1 Introduction

1.1 Partial Differential Equations

Definition 1.1. A partial differential equation is a relation of the following type:

$$F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n},, u_{x_1x_2} \cdots) = 0, \tag{1.1.1}$$

where the unknown $u = u(x_1, \cdots, x_n)$ is a function of n variables and $u_{x_j}, \cdots, u_{x_ix_j}, \cdots$ are its partial derivatives. The highest order of differentiation occuring in the equation is the *order* of the equation.

A first important distinction is between *linear* and *nonlinear* equations.

Definition 1.2. Equation (1.1.1) is *linear* if F is linear w.r.t. u and all its derivatives, otherwise it is *nonlinear*.

A second distinction concerns the types of nonlinearitly.

Definition 1.3.
1. *Semilinear* equations when F is nonlinear only w.r.t. u but is linear w.t.r. to all its derivatives, with coefficients depending only on $\mathbf{x} = (x_1, \cdots, x_n)$.
2. *Quasi-linear* equations when F is linear w.r.t. the highest order derivatives of u, with coefficients depending only on $\mathbf{x}, u$ and lower order derivatives.
3. *Fully nonlinear* equations when F is nonlinear w.r.t. the highest order derivatives of u.

1.2 Well Posed Problems

2 Linear PDE

2.1 Transport Equation

We write $Du = D_x u = (u_{x_1}, \cdots, u_{x_n})$ for the gradient of u w.r.t. spatial variables x. The *transport equation* with constant coefficients, is the PDE

$$u_t + b \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0, \infty), \tag{2.1.1}$$

where b is a fixed vector in $\mathbb{R}^n$, $b = (b_1, \cdots, b_n)$, and $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the unknown.

Define $z(s) := u(x + sb, t + s)(s \in \mathbb{R})$, by (2.1.1), we then calculate

$$\dot{z}(s) = Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = 0. \tag{2.1.2}$$

Initial-value Problem For definition therefore, let us consider the initial-value problem

$$\begin{cases} u_t + b \cdot Du & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times 0. \end{cases} \tag{2.1.3}$$

From (2.1.2), we know that $u(x - tb, 0) = g(x - tb)$, thus

$$u(x, t) = g(x - tb)(x \in \mathbb{R}^n, g \geq 0). \tag{2.1.4}$$

Nonhomogeneous Problem Next let us look at the associated nonhomogeneous problem

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n(0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times 0. \end{cases} \tag{2.1.5}$$

Still, we set $z(s) := u(x+sb, t+s)$ for $s \in \mathbb{R}$, then

$$\dot{z}(s) = Du(x+sb, t+s) \cdot b + u_t(x+sb, t+s) = f(x+sb, t+s).$$

Consequently,

$$\begin{aligned} u(x,t) - g(x-tb) = z(0) - z(-t) &= \int_{-t}^{0} \dot{z}(s)\,\mathrm{d}s \\ &= \int_{-t}^{0} f(x+sb, t+s)\,\mathrm{d}s \\ &= \int_{0}^{t} f(x+(s-t)b, s)\,\mathrm{d}s, \end{aligned}$$

and so

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\,\mathrm{d}s (x \in \mathbb{R}^n, t \geq 0) \tag{2.1.6}$$

solves the initial-value problem (2.1.5).

2.2 Laplace's Equation

Among the most important of all partial differential equations are undoubtedly *Laplace's equation*

$$\Delta u = 0 \tag{2.2.1}$$

and *Poisson's equation*

$$-\Delta u = f. \tag{2.2.2}$$

In both (2.2.1) and (2.2.2), $x \in U$ and unknown is $u : \overline{U} \to \mathbb{R}, u = u(x)$, where $U \subseteq \mathbb{R}^n$ is a given open set. In (2.2.2), the function $f : U \to \mathbb{R}$ is also given.

Definition 2.1. A C^2 function u satisfying (2.2.1) is called a harmonic function.

Fundamental Solution We attempt to find a solution u of Laplace's equation (2.2.1) in $U = \mathbb{R}^n$, having the form

$$u(x) = v(r),$$

where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and v is to be selected so that $\Delta u = 0$ holds. First note for $i = 1, \cdots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-1/2} 2x_i = \frac{x_i}{r}.$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, \quad u_{x_ix_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1, \cdots, n$, and so

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r).$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0. \tag{2.2.3}$$

If $v' \neq 0$, we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r},$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b\log r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3, \end{cases}$$

where b and c are constants. Hence we define

Definition 2.2. The function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3, \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the *fundamental solution* of Laplace's equation.

Remark 2.3. Observe that we have the estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \quad \left|D^2\Phi(x)\right| \leq \frac{C}{|x|^n}(x \neq 0)$$

for some constant $C > 0$.

Poisson's Equation If we shift the origin to a new point y, the Laplace's equation (2.2.1) is unchanged. And so $x \mapsto \Phi(x-y)$ is also harmonic as a function of x, $x \neq y$. Now take $f : \mathbb{R}^n \to \mathbb{R}$, and note that the mapping $x \mapsto \Phi(x-y)f(y) \quad (x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points y.

What above discussed suggests that the convolution

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y \\ &= \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log(|x-y|)f(y)\,\mathrm{d}y & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}\,\mathrm{d}y & (n \geq 3) \end{cases} \end{aligned} \tag{2.2.4}$$

will solve (2.2.1). But this is wrong: $D^2\Phi(x-y)$ may be ∞. However, when we assume that $f \in C_c^2(\mathbb{R}^n)$, (2.2.4) gives a solution for (2.2.2).

Theorem 2.4. *Define u by (2.2.4) with $f \in C_c^2(\mathbb{R}^n)$. Then*

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$ *in* $\mathbb{R}^n$.

We consequently see that (2.2.4) provides us with a formula for a solution of Poisson's equation (2.2.2) in $\mathbb{R}^n$.

Proof. By direct computation(definition), we can find

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y)\,\mathrm{d}y \text{ and } u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y)\,\mathrm{d}y,$$

for $i, j = 1, \cdots, n$. Hence $u \in C^2(\mathbb{R}^n)$.

In what follows, we use C to denote some constants if without ambiguity. Fix $\varepsilon > 0$, then

$$\begin{aligned}\Delta u(x) &= \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y + \int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y \\ &=: I_\varepsilon + J_\varepsilon.\end{aligned}$$

One can check that

$$|I_\varepsilon| \leq \begin{cases} C\varepsilon^2 \left|\log \varepsilon\right| & n = 2 \\ C\varepsilon^2 & n \geq 3. \end{cases}$$

By Green's identity,

$$\begin{aligned}J_\varepsilon &= \int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^n - B(0,\varepsilon)} D\Phi(y)\cdot D_y f(x-y)\,\mathrm{d}y + \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)\,\mathrm{d}S(y) \\ &=: K_\varepsilon + L_\varepsilon,\end{aligned}$$

where ν denote the inward pointing unit normal along $\partial B(0,\varepsilon)$. We readily check

$$|L_\varepsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\,\mathrm{d}S(y) \leq \begin{cases} C\varepsilon \left|\log\varepsilon\right| & n = 2 \\ C\varepsilon & n \geq 3. \end{cases}$$

We use the Green identity one again

$$\begin{aligned}K_\varepsilon &= \int_{\mathbb{R}^n - B(0,\varepsilon)} \Delta\Phi(y) f(x-y)\,\mathrm{d}y - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y) \\ &= -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y),\end{aligned}$$

since Φ is harmonic away from the origin. Now $D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$, $y \neq 0$, and $\nu = \frac{-y}{|y|} = -\frac{y}{\varepsilon}$ on $\partial B(0,\varepsilon)$. Consequently, $\frac{\partial\Phi}{\partial\nu}(y) = \nu \cdot D\Phi(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$ on $\partial B(0,\varepsilon)$. Since

$n\alpha(n)\varepsilon^{n-1}$ is the surface area of the sphere $\partial B(0,\varepsilon)$, we have

$$\begin{aligned} K_\varepsilon &= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\,\mathrm{d}S(y) \\ &= -\fint_{\partial B(x,\varepsilon)} f(y)\,\mathrm{d}S(y) \to -f(x) \text{ as } \varepsilon \to 0. \end{aligned}$$

□

Mean-Value Formulas

Theorem 2.5 (Mean-value formula for Laplace equation)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set, and $B(x,r) \subseteq U$. If $u \in C^2(U)$ is harmonic, then*

$$u(x) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u\,\mathrm{d}S = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u\,\mathrm{d}y \tag{2.2.5}$$

for each ball $B(x,r) \subseteq U$.

Proof. Set

$$\phi(r) := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)\,\mathrm{d}S(z).$$

Then

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} Du(x+rz)\cdot z\,\mathrm{d}S(z),$$

and consequently, using Green's formulas, we compute

$$\begin{aligned} \phi'(r) &= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} Du(y)\cdot \frac{y-x}{r}\,\mathrm{d}S(y) \\ &= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\,\mathrm{d}S(y) \\ &= \frac{r}{n}\frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\,\mathrm{d}y = 0 \end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0}\phi(t) = \frac{1}{n\alpha(n)r^{n-1}} \lim_{t\to 0} \int_{\partial B(x,t)} u(y)\,\mathrm{d}S(y) = u(x).$$

Then

$$\begin{aligned} \int_{B(x,r)} u\,\mathrm{d}y &= \int_0^r \left(\int_{\partial B(x,s)} u\,\mathrm{d}S \right) \mathrm{d}s \\ &= u(x) \int_0^r n\alpha(n)s^{n-1}\,\mathrm{d}s = \alpha(n)r^n u(x). \end{aligned}$$

□

Its inverse is also true.

Theorem 2.6. *If $u \in C^2(U)$ satisfies*

$$u(x) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u \,\mathrm{d}S$$

for each ball $B(x,r) \subseteq U$, then u is harmonic.

Proof. If $\Delta u \not\equiv 0$, there exists some ball $B(x,r) \subseteq U$ such that, say, $\Delta u > 0$ within $B(x,r)$. But

$$0 = \phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \frac{r}{n} \Delta u(y) \,\mathrm{d}y > 0,$$

a contradiction. □

Properties of Harmonic Functions

Theorem 2.7 (Strong Maximum Principle). *Suppose $u \in C^2(U) \cap C(\overline{U})$ is harmonic within U.*

1. *Then*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

2. *Furthermore, if U is connected and there exists a point $x_0 \in U$ such that*

$$u(x_0) = \max_{\overline{U}} u,$$

then u is constant within U.

Replacing u by $-u$, we recover also similar assertions with "min" replacing "max".

Proof. Suppose there exists a point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u =: M$. Then for $0 < r < \mathrm{dist}(x_0, \partial U)$, the mean-value property asserts

$$M = u(x_0) = \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u \,\mathrm{d}y \leq M.$$

As equality holds only if $u \equiv M$ within $B(x_0, r)$. Hence, (2) is proved. And (1) follows from (2). □

Corollary 2.8. *If U is connected and $u \in C^2(U) \cap C(\overline{U})$ satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

where $g \geq 0$, then u is positive everywhere in U if g is positive somewhere on ∂U.

Theorem 2.9 (Uniqueness). *Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the boundary-value problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Proof. Apply 2.7. □

Theorem 2.10 (Smoothness). *If $u \in C(U)$ satisfies the mean-value property (2.2.5) for each ball $B(x,r) \subseteq U$, then*

$$u \in C^\infty(U).$$