ . In particular, since we are working with discrete random variables, there is a probability of larger zero that  $|S \cup T|$  is at most the floored value of our final upper bound for the expected value (also see [Lemma 1.4](#)).  $\square$

**Exercise 1.3.** Show that if

$$4 \binom{k}{2} \binom{n}{k-2} 2^{1-\binom{k}{2}} \leq 1$$

then the  $k$ th symmetric Ramsey number satisfies  $R_k > n$ .

*Proof.* Let  $n, k \in \mathbb{N}$  such that the inequality is satisfied, and  $G \sim \mathcal{G}_{n,p}$ . Define for every subset  $S \subseteq V(G)$  with  $|S| = k$  an event  $E_S$  that  $S$  is a  $k$ -clique or  $k$ -independent set, i.e. induces a complete or empty graph. As shown in the lecture (see (1)),  $P(E_S) = 2^{1-\binom{k}{2}}$ . Consider the dependency graph for these events. For  $S, T \subseteq V(G)$  of size  $k$ , it holds that  $E_S, E_T$  are independent iff  $|S \cap T| \leq 1$ , since only in this case they do not share an edge. In particular,  $E_S$  is mutually independent of all  $E_T$  with  $|S \cap T| \leq 1$  since edges are existing independently from each other.

Therefore, a trivial upper bound of the degree of  $E_S$  is given as  $\binom{k}{2} \binom{n}{k-2}$ , i.e. choose at least 2 vertices from  $S$  to guarantee  $|S \cap T| \geq 2$ , and choose any  $k-2$  vertices from all the other vertices (in fact, we could reduce  $n$  to  $n-2$  to get a tighter upper bound, but that's not what the exercise wants). Since  $4 \cdot \binom{k}{2} \binom{n}{k-2} \cdot P(E_S) \leq 1$  by assumption, we can apply Lovász Local Lemma (Theorem 1.14) and deduce that  $P(\bigcap_S \overline{E_S}) > 0$ , i.e. with non-zero probability there exists a graph with  $n$  nodes that does not contain any  $k$ -cliques or  $k$ -independent sets, proving  $R_k > n$ .  $\square$

**Exercise 1.4.** Consider the general form of Lovász Local Lemma:

**Theorem A.1.** Let  $E_1, \dots, E_n$  be a set of events in some probability space and  $G = (V, E)$  their dependency graph. If there exist  $x_1, \dots, x_n \in [0, 1]$  such that

$$P(E_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j),$$

then it holds that

$$P\left(\bigcap_{i=1}^n \overline{E_i}\right) \geq \prod_{i=1}^n (1 - x_i).$$

Use this to prove that you can replace the condition  $4dp \leq 1$  in the symmetric version (Theorem 1.14) by the weaker condition  $ep(d+1) \leq 1$  (where  $e$  denotes Euler's number).

*Proof.* Assume  $P(E_i) \leq p$  such that the maximum degree of the dependency event

graph  $G = (V, E)$  is  $d$ , and  $ep(d+1) \leq 1$ . Let  $x_i := \frac{1}{d+1}$  for every  $i$ . Then

$$\begin{aligned} x_i \prod_{(i,j) \in E} (1 - x_j) &= \frac{1}{d+1} \prod_{(i,j) \in E} \left(1 - \frac{1}{d+1}\right) \quad | \max_{v \in V} \deg(v) \leq d \\ &\geq \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d \quad | \text{Euler's inequality } e^x > \left(1 + \frac{x}{n+1}\right)^n \\ &> \frac{1}{d+1} \frac{1}{e} \geq p \geq P(E_i). \end{aligned}$$

By the general Local Lemma,  $P(E_i) \geq \prod_{i=1}^n \frac{d}{d+1} > 0$ . □



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