1 The Exponential Map

1.1 One-Parameter Subgroup and the Exponential Map

One-Parameter Subgroups

Definition 1.1. A *one-parameter subgroup of G* is defined to be a Lie homomorphism $g : \mathbb{R} \to G$ with $\mathbb{R}$ considered as a Lie group under addition.

Theorem 1.2. *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector field starting at the identity.*

Definition 1.3. Given $X \in \mathrm{Lie}(G)$, the one-parameter subgroup determined by X in this way is called the *one-parameter subgroup generated by X*.

The one-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$ are not hard to compute explicitly.

Proposition 1.4. *For any $A \in \mathfrak{gl}(n, \mathbb{R})$, let*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \cdots .$$

This series converges to an invertible matrix $e^A \in \mathrm{GL}(n, \mathbb{R})$, and the one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by $A \in \mathfrak{gl}(n, \mathbb{R})$ is $\gamma(t) = e^{tA}$.

We would like to compute the one-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$, such as $O(n)$.

Proposition 1.5. *Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in $T_e H$.*

The Exponential Map

Definition 1.6. Given a Lie group G with Lie algebra $\mathfrak{g}$, we define a map $\exp : \mathfrak{g} \to G$, called the *exponential map* of G, as follows: for any $X \in \mathfrak{g}$, we set

$$\exp X = \gamma(1),$$

where γ is the one-parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity.

Proposition 1.7. *Let G be a Lie group. For any $X \in \mathrm{Lie}(G), \gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X.*

Proposition 1.8. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

1. *The exponential map is a smooth map from $\mathfrak{g}$ to G.*
2. *For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}, \exp(s + t)X = \exp sX \exp tX$.*
3. *For any $X \in \mathfrak{g}, (\exp X)^{-1} = \exp(-X)$.*

4. *For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}, (\exp X)^n = \exp(nX)$.*

5. *The differential $(\mathrm{d}\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself.*

6. *The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.*

7. *If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi : G \to H$ is a Lie group homomorphism, the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"'] & H \end{tikzcd}$$

8. *The flow θ of a left-invariant vector field X is given by $\theta_t = R_{\exp tX}$ which is the right multiplication by $\exp tX$.*

Proposition 1.9. *Let G be a Lie group, and let $H \subseteq G$ be a Lie subgroup. With $\mathrm{Lie}(H)$ considered as a subalgebra of $\mathrm{Lie}(G)$ in the usual way, the exponential map of H is the restriction to $\mathrm{Lie}(H)$ of the exponential map of G, and*

$$\mathrm{Lie}(H) = \{X \in \mathrm{Lie}(G) : \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$