1 Topological Vector Spaces

Example 1.1. Assume U is an open subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. If $f : U \to \mathbb{R}$ is measurable, we define

$$\|f\|_{L^p(U)} := \begin{cases} \left(\int_u |f|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_U |f| & \text{if } p = \infty. \end{cases}$$

We define $L^p(U)$ to be the linear space of all measurable functions $f : U \to \mathbb{R}$ for which $\|f\|_{L^p(U)} < \infty$. Then $L^p(U)$ is a Banach space, provided we identify two functions which agree a.e.

Example 1.2. The space $L^2(U)$ is a Hilbert space, with

$$(f, g) = \int_U fg \, \mathrm{d}x.$$

Definition 1.3. Let H be a Hilbert spaces.

1. Two elements $u, v \in H$ are *orthogonal* if $(u, v) = 0$.
2. A countable basis $\{w_k\}_{k=1}^{\infty} \subseteq H$ is called *orthonormal* if

$$\begin{cases} (w_k, w_l) = 0 & k, l = 1, \cdots; l \neq l \\ \|w_k\| = 1 & k = 1, \cdots. \end{cases}$$

Let H be a Hilbert spaces. If $u \in H$ and $\{w_k\}_{k=1}^{\infty} \subseteq H$ is an orthonormal basis, we can write

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k,$$

the series coverging in H. In addition

$$\|u\|^2 = \sum_{k=1}^{\infty} (u, w_k)^2.$$

Definition 1.4. If S is a subspace of H, $S^{\perp} := \{u \in H : (u, v) = 0 \text{ for all } v \in S\}$ is the subspace orthogonal to S.

Definition 1.5. Let X and Y be real Banach spaces.

1. A mapping $A : X \to Y$ is a *linear operator* provided

$$A[\lambda u + \mu v] = \lambda A u + \mu A v$$

   for all $u, v \in X$, $\lambda, \mu \in \mathbb{R}$.

2. The *range* of A is $\mathcal{R}(A) := \{v \in Y : v = Au \text{ for some } u \in X\}$ and the *null space* of A is $\mathcal{A} := \{u \in X : Au = 0\}$.

Definition 1.6. A linear operator $A : X \to Y$ is *bounded* if

$$\|A\| := \sup \{\|Au\|_Y : \|u\|_X \leq 1\} < \infty.$$

Proposition 1.7. *A linear operator $A : X \to Y$ is bounded if and only if it is continuous.*

Theorem 1.8 (Closed Graph Theorem)**.** *Let $A : X \to Y$ be a closed, linear operator. Then A is bounded.*

Definition 1.9. Let $A : X \to X$ be a bounded linear operator.

1. The *resolvent set* of A is

$$\rho(A) = \{\eta \in \mathbb{R} : (A - \eta I) \text{ is one-to-one and onto } \}.$$

2. The spectrum of A is

$$\sigma(A) = \mathbb{R} - \rho(A).$$

If $\eta \in \rho(A)$, 1.8 then implies that the inverse $(A - \eta I)^{-1} : X \to X$ is a bounded linear operator.

Definition 1.10. 1. We say $\eta \in \sigma(A)$ is an eigenvalue of A provided

$$N(A - \eta I) \neq \{0\}.$$

We write $\sigma_p(A)$ to denote the collection of eigenvalues of A; $\sigma_p(A)$ is the point spectrum.

2. If η is an eigenvalue and $w \neq 0$ satisfies

$$Aw = \eta w,$$

we say w is an associated eigenvector.

Definition 1.11. 1. A bounded linear operator $u^* : X \to \mathbb{R}$ is called a *bounded linear functional* on X.

2. We write X^* to denote the collection of all bounded linear functionals on X; X^* is the *dual space* of X.

Definition 1.12. 1. If $u \in X, u^* \in X^*$, we write

$$\langle u^*, u \rangle$$

to denote the real number $u^*(u)$. The symbol $\langle \cdot, \cdot \rangle$ denotes the pairing of X^* and X.

2. We define

$$\|u^*\| := \sup \{\langle u^*, u \rangle : \| \leq 1\|\}$$

3. A Banach space is *reflexive* if $(X^*)^* = X$. More precisely, this means that for each $u^{**} \in (X^*)^*$, there exists $u \in X$ such that

$$\langle u^{**}, u^* \rangle = \langle u^*, u \rangle \text{ for all } u^* \in X^*.$$

Theorem 1.13 (Riesz Representation Theorem)**.** *Let H be a real Hilbert space, with inner product $(\cdot,\cdot)$. H^* can be canonically identitfied with H; more precisely, for each $u^* \in H^*$ there exists a unique element $u \in H$ such that*

$$\langle u^*, v\rangle = (u, v) \text{ for all } v \in H$$

The mapping $u^* \mapsto u$ is a linear isomorphism of H^* onto H.

Definition 1.14. Let H be a real Hilbert space.

1. If $A : H \to H$ is a bounded, linear operator, its adjoint $A^* : H \to H$ satisfies
$$(Au, v) = (u, A^*v)$$
for all $u, v \in H$.
2. A is symmetric if $A^* = A$.

Definition 1.15. Let X denote a real Banach space. We say a sequence $\{u_k\}_{k=1}^{\infty} \subseteq X$ converges weakly to $u \in X$, written

$$u_k \rightharpoonup u,$$

if

$$\langle u^*, u_k\rangle \to \langle u^*, u\rangle$$

for each bounded linear functional $u^* \in X^*$.

It is easy to check if $u_k \to u$, then $u_k \rightharpoonup u$. It is also true that any weakly convergent sequence is bounded. In addition, if $u_k \rightharpoonup u$, then

$$\{u\} \leq \liminf_{k\to\infty} \|u_k\|\,.$$

Theorem 1.16. *Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subseteq X$ is bounded. Then there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty} \subseteq \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that*

$$u_{k_j} \rightharpoonup u.$$

Example 1.17. We will most often employ weak convergence ideas in the following context. Take $U \subseteq \mathbb{R}^n$ to be open, $X = L^p(U)$, and assume $1 \leq p < \infty$, then

$$X^* = L^q(U),$$

where $\frac{1}{p} + \frac{1}{q} = 1, 1 < q \leq \infty$. More precisely, each bounded linear functional on $L^p(U)$ can be represented as $f \mapsto \int_U gf \,\mathrm{d}x$ for some $g \in L^q(U)$. Therefore

$$f_k \rightharpoonup f \text{ weakly in } L^p(U)$$

means

$$\int_U gf_k \,\mathrm{d}x \to \int_U gf \,\mathrm{d}x \text{ as } k \to \infty, \text{ for all } g \in L^q(U).$$

Now the identification of $L^q(U)$ as the dual of $L^p(U)$ shows that

$$L^p(U) \text{ is reflexive if } 1 < p\infty.$$

Definition 1.18. Let X and Y be real Banach spaces. A bounded linear operator

$$K : X \to Y$$

is called *compact* provided for each bounded sequence $\{u_k\}_{k=1}^{\infty} \subseteq X$, the sequence $\{Ku_k\}_{k=1}^{\infty}$ is precompact in Y; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{Ku_{k_j}\}_{j=1}^{\infty}$ converges in Y.

Theorem 1.19. *Let H denote a real Hilbert space, with inner product $(\cdot,\cdot)$. If $K : H \to H$ is compact, so is $K^* : H \to H$.*

Theorem 1.20 (supporting hyperplane theorem)**.** *If S is a convex set in the topological vector space $X = \mathbb{R}^n$, and x_0 is a point on the boundary of S, then there exsits a supporting hyperplane containing x_0. If $x^* \in X^* \setminus 0$, where X^* is the dual space of X, x^* is a nonzero linear functional, such that $x^*(x_0) \geq x^*(x)$ for all $x \in S$ then*

$$H = \{x \in X : x^*(x) = x^*(x_0)\}$$

defines a supporting hyperplane.

Conversely, if S is a closed set with nonempty interior such that every point on the boundary has a supporting hyperplane, then S is a convex set, and is the intersection of all its supporting closed half-spaces.

Theorem 1.21 (Supporting hyperplanes)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then for each $x \in \mathbb{R}^n$ there exists $r \in \mathbb{R}^n$ such that the inequality*

$$f(y) \geq f(x) + r \cdot (y - x)$$

holds for all $y \in \mathbb{R}^n$.

Theorem 1.22 (Hyperplane separation theorem)**.** *Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exist a nonzero vector v and a real number c such that*

$$\langle x, v\rangle \geq c \text{ and } \langle y, v\rangle \leq c$$

for all x in A and y in B; i.e. the hyperplane $\langle \cdot, v\rangle = c$, v the normal vector, separates A and B.

If both sets are closed, and at least one of them is compact, then the separation can be strict, that is, $\langle x, v\rangle > c_1$ and $\langle y, v\rangle < c_2$ for some $c_1 > c_2$.

Proof. Hahn-Banach and Riez representation theorem Wikipedia □

1.1 Quotient Spaces

Definition 1.23. Let N be a subspace of vector space X. For every $x \in X$, let $\pi(x)$ be the coset of N that contains x; thus

$$\pi(x) = x + N.$$

These cosets are the elements of a vector space X/N, called the *quotient space of X modulo N*, in which addition and scalar multiplication are defined by

$$\pi(x) + \pi(y) = \pi(x + y), \quad \alpha\pi(x) = \pi(\alpha x).$$

π is often called the *quotient map of X onto X/N*.

1. Suppose now that τ is a vector topology on X and that N is closed subspace of X.

2. Let τ_N be the collection of all sets $E \subseteq X/N$ for which $\pi^{-1}(E) \in \tau$. Then τ_N turns out to be a topology on X/N, called the *quotient topology.*

Remark 1.24. The origin of X/N is $\pi(0) = N$. π is a linear mapping of X onto X/N with N as its null space.

Theorem 1.25. *Let N be a closed subspace of a topological vector space X. Let τ be the topology of X and then τ_N is a topology of X/N.*

1. *The quotient map $\pi : X \to X/N$ is linear, continuous, open.*

2. *If $\mathcal{B}$ is a local base for τ, then the collection of all sets $\pi(V)$ with $V \in \mathcal{B}$ is a local base for τ_N.*

3. *Each of the following properties of X is inherited by X/N: local convexity, local boundedness, metrizability, normalbility.*

4. *If X is an F-space, or a Freéchet space, or a Banach space, so is X/N.*

Proof. (1) and (2) are directly checked. (3) follows from (2), and in fact

1. the metric ρ induced by X in X/N is $\rho(\pi(x), \pi(y)) = \inf \{d(x - y, z) : z \in N\}$.

2. $\|\pi(x)\| = \inf \{\|x - z\| : z \in N\}$.

To prove (4), by (3), it suffices to show that τ_N inherits completeness, which can also be checked directly. □

2 completeness

2.1 Baire Category Theorem

2.2 Banach-Steinhaus Theorem

Theorem 2.1. *Suppose X and Y are topological vector space, Γ is an equicontinuous collection of linear mappings from X into Y, and E is a bounded subset of X. Then Y has a bounded subset $F = \bigcup \Lambda(E)$ such that $\Lambda(E) \subseteq F$ for every $\Lambda \in \Gamma$. In particular, $\bigcup \Lambda(E)$ is bounded.*

Theorem 2.2 (Banach-Steinhaus)**.** *Suppose X and Y are topological vector spaces, Γ is a collection of continuous linear mappings from X into Y, and B is the set of all $x \in X$ whose orbits*

$$\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$$

are bounded in Y.

If B is of second category in X, then $B = X$ and Γ is equicontinuous.

Proof. Let W be a neighborhood of 0 in Y, we can choose open set $U \subsetneq Y$, such that $\overline{U} + \overline{U} \subseteq W$ and $0 \in U$. Now we consider

$$E = \bigcap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U}).$$

Suppose $x \in B$, then there exists $n \in \mathbb{N}$ such that $\Gamma(x) \subseteq nU$ which implies $x \in nE$. Hence $B \subseteq \bigcup_n nE$. By hypothesis, B is of second category, hence one of nE is so. But $x \mapsto nx$ is a homeomorphism, it implies that E is of second category. By construction, E is closed, so E has at least one interior point x_0 with its neighborhood $V \subseteq E$. Now

$$\Lambda(V - x) \subseteq \Lambda(E) + \Lambda(E) \subseteq \overline{U} + \overline{U} \subseteq W,$$

and noting that $V - x$ is a neighborhood of 0, Γ is equicontinuous. The statement $B = X$ follows from 2.1. □

Corollary 2.3 (Banach-Steinhaus)**.** *If Γ is a collection of continuous linear mappings from an F-space X into a topological vector space Y, and if the sets*

$$\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$$

are bounded in Y, for every $x \in X$, then Γ is equicontinuous.

By replacing “bounded sequence” by “Cauchy sequence” or “Convergent sequence” in 2.2, we have similar conclusions.

Theorem 2.4. *Suppose X and Y are topological vector space, and $\{\Lambda_n\}$ is a sequence of continuous linear mappings of X into Y.*

1. *If C is the set of all $x \in X$ for which $\Lambda_n x$ is a Cauchy sequence in Y, and if C is of the second category in X, then $C = X$.*
2. *If L is the set of all $x \in X$ at which*

$$\Lambda x = \lim_{n\to\infty} \Lambda_n x$$

exists; if L is of the second category in X, and if Y is an F-space, then $L = X$ and $\Lambda : X \to Y$ is continuous.

Moreover, (2) in 2.4 can be modified as the following.

Theorem 2.5. *If $\{\Lambda_n\}$ is a sequence of continuous linear mappings from a F-space X into a topological space Y, and if*

$$\Lambda x = \lim_{n\to\infty} \Lambda_n x$$

exists for every $x \in X$, then Λ is continuous.

In the following variant of the Banach-Steinhaus theorem 2.2, the category argument is applied to a compact set.

Theorem 2.6. *Suppose X and Y are topological vector space, K is a compact convex set in X, Γ is a collection of continuous linear mappings of X into Y, and the orbits*

$$\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$$

are bounded subsets of Y, for every $x \in K$. Then there is a bounded set $B \subseteq Y$ such that $\Lambda(K) \subseteq B$ for every $\Lambda \in \Gamma$.

2.3 The Open Mapping Theorem

Theorem 2.7. *Suppose*

1. X *is an* F*-space,*
2. Y *is a topological vector space,*
3. $\Lambda : X \to Y$ *is continuous and linear, and*
4. $\Lambda(X)$ *is of the second category in* Y*.*

Then

1. $\Lambda(X) = Y$*,*
2. Λ *is an open mapping, and*
3. Y *is an* F*-space.*

Corollary 2.8.
1. *If* Λ *is a continuous linear mapping of an* F*-space* X *onto an* F*-space* Y*, then* Λ *is open.*
2. *If* Λ *satisfies (1) and is one-to-one, then* $\Lambda^{-1} : Y \to X$ *is continuous.*
3. *If* X *and* Y *are Banach spaces, and if* $\Lambda : X \to Y$ *is continuous, linear, one-to-one, and onto, then there exist positive real numbers* a *and* b *such that*
$$a\,\|x\| \leq \|\Lambda x\| \leq b\,\|x\|$$
for every $x \in X$*.*
4. *If* $\tau_1 \subseteq \tau_2$ *are vector topologies on a vector space* X *and if both* (X, τ_1) *and* (X, τ_2) *are* F*-spaces, then* $\tau_1 = \tau_2$*.*

2.4 The Closed Graph Theorem

We know that

Proposition 2.9. *If* X *is a topological space,* Y *is a Hausdorff space, and* $f : X \to Y$ *is continuous, then the graph* G *is closed.*

The following kind of inverse of 2.9 is true, called the closed graph theorem.

Theorem 2.10 (The Closed Graph Theorem)**.** *Suppose*

1. X *and* Y *are* F*-spaces,*
2. $\Lambda : X \to Y$ *is linear,*
3. $G = \{(x, \Lambda x) : x \in Y\}$ *is closed in* $X \times Y$*.*

Then Λ *is continuous.*

2.5 Bilinear Mappings

Definition 2.11. Let X, Y, Z are topological vector spaces and $B : X \times Y \to Y$ is a map. if every $B_x(y) := B(x, y)$ and every $B^y := B(x, y)$ is continuous, then B is said to be *separately continuous*.

Theorem 2.12. *Suppose $B : X \times Y \to Z$ is bilinear and separately continuous, X is an F-space, and Y and Y are topological vector spaces. Then*

$$B(x_n, y_n) \to B(x_0, y_0) \text{ in } Z$$

whenever $x_n \to x_0$ in X and $y_n \to y_0$ in Y. If Y is metrizable, it follows that B is continuous.

Proof. Application of 2.2 by constructing $b_n(x) = B(x, y_n)$ and noting that

$$B(x_n, y_n) - B(x_0, y_0) = b_n(x_n - x_0) + B(x_0, y_n - y_0).$$

$\square$

3 Convexity

3.1 The Hahn-Banach Theorems

Definition 3.1. The *dual space* of a topological vector space X is the vector space X^* whose elements are the *continuous* linear functionals on X.

Remark 3.2. It will be convenient to use the following terminology: An additive functional Λ on a complex vector space X is called *real-linear(complex-linear)* if $\Lambda(\alpha x) = \alpha\Lambda x$ for every $x \in X$ and for every real(complex) scalar α. If u is the real part of a complex-linear functional f on X, then u is real-linear and

$$f(x) = u(x) - iu(ix) \quad (x \in X) \tag{3.1.1}$$

because $z = \operatorname{Re} z - i \operatorname{Re}(iz)$ for every $z \in \mathbb{C}$.

Conversely, if $u : X \to \mathbb{R}$ is real-linear on a complex vector space X and if f is defined by (3.1.1), a straightforward computation shows that f is complex linear.

Extension We will give some theorems about extending functions of subspace into the whole space with some controls.

Theorem 3.3. *Suppose*

1. *M is a subspace of a real vector space X,*
2. *$p : X \to \mathbb{R}$ satisfies*

   $$p(x + y) \le p(x) + p(y) \text{ and } p(tx) = p(x)$$

   if $x \in X$, $y \in X$, $t \ge 0$,

3. *$f : M \to \mathbb{R}$ is linear and $f(x) \le p(x)$ on M.*

Then there exists a linear $\Lambda : X \to \mathbb{R}$ such that

$$\Lambda x = f(x)(x \in M)$$

and

$$-p(-x) \le \Lambda x \le p(x)(x \in X).$$

Theorem 3.4. *Suppose M is a subspace of a vector space X, p is a seminorm on X, and f is a linear functional on M such that*

$$|f(x)| \le p(x) \quad (x \in M).$$

Then f extends to a linear functional Λ on X that satisfies

$$|\Lambda x| \le p(x) \quad (x \in X).$$

Proof. If the scalar field is $\mathbb{R}$, this is contained in 3.3. Assume that the scalar field is $\mathbb{C}$. Put $u = \operatorname{Re} f$, then by 3.3, there is a real-linear g on X such that

$$g = u \text{ on } M \text{ and } g \le p \text{ on } X.$$

Let Λ be the complex-linear functional on X whose real part is g, then by (3.1.1), $\Lambda = f$ on M.

Finally, to every $x \in X$ corresponds an $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that $\alpha \Lambda x = |\Lambda x|$. Hence

$$|\Lambda x| = \Lambda(\alpha x) = g(\alpha x) \le p(\alpha x) = p(x).$$

□

Corollary 3.5. *If X is a normed space and $x_0 \in X$, there exists $\Lambda \in X^*$ such that*

$$\Lambda x_0 = \|x_0\| \text{ and } |\Lambda x| \le \|x\| \text{ for all } x \in X.$$

Proof. If $x_0 = 0$, take $\Lambda = 0$. If $x_0 \neq 0$, apply 3.4, with $p(x) = \|x\|$, M the one-dimensional space generated by x_0, and $f(\alpha x_0) = \alpha \|x_0\|$ on M. □

Separation Here we will apply what we discuss before to show some disjoint convex sets in a topological vector space can be separated.

Theorem 3.6. *Suppose A and B are disjoint, nonempty, convex sets in a topological vector space X.*

1. *If A is open there exist $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that*

$$\operatorname{Re} \Lambda x < \gamma \le \operatorname{Re} \Lambda y$$

for every $x \in A$ and for every $y \in B$.

2. *If A is compact, B is closed, and X is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1 \in \mathbb{R}$, $\gamma_2 \in \mathbb{R}$, such that*

$$\operatorname{Re} \Lambda x < \gamma_1 < \gamma_2 < \operatorname{Re} \Lambda y$$

for every $x \in A$ and for every $y \in B$.

Proof. By (3.1.1), it suffices to prove this for real scalar. (1) Fix $a_0 \in A$, $b_0 \in B$, and put $x_0 = b_0 - a_0$, $C = A - B + x_0$. Then C is a convex neighborhood of 0 in X. Let $p = p(x) := \inf \{t \in \mathbb{R} : x \in tC\}$ be the Minkowski functional of C, where $x \in X$. Since A is open, A is obsorbing, then by theorem 1.35, p satisfies the hypothesis of 3.3. Since $A \cap B = \varnothing$, we know $0 \notin A - B$ which implies that $x_0 \notin C$, and so $p(x_0) \geq 1$.

Define $f(tx_0) = t$ on the subspace M of X generated by x_0. Then

$$\begin{cases} f(tx_0) = t \leq tp(x_0) = p(tx_0) & \text{if } t \geq 0, \\ f(tx_0) < 0 \leq p(tx_0) & \text{if } t < 0. \end{cases}$$

Thus $f \leq p$ on M. By 3.3, f extends to a linear functional Λ on X that also satisfies $\Lambda \leq p$.

In particular, $\Lambda \leq p < 1$ on C and hence $\Lambda \geq -1$ on $-C$, so that $|\Lambda| \leq 1$ on the neighborhood $C \cap (-C)$ of 0. For linear funtionals, boundedness is equivalent to continuity, so $\Lambda \in X^*$.

If now $\alpha \in A$ and $b \in B$, we have

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + x_0) \leq p(a - b + x_0) < 1$$

since $\Lambda x_0 = 1$, $a - b + x_0 \in C$, and C is open.

It follows that $\Lambda(A)$ and $\Lambda(B)$ are disjoint convex subsets of R, with $\Lambda(A)$ to the left of $\Lambda(B)$. Also $\Lambda(A)$ is an open set since A is open and sinceevery nonconstant linear functional on X is an open mapping. Let γ be the right end point of $\Lambda(A)$ to get the conclusion of (1).

(2) There is a convex neighborhood V of 0 in X such that $(A + V) \cap B = \varnothing$. Then we can apply (1), and noting that $\Lambda(A)$ is a compact subset of $\Lambda(A + V)$, it can attain its upper boundary. □

Corollary 3.7. *If X is a locally convex space then X^* separates points on X.*

Theorem 3.8. *Suppose M is a subspace of a locally convex space X, and $x_0 \in X$. If x_0 is not in the closure of M, then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ but $\Lambda x = 0$ for every $x \in M$.*

Proof. 3.6, there exists $\Lambda \in X^*$ such that the real parts of Λx_0 and $\Lambda(M)$ are disjoint. But $\Lambda(M)$ is a subspace of the scalar field, this forces $\Lambda(M) = \{0\}$ and $\Lambda x_0 \neq 0$. □

Theorem 3.9. *Suppose B is a convex, balanced, closed set in a locally convex space X, $x_0 \in X$, but $x_0 \notin B$. Then there exists $\Lambda \in X^*$ such that*

$$|\Lambda x| \leq 1 \textit{ for all } x \in B, \textit{ but } \Lambda x_0 > 1.$$

Proof. By 3.6, we can obtain $\Lambda_1 \in X^*$ with $\Lambda_1 x_0 = re^{i\theta}$ lies outside the closure K of $\Lambda_1(B)$. K is balanced, since B is balanced. Hence there exists s, $0 < s < r$, so that $|z| \leq s$ for all $z \in K$. The functional $\Lambda = s^{-1}e^{-i\theta}\lambda_1$ is the desired. □

3.2 Weak Topologies

Theorem 3.10. *If $\tau_1 \subset \tau_2$ are topologies on a set X, if τ_1 is Hausdorff, and if τ_2 is compact, then $\tau_1 = \tau_2$.*

Example 3.11. Consider the quotient topology τ_N of X/N and the quotient map $\pi : X \to X/N$. By definition, τ_N is the strongest topology on X/N that makes π continuous, and it is the weakest one that makes π an open mapping.

Definition 3.12. Suppose next that X is a set and $\mathcal{F}$ is a nonemty family of mappings $f : X \to Y_f$, where each Y_f is a topological space. Let τ be the collection of all unions of finite intersections of sets $f^{-1}(V)$, with $f \in \mathcal{F}$ and V open in Y_f. Then τ is a topology is a topology on X, and it is in fact the weakest topology on X that makes every $f \in \mathcal{F}$ continuous. This τ is called the *weak topology* on X induced by $\mathcal{F}$, or, $\mathcal{F}$-topology of X.

4 Duality In Banach Spaces

Theorem 4.1. *Suppose X and Y are Banach spaces, and $T \in \mathcal{B}(X,Y)$. Then*

$$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp \text{ and } \mathcal{N}(T) = {}^\perp\mathcal{R}(T^*).$$

Proof. For the first statement:

$$\begin{aligned} y^* \in \mathcal{N}(T^*) &\iff T^*y^* = 0 \iff \langle x, T^*y^* \rangle = 0 \text{ for all } x \\ &\iff \langle Tx, y^* \rangle = 0 \text{ for all } x \iff y^* \in \mathcal{R}(T)^\perp. \end{aligned}$$

For the second statement:

$$\begin{aligned} x \in \mathcal{N}(T) &\iff Tx = 0 \iff \langle Tx, y^* \rangle = 0 \text{ for all } y^* \\ &\iff \langle x, T^*y^* \rangle = 0 \text{ for all } y^* \iff x \in {}^\perp\mathcal{R}(T^*). \end{aligned}$$

□

Corollary 4.2.
1. *$\mathcal{N}(T^*)$ is weak*-closed in Y^*.*
2. *$\mathcal{R}(T)$ is dense in Y if and only if T^* is one-to-one.*
3. *T is one-to-one if and only if $\mathcal{R}(T^*)$ is weak*-dense in X^*.*

Proof. □

Theorem 4.3. *Let U and V be the open unit balls in the Banach spaces X and Y, respectively. If $T \in \mathcal{B}(X,Y)$ and $\delta > 0$, then the implications*

$$(1) \implies (2) \implies (3) \implies (4)$$

hold among the following statements:

1. *$\|T^*y^*\| \geq \delta \|y^*\|$ for every $y^* \in Y^*$.*
2. *$\overline{T(U)} \supseteq \delta V$.*

3. $T(U) \supseteq \delta V$.

4. $T(X) = Y$.

Moreover, if (4) holds, then (1) folds for some $\delta > 0$.

Theorem 4.4. *If X and Y are Banach spaces and if $T \in \mathcal{B}(X, Y)$, then the followings are equivalent:*

1. *$\mathcal{R}(T)$ is closed in Y.*

2. *$\mathcal{R}(T^*)$ is weak*-closed in X^*.*

3. *$\mathcal{R}(T^*)$ is norm-closed in X^*.*

Proof. (2) $\implies$ (3) is clear. Now we show that (1) $\implies$ (2). Suppose (1) holds. ☐

Theorem 4.5. *Suppose X and Y are Banach spaces, and $T \in \mathcal{B}(X, Y)$. Then $\mathcal{R}(T) = Y$ if and only if T^* is one-to-one and $\mathcal{R}(T^*)$ is norm-closed.*

Proof. If $\mathcal{R}(T) = Y$, then T^* is one-to-one by 4.2. By 4.3, T^* is a dilation. By completeness, $\mathcal{R}(T^*)$ is closed.

If the latter statement holds, then $\mathcal{R}(T)$ is dense in Y by 4.2. $\mathcal{R}(T)$ is closed by 4.4. ☐

Definition 4.6. Suppose X and Y are Banach spaces and U is the open unit ball in X. A linear map $T : X \to Y$ is said to be *compact* if the closure of $T(U)$ is compact in Y. It is clear that T is then bounded. Thus $T \in \mathcal{B}(X, Y)$.

Definition 4.7.

1. Suppose X is a Banach space. Then $\mathcal{B}(X)$ is not merely a Banach space but also an algebra: If $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$, one defines $ST \in \mathcal{B}(X)$ by

   $$(ST)(x) := S(T(x)) \quad \text{where } x \in X.$$

   The inequality

   $$\|ST\| \leq \|S\| \, \|T\|$$

   is trivial to verify.

   In particular, powers of $T \in \mathcal{B}(X)$ can be defined: $T^0 = \mathrm{Id}$, the identity mapping on X, given by $\mathrm{Id}\, x = x$, and $T^n = TT^{n-1}$, for $n = 1, 2, 3, \cdots$.

2. An operator $T \in \mathcal{B}(X)$ is said to be *invertible* if there exists $S \in \mathcal{B}(X)$ such that

   $$ST = \mathrm{Id} = TS.$$

   In this case, we write $S = T^{-1}$. By the open mapping theorem, this happens if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = X$.

3. The *spectrum* $\sigma(T)$ of an operator $T \in \mathcal{B}(X)$ is the set of all scalars λ such that $T - \lambda \,\mathrm{Id}$ is not invertible. Thus $\lambda \in \sigma(T)$ if and only if at least one of the following statements is true:

   (a) The range of $T - \lambda \,\mathrm{Id}$ is not all of X.

(b) $T - \lambda\,\mathrm{Id}$ is not one-to-one.

If (2) holds, λ is said to be an *eigenvalue* of T; the corresponding eigenspace is $\mathcal{N}(T - \lambda\,\mathrm{Id})$; each $x \in \mathcal{N}(T - \lambda\,\mathrm{Id})$ is an *eigenvector* of T; it satisfies the equation

$$Tx = \lambda x.$$

Theorem 4.8. *Let X and Y be Banach spaces.*

1. *If $T \in \mathcal{B}(X, Y)$ and $\dim \mathcal{R}(T) < \infty$, then T is compact.*

2. *If $T \in \mathcal{B}(X, Y)$, T is compact, and $\mathcal{R}(T)$ is closed, then $\dim \mathcal{R}(T) < \infty$.*

Proof. (1) (2) □