大学数学理一(II)期末试卷(A卷)参考答案 (2012.6.20)

一、1. $\pi a(a^2-h^2)$.

2. $P=(y-z)x,\ Q=0,\ R=x-y$, 由高斯公式，

$$\text{原式}=\iiint\limits_V\Big(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\Big)\,\mathrm{d}V=\iiint\limits_V(y-z)\,\mathrm{d}V=-\iiint\limits_V z\,\mathrm{d}V=\int_0^3 z\,\mathrm{d}z\iint\limits_{x^2+y^2\le 1}\mathrm{d}x\mathrm{d}y=-\frac{9}{2}\pi.$$

3. $$\frac{n}{(2n-1)^2(2n+1)^2}=\frac{1}{8}\Big(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\Big)$$

$$S_n=\frac{1}{8}\bigg(\Big(1-\frac{1}{3^2}\Big)+\Big(\frac{1}{3^2}-\frac{1}{5^2}\Big)+\cdots+\Big(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\Big)\bigg)=\frac{1}{8}\Big(1-\frac{1}{(2n+1)^2}\Big),\quad \text{原式}=\lim_{n\to\infty}S_n=\frac{1}{8}.$$

4. $l=\lim\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n+1}}{\frac{1}{n}}=1$, 所以收敛半径 $R=\dfrac{1}{l}=1$. $x=1$ 时级数为 $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n}$，这是莱布尼茨型的交错级数，收敛；当 $x=-1$ 时，级数为 $\sum\limits_{n=1}^{\infty}-\dfrac{1}{n}$, 此级数发散，故收敛域为 $(-1,1]$.

5. 通解为 $y=C_1\cos x+C_2\sin x+x^2-2$

6. 原方程化为 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{y}{x}-1}{\frac{y}{x}+1}$, 这是一个齐次方程. 令 $u=\dfrac{y}{x}$, 则 $y=ux$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=u+x\dfrac{\mathrm{d}u}{\mathrm{d}x}$, 原方程化为 $u+x\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{u-1}{u+1}$, 分离变量得 $\big(\dfrac{u+1}{1+u^2}\big)\mathrm{d}u=-\dfrac{1}{x}\mathrm{d}x$, 两边积分得 $\ln(1+u^2)+2\arctan u=-2\ln|x|+\ln|C|$, 所以原方程的通积分为 $x^2+y^2=Ce^{-2\arctan\frac{y}{x}}$.

7. $\ln(1+x)=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n}x^n\quad(-1<x\le 1)$, $\ln(1-x)=\sum\limits_{n=1}^{\infty}-\dfrac{1}{n}x^n\quad(-1\le x<1)$, 因此

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)=\sum_{n=1}^{\infty}\frac{2}{2n-1}x^{2n-1}\quad(-1<x<1),$$

8. $\lim\limits_{x\to+\infty}x^p\cdot\dfrac{\arctan x}{1+x^p}=\dfrac{\pi}{2}$, 所以 $p>1$ 时原广义积分收敛，$0<p\le 1$ 时，原广义积分发散.

9. 方法1： 设曲线 C 的参数方程为 $x=a\sin\theta\cos\theta, y=a\sin^2\theta\ (0\le\theta\le\pi)$, 则

$$\int_C\sqrt{x^2+y^2}\mathrm{d}s=\int_0^{\pi}\sqrt{a^2\sin^2\theta}\sqrt{a^2(\cos^2\theta-\sin^2\theta)^2+4a^2\sin^2\theta\cos^2\theta}\,\mathrm{d}\theta$$
$$=\int_0^{\pi}a\sin\theta\sqrt{a^2(\cos^2\theta+\sin^2\theta)^2}\,\mathrm{d}\theta=\int_0^{\pi}a^2\sin\theta\,\mathrm{d}\theta=2a^2.$$

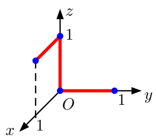
方法2： 设曲线 C 的参数方程为 $x=\dfrac{a}{2}\cos\theta, y=\dfrac{a}{2}+\dfrac{a}{2}\sin\theta\ (0\le\theta\le 2\pi)$, 则

$$\int_C\sqrt{x^2+y^2}\mathrm{d}s=\int_0^{2\pi}\sqrt{\frac{a^2}{2}(1+\sin\theta)}\sqrt{\frac{a^2}{4}\sin^2\theta+\frac{a^2}{4}\cos^2\theta}\,\mathrm{d}\theta$$
$$=\int_0^{2\pi}\frac{a}{2}\sqrt{\frac{a^2}{2}\Big(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\Big)^2}\,\mathrm{d}\theta=\frac{a^2}{2\sqrt{2}}\int_0^{2\pi}\Big|\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\Big|\mathrm{d}\theta$$
$$=\frac{a^2}{2\sqrt{2}}\int_0^{\frac{3\pi}{2}}\Big(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\Big)\mathrm{d}\theta+\frac{a^2}{2\sqrt{2}}\int_{\frac{3\pi}{2}}^{2\pi}\Big(-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\Big)\mathrm{d}\theta=2a^2.$$

10. 采用柱坐标，原式 $=\displaystyle\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}\rho\int_{2\rho}^2\rho^2\sin^2\theta\cdot\rho\,\mathrm{d}z=\frac{\pi}{10}$.

二、 $\sin x = x - \dfrac{x^3}{6} + o(x^3)$, 所以 $\left(\dfrac{1}{n} - \sin\dfrac{1}{n}\right)^p = \left(\dfrac{1}{n} - \dfrac{1}{n} + \dfrac{1}{6n^3} + o(\dfrac{1}{n^3})\right)^p \sim \dfrac{1}{6^p n^{3p}}$, 所以原式当 $3p > 1$ 即 $p > \dfrac{1}{3}$ 时收敛, 当 $3p \le 1$ 即 $p \le \dfrac{1}{3}$ 时发散.

三、 $P = (x^2 - f(x))y + \dfrac{1}{2}g(x)y^2$, $Q = f(x)y - g(x)$, $R = 1$, 积分与路径无关的充要条件是 $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, 即 $f'(x)y - g'(x) = x^2 - f(x) + g(x)y$, 整理得 $(f'(x) - g(x))y = g'(x) - f(x) + x^2$ 此式对所有的 x, y 都成立，必有 $f'(x) - g(x) = 0, g'(x) - f(x) + x^2 = 0$. 整理得 $f''(x) - f(x) = -x^2$, 这是二阶非齐次线性常系数微分方程，且有初始条件 $f(0) = 0, g(0) = f'(0) = 0$, 解得 $f(x) = -e^{-x} - e^x + x^2 + 2, g(x) = f'(x) = e^{-x} - e^x + 2x$.



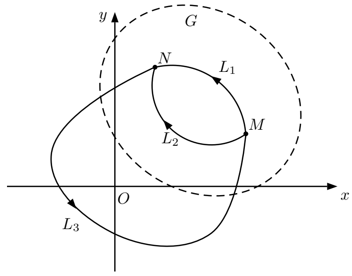
因为积分与路径无关，沿如图所示折线积分，可得

原式 $= \displaystyle\int_0^1 0\,\mathrm{d}y + \int_0^1 \mathrm{d}z + \int_0^1 0\,\mathrm{d}x = 1.$

四、 1. 显然 $f(x)$ 是偶函数，且 $f(x)$ 连续. 所以 $b_n = 0\ (n = 1, 2, \cdots)$, $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi (\pi^2 - x^2)\,\mathrm{d}x = \dfrac{4}{3}\pi^2$, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi (\pi^2 - x^2)\cos nx\,\mathrm{d}x = (-1)^{n+1}\dfrac{4}{n^2}$, 故 $f(x) = \pi^2 - x^2 = \dfrac{2\pi^2}{3} + \displaystyle\sum_{n=1}^\infty (-1)^{n+1}\dfrac{4}{n^2}\cos nx$, $(-\pi \le x \le \pi)$.

在上式中分别令 $x = 0$, $x = \pi$ 可得 $\displaystyle\sum_{n=1}^\infty (-1)^{n-1}\frac{1}{n^2} = \frac{\pi^2}{12}$, $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

五、 证明：(1) 如图所示，设 M, N 是 G 内任意两点， L_1, L_2 是 G 内连接 M, N 的任意两条曲线，只需要证明



$$\int_{L_1} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \int_{L_2} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x)$$

取 L_3 为连接 M, N 的曲线，使得 $L_1 + L_3$ 为包围原点的简单闭曲线，则 $L_2 + L_3$ 也是包围原点的简单闭曲线，据题意可知

$$\int_{L_1 + L_3} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \int_{L_2 + L_3} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = A,$$

$$\text{所以} \quad \int_{L_1} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \int_{L_2} \frac{1}{f(x) + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x).$$

(2) $P = \dfrac{-y}{f(x) + 8y^2}$, $Q = \dfrac{x}{f(x) + 8y^2}$, 因为积分和路径无关，所以 $Q'_x = P'_y$, 即

$$\frac{8y^2 - f(x)}{(f(x) + 8y^2)^2} = \frac{f(x) + 8y^2 - xf'(x)}{(f(x) + 8y^2)^2},$$

可得 $xf'(x) = 2f(x)$, 分离变量解得 $f(x) = Cx^2$, 又 $f(1) = 1$, 所以 $f(x) = x^2$.

取 $l : x^2 + 8y^2 = \varepsilon^2$, 即 $x = \varepsilon\cos\theta, y = \frac{1}{\sqrt{8}}\varepsilon\sin\theta$, $0 \le \theta \le 2\pi$, 则

$$A = \int_L \frac{1}{x^2 + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \int_l \frac{1}{x^2 + 8y^2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \int_0^{2\pi} \frac{1}{\sqrt{8}}\mathrm{d}\theta = \frac{\pi}{\sqrt{2}}.$$

六、见教材163页例7.6.6.