1 Multi-Variables

1.1 Inequality

Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* provided

$$f(\tau x + (1-\tau)y) \le \tau f(x) + (1-\tau) f(y)$$

for all $x, y \in \mathbb{R}^n$ and each $0 \le \tau \le 1$.

Theorem 1.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then for each $x \in \mathbb{R}^n$ there exists $r \in \mathbb{R}^n$ such that the inequality*

$$f(y) \ge f(x) + r \cdot (y - x)$$

holds for all $y \in \mathbb{R}^n$.

1.2 Theorems

Definition 1.3. We say the boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that – upon relabeling and reorienting the coordinates axes if necessary – we have

$$U \cap B(x^0, r) = \left\{ x \in B(x^0, r) : x_n > \gamma(x_1, \cdots, x_{n-1}) \right\}.$$

Likewise, ∂U is C^∞ if ∂U is C^k for $k = 1, 2, \cdots$, and ∂U is analytic if the mapping γ is analytic.

Definition 1.4.

1. If ∂U is C^1, then along ∂U is defined the *outward pointing unit normal vector field*

   $$\nu = (\nu^1, \cdots, \nu^n).$$

   The unit normal at any point $x^0 \in \partial U$ is $\nu(x^0) = \nu = (\nu_1, \cdots, \nu_n)$.

2. Let $u \in C^1(\overline{U})$. We call

   $$\frac{\partial u}{\partial \nu} := \nu \cdot Du$$

   the outward normal derivative of u.

Theorem 1.5 (Gauss-Green)**.**

1. *Suppose $u \in C^1(\overline{U})$. Then*

   $$\int_U u_{x_i} \, \mathrm{d}x = \int_{\partial U} u \nu^i \, \mathrm{d}S$$

2. *(Divergence) We have*

   $$\int_U \operatorname{div} u \, \mathrm{d}x = \int_{\partial U} u \cdot v \, \mathrm{d}S$$

   for each vector field $u \in C^1(\overline{U}; \mathbb{R}^n)$.

Proof. (1) follows from (2): apply (2) to $w = (0, \cdots, u_{x_i}, \cdots, 0)$. □

Theorem 1.6 (Integration by parts formula)**.** *Let $u, v \in C^1(\overline{U})$. Then*

$$\int_U u_{x_i} v \,\mathrm{d}x = -\int_U u v_{x_i} \,\mathrm{d}x + \int_{\partial U} u v \nu^i \,\mathrm{d}S, \quad (i = 1, \cdots, n). \tag{1.2.1}$$

Proof. Apply 1.5 (1) to uv. □

Theorem 1.7 (Green)**.** *Let $u, v \in C^2(\overline{U})$. Then*

1. $\int_U \Delta u \,\mathrm{d}x = \int_{\partial U} \frac{\partial u}{\partial \nu} \,\mathrm{d}S$,
2. $\int_U Dv \cdot Du \,\mathrm{d}x = -\int_U u \Delta v \,\mathrm{d}x + \int_{\partial U} \frac{\partial u}{\partial \nu} u \,\mathrm{d}S$,
3. $\int_U u\Delta v - v\Delta u \,\mathrm{d}x = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \,\mathrm{d}S$.

Proof. Using (1.2.1), with u_{x_i} in place of u and $v \equiv 1$, we see

$$\int_U u_{x_i x_i} \,\mathrm{d}x = \int_{\partial U} u_{x_i} \nu^i \,\mathrm{d}S.$$

Sum $i = 1, \cdots, n$ to establish (1).

To derive (2), we employ (1.2.1) with v_{x_i} replacing v. (3) follows directly from (2). □

Theorem 1.8 (Coarea formula)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous and assume that for a.e. $r \in \mathbb{R}$ the level set*

$$\{x \in \mathbb{R}^n : u(x) = r\}$$

is smooth, $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$. Suppose also $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and summable. Then

$$\int_{\mathbb{R}^n} f\,|Du| \,\mathrm{d}x = \int_{-\infty}^{\infty} \left(\int_{u=r} f \,\mathrm{d}S \right) \mathrm{d}r.$$

Theorem 1.9 (Polar coordinates)**.**

1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

$$\int_{\mathbb{R}^n} f \,\mathrm{d}x = \int_0^{\infty} \left(\int_{\partial B(x_0, r)} f \,\mathrm{d}S \right) \mathrm{d}r$$

   for each point $x_0 \in \mathbb{R}^n$.

2. *In particular,*

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(\int_{B(x_0, r)} f \,\mathrm{d}x \right) = \int_{\partial B(x_0, r)} f \,\mathrm{d}S$$

   for each $r > 0$.

Proof. (1) follows directly from 1.8. □

Theorem 1.10. *Consider a family of smooth, bounded regions $U(\tau) \subseteq \mathbb{R}^n$ that depend smoothly upon the parameter $\tau \in \mathbb{R}$. Write v for the velocity of the moving boundary $\partial U(\tau)$ and ν for the outward pointing unit normal. If $f = f(x, \tau)$ is a smooth function, then*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{U(\tau)} f \,\mathrm{d}x = \int_{\partial U(\tau)} f v \cdot \nu \,\mathrm{d}S + \int_{U(\tau)} f_\tau \,\mathrm{d}x.$$

If $U \subseteq \mathbb{R}^n$ is open and $\varepsilon > 0$, we write

$$U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Definition 1.11. 1. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

the constant $C > 0$ selected so that $\int_{\mathbb{R}^n} \eta \, \mathrm{d}x = 1$.

2. For each $\varepsilon > 0$, set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

We call η the *standard mollifier.* The function η_ε are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\varepsilon \, \mathrm{d}x = 1, \quad \operatorname{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon).$$

Definition 1.12. If $f : U \to \mathbb{R}$ is locally integrable, define its *mollification*

$$f^\varepsilon := \eta_\varepsilon * f \text{ in } U_\varepsilon.$$

That is,

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x - y) f(y) \, \mathrm{d}y = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x - y) \, \mathrm{d}y$$

for $x \in U_\varepsilon$.

Theorem 1.13. *1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.*

2. $f^\varepsilon \to f$ a.e. as $\varepsilon \to 0$.

3. If $f \in C(U)$, then $f^\varepsilon \to f$ uniformly on compact subsets of U.

4. If $1 \leq p < \infty$ and $f \in L^p_{\mathrm{loc}}(U)$, then $f^\varepsilon \to f$ in $L^p_{\mathrm{loc}}(U)$.

Proof. 1. Fix $x \in U_\varepsilon$, $i \in \{1, \cdots, n\}$, and h so small that $x + he_i \in U_\varepsilon$ so small that $x + he_i \in U_\varepsilon$. Then

$$\begin{aligned} \frac{f^\varepsilon(x + he_i) - f^\varepsilon(x)}{h} &= \frac{1}{\varepsilon^n} \int_U \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{\varepsilon} - \eta\left(\frac{x - y}{\varepsilon}\right)\right)\right] f(y) \, \mathrm{d}y \\ &= \frac{1}{\varepsilon^n} \int_V \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{\varepsilon} - \eta\left(\frac{x - y}{\varepsilon}\right)\right)\right] f(y) \, \mathrm{d}y \end{aligned}$$

for some open set $V \subset U$. As

$$\frac{1}{h}\left[\eta\left(\frac{x + he_i - y}{\varepsilon}\right) - \eta\left(\frac{x - y}{\varepsilon}\right)\right] \to \frac{1}{\varepsilon} \eta_{x_i}\left(\frac{x - y}{\varepsilon}\right)$$

uniformly on V, the partial derivative $f^\varepsilon_{x_i}(x)$ exists and equals

$$\int_U \eta_{\varepsilon,x_i}(x-y)f(y)\,\mathrm{d}y.$$

A similar argument shows that $D^\alpha f^\varepsilon(x)$ exists and

$$D^\alpha f^\varepsilon(x) = \int_U D^\alpha \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y, \quad (x \in U_\varepsilon),$$

for each multiindex α.

2. By Lebesgue's Differentiation Theorem,

$$\lim_{r\to 0} \frac{1}{\alpha(n)r^n} |f(y)-f(x)|\,\mathrm{d}y = 0$$

for a.e. $x \in U$. Fix such a point x. Then

$$\begin{aligned} |f^\varepsilon(x)-f(x)| &= \left| \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)[f(y)-f(x)]\,\mathrm{d}y \right| \\ &\le \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) |f(y)-f(x)|\,\mathrm{d}y \\ &\le C\frac{1}{\alpha(n)r^n} \int_{B(x,\varepsilon)} |f(y)-f(x)|\,\mathrm{d}y \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

$\square$