1 Existence and Uniqueness

1.1 Basic Concepts

Phase Space

Definition 1.1. A process is called *deterministic* if its entire future course and its entire past are uniquely determined by its state at the present time. The set of all states of the process is called the *phase space*.

Definition 1.2. A process is called *finite-dimensional* if its phase space is finite-dimensional, i.e., if the number of parameters needed to describe its states is finite.

Definition 1.3. A process is called *differentiable* if its phase space has the structure of a differentiable manifold, and the change of state with time is described by differentiable functions.

1.2 First Order Equation

Theorem 1.4. *Given a differential equation*

$$\dot{x} = f(t, x) \tag{1}$$

Suppose $f(t,x)$ is defined on an open set Γ in the plane P spanned by t and x, and f and $\frac{\partial f}{\partial x}$ is continuous over t and x, then

1. *For any $(t_0, x_0) \in \Gamma$, there is a solution $x = \varphi(x)$ of (1) satisfying the initial condition*
$$\varphi(t_0) = x_0.$$
2. *If $x = \psi(t)$ and $x = \chi(t)$ are solutions of (1) coincides at some t_0, i.e.*
$$\psi(t_0) = \chi(t_0),$$
then the they are equal everywhere.

2 Constant Coefficients ODE

2.1 Integrable Cases

Consider

$$y' = f(x, y) \tag{2}$$

The right-hand side $f(x,y)$ of the equation is assumed to be defined as a real-valued function on a set D in the xy-plane.

Definition 2.1. Let J be an interval. A function: $y(x) : J \to \mathbb{R}$ is called a *solution* to the differential equation 2 if y is differentiable in J, the graph of y is a subset of D, and 2 holds, i.e., if

$$(x, y(x)) \in D \text{ and } y'(x) = f(x, y(x)) \text{ for all } x \in J.$$

Definition 2.2. A numerical triple (x, y, p) is called a *line element*: (x, y) gives a point in the plane, and the third component p gives the slope of a line through the point (x, y). The collection of all line elements of the form $(x, y, f(x, y))$ is called a *direction field*.

Definition 2.3. Let a function $f(x, y)$, defined on a set D in the (x, y)-plane, and a fixed point $(\xi, \eta) \in D$ be given. A function $y(x)$ is sought that is differentiable in an interval J with $\xi \in J$ such that

$$y'(x) = f(x, y(x)) \in J, \tag{3}$$

$$y(\xi) = \eta. \tag{4}$$

Case $y' = f(x)$ Suppose the function $f(x)$ is continuous in an interval J and the set D is a strip $J \times \mathbb{R}$. One can prove that all of the solutions can be obtained by translating any particular solution in the direction indicated by the y-axis.

Case $y' = g(y)$ A formal calculation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y) \iff \frac{\mathrm{d}y}{g(y)} = \mathrm{d}x$$

and h