Contents

1 Introduction

1.1 The Euclidean Plane

In most people's experience, geometry is concerned with properties such as distances, lengths, angles, areas, volumes, and curvature.

Definition 1.1. Two plane figures are *congruent* if one can be transformed into the other by a *rigid motion of the plane*, which is a bijective transformation from the plane to itslef that preserves distances.

In what follows, we will give some theorems we may met before in the fashion "classification theorem" and "local-to-global theorem" to illustrate what we will study in this chapter of the note.

Triangles

Theorem 1.2. *The Euclidean triangles are congruent if and only if the lengths of their corresponding sides are equal.*

Theorem 1.3. *The sum of the interior angles of a Euclidean triangle is* π.

Circles

Theorem 1.4. *Two circles in the Euclidean plane are congruent if and only if they have the same radius.*

Theorem 1.5. *The circumference of a Euclidean circle of radius* R *is* $2\pi R$.

Plane Curve

Definition 1.6. The *curvature* of a plane curve γ is defined to be $\kappa(t) = |\gamma''(t)|$, the length of the acceleration vector, when γ is given a unit-speed parametrization.

It is often convenient for some purposes to extend the definition of the curvature of a plane curve, allowing it to take on both positive and negative values, and

Definition 1.7. We call the generalizaion in the following manner *signed curvature*: by choosing a continuous unit normal vector field N along the curve, and assigning the curvature a positive sign if the curve is turning toward the chosen normal or a negative sign if it is turning away from it.

Theorem 1.8. *Suppose* γ *and* $\tilde{\gamma} : [a, b] \to \mathbb{R}^2$ *are smooth, unit-speed plane curves with unit normal vector fields* N *and* $\tilde{N}$, *and* $\kappa_N(t), \kappa_{\tilde{N}}(t)$ *represent the signed curvatures at* $\gamma(t)$ *and* $\tilde{\gamma}(t)$, *respectively. Then* γ *and* $\tilde{\gamma}$ *are congruent by a direction-preserving congruence if and only if* $\kappa_N(t) = \kappa_{\tilde{N}}(t)$ *for all* $t \in [a, b]$.

Theorem 1.9. *If* $\gamma : [a, b] \to \mathbb{R}^2$ *is a unit-speed simple closed curve such that* $\gamma'(a) = \gamma'(b)$, *and* N *is the inward-pointing normal, then*

$$\int_a^b \kappa_N(t)\,\mathrm{d}t = 2\pi.$$

1.2 Surfaces in Space

Which properties of a surface are intrinsic? Roughly speaking, intrinsic properties are those that could in principle be measured or computed by a 2-dimensional being living entirely within the surface. More precisely, a property of surfaces in $\mathbb{R}^3$ is called *intrinsic* if it is preserved by *isometries*.

Definition 1.10. The curvature of surface in space is described by two numbers at each point, called the *principal curvature*. Here is an informal recipe for computing them. Suppose S is a surface in $\mathbb{R}^3$, p is a point in S, and N is a unit normal vector to S at p.

1. Choose a plane Π passing through p and parallel to N. The intersection of Π with a neighborhood of p in S is a plane curve $\gamma \subseteq \Pi$ containing p.

2. Compute the signed curvature κ_N of γ at p w.r.t. the chosen unit normal N.

3. Repeat this for all normal planes Π. The *principal curvatures* of S at p, denoted by κ_1 and κ_2, are the minimum and maximum signed curvatures so obtained.

Principal curvatures are NOT intrinsic:

Example 1.11. Consider

$$S_1 = \{(x, y, 0) : 0 < x < \pi, 0 < y < \pi\} \text{ and } S_2 = \left\{(x, y, z) : 0 < x < \pi, |y| < 1, z = \sqrt{1 - y^2}\right\}$$

One can check $\kappa_1 = \kappa_2 = 0$ for S_1 and $\kappa_1 = 0, \kappa_2 = 1$ for S_2. But $(x, y, 0) \mapsto (x, \cos y, \sin y)$ is an isometry from S_1 to S_2.

However, the great German mathematician Carl Friedrich Gauss made the surprising discovery that $K = \kappa_1\kappa_2$ is intrinsic, which is now called the *Gaussian curvature*. He thought this result was so amazing that he named it *Theorema Egregium*.

The model spaces of surface theory are the surfaces with constant Gaussian curvature.

1. $K = 0$. The *Euclidean plane* $\mathbb{R}^2$.

2. $K > 0$. The *sphere* of radius R(in this case $K = 1/R^2$).

3. $K < 0$. The *hyperbolic plane*.

We can give a classification theorem and a local-to-global theorem as before:

Theorem 1.12. *Every connected 2-manifold is diffeomorphic to a quotient of one of the constant curvature model surfaces by a discrete group of isometries without fixed points. Thus every connected 2-manifold has a complete Riemannian metric with contant Gaussian curvature.*

Theorem 1.13. *Suppose S is a compact Riemannian 2-manifold. Then*

$$\int_S K \, \mathrm{d}A = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

1.3 Curvature in Higher Dimensions

TODO

2 Riemannian Metrics

2.1 Riemannian Mainfolds

Definition 2.1. Let M be a smooth manifold. A *Riemannian metric* on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2 M$ whose value g_p at each $p \in M$ is an inner product on $g_p(v, v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$. A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g is a specific choice of Riemannian metric on M.

Proposition 2.2. *Every smooth manifold admits a Riemannian metric.*

Definition 2.3. A *Riemannian manifold with boundary* is a pair (M, g), where M is a smooth manifold with boundary and g is a Riemannian metric on M.

Let g be a Riemannian metric on a smooth manifold M with or without boundary. We often use the following angle-bracket notation for $v, w \in T_pM$:

$$\langle v, w \rangle_g = g_p(v, w).$$

2.2 Isometries

Definition 2.4. Suppose (M, g) and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds with or without boundary. An *isometry from* (M, g) to $(\tilde{M}, \tilde{g})$ is a diffeomorphism $\phi : M \to \tilde{M}$ such that $\phi^* \tilde{g} = g$. We say (M, g) and $(\tilde{M}, \tilde{g})$ are *isomorphic* if there exists an isometry between them.

Remark 2.5. To say ϕ is an isometry from (M, g) to $(\tilde{M}, \tilde{g})$ is equivalent to say ϕ is a smooth bijection and each differential $\mathrm{d}\phi_p : T_pM \to T_{\phi(p)}\tilde{M}$ is a linear isometry.

Definition 2.6. If (M, g) are $(\tilde{M}, \tilde{g})$ are Riemannian manifolds, a map $\phi : M \to \tilde{M}$ is a *local isometry* if each point $p \in M$ has a neighborhood U such that $\phi \mid_U$ is an isometry onto an open subset on $\tilde{M}$.

Definition 2.7. A Riemannian n-manifold is said to be *flat* if it is locally isometric to a Euclidean space, that is, if every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric.

Definition 2.8. An isometry from (M, g) to itself is called an *isometry of* (M, g). The set of all isometries of (M, g) is a group under composition, called the *isometry group of* (M, g).

2.3 Local Representations for Metrics

Suppose (M, g) is a Riemannian manifold with or without boundary. If $(x^1, \cdots, x^n)$ are any smooth local coorinates on an open subset $U \subseteq M$, then g can be written locall in U as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

for some collection of n^2 smooth functiosn g_{ij} for $i, j = 1, \cdots, n$.

Using the symmetry of g_{ij}, we compute

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$$

Definition 2.9. A local frame (E_i) for M on open set U is said to be an orthonormal frame if the vectors $E_1 \mid_p, \cdots, E_n \mid_p$ are an orthonormal basis for T_pM at each $p \in U$.

Remark 2.10. Equivalenly, (E_i) is an orthonormal frame if and only if

$$\langle E_i, E_j \rangle = \delta_{ij},$$

in which case g has the local expression

$$g = (\varepsilon^1)^2 + \cdots (\varepsilon^n)^2,$$

where $(\varepsilon^i)^2$ denotes the symmetric product $\varepsilon^i \varepsilon^i = \varepsilon^i \otimes \varepsilon^i$.

Proposition 2.11. *Let (M, g) be a Riemannian n-manifold with or without boundary. If (X_j) is any smooth local frame for TM over an open subset $U \subseteq M$, then there is a smooth orthonormal frame (E_j) over U such that* $\operatorname{span}(E_1 \mid_p, \cdots, E_k \mid_p) = \operatorname{span}(X_1 \mid_p, \cdots, X_k \mid_p)$ *for each $k = 1, \cdots, n$ and each $p \in U$. In particular, for every $p \in M$, there is a smooth orthonormal frame (E_j) defined on some neighborhood of p.*

Definition 2.12. For a Riemannian manifold (M, g) with or without boundary, we definte the *unit tangent bundle* to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM := \left\{ (p, v) \in TM : |v|_g = 1 \right\}.$$

Proposition 2.13. *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a smooth, properly embedded codimension-1 submanifold with boundary in TM, with $\partial(UTM) = \pi^{-1}(\partial M)$, where $\pi : UTM \to M$ is the canonical projection. THe unit tangent bundle is connected if and only if M is connected, and compact if and only if M is compact.*

3 The Levi-Civita Connection

3.1 Eucliean Case

Suppose $\gamma : I \to M$ is a smooth curve. Then γ can be regarded as either a smooth curve in M or a smooth curve in $\mathbb{R}^N$, and a smooth vector field V along γ that takes its values in TM can be regarded as either a vector field along γ in M or a vector field along γ in $\mathbb{R}^n$. Let $\overline{D}_t V$ denote the covariant derivative of V along γ w.r.t. the Euclidean connection $\overline{\nabla}$, and let $D_t^\top V$ denote its covariant derivative along γ denote its covariant derivative along γ w.r.t. the tangential connection $\nabla^\top$.

Proposition 3.1. *Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold, $\gamma : I \to M$ a smooth curve in M, and V a smooth vector field along γ that takes its values in TM. Then for each $t \in I$,*

$$D_t^\top V(t) = \pi^\top(\overline{D}_t V(t)).$$

Corollary 3.2. *Suppose $M \subseteq \mathbb{R}^n$ is an embedded submanifold. A smooth curve $\gamma : I \to M$ is a geodesic w.r.t. the tangential connection on M if and only if its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.*

3.2 Connections on Abstract Riemannian Manifolds

As we can verify that the Euclidean connection on $\mathbb{R}^n$ has one nice property w.r.t. Euclidean metric: it satisfies the product rule

$$\overline{\nabla}_X \langle Y, Z\rangle = \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Y\right\rangle .$$

We define the following:

Definition 3.3. A connection ∇ on TM is said to be *compatible with* g, or to be a *metric connection*, if for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

Proposition 3.4. *Let (M, g) be a Riemannian manifold, and let ∇ be a connection on TM. The following conditions are equivalent:*

1. *∇ is compatible with g:*

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

2. *g is* parallel *w.r.t. ∇: $\nabla g \equiv 0$.*

3. *In terms of any smooth local frame (E_i), the connection coefficients of ∇ satisfy*

$$\Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} = E_k(g_{ij}).$$

4. *If V, W are smooth vector fields along any smooth curve γ, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle .$$

5. *If V, W are parallel vector fields along a smooth curve γ in M, then $\langle V, W\rangle$ is constant along γ.*

6. *Given any smooth curve γ in M, every parallel transport map along γ is a linear isometry.*

7. *Given any smoth curve γ in M, every orthonormal basis at a point γ can be extended to a parallel orthonormal frame along γ.*

Corollary 3.5. *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold with or without boundary, ∇ is a metric connection on M, and $\gamma : I \to M$ is a smooth curve.*

1. *$|\gamma'(t)|$ is constant if and only if $D_t\gamma'(t)$ is orthogonal to $\gamma'(t)$ for all $t \in I$.*

2. *If γ is a geodesic, then $|\gamma'(t)|$ is constant.*

Proposition 3.6. *If M is an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the tangential connection on M is compatible with the induced Riemannian or pseudo-Riemannian metric.*

3.3 Symmetric Connections

As we can verify that the Euclidean connection on $\mathbb{R}^n$ has an another nice property w.r.t. Euclidean metric: it satisfies the product rule

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

Thus, we naturally define the following:

Definition 3.7. We say that a connection ∇ on the tangent bundle of a smooth manifold M is *symmetric* if

$$\nabla_X Y - \nabla_Y X \equiv [X, Y], \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

To study this, we define

Definition 3.8. The *torsion tensor* is the smooth $(1, 2)$-tensor field $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

Proposition 3.9. *If M is an embeded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space, then the tangential connection on M is symmetric.*

Theorem 3.10. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold(with or without boundary). There exists a unique connection ∇ on TM that is compatible with g and symmetric. It is called the* Levi-Civita connection of g.

4 Curvature

4.1 The Curvature Tensor

Definition 4.1. Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and define a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then R is a $(1, 3)$-tensor field on M. For each pair of vector fields $X, Y \in \mathfrak{X}(M)$, then map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X, Y)Z$ is a smooth bundle endomorphism of TM, called the *curvature endomorphism determined by* X and Y. The tensor field R itself is called the *(Riemannian) curvature endomorphism* or the $(1, 3)$-*curvature tensor.*

Remark 4.2. As a $(1, 3)$-tensor field, the curvature endomorphism can be wriiten in terms of any local frame with one upper and three lower indices. For example,

$$R = R_{ijk}{}^l \, \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \partial_l,$$

where the coefficients $R_{ijk}{}^l$ are defined by

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial.$$

Proposition 4.3. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. In terms of any smooth local coorinates, the components of the $(1,3)$-curvature tensor are given by*

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{ik} + \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

Proposition 4.4. *Suppose (M, g) is a smooth Riemannian or pseudo-Riemannian manifold and $\Gamma : J \times I \to M$ is a smooth one-parameter family of curves in M. Then for every smooth vector field V along Γ,*

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \partial_t \Gamma)V.$$

Definition 4.5. We define the (Riemannian) curvature tensor to be the $(0,4)$-tensor field $Rm = R^\flat$ obtained from the $(1,3)$-curvature tensor R by lower its last index.

Remark 4.6. Its action on vector fields is given by

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle_g .$$

In terms of any smooth local coorinates it is written

$$Rm = R_{ijkl}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l,$$

where $R_{ijkl} = R_{lm}R_{ijk}{}^{m}$. Thus 4.3 yields

$$R_{ijkl} = g_{lm}(\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}).$$

Proposition 4.7. *The curvature tensor is a local isometry invariant: if (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\phi : M \to \widetilde{M}$ is a local isometry, then $\phi^*\widetilde{Rm} = Rm$.*

4.2 Flat Manifolds

Definition 4.8. We say that a connection ∇ on a smooth manifold M satisfies the *flatness criterion* if whenever X, Y, Z are smooth vector fields defined on an open set of M, the following identity holds:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

Lemma 4.9. *Suppose M is a smooth mainfold, and ∇ is any connection on M satisfying the flatness criterion. Given $p \in M$ and any vector $v \in T_pM$, there exists a parallel vector field V on a neighborhood of p such that $V_p = v$.*

Theorem 4.10. *A Riemannian or pseudo-Riemannian manifold if flat if and only if its curvature tensor vanishes identically.*

5 The Gauss-Bonnet Theorem

5.1 Some Plane Geometry

Troughout this subsection $\gamma : [a, b] \to \mathbb{R}^2$ is an admissible curve in the plane.

Definition 5.1. We say that γ is a *simple closed curve* if $\gamma(a) = \gamma(b)$ but γ is injective on $[a, b)$. We define the *unit tangent vector field of* γ as the vector field T along each smooth segment of γ given by

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Definition 5.2. If γ is smooth, we define a *tangent angle function for* γ to be a continuous function $\theta : [a, b] \to \mathbb{R}$ such that $T(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in [a, b]$.(Not necessarily unique, but deterimined once its value at any single point is determined) If γ is a continuously differentiable simple closed curve such that $\gamma'(a) = \gamma(b)$, we define the *rotation index* of γ to be the following integer

$$\rho(\gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any tangent angle function for γ.

We would like to extend the definition of the rotation index to certain piecewise regular closed curves.

Definition 5.3. Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is an admissible simple closed curve, and let $(a_0, \cdots, a_k)$ be an admissible partition of $[a, b]$. The points $\gamma(a_i)$ are called the *vertices* of γ, and the curve segments $\gamma\ |_{[a_{i-1},a_i]}$ are called its *edges*.

At each vertex $\gamma(a_i)$ we use $\gamma'(a_i^-)$ and $\gamma'(a_i^+)$ to denote left-hand and right-hand velocity vectors, respectively; let $T(a_i)^-$ and $T_(a_i)^+$ denote the corresponding unit vectors. We classify each vertex into the following manners:

- If $T(a_i^-) \neq \pm T(a_i^+)$, then $\gamma(a_i)$ is an *ordinary vertex*.
- If $T(a_i^-) = T(a_i^+)$, then $\gamma(a_i)$ is an *flat vertex*.
- If $T(a_i^-) = -T(a_i^+)$, then $\gamma(a_i)$ is an *cusp vertex*.

At each ordinary vertex, define the *exterior angle at* $\gamma(a_i)$ to be the oriented measure ε of the angle from $T(a_i^-)$ to $T(a_i^+)$, chosen to be in the interval $(-\pi, \pi)$, with a positive sign if $(T(a_i^-), T(a_i^+))$ is an oriented basis for $\mathbb{R}^2$, and a negative sign otherwise. At cusp vertex, we leave the exterior angle undefined, since there is no simple way to choose unambiguously.

If $\gamma(a_i)$ is an ordinary or a flat vertex, the *interior angle* at $\gamma(a_i)$ is defined to be $\theta_i = \pi - \varepsilon_i$.

Definition 5.4. A *curved polygon* in the plane is an admissible simple closed curve without cusp vertices, whose image is the boundary of a precompact open set $\Omega \subseteq \mathbb{R}^2$. The set Ω is called the *interior* of γ.

Definition 5.5. Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is a curved polygon. If γ is parametrized so that at smooth points, γ' is positively oriented w.r.t. the induced orientation on $\partial\Omega$ in the sense of Stokes's theorem, we say that γ is *positively oriented*.

Definition 5.6. We define a *tangent angle function* for a curved polygon γ to be a piecewise continuous function $\theta : [a, b] \to \mathbb{R}$ that satisfies $T(t) = (\cos\theta(t), \sin\theta(t))$ at each point t where γ is smooth, that is continuous from the right at each α_i with

$$\theta(a_i) = \lim_{t \nearrow a_i} \theta(t) + \varepsilon_i,$$

and that satisfies

$$\theta(b) = \lim_{t \nearrow b} \theta(t) + \varepsilon_k,$$

where ε_k is the exterior angle at $\gamma(b)$.

We define the *rotation-index* of γ to be $\rho(\gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a))$.

Theorem 5.7. *The rotation index of a positively oriented curved polygon in the plane is* $+1$.

5.2 The Gauss-Bonnet Formula

Troughout this subsection, (M, g) is an oriented 2-manifold. Like what we do in the previous subsection, we define the followings:

Definition 5.8. An admissible simple closed curve $\gamma : [a, b] \to M$ is called a *curved polygon* in M if the image of γ is the boundary of a precompact open set $\Omega \subseteq M$, and there is an oriented smooth coorinate disk containing $\overline{\Omega}$ under whose image γ is a curved polygon in the plane.

A curved polygon whose edges are all geodesic segments is called a *geodesic polygon*.

For a curved polygon γ in M, our previous definitions go through almost unchanged. We say that γ is *positively oriented* if it is parametrized in the direction of its Stokes orientation as the boundary of Ω.

On each smooth segment of γ, we define the *unit tangent vector field* $T(t) = \gamma'(t)/\left|\gamma'(t)\right|_g$. If $\gamma(a_i)$ is an ordinary or flat vertex, we define the *exterior angle of* γ at $\gamma(a_i)$ to be

$$\varepsilon := \frac{\mathrm{d}V_g(T(a_i^-), T(a_i^+))}{\left|\mathrm{d}V_g(T(a_i^-), T(a_i^+))\right|} \arccos \left\langle T(a_i^-), T(a_i^+) \right\rangle_g.$$

The corresponding *interior angle* of γ at $\gamma(a_i)$ is $\theta_i = \pi - \varepsilon_i$.

Definition 5.9. Suppose $\gamma : [a, b] \to M$ is a curved polygon and Ω is its interior, and let (U, ϕ) be an oriented smooth chart such that U contains $\overline{\Omega}$. Applying the Gram-Schmidt algorithm to (∂_x, ∂_y), we can find an oriented orthonormal frame (E_1, E_2).

We define a *tangent angle function for* γ to be a piecewise continuous function $\theta : [a, b] \to \mathbb{R}$ that satisfies

$$T(t) = \cos\theta(t) E_1 \mid_{\gamma(t)} + \sin\theta(t) E_2 \mid_{\gamma(t)}$$

at each t where γ' is continuous, and that is continuous from the right like 5.6.

The *rotation index* of γ is $\rho(\gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a))$.

6 Jacobi Fields

6.1 Conjugate Points

Definition 6.1. Given a geodesic segment $\gamma : [a, c] \to M$, we say that γ has a *conjugate point* if there is some $b \in (a, c]$ such that $\gamma(b)$ is conjugate to $\gamma(a)$ along γ, and γ has an *interior conjugate point* if there is such a $b \in (a, c)$.

Theorem 6.2. *Let (M, g) be a Riemannian manifold and $p, q \in M$. If γ is a unit-speed geodesic segment from p to θ that has an interior conjugate point, then there exists a proper normal vector field X along γ such that $I(X, X) < 0$. Therefore, γ is not minimizing.*

Lemma 6.3. *Let $\gamma : [a, b] \to M$ be a geodesic segment, and suppose J_1 and J_2 are Jacobi fields along γ. Then $\langle D_t J_1(t), J_2(t)\rangle - \langle J_1(t), D_t J_2(t)\rangle$ is constant along γ.*

Theorem 6.4. *Let (M, g) be a Riemannian manifold. Suppose $\gamma : [a, b] \to M$ is a unit-speed geodesic segment without interior conjgate points. If V is any proper normal piecewise smooth vector field along γ, then $I(V, V) \geq 0$, with equality if and only if V is a Jacobi field. In particular, if $\gamma(b)$ is not conjugate to $\gamma(a)$ along γ, then $I(V, V) > 0$.*

Proof. After replacing t by $t - a$, assume that $a = 0$. Let $p = \gamma(0)$, and let $(w_1, \cdots, w_n)$ be an orthonormal basis for T_pM such that $w_1 = \gamma'(0)$. For each $\alpha = 2, \cdots, n$, let J_α be the unique normal Jacobi field along γ satisfying $J_\alpha(0) = 0$ and $D_t J_\alpha(0) = w_\alpha$. By assumption, no nonzero linear combination of the $J_\alpha(t)$'s can vanish for any $t \in (0, b)$, and thus $(J_\alpha(t))$ forms a basis for the orthogonal complement of $\gamma'(t)$ in $T_{\gamma(t)}M$ for each such t. Hence, we can write

$$V(t) = v^\alpha(t) J_\alpha(t)$$

for some piecewise smooth functions $v^\alpha : (0, b) \to M$.

In fact, each function v^α has a piecewise smooth extension to $[0, b]$. Let (x^i) be the normal coorinates centered at p determined by the basis (w_i). For sufficiently small $t > 0$, we can express $J_\alpha(t)$ and $V(t)$ in normal coorinates as

$$\begin{aligned} J_\alpha(t) &= t\frac{\partial}{\partial x_\alpha} |_{\gamma(t)}, \quad \alpha = 2, \cdots, n \\ V(t) &= v^\alpha(t) J_\alpha(t) = t v^\alpha(t) \frac{\partial}{\partial x_\alpha} |_{\gamma(t)} . \end{aligned}$$

Because V is smooth on $[0, \delta)$ for some $\delta > 0$ and $V(0) = 0$, it follows from Taylor's theorem that the component of extend smoothly to $[0, \delta)$, which shows that v^α is smooth here. For b, it follows similarly.

Let $(a_0, \cdots, a_k)$ be an admissible partition for V. On each subinterval (a_{i-1}, a_i) where V is smooth, define vector fields X and Y along γ by

$$X = v^\alpha D_t J_\alpha, \quad Y = \dot{v}^\alpha J_\alpha.$$

Then $D_t V = X + Y$ for each intervals. And the fact that V is piecewise smooth implies that $D_t V, X$ and Y extend smoothly to $[a_{i-1}, a_i]$ for each i, with one-sided derivatives at the enpoints.

To compute $I(V,V)$, we need one more equation on each subintervals $[a_{i-1},a_i]$:

$$|D_tV|^2 - Rm(V,\gamma',\gamma',V) = \frac{\mathrm{d}}{\mathrm{d}t}\langle V,X\rangle + |Y|^2 .$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V,X\rangle = \langle D_tV,X\rangle + \langle V,D_tX\rangle = \langle X+Y,X\rangle + \langle V,D_tX\rangle .$$

The Jacobi equation gives

$$\begin{aligned} D_tX &= \dot{v}^\alpha D_tJ_\alpha + v^\alpha D_t^2 J_\alpha = \dot{v}^\alpha D_tJ_\alpha - v^\alpha R(J_\alpha,\gamma')\gamma' \\ &= \dot{v}^\alpha D_tJ_\alpha - R(V,\gamma')\gamma'. \end{aligned}$$

Therefore, $\langle D_tX,V\rangle = \left\langle \dot{v}^\alpha D_tJ_\alpha, v^\beta J_\beta\right\rangle - Rm(V,\gamma',\gamma',V)$. By 6.3 and $\langle D_tJ_\alpha,J_\beta\rangle = \langle J_\alpha, D_tJ_\beta\rangle = 0$ at $t=0$, we know $\langle D_tJ_\alpha,J_\beta\rangle = \langle J_\alpha,D_tJ_\beta\rangle = 0$ all along γ. Hence

$$\begin{aligned} \left\langle \dot{v}^\alpha D_tJ_\alpha, v^\beta J_\beta\right\rangle &= \dot{v}^\alpha v^\beta \langle D_tJ_\alpha,J_\beta\rangle = \dot{v}^\alpha v^\beta \langle J_\alpha, D_tJ_\beta\rangle \\ &= \left\langle \dot{v}^\alpha J_\alpha, v^\beta D_tJ_\beta\right\rangle = \langle Y,X\rangle . \end{aligned}$$

So $\langle D_tX,V\rangle = \langle Y,X\rangle - Rm(V,\gamma',\gamma',V)$, and then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle V,X\rangle &= \langle X+Y,X\rangle + \langle Y,X\rangle - Rm(V,\gamma',\gamma',V) \\ &= |X+Y|^2 - |Y|^2 - Rm(V,\gamma',\gamma',V), \end{aligned}$$

which is the equation we want.

Now, by the fundamental theorem of calculus to compute

$$\begin{aligned} I(V,V) &= \sum_{i=1}^k \int_{a_{i-1}}^{a_i} \left(|D_tV|^2 - Rm(V,\gamma',\gamma',V)\right)\mathrm{d}t \\ &= \sum_{i=1}^k \langle V,X\rangle \Big|_{t=a_{i-1}}^{t=a_i} + \int_0^b |Y|^2\,\mathrm{d}t, \end{aligned}$$

where the first term on the RHS cancels by the continuity of V and X. If $I(V,V)=0$ then $Y\equiv 0$ on $(0,b)$. Then $\dot{v}^\alpha \equiv 0$, so each v^α is constant. V is a linear combination of Jacobi fields, and hence it is so. □

Corollary 6.5. *Let (M,g) be a Riemannian manifold, and let $\gamma:[a,b]\to M$ be a unit speed geodesic segment.*

1. *If γ has an interior conjugate point, then it is not minimizing.*
2. *If $\gamma(a)$ and $\gamma(b)$ are conjugate but γ has no interior conjugate points, then for every proper normal variation Γ of γ, the curve Γ_s is strictly longer than γ for all sufficiently small nonzero s unless the variation field of Γ is a Jacobi field.*
3. *If γ has no conjugate points, then for every proper normal variation Γ of γ, the curve Γ_s is strictly longer than γ for all sufficiently small nonzero s.*

6.2 Cut Points

Definition 6.6. Suppose (M, g) is a complete, connected Riemannian manifold, p is a point of M, and $v \in T_pM$. Define the *cut time of* (p, v) by

$$t_{\text{cut}}(p, v) = \sup\{b > 0 : \text{the restriction of } \gamma_v \text{ to } [0, b] \text{ is minimizing}\},$$

where γ_v is the maximal geodesic starting at p with initial velocity v.

If $t_{\text{cut}}(p, v) < \infty$, the *cut point of p along γ_v* is the point $\gamma_v(t_{\text{cut}}(p, v)) \in M$. The *cut locus of p*, denoted by $\text{Cut}(p)$, is the set of all $q \in M$ such that q is the cut point of p along some geodesic.

Proposition 6.7. *Suppose (M, g) is a complete, connected Riemannian manifold, $p \in M$, and v is a unit vector in T_pM. Let $c = t_{cut}(p, v) \in (0, \infty]$.*

1. *If $0 < b < c$, then $\gamma_v \mid_{[0,b]}$ has no conjugate points and is the unique unit-speed minimizing curve between its endpoints.*

2. *If $c < \infty$, then $\gamma_v \mid_{[0,c]}$ is minimizing, and one or both of the following conditions are true.*

   - *$\gamma_v(c)$ is conjugate to p along γ_v.*
   - *There are two or more unit-speed minimizing geodesics from p to $\gamma_v(c)$.*

Proof. Suppose first that $0 < b < c$. Then the minimizing statement follows from 6.2. To see that $\gamma_v \mid_{[0,b]}$ is the unique unit-speed minimizing curve, suppose for the sake of contradiction that $\sigma : [0, b] \to M$ is another. Then $\sigma'(b) \neq \gamma_v'(b)$ by the uniqueness of the geodesics. Define a new unit-speed admissble curve $\tilde{\gamma} : [0, b'] \to M$ for some $b < b' < c$ that is equal to $\gamma(t)$ for $t \in [0, b]$ and equal to $\gamma_v(t)$ for $t \in [b, b']$. Then $\tilde{\gamma}$ is minimizing but not smooth, and hence not geodesic, a contradiction.

Suppose $c < \infty$. □

Theorem 6.8. *Suppose (M, g) is a complete, connected Riemannian manifold. The function $t_{cut} : UTM \to (0, \infty]$ is continuous.*

Definition 6.9. Given $p \in M$, we define two subsets of T_pM as follows: the *tangent cut locus of p* is the set

$$\text{TCL}(p) = \{v \in T_pM : |v| = t_{\text{cut}}(p, v/|v|)\},$$

and the *injective domain of p* is

$$\text{ID}(p) = \{v \in T_pM : |v| < t_{\text{cut}}(p, v/|v|)\}.$$

It is immediate that $\text{TCL}(p) = \partial\,\text{ID}(p)$ is $\text{Cut}(p) = \exp_p(\text{TCL}(p))$.

Theorem 6.10. *Let (M, g) be a complete, connected Riemannian manifold and $p \in M$.*

1. *The cut locus of p is a closed subset of M of measure zero.*

2. *The restriction of $\exp_p$ to $\overline{\text{ID}(p)}$ is surjective.*

3. *The restriction of* $\exp_p$ *to* $\mathrm{ID}(p)$ *is a diffeomorphism onto* $M \setminus \mathrm{Cut}(p)$.

Proof. The closeness of $\mathrm{Cut}(p)$ follows from 6.8 and the preimages of the points of the sequence in $\mathrm{Cut}(p)$ is bounded. The fact that $\mathrm{Cut}(p)$ is measure zero follows from $\mathrm{TCL}(p)$ can expressed locally as the graph of the continuous function $r = t_{\mathrm{cut}}(p, (\theta^1, \cdots, \theta^{n-1}))$ in polar coorinates and the Sard'theorem.

The second statement follows from the lemma of Hopf-Rinow's theorem.

For the third statement, the local diffeomorphism follows from the equivalent definition of conjugate points and the injectivity follows from 6.7. □

Corollary 6.11. *Every compact, connected, smooth* n*-manifold is homeomorphic to a quotient space of* $\overline{\mathbb{B}}^n$ *by an equivalence relation that identifies only points on the boundary.*

7 Curvature and Topology

7.1 Cartan-Hadamard Theorem

Theorem 7.1 (Cartan-Hadamard)**.** *If* (M, g) *is a complete, connected, Riemannian manifold with nonpositive sectional curvature, then for every point* $p \in M$*, the map* $\exp_p : T_pM \to M$ *is a smooth covering map. Thus the universal covering space of* M *is diffeomorphic to* $\mathbb{R}^n$*, and if* M *is simply connected, then* M *itself is diffeomorphic to* $\mathbb{R}^n$.

Hence, we can define

Definition 7.2. A complete, simply connected Riemannian manifold with nonpositive sectional curvature is called a *Cartan-Hadamard manifold.*

Definition 7.3. A *line* in a Riemannian manifold is the image of a nonconstant geodesic that is defined on all of $\mathbb{R}$ and restricts to a minimizing segment between any two of its points.

Proposition 7.4. *Suppose* (M, g) *is a Cartan-Hadamard manifold.*

1. *The injectivity radius of* M *is infinite.*

2. *The image of every nonconstant maximal geodesic in* M *is a line.*

3. *Any two distinct points in* M *are contained in a unique line.*

4. *Every open or closed metric ball in* M *is a geodesic ball.*

5. *For every point* $q \in M$*, the function* $r(x) = d_g(q, x)$ *is smooth on* $M \setminus \{q\}$ *and* $r(x)^2$ *is smooth on all of* M.

Definition 7.5. Suppose A, B, C are three points in a Cartan-Hadamard manifold that are *noncollinear*, meaning that they are not all contained in a single line. The *geodesic triangle* $\triangle ABC$ determined by the points is the union of the images of the geodesic segments connecting the three points.

If $\triangle ABC$ is a geodesic triangle, we denote the angle in T_AM formed by the initial velocities of the geodesic segments from A to B and A to C by $\angle A$ or $\angle CAB$ if necessary to avoind ambiguity, and similarly for the other angles.

Proposition 7.6. *Suppose $\triangle ABC$ is a geodesic triangle in a Cartan-Hadamard manifold (M, g), and let a, b, c denote the lengths of the segments opposite the vertices A, B and C, respectively. The following inequalities hold:*

1. $c^2 \geq a^2 + b^2 - 2ab\cos\angle C$.
2. $\angle A + \angle B + \angle C \leq \pi$.

Corollary 7.7. *No simply connected compact manifold admits a metric of nonpositive sectional curvature.*

Corollary 7.8. *Suppose M and N are positive-dimensional compact, connected smooth manifolds, at least one of which is simply connected. Then $M \times N$ does not admit any Riemannian metric of nonpositive sectional curvature.*

7.2 Cartan's Torsion Theorem

Lemma 7.9. *Suppose (M, g) is a Cartan-Hadamard manifold. Given $q \in M$, let $f : M \to [0, \infty)$ be the function $f(x) = \frac{1}{2}d_g(x, q)^2$. Then f is strictly goedesically convex, in the sense that for every geodesic segment $\gamma : [0, 1] \to M$, the following inequality holds for all $t \in (0, 1)$:*

$$f(\gamma(t)) < (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Lemma 7.10. *Suppose (M, g) is a Cartan-Hadamard manifold and S is a compact subset of M containing more than one point. Then there is a unique closed ball of minimum radius containing S.*

Let us call the center of the smallest enclosing ball the 1-*center* of the set S in 7.10.

Theorem 7.11 (Cartan's Fixed-Point Theorem). *Suppose (M, g) is a Cartan-Hadamard manifold and G is a compact Lie group acting smoothly and isometrically on M. Then G has a fixed point in M, that is, a point $p_0 \in M$ such that $\phi.p_0 = p_0$ for all $\phi \in G$.*

Proof. Let $q_0 \in M$ be arbitrary, and let $S = G.q_0$. If $\#S = 1$, then q_0 is a fixed point. So we may assume $\#S > 1$. S is a continuous image of a compact set, hence S is compact. By 7.10, there exists a unique smallest closed geodesic ball containing S, denoted its center by p_0 and its radius by c_0.

Now let $\phi_0 \in G$ be arbitrary, then $\phi_0.S = S$. G acts by isometries, so $\phi_0.\overline{B}_{c_0}(p_0) = \overline{B}_{c_0}(\phi_0.p_0)$. By the uniqueness of S, $\phi_0 p_0 = p_0$. p_0 is fixed, since ϕ_0 is arbitrary. □

Corollary 7.12 (Cartan's Torsion Theorem). *Suppose (M, g) is a complete, connected Riemannian manifold with nonpositive sectional sectional curvature. Then $\pi_1(M)$ is torsion-free.*

7.3 Preissman's Theorem

Definition 7.13. Suppose (M, g) is a complete Riemannian manifold and $\phi : M \to M$ is an isometry. A geodesic $\gamma : \mathbb{R} \to M$ is called an *axis for ϕ* if ϕ restricts to a nontrivial translation along γ, that is, if there is a nonzero constant c such that $\phi(\gamma(t)) = \gamma(t + c)$ for all $t \in \mathbb{R}$. An isometry with no fixed points that has an axis is said to be *axial*.

Lemma 7.14. *Suppose (M, g) is a compact, connected Riemannian manifold, and $\pi : \widetilde{M} \to M$ is its universal covering manifold endowed with the metric $\tilde{g} = \pi^* g$. Then every covering automorphism of π has an axis, which restricts to a lift of a closed geodesic in M that is the shortest admissible path in its free homotopy class.*

Lemma 7.15. *Suppose (M, g) is a Cartan-Hadamard manifold with strictly negative sectional curvature. If $\phi : M \to M$ is an axial isometry, then its axis is unique up to reparametrization.*

Theorem 7.16 (Preissman)**.** *If (M, g) is a compact, connected Riemannian manifold with strictly nagative sectional curvature, then every nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

Corollary 7.17. *No product of positive-dimensional connected compact manifolds admits a metric of strictly negative sectional curvature.*

7.4 Theorems about Positive Curvature

Theorem 7.18 (Myers)**.** *Let (M, g) be a complete, connected Riemannian n-manifold, and suppose that there is a positive constant R such that the Ricci curvature of M satisfies $Rc(v, v) \geq (n-1)/R^2$ for all unit vectors v. Then M is compact, with diameter less than or equal to πR, and its fundamental group is finite.*

Corollary 7.19. *Suppose (M, g) is a compact, connected Riemannian n-manifold whose Ricci tensor is positive definite everywhere. Then M has finite fundamental group.*

Proof. By condition, the unit tangent bundle of M is compact, and hence $Rc(v, v) \geq c$ for all unit tangent vectors v. Apply 7.18. $\square$

Corollary 7.20. *If (M, g) is a complete Einstein manifold with positive scalar curvature, then M is compact.*

Theorem 7.21 (Cheng's Maximal Diameter Theorem)**.** *Let (M, g) be a complete, connected Riemannian n-manifold, and suppose that there is a positive constant R such that the Ricci curvature of M satisfies $Rc(v, v) \geq (n-1)/R^2$ for all unit vectors v. And* $\operatorname{diam}(M) = \pi R$. *Then (M, g) is isometric to $(\mathbb{S}^n(R), g^\circ_R)$.*

Theorem 7.22 (Synge)**.** *Suppose (M, g) is a compact, connected Riemannian n-manifold with strictly positive sectional curvature.*

1. *If n is even and M is orientable, then M is simply connected.*

2. *If n is odd, then M is orientable.*