1 Arithmetic property of Complex Number

For convenience, we assume that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Addition

$$(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta).$$

Multiplication

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma).$$

Division Provided that $\gamma + i\delta \neq 0$, then

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{(\alpha + i\beta)(\gamma - i\delta)}{(\gamma + i\delta)(\gamma - i\delta)} = \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2},$$

in particular,

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}.$$

Square Roots

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right) + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}},$$

since if $(x + iy)^2 = \alpha + i\beta$ then

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta. \end{aligned}$$

Hence $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2$, which implies $x^2 + y^2 = \sqrt{\alpha^2 + \beta^2}$. Now we can solve x and y.

Conjugation The transformation which replaces $\alpha + i\beta$ by $\alpha - i\beta$ is called *complex conjugation*, and $\alpha - i\beta$ is the *conjugate* of $\alpha + i\beta$.

- The conjagtion is an *involutory* transformation: this means that $\overline{\overline{a}} = a$.
- $\operatorname{Re} a = \frac{a+\overline{a}}{2}, \quad \operatorname{Im} a = \frac{a-\overline{a}}{2i}$.
- $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{a} \cdot \overline{b}$, $(\overline{b/a}) = \overline{b}/\overline{a}$. More generally, let $R(a, b, c, \cdots)$ stand for any rational operation applied to the complex number $a, b, c, \cdots$. Then

$$\overline{R(a, b, c, \cdots)} = R(\overline{a}, \overline{b}, \overline{c}, \cdots).$$

Absolute Value The product $a\bar{a} = \alpha^2 + \beta^2$ is always positive or zero. Its nonnegative square root is called the *modulus* or *absolute value* of the complex number a; it is denoted by $|a|$.

- The absolute value of a product is equal to the product of the absolute values of the factors:
$$|a_1 a_2 \cdots a_n| = |a_1| \cdot |a_2| \cdots |a_n| .$$
- $|a+b|^2 = |a|^2 + |b|^2 + 2\operatorname{Re} a\bar{b}$, $|a-b|^2 = |a|^2 + |b|^2 - 2\operatorname{Re} a\bar{b}$. By addition we obtain the identity
$$|a+b|^2 + |a-b|^2 = 2\left(|a|^2 + |b|^2\right) .$$

Inequality

Geotric Addtion and Multiplication For $a_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $a_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$, we have
$$a_1 a_2 = r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)].$$
The argument of a product is equal to the sum of the arguments of the factors.

The Binomial Equation

$$a^n = r^n(\cos n\varphi + i\sin n\varphi),$$

for $n \in \mathbb{Z}$.

To find the nth root of a complex number a we have to solve the equation

$$z^n = a.$$

Suppose that $a \neq 0$ we write $a = r(\cos\varphi + i\sin\varphi)$ and $z = \rho(\cos\theta + i\sin\theta)$. Then

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right)\right], \quad k = 0, 1, \cdots, n-1.$$

The Spherical Representation

2 Complex Functions

2.1 Analytic Functions

2.2 Polynomials and Rational Functions

2.3 Power Series

2.4 The Exponential and Trigonometric Functions

3 Analytic Functions

3.1 Conformal Mappings

3.2 Linear Fractional Transformation

4 Complex Integration

4.1 Path Integration

4.2 Cauchy's Theorem and Cauchy's Formula

4.3 Taylor Theorems

4.4 The General Form of Cauchy's Theorem

5 Series and Product Development

6 Conformal Mapping and Dirichlet's Problem

6.1 The Riemann Mapping Theorem

6.2 Harmonic Functions

7 Elliptic Functions