# Finite Element Method Homework

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日期：2025 年 2 月 5 日

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## 目录

# 1 Gauss-Legendre quadrature

## 1.1 实现 Gauss-Legendre 求积公式

首先，我们编写求解 $n$ 阶 Gauss-Legendre 型求积节点与求积系数的子函数 **get_gauss_point_weight**，这里求积节点 $x_k(k=1,\cdots,n)$ 与求积系数 $A_k(k=1,\cdots,n)$ 满足:

$$P_{n+1}(x_k)=0, A_k=\frac{2}{(1-x_k^2)[P'_{n+1}(x_k)]^2}.$$

其中

$$P_{n+1}(x)=\frac{1}{(n+1)!2^{n+1}}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{x+1}}(x^2-1)^{n+1}.$$

**Listing 1:** get_gauss_point_weight

```
% n 为求积阶数
function [point,weight] = get_gauss_point_weight(n)
syms x;
N = 10;% 有效数字位数
Poly = diff((x.^2 - 1).^(n + 1),x,n + 1) ./ (factorial(n + 1) .* 2 .^ (n + 1));
Poly_d = matlabFunction(diff(Poly,x));
point = vpa(solve(Poly),N);
weight = vpa(2 ./ ((1 - point.^2).*(Poly_d(point).^2)),N);
end
```

调用 **get_gauss_point_weight** 得到对应求积节点与系数后, 下面开始实现 1 维的 Gauss-Legendre 求积函数 **gauss_1d**:

$$\int_a^b f(x)\mathrm{d}x\approx\frac{b-a}{2}\sum_{k=0}^{n}A_k f(\frac{b-a}{2}x_k+\frac{b+a}{2}).$$

**Listing 2:** gauss_1d

```
% func need to be a function handle which can feed in x.
function result = gauss_1d(func,vertices)
[xmax,xmin] = deal(max(vertices),min(vertices));
gauss_points =
    [0.9602898565,-0.9602898565,0.7966664774,-0.7966664774,0.5255324099,-0.5255324099,0.18343464

gauss_weights =
    [0.1012285363,0.1012285363,0.2223810345,0.2223810345,0.3137066459,0.3137066459,0.3626837834,

mapped_gauss_points = (xmax - xmin) ./ 2 .* gauss_points + (xmax + xmin) ./ 2;
result = (xmax - xmin) ./ 2 * gauss_weights * func(mapped_gauss_points)';
end
```

下面取求积节点与求积系数的有效数字位数为默认值 10.

**PS:** 如遇收敛阶误差突变或有不符, 可能是由于求积权重及求积节点处的有效数字截断所导致, 提高对应有效数字, 重新计算即可.

# 2 FEM for two-point boundary value problem

## 2.1 线性元求解一维二阶椭圆方程 (Dirichlet 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^x \dfrac{du(x)}{dx}\right) = -e^x[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)]\,, x \in \Omega, \\ u(0) = 0, u(1) = \cos(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

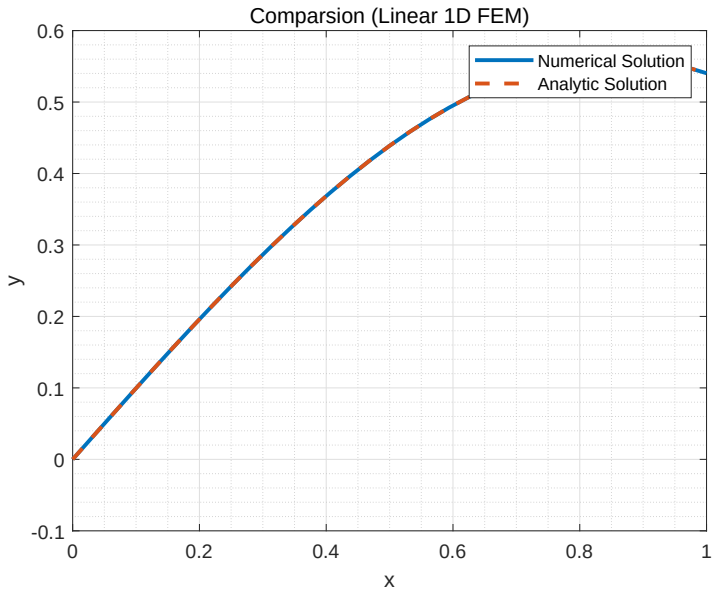


**图 1:** $\frac{1}{32}$ 网格下解析解与数值解对比

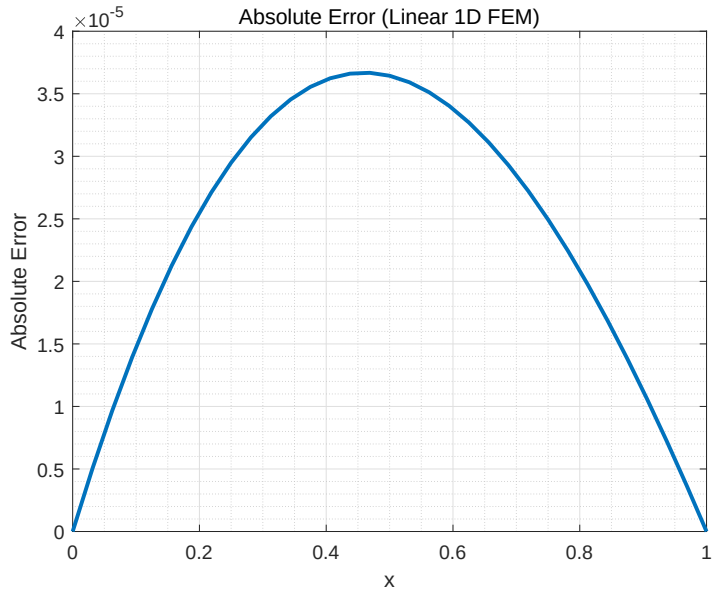


**图 2:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.1.1 收敛阶及各误差计算

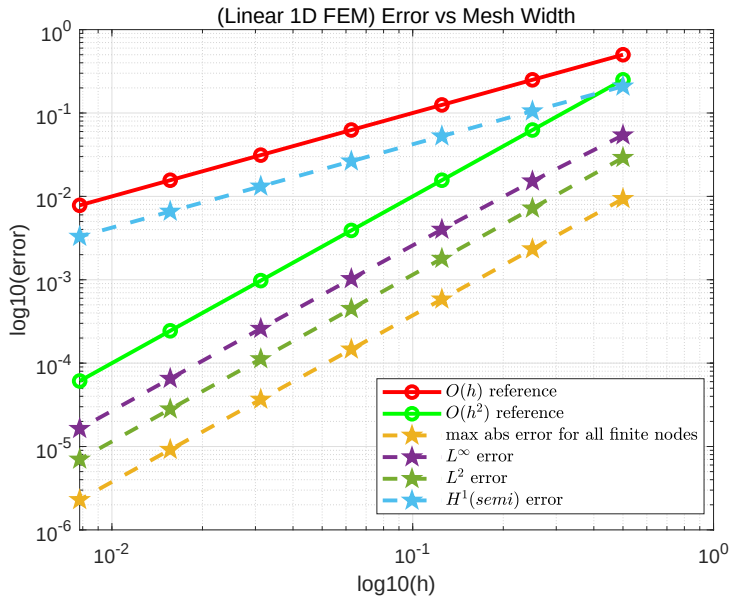


**图 3:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.45504180 s, Total: 7
Completed task num: 2, time passed: 0.81365030 s, Total: 7
Completed task num: 3, time passed: 0.86567780 s, Total: 7
Completed task num: 4, time passed: 0.94708620 s, Total: 7
Completed task num: 5, time passed: 1.02182640 s, Total: 7
Completed task num: 6, time passed: 1.10857340 s, Total: 7
Completed task num: 7, time passed: 1.20531670 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2            0.0093567           0.054162        0.029088         0.20889
    1/4             0.002334           0.015207       0.0071969         0.10528
    1/8           0.00058317          0.0040047       0.0017951        0.052731
    1/16          0.00014645          0.0010256      0.00044854        0.026376
    1/32          3.6675e-05          0.00025936     0.00011212        0.013189
    1/64            9.17e-06          6.5204e-05     2.8029e-05       0.0065949
    1/128         2.2929e-06          1.6346e-05     7.0072e-06       0.0032975

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          2.0032              1.8326          2.015          0.98847
          2.0008              1.9249         2.0033          0.99753
          1.9936              1.9653         2.0008          0.99941
          1.9975              1.9834         2.0002          0.99985
          1.9998              1.9919              2          0.99996
          1.9997               1.996              2          0.99999
```

## 2.2 二次元求解一维二阶椭圆方程 (Dirichlet 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^{x}\dfrac{du(x)}{dx}\right) = -e^{x}\left[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)\right], x \in \Omega, \\ u(0) = 0, u(1) = \cos(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

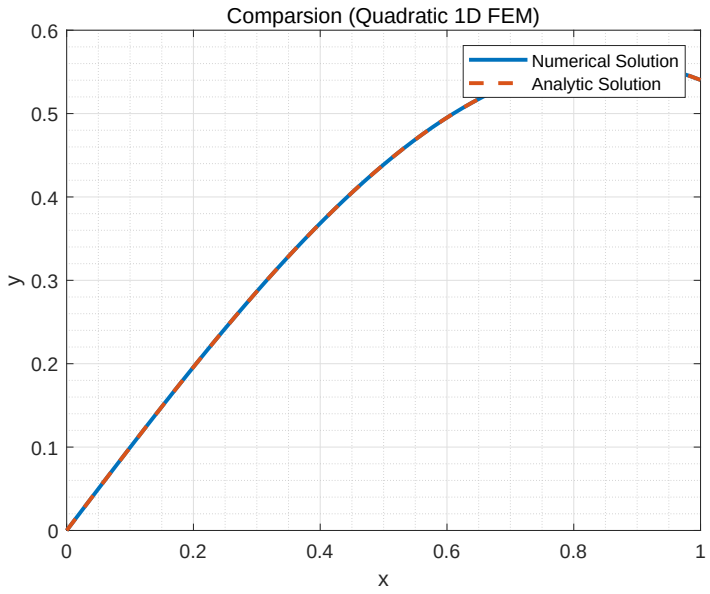


**图 4:** $\frac{1}{32}$ 网格下解析解与数值解对比

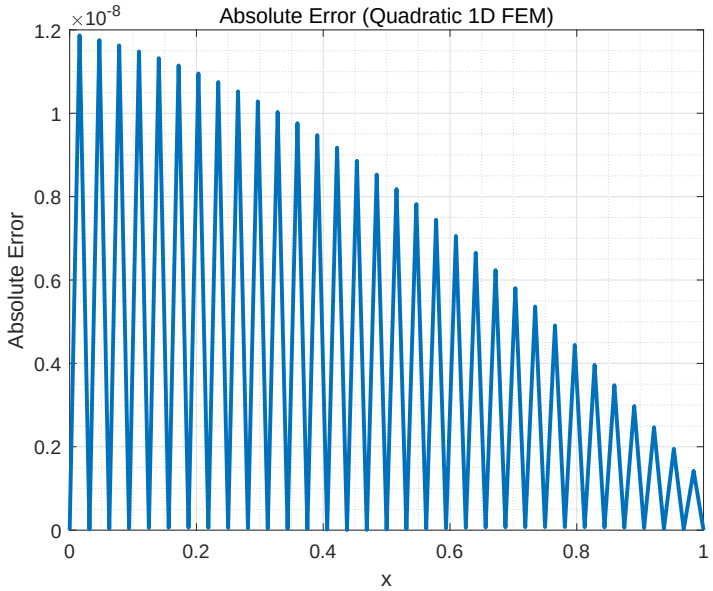


**图 5:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.2.1 收敛阶及各误差计算

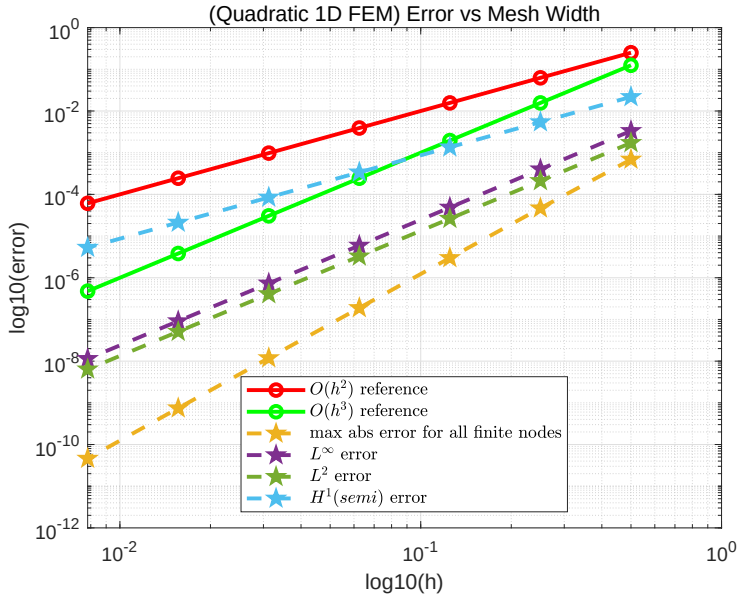


**图 6:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.04481590 s, Total: 7
Completed task num: 2, time passed: 0.08635460 s, Total: 7
Completed task num: 3, time passed: 0.13669320 s, Total: 7
Completed task num: 4, time passed: 0.21603380 s, Total: 7
Completed task num: 5, time passed: 0.30359780 s, Total: 7
Completed task num: 6, time passed: 0.39032410 s, Total: 7
Completed task num: 7, time passed: 0.49338200 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2           0.00068819           0.0033372       0.0017218        0.021808
    1/4           4.6597e-05           0.00040105       0.0002104       0.0054212
    1/8           2.9918e-06           4.8471e-05      2.6144e-05       0.0013534
    1/16          1.8901e-07           5.9371e-06      3.2631e-06      0.00033823
    1/32          1.1869e-08             7.34e-07      4.0774e-07       8.455e-05
    1/64          7.4349e-10           9.1226e-08      5.0962e-08      2.1137e-05
    1/128         4.5985e-11           1.1369e-08      6.3701e-09      5.2842e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          3.8845               3.0567         3.0327           2.0082
          3.9611               3.0486         3.0085            2.002
          3.9845               3.0293         3.0022           2.0005
          3.9932               3.0159         3.0005           2.0001
          3.9968               3.0083         3.0001                2
          4.0151               3.0043              3                2
```

## 2.3 线性元求解一维二阶椭圆方程 (Neumann 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^x \dfrac{du(x)}{dx}\right) = -e^x[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)]\,, x \in \Omega, \\ u(0) = 0, u'(1) = \cos(1) - \sin(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

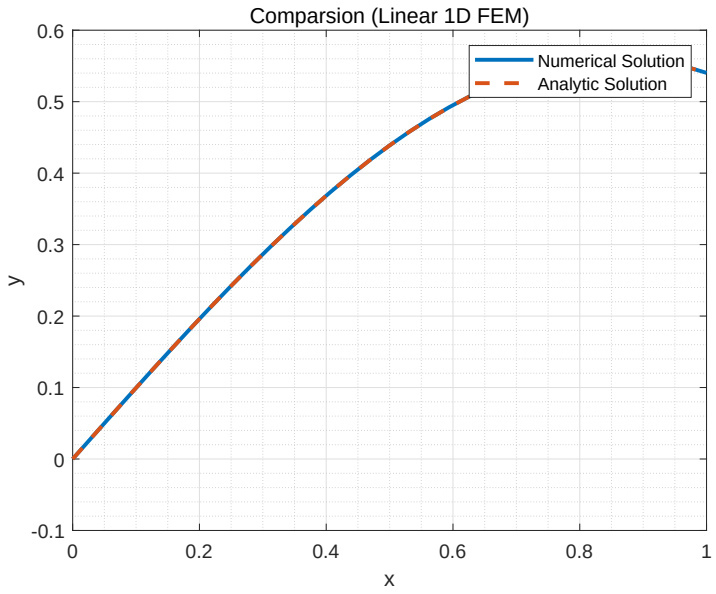


**图 7:** $\frac{1}{32}$ 网格下解析解与数值解对比

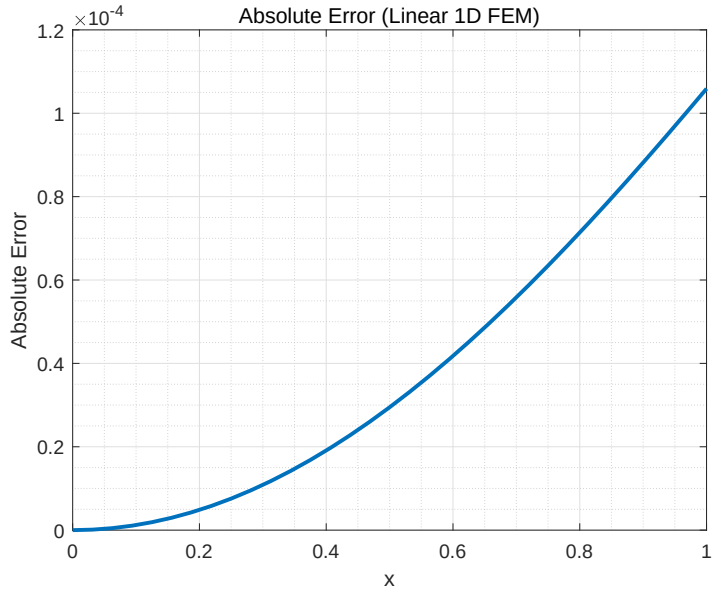


**图 8:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.3.1 收敛阶及各误差计算

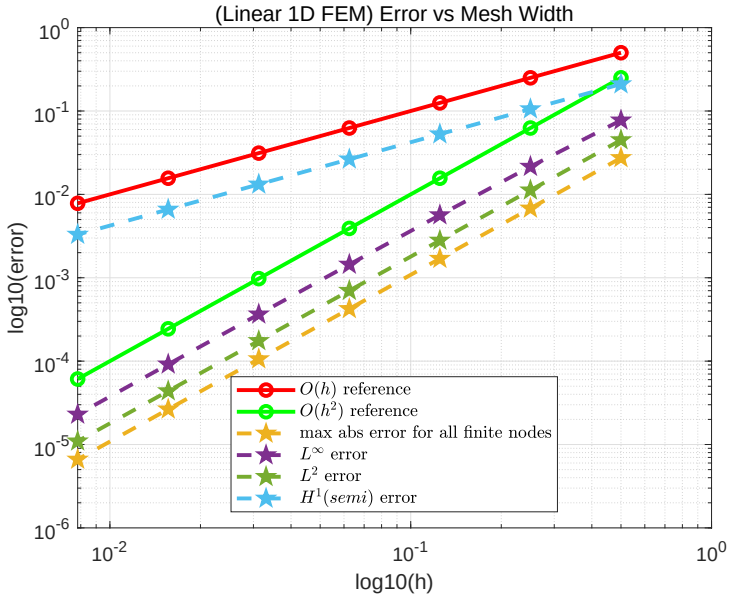


**图 9:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.16174400 s, Total: 7
Completed task num: 2, time passed: 0.30225570 s, Total: 7
Completed task num: 3, time passed: 0.35420350 s, Total: 7
Completed task num: 4, time passed: 0.43063220 s, Total: 7
Completed task num: 5, time passed: 0.52794800 s, Total: 7
Completed task num: 6, time passed: 0.63661270 s, Total: 7
Completed task num: 7, time passed: 0.75037850 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2             0.027383           0.077324        0.044866         0.21018
    1/4             0.006794           0.021542        0.011205         0.10542
    1/8            0.0016953          0.0056463       0.0028009        0.052748
    1/16         0.00042362          0.0014427       0.0007002        0.026378
    1/32         0.00010589          0.00036445      0.00017505         0.01319
    1/64         2.6473e-05          9.1578e-05      4.3762e-05       0.0065949
    1/128        6.6181e-06          2.2952e-05       1.094e-05       0.0032975

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          2.0109              1.8438          2.0014          0.99545
          2.0027              1.9318          2.0002          0.99901
          2.0007              1.9686          2.0001          0.99976
          2.0002               1.985               2          0.99994
               2              1.9927               2          0.99999
               2              1.9964               2                1
```

## 2.4 二次元求解一维二阶椭圆方程 (Neumann 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^x \dfrac{du(x)}{dx}\right) = -e^x[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)]\,, x \in \Omega, \\ u(0) = 0, u'(1) = \cos(1) - \sin(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

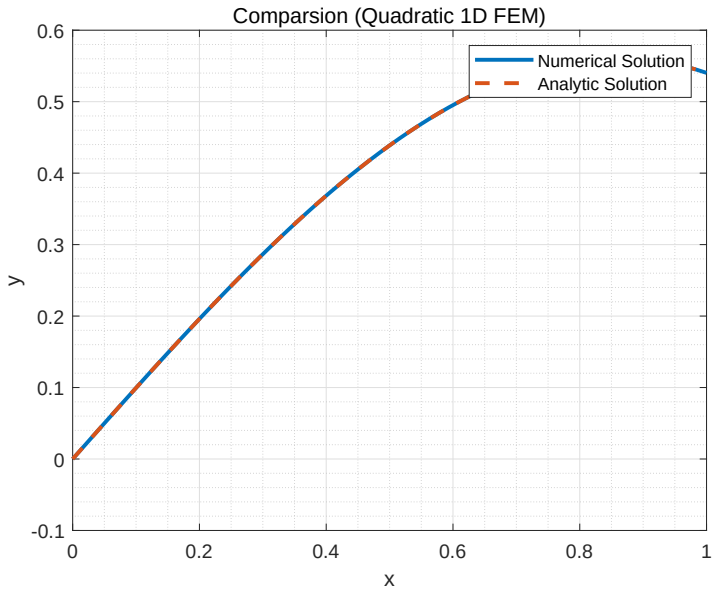


**图 10:** $\frac{1}{32}$ 网格下解析解与数值解对比

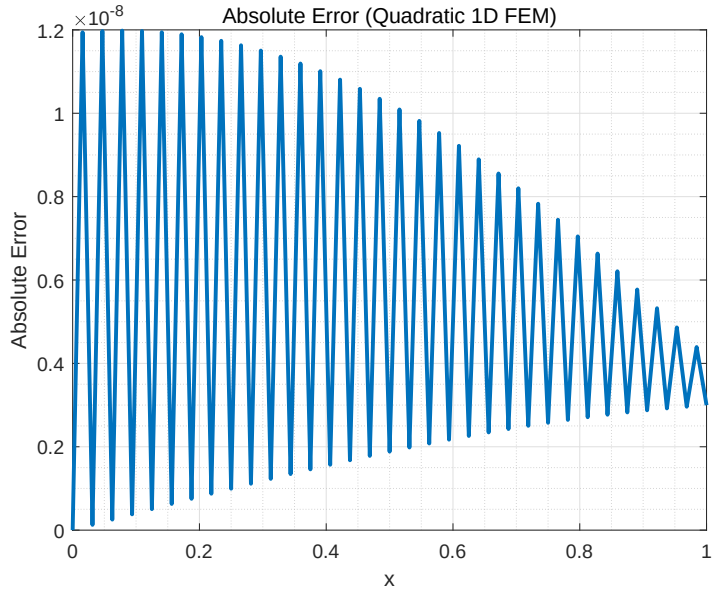


**图 11:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.4.1 收敛阶及各误差计算

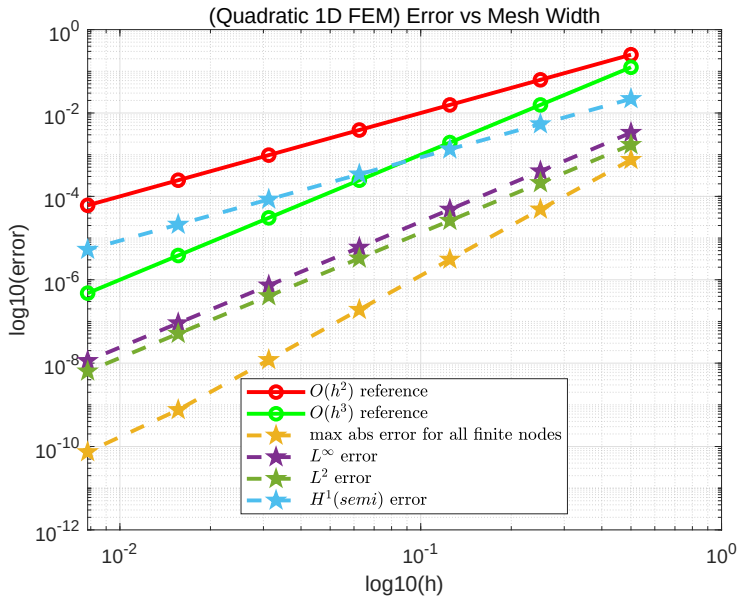


**图 12:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.03712940 s, Total: 7
Completed task num: 2, time passed: 0.07630160 s, Total: 7
Completed task num: 3, time passed: 0.12439200 s, Total: 7
Completed task num: 4, time passed: 0.19806800 s, Total: 7
Completed task num: 5, time passed: 0.27941540 s, Total: 7
Completed task num: 6, time passed: 0.37548220 s, Total: 7
Completed task num: 7, time passed: 0.48702320 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2           0.00075606           0.0033712       0.0017257        0.021813
    1/4           4.8842e-05          0.00040215       0.0002105       0.0054213
    1/8           3.0641e-06           4.8506e-05      2.6147e-05       0.0013534
    1/16          1.9155e-07           5.9382e-06      3.2632e-06      0.00033823
    1/32          1.1982e-08           7.3403e-07      4.0774e-07       8.455e-05
    1/64          7.5654e-10           9.1227e-08      5.0963e-08      2.1137e-05
    1/128          7.404e-11            1.137e-08      6.3703e-09      5.2842e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          3.9523              3.0675          3.0353           2.0085
          3.9946              3.0515          3.0091           2.0021
          3.9997              3.0301          3.0023           2.0005
          3.9987              3.0161          3.0006           2.0001
          3.9853              3.0083          3.0001                2
           3.353              3.0042               3                2
```

## 2.5 线性元求解一维二阶椭圆方程 (Robin 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^x \dfrac{du(x)}{dx}\right) = -e^x[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)]\,, x \in \Omega, \\ u'(0) + u(0) = 1, u(1) = \cos(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

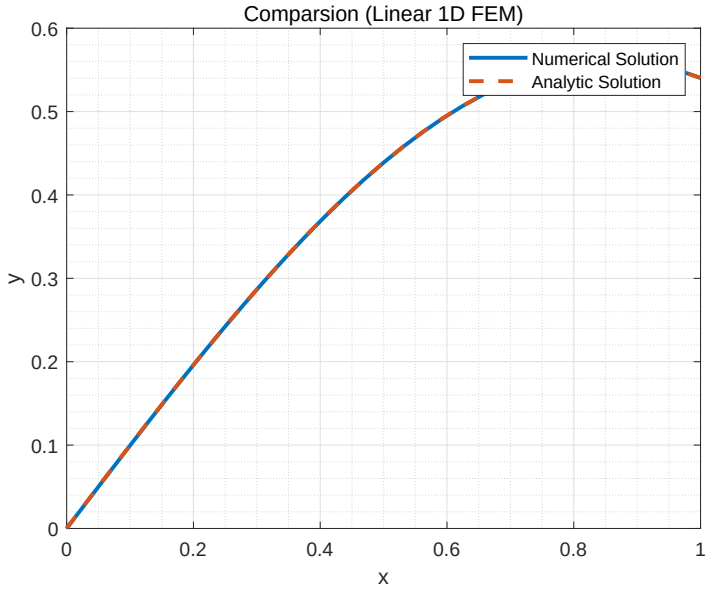


**图 13:** $\frac{1}{32}$ 网格下解析解与数值解对比

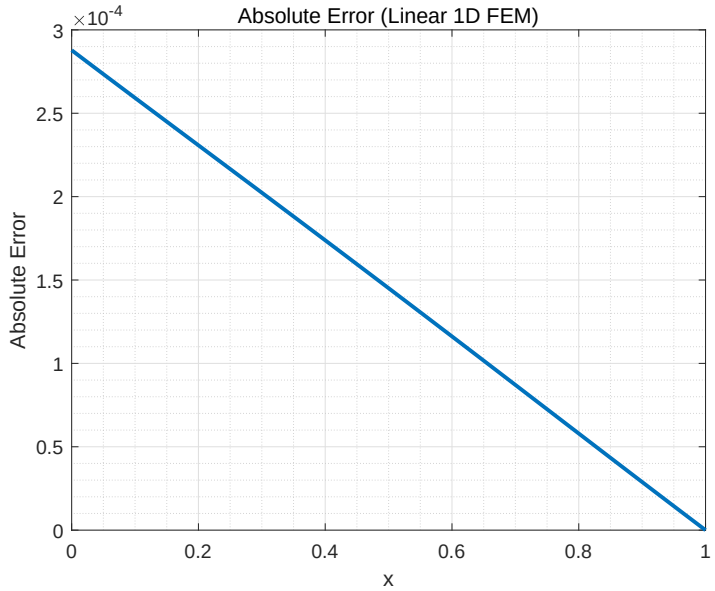


**图 14:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.5.1 收敛阶及各误差计算

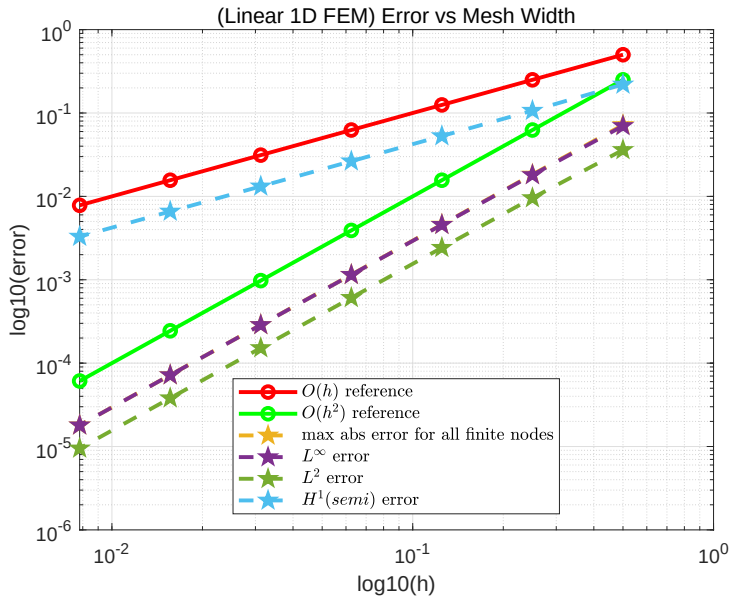


**图 15:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.03884990 s, Total: 7
Completed task num: 2, time passed: 0.07852350 s, Total: 7
Completed task num: 3, time passed: 0.13343780 s, Total: 7
Completed task num: 4, time passed: 0.20902300 s, Total: 7
Completed task num: 5, time passed: 0.29120930 s, Total: 7
Completed task num: 6, time passed: 0.37522040 s, Total: 7
Completed task num: 7, time passed: 0.47304150 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2             0.071892           0.069974        0.036087         0.22012
    1/4             0.018305           0.018061       0.0095582         0.10673
    1/8             0.004598          0.0045673       0.0024222        0.052914
    1/16           0.0011509            0.001147      0.00060759        0.026399
    1/32          0.00028781          0.00028733      0.00015202        0.013192
    1/64          7.1958e-05          7.1898e-05      3.8014e-05       0.0065953
    1/128          1.799e-05           1.7982e-05      9.5041e-06       0.0032975

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          1.9736               1.954          1.9167           1.0443
          1.9931              1.9834          1.9804           1.0123
          1.9983              1.9934          1.9952           1.0032
          1.9996              1.9972          1.9988           1.0008
          1.9999              1.9987          1.9997           1.0002
               2              1.9994          1.9999                1
```

## 2.6 二次元求解一维二阶椭圆方程 (Robin 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\dfrac{d}{dx}\left(e^x \dfrac{du(x)}{dx}\right) = -e^x[\cos(x) - 2\sin(x) - x\cos(x) - x\sin(x)]\,, x \in \Omega, \\ u'(0) + u(0) = 1, u(1) = \cos(1), \\ \Omega = [0, 1]. \end{cases}$$

该问题的解析解为:$u(x) = x\cos(x)$.

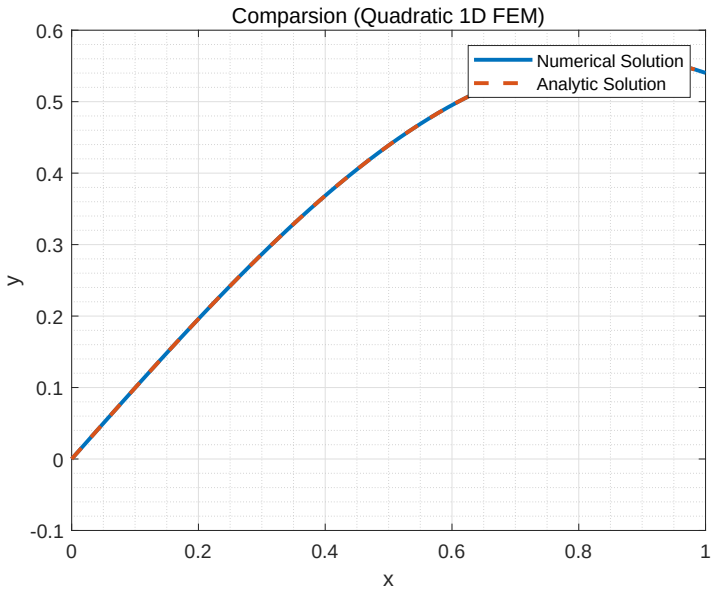


**图 16:** $\frac{1}{32}$ 网格下解析解与数值解对比

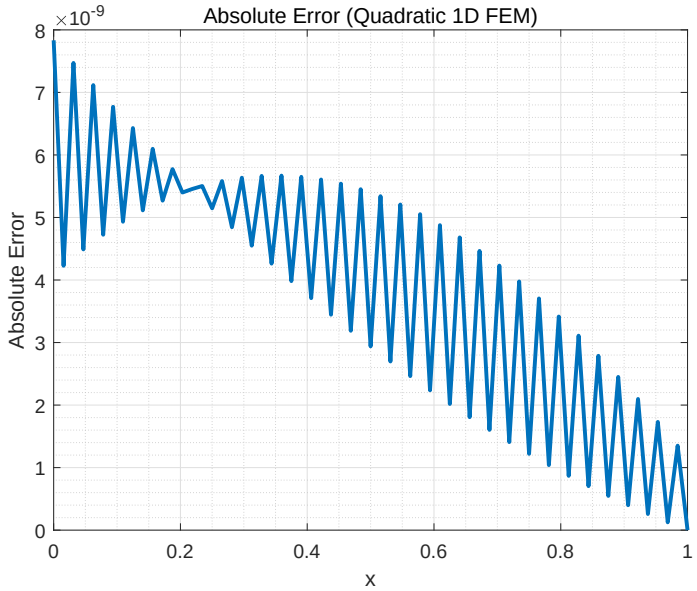


**图 17:** $\frac{1}{32}$ 网格下有限元节点误差分析

### 2.6.1 收敛阶及各误差计算

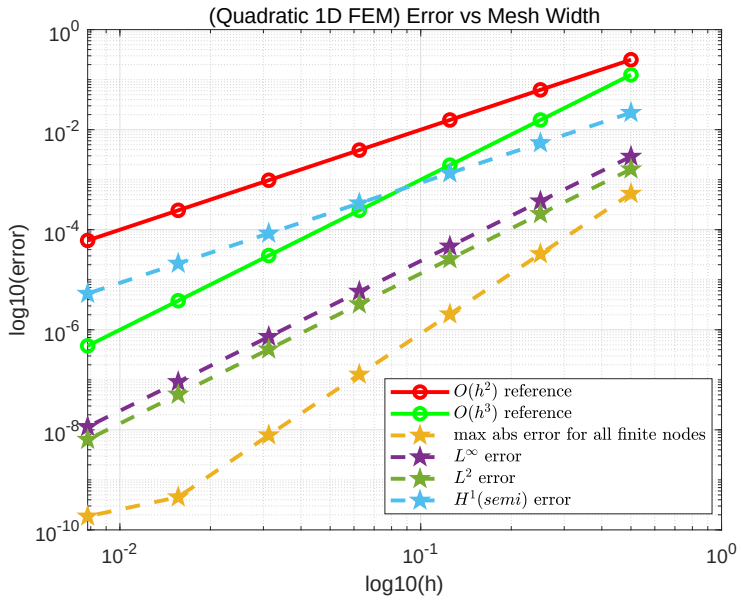


**图 18:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.03326900 s, Total: 7
Completed task num: 2, time passed: 0.07919760 s, Total: 7
Completed task num: 3, time passed: 0.13254820 s, Total: 7
Completed task num: 4, time passed: 0.20414290 s, Total: 7
Completed task num: 5, time passed: 0.27777370 s, Total: 7
Completed task num: 6, time passed: 0.38222060 s, Total: 7
Completed task num: 7, time passed: 0.48460520 s, Total: 7
      h       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     ------------------    -------------    ----------    ---------------

    1/2           0.00052753           0.0029022       0.0016195        0.021825
    1/4           3.2834e-05          0.00037121      0.00020714       0.0054215
    1/8           2.0498e-06          4.6516e-05      2.6042e-05       0.0013534
    1/16          1.2792e-07          5.8121e-06      3.2599e-06      0.00033823
    1/32          7.8332e-09          7.2626e-07      4.0764e-07       8.455e-05
    1/64          4.5204e-10            9.09e-08       5.096e-08      2.1137e-05
    1/128         1.8668e-10          1.1511e-08      6.3711e-09      5.2842e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

           4.006              2.9669          2.9669           2.0092
          4.0016              2.9964          2.9917           2.0021
          4.0022              3.0006          2.9979           2.0005
          4.0295              3.0005          2.9995           2.0001
          4.1151              2.9981          2.9998                2
          1.2759              2.9813          2.9998                2
```

# 3 FEM for 2D second order elliptic equation

## 3.1 线性元求解二维二阶椭圆方程 (Dirichlet 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -y(1-y)(1-x-\frac{x^2}{2})e^{x+y} - x(1-\frac{x}{2})(-3y-y^2)e^{x+y}, x\in\Omega, \\ u = -1.5y(1-y)e^{-1+y}, x=-1, \\ u = 0.5y(1-y)e^{1+y}, x=1, \\ u = -2x(1-\frac{x}{2})e^{x-1}, y=-1, \\ u = 0, y=1, \\ \Omega = [-1,1]^2. \end{cases}$$

该问题的解析解为:$u(x,y) = xy(1-\frac{x}{2})(1-y)e^{x+y}$.

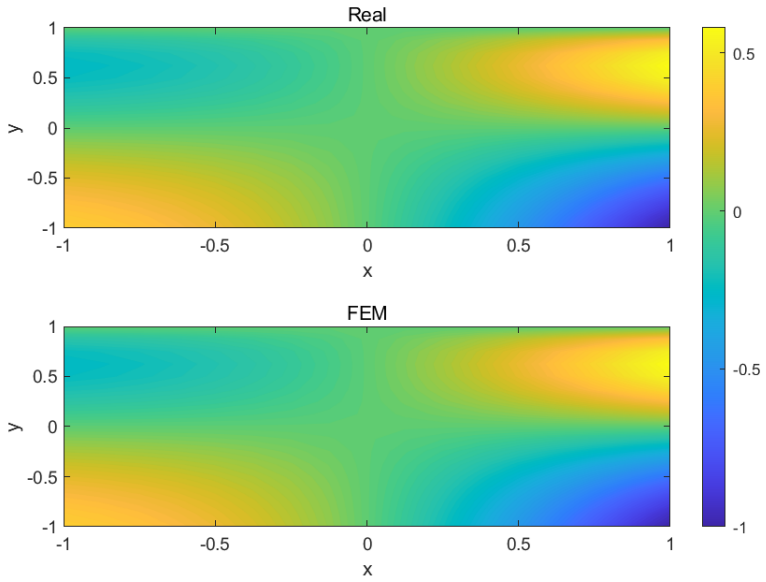


**图 19:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

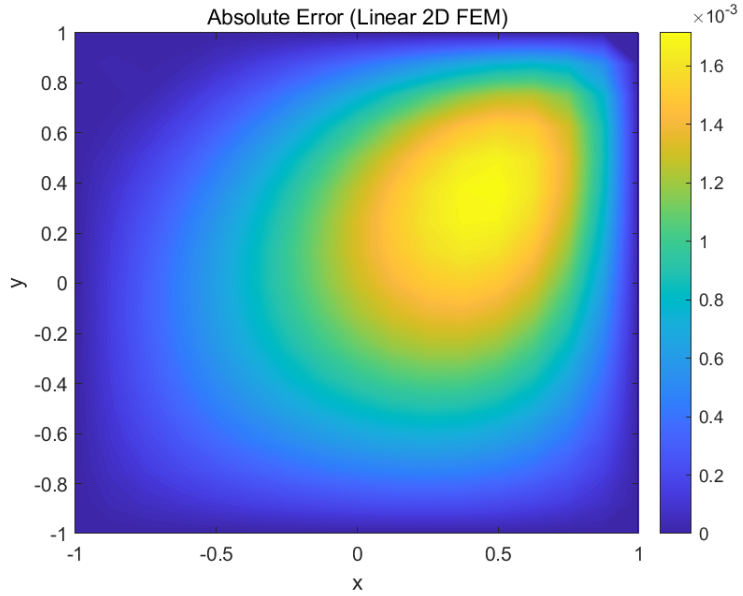


**图 20:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.1.1 收敛阶及各误差计算

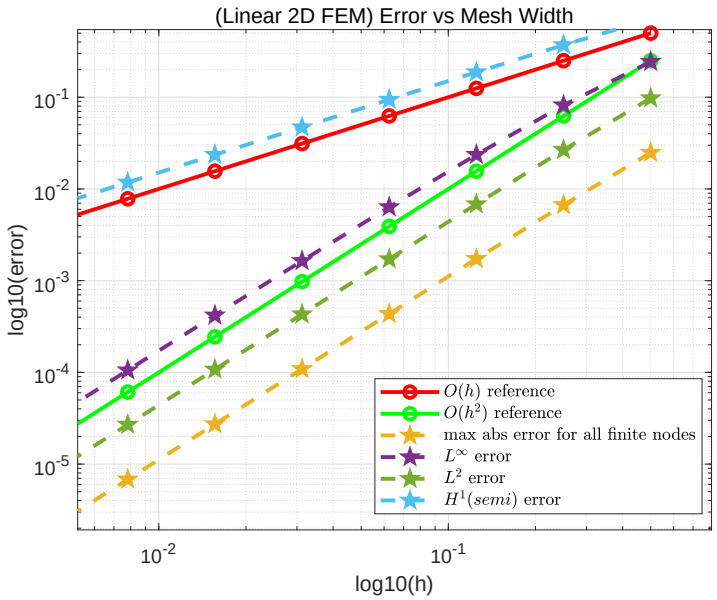


**图 21:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.28129400 s, Total: 8
Completed task num: 2, time passed: 0.55425210 s, Total: 8
Completed task num: 3, time passed: 0.93234800 s, Total: 8
Completed task num: 4, time passed: 1.73248420 s, Total: 8
Completed task num: 5, time passed: 3.39934570 s, Total: 8
Completed task num: 6, time passed: 6.94384890 s, Total: 8
Completed task num: 7, time passed: 19.95738220 s, Total: 8
Completed task num: 8, time passed: 63.32076950 s, Total: 8
     hx        hy       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     -----     ------------------    -------------    ----------    ---------------

    1/2       1/2             0.024879            0.24398        0.097643         0.70659
    1/4       1/4            0.0066922           0.081802        0.026648          0.3708
    1/8       1/8              0.00173            0.02362         0.00683         0.18774
    1/16      1/16          0.00043521          0.0063421       0.0017189        0.094167
    1/32      1/32          0.00010902           0.001643      0.00043049        0.047121
    1/64      1/64           2.727e-05          0.0004181      0.00010767        0.023565
    1/128     1/128         6.8179e-06          0.00010546     2.6922e-05        0.011783
    1/256     1/256         1.7046e-06           2.6481e-05     6.7308e-06       0.0058916

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          1.8944              1.5766          1.8735          0.93022
          1.9517              1.7922           1.964          0.98193
           1.991              1.8969          1.9904          0.99544
          1.9971              1.9487          1.9975          0.99886
          1.9992              1.9744          1.9993          0.99971
          1.9999              1.9872          1.9998          0.99993
```

```
    1.9999          1.9936          1.9999          0.99998
```

### 3.2 二次元求解二维二阶椭圆方程 (Dirichlet 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -y(1-y)(1-x-\frac{x^2}{2})e^{x+y} - x(1-\frac{x}{2})(-3y-y^2)e^{x+y}, x\in\Omega, \\ u = -1.5y(1-y)e^{-1+y}, x=-1, \\ u = 0.5y(1-y)e^{1+y}, x=1, \\ u = -2x(1-\frac{x}{2})e^{x-1}, y=-1, \\ u = 0, y=1, \\ \Omega = [-1,1]^2. \end{cases}$$

该问题的解析解为:$u(x,y) = xy(1-\frac{x}{2})(1-y)e^{x+y}$.

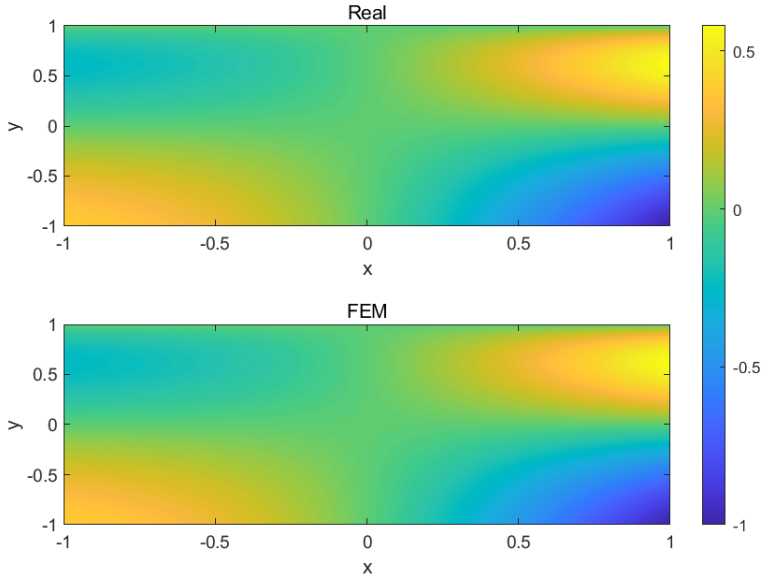


**图 22:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

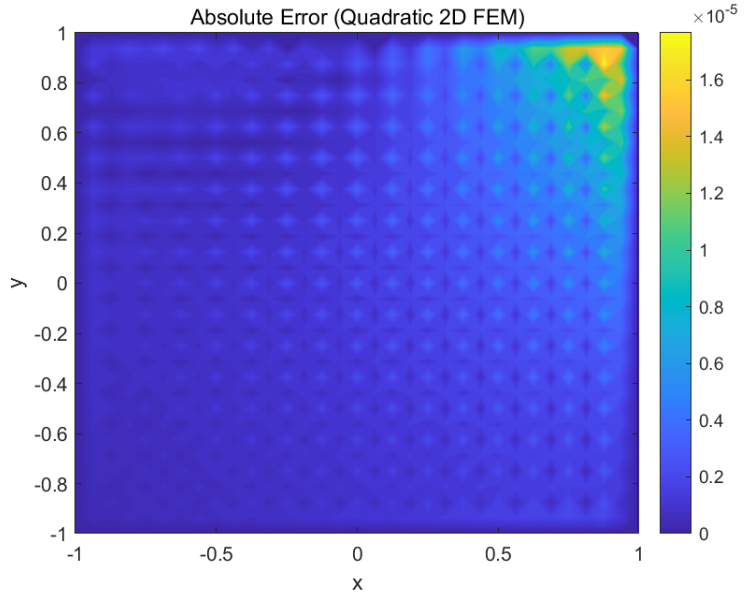


**图 23:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.2.1 收敛阶及各误差计算

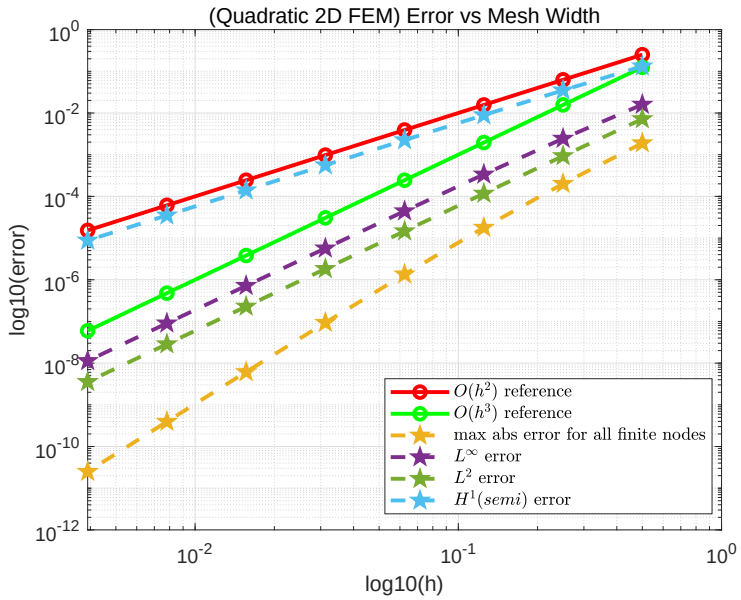


**图 24:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.26736150 s, Total: 8
Completed task num: 2, time passed: 0.49815530 s, Total: 8
Completed task num: 3, time passed: 0.82082410 s, Total: 8
Completed task num: 4, time passed: 1.46083190 s, Total: 8
Completed task num: 5, time passed: 2.98188070 s, Total: 8
Completed task num: 6, time passed: 8.43923400 s, Total: 8
Completed task num: 7, time passed: 31.80906340 s, Total: 8
Completed task num: 8, time passed: 154.40946730 s, Total: 8
     hx        hy       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     -----     ------------------    -------------    ----------    ---------------

    1/2       1/2            0.0018989           0.015945       0.0071914         0.13206
    1/4       1/4           0.00020116          0.0024361      0.00093065         0.03507
    1/8       1/8           1.7836e-05          0.00033678     0.00011705        0.0089192
    1/16      1/16          1.3528e-06          4.4273e-05     1.4637e-05        0.0022414
    1/32      1/32          9.3471e-08          5.6752e-06     1.8289e-06       0.00056131
    1/64      1/64          6.1462e-09          7.1839e-07     2.2853e-07       0.00014042
    1/128     1/128         3.9406e-10          9.0366e-08      2.856e-08       3.5114e-05
    1/256     1/256         2.4944e-11          1.1331e-08     3.5695e-09       8.7795e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          3.2388              2.7104           2.95           1.9129
          3.4955              2.8547         2.9911           1.9753
          3.7208              2.9273         2.9994           1.9925
          3.8553              2.9637         3.0006           1.9975
          3.9267              2.9818         3.0005           1.9991
          3.9632              2.9909         3.0003           1.9996
```

```
        3.9816          2.9955          3.0002          1.9998
```

### 3.3 线性元求解二维二阶椭圆方程 (Neumann 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -2e^{x+y}, x\in\Omega, \\ u = e^{-1+y}, x=-1, \\ u = e^{1+y}, x=1, \\ \nabla u\cdot\vec{n} = -e^{x-1}, y=-1, \\ u = e^{x+1}, y=1, \\ \Omega = [-1,1]^2, \end{cases}$$

该问题的解析解为:$u(x,y)=e^{x+y}$.

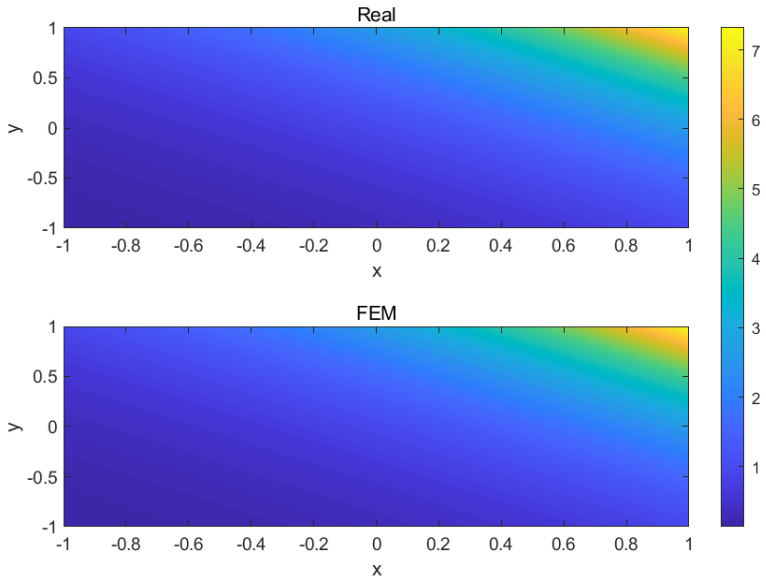


**图 25:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

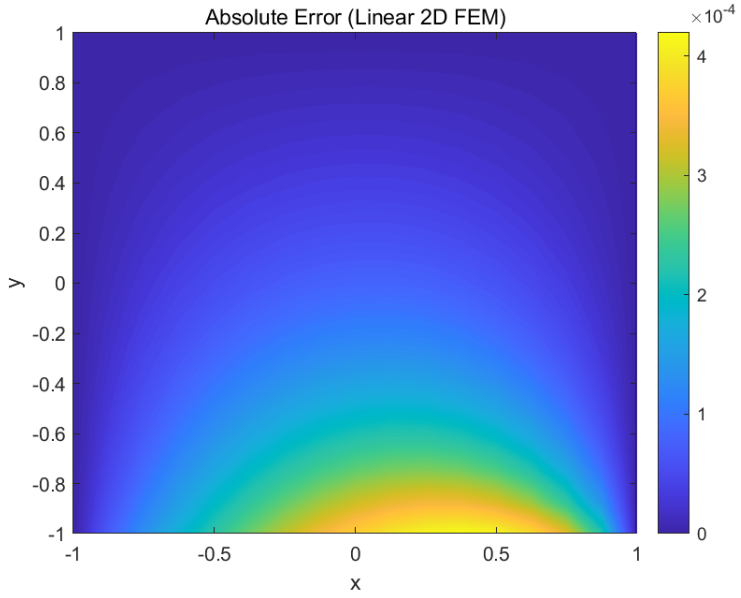


**图 26:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.3.1 收敛阶及各误差计算

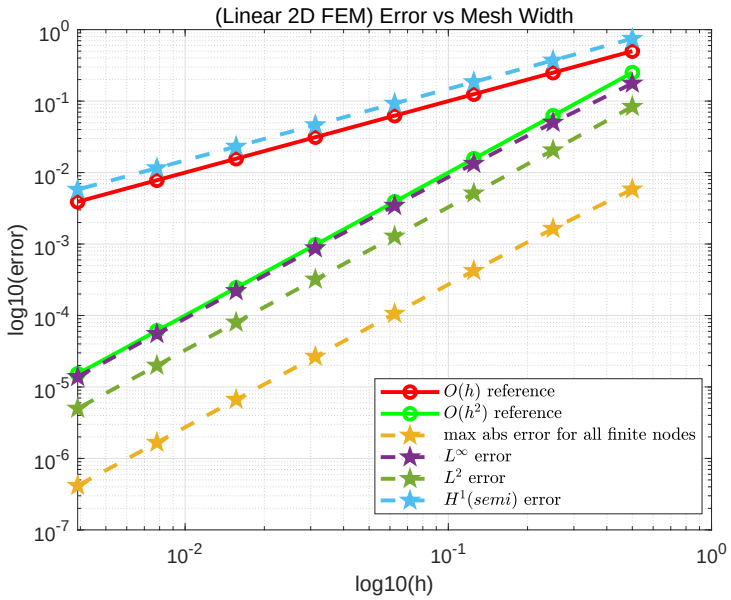


**图 27:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.30794330 s, Total: 8
Completed task num: 2, time passed: 0.57837320 s, Total: 8
Completed task num: 3, time passed: 1.05729290 s, Total: 8
Completed task num: 4, time passed: 1.59865150 s, Total: 8
Completed task num: 5, time passed: 2.68071490 s, Total: 8
Completed task num: 6, time passed: 5.51959700 s, Total: 8
Completed task num: 7, time passed: 16.08474890 s, Total: 8
Completed task num: 8, time passed: 57.60809370 s, Total: 8
```

| hx | hy | Max Absolute Error | L^infty Error | L^2 Error | H^1(semi) Error |
|---|---|---|---|---|---|
| 1/2 | 1/2 | 0.0058377 | 0.17681 | 0.083561 | 0.74881 |
| 1/4 | 1/4 | 0.0016426 | 0.050127 | 0.020574 | 0.37129 |
| 1/8 | 1/8 | 0.0004235 | 0.013358 | 0.0051224 | 0.18523 |
| 1/16 | 1/16 | 0.00010633 | 0.0034487 | 0.0012793 | 0.092559 |
| 1/32 | 1/32 | 2.6609e-05 | 0.00087622 | 0.00031973 | 0.046273 |
| 1/64 | 1/64 | 6.6554e-06 | 0.00022084 | 7.9928e-05 | 0.023136 |
| 1/128 | 1/128 | 1.664e-06 | 5.5433e-05 | 1.9982e-05 | 0.011568 |
| 1/256 | 1/256 | 4.1599e-07 | 1.3886e-05 | 4.9954e-06 | 0.0057838 |

Convergence Order:

| Max Absolute Error | L^infty Error | L^2 Error | H^1(semi) Error |
|---|---|---|---|
| 1.8294 | 1.8186 | 2.022 | 1.012 |
| 1.9556 | 1.9079 | 2.0059 | 1.0033 |
| 1.9939 | 1.9536 | 2.0015 | 1.0008 |
| 1.9985 | 1.9767 | 2.0004 | 1.0002 |
| 1.9993 | 1.9883 | 2.0001 | 1.0001 |
| 1.9999 | 1.9942 | 2 | 1 |

```
2                1.9971                2                1
```

### 3.4 二次元求解二维二阶椭圆方程 (Neumann 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -2e^{x+y}, x\in\Omega, \\ u = e^{-1+y}, x=-1, \\ u = e^{1+y}, x=1, \\ \nabla u\cdot\vec{n} = -e^{x-1}, y=-1, \\ u = e^{x+1}, y=1, \\ \Omega = [-1,1]^2, \end{cases}$$

该问题的解析解为:$u(x,y)=e^{x+y}$.

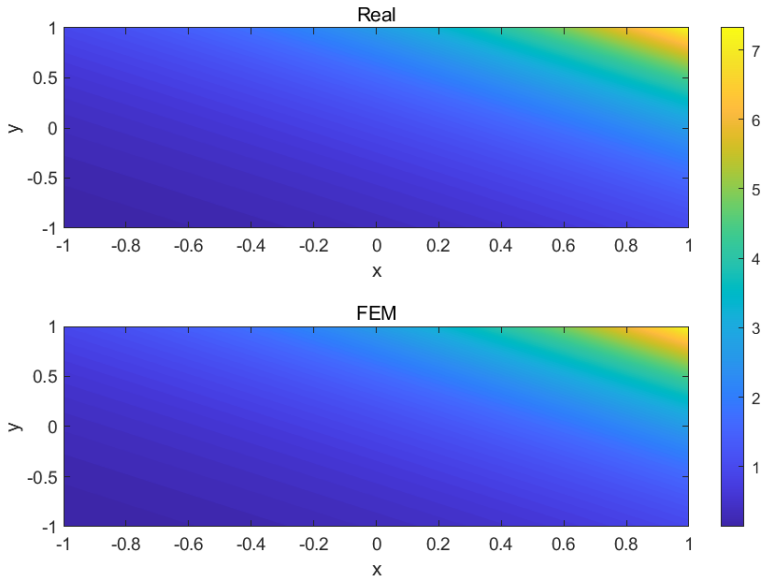


**图 28:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

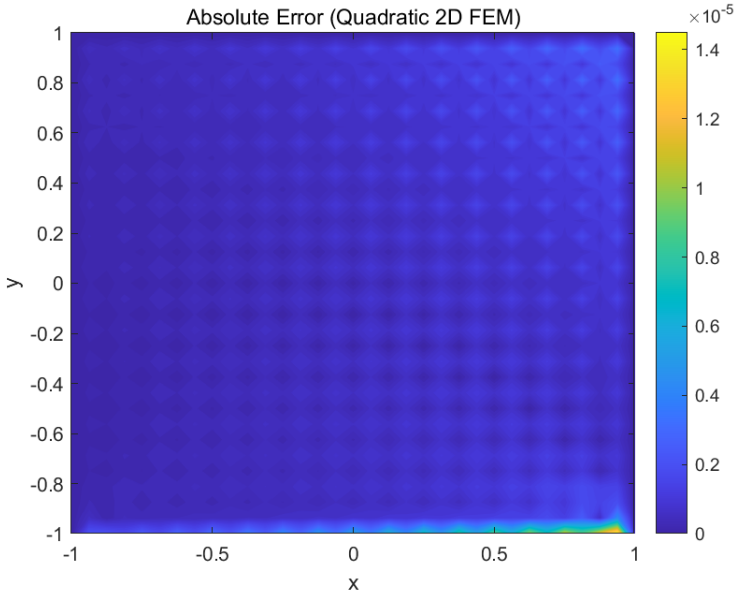


**图 29:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.4.1 收敛阶及各误差计算

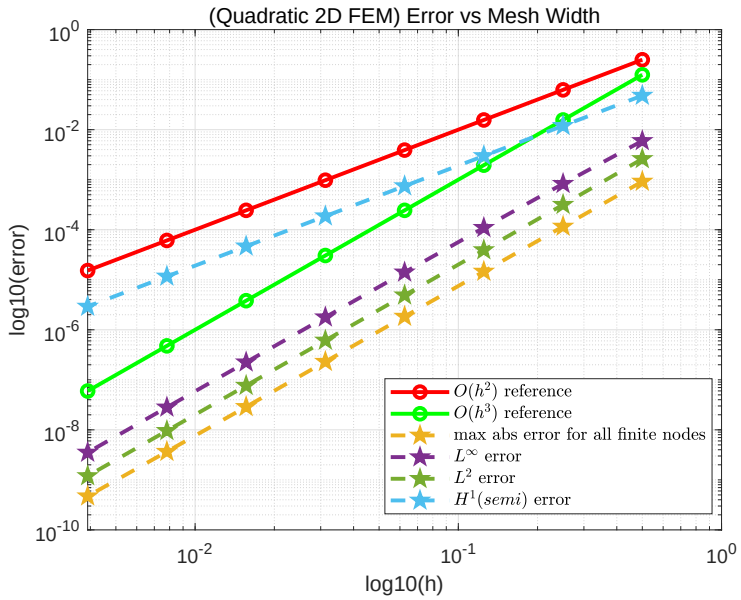


**图 30:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.24310570 s, Total: 8
Completed task num: 2, time passed: 0.49908680 s, Total: 8
Completed task num: 3, time passed: 0.83317690 s, Total: 8
Completed task num: 4, time passed: 1.56816370 s, Total: 8
Completed task num: 5, time passed: 3.17896920 s, Total: 8
Completed task num: 6, time passed: 8.72911110 s, Total: 8
Completed task num: 7, time passed: 30.10485850 s, Total: 8
Completed task num: 8, time passed: 120.76316790 s, Total: 8
     hx        hy       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     -----     ------------------    -------------    ----------    ---------------

    1/2       1/2           0.00092076          0.0059728       0.0026061        0.048064
    1/4       1/4           0.00011609         0.00083036      0.00031672        0.011963
    1/8       1/8            1.463e-05         0.00010956      3.9285e-05       0.0029874
    1/16      1/16          1.8384e-06          1.4074e-05      4.9015e-06      0.00074668
    1/32      1/32          2.3048e-07          1.7835e-06      6.1243e-07      0.00018667
    1/64      1/64          2.8854e-08          2.2447e-07      7.6549e-08      4.6667e-05
    1/128     1/128         3.6636e-09          2.8155e-08      9.5686e-09      1.1667e-05
    1/256     1/256         4.7139e-10          3.5254e-09      1.1961e-09      2.9167e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          2.9875              2.8466          3.0406           2.0064
          2.9883               2.922          3.0111           2.0016
          2.9924              2.9607          3.0027           2.0003
          2.9958              2.9802          3.0006                2
          2.9978              2.9901          3.0001                2
          2.9775               2.995               3                2
```

```
        2.9583              2.9975              3                   2
```

### 3.5 线性元求解一维二阶椭圆方程 (Robin 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -2e^{x+y}, x\in\Omega, \\ u = e^{-1+y}, x=-1, \\ u = e^{1+y}, x=1, \\ \nabla u\cdot\vec{n}+u=0, y=-1, \\ u = e^{x+1}, y=1, \\ \Omega=[-1,1]^2. \end{cases}$$

该问题的解析解为:$u(x,y)=e^{x+y}$.

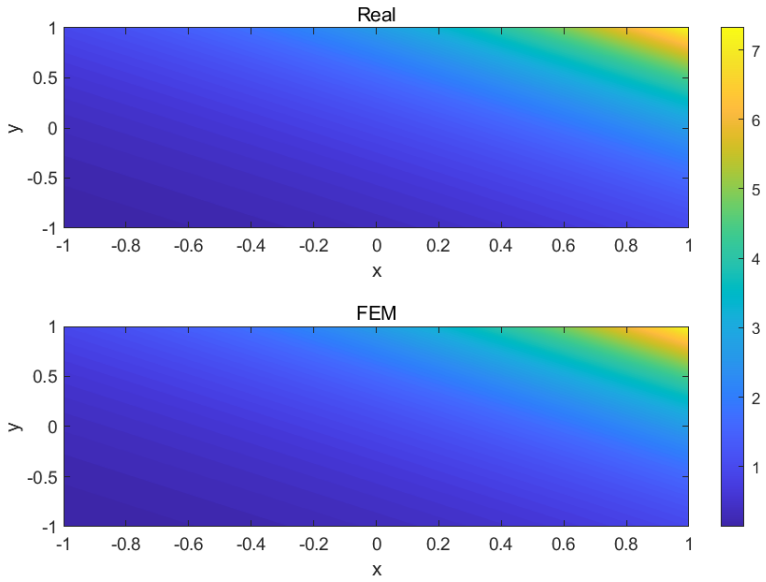


**图 31:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

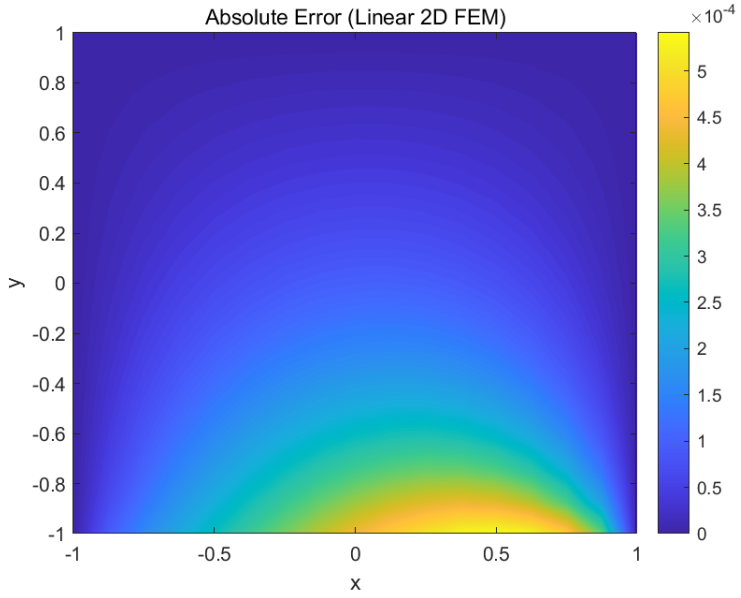


**图 32:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.5.1 收敛阶及各误差计算

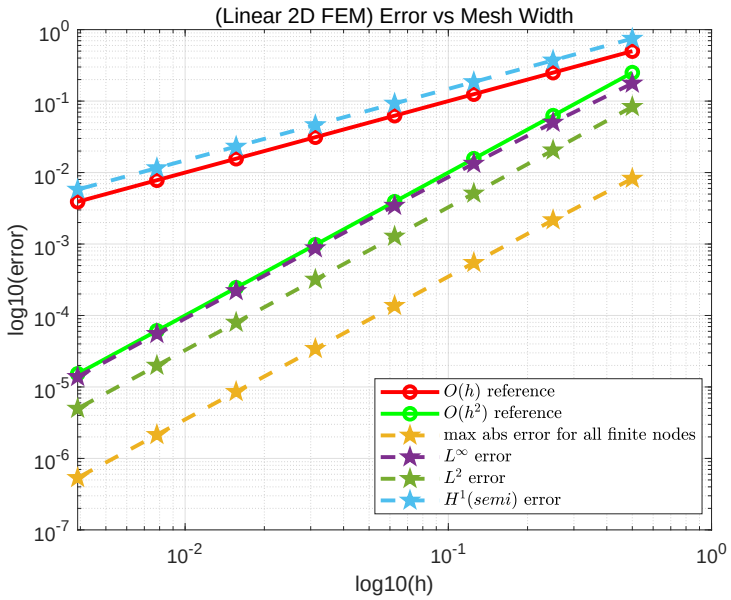


**图 33:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.26776540 s, Total: 8
Completed task num: 2, time passed: 0.56631140 s, Total: 8
Completed task num: 3, time passed: 0.95989400 s, Total: 8
Completed task num: 4, time passed: 1.61656230 s, Total: 8
Completed task num: 5, time passed: 3.04340740 s, Total: 8
Completed task num: 6, time passed: 6.11554930 s, Total: 8
Completed task num: 7, time passed: 16.75621310 s, Total: 8
Completed task num: 8, time passed: 58.80878620 s, Total: 8
     hx        hy       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     -----     ------------------    -------------    ----------    ---------------

    1/2       1/2            0.0082415            0.17681        0.083324         0.74881
    1/4       1/4            0.0021647           0.050127         0.02052         0.37129
    1/8       1/8           0.00054686           0.013358       0.0051094         0.18523
    1/16      1/16          0.00013691          0.0034487        0.001276        0.092559
    1/32      1/32          3.4236e-05          0.00087622      0.00031893        0.046273
    1/64      1/64          8.5614e-06          0.00022084      7.9727e-05        0.023136
    1/128     1/128         2.1404e-06          5.5433e-05      1.9932e-05        0.011568
    1/256     1/256         5.3509e-07          1.3886e-05      4.9829e-06       0.0057838

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          1.9287              1.8186         2.0217            1.012
          1.9849              1.9079         2.0058           1.0033
           1.998              1.9536         2.0015           1.0008
          1.9996              1.9767         2.0004           1.0002
          1.9996              1.9883         2.0001           1.0001
               2              1.9942              2                1
```

```
2                1.9971                2                1
```

### 3.6 二次元求解一维二阶椭圆方程 (Robin 边界)

下面使用 FEM 求解如下问题，并给出对应误差及收敛阶:

$$\begin{cases} -\nabla\cdot(\nabla u) = -2e^{x+y}, x\in\Omega, \\ u = e^{-1+y}, x=-1, \\ u = e^{1+y}, x=1, \\ \nabla u\cdot\vec{n}+u=0, y=-1, \\ u = e^{x+1}, y=1, \\ \Omega=[-1,1]^2. \end{cases}$$

该问题的解析解为:$u(x,y)=e^{x+y}$.

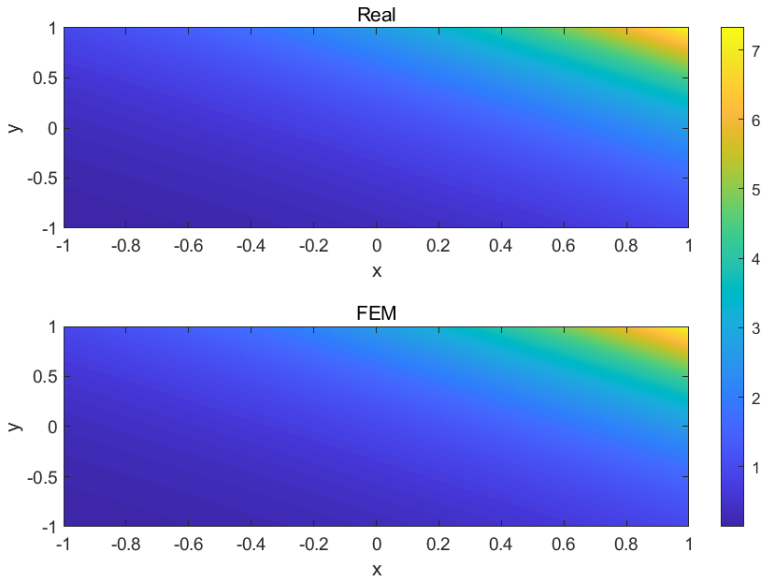


**图 34:** $(\frac{1}{8},\frac{1}{8})$ 网格下解析解与数值解对比

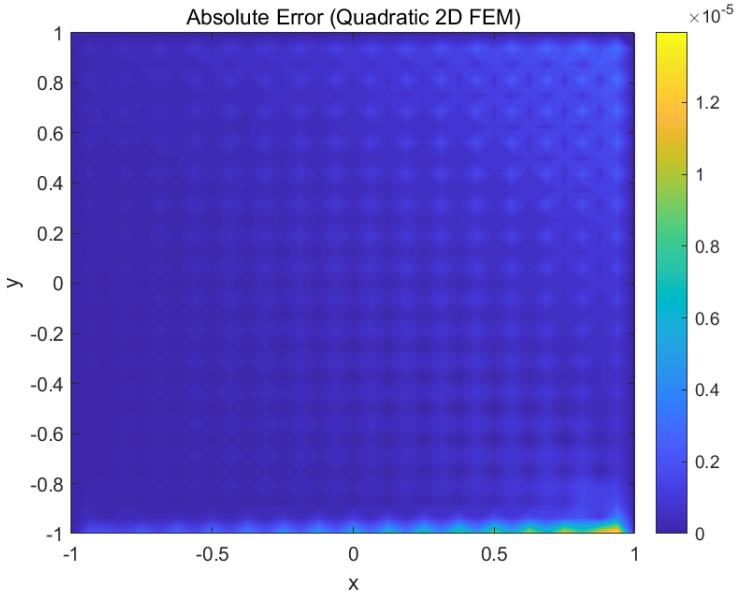


**图 35:** $(\frac{1}{8},\frac{1}{8})$ 网格下有限元节点误差分析

### 3.6.1 收敛阶及各误差计算

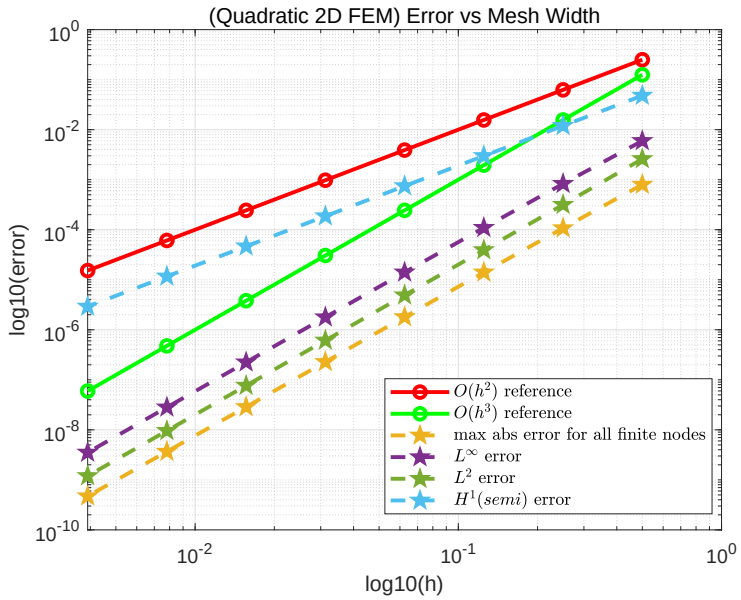


**图 36:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.25604020 s, Total: 8
Completed task num: 2, time passed: 0.53160370 s, Total: 8
Completed task num: 3, time passed: 0.87748830 s, Total: 8
Completed task num: 4, time passed: 1.58005730 s, Total: 8
Completed task num: 5, time passed: 3.17873810 s, Total: 8
Completed task num: 6, time passed: 8.95627680 s, Total: 8
Completed task num: 7, time passed: 32.08685930 s, Total: 8
Completed task num: 8, time passed: 132.06159540 s, Total: 8
     hx        hy       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    -----     -----     ------------------    -------------    ----------    ---------------

    1/2       1/2           0.00079892          0.0059729       0.0025993        0.048064
    1/4       1/4           0.00010773         0.00083036      0.00031649        0.011963
    1/8       1/8           1.4076e-05         0.00010956      3.9278e-05       0.0029874
    1/16      1/16          1.8025e-06         1.4074e-05      4.9012e-06      0.00074668
    1/32      1/32          2.2819e-07         1.7835e-06      6.1243e-07      0.00018667
    1/64      1/64          2.8709e-08         2.2447e-07      7.6549e-08      4.6667e-05
    1/128     1/128         3.6709e-09         2.8155e-08      9.5686e-09      1.1667e-05
    1/256     1/256         4.7146e-10         3.5254e-09      1.1961e-09      2.9167e-06

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          2.8906              2.8466          3.0379           2.0064
          2.9361               2.922          3.0104           2.0016
          2.9652              2.9607          3.0025           2.0003
          2.9817              2.9802          3.0005                2
          2.9906              2.9901          3.0001                2
          2.9673               2.995               3                2
```

```
        2.9609              2.9975              3                2
```

# 4 FEM for 2D heat equation

## 4.1 线性元求解二维二阶含时抛物方程,Backward Euler 格式

$$\begin{cases} u_t - \nabla \cdot (2\nabla u) = -3e^{x+y+t}, x \in \Omega \times T, \\ u(\partial\Omega, t = 0) = e^{x+y}, \\ u = e^{y+t}, x = 0, \\ u = e^{2+y+t}, x = 2, \\ u = e^{x+t}, y = 0, \\ u = e^{x+1+y}, y = 1, \\ \Omega = [0, 2] \times [0, 1], T = [0, 1]. \end{cases}$$

取差分格式为:

$$\left[\frac{M}{\Delta t} + \theta A\right] X^{m+1} = \theta b(t_{m+1}) + (1-\theta)b(t_m) + \left[\frac{M}{\Delta t}X^m - (1-\theta)A\right] X^m.$$

其中 $\theta = 1, \Delta t = 4h^2$, 该问题的解析解为:$u(x, y) = e^{x+y+t}$.

### 4.1.1 收敛阶及各误差计算

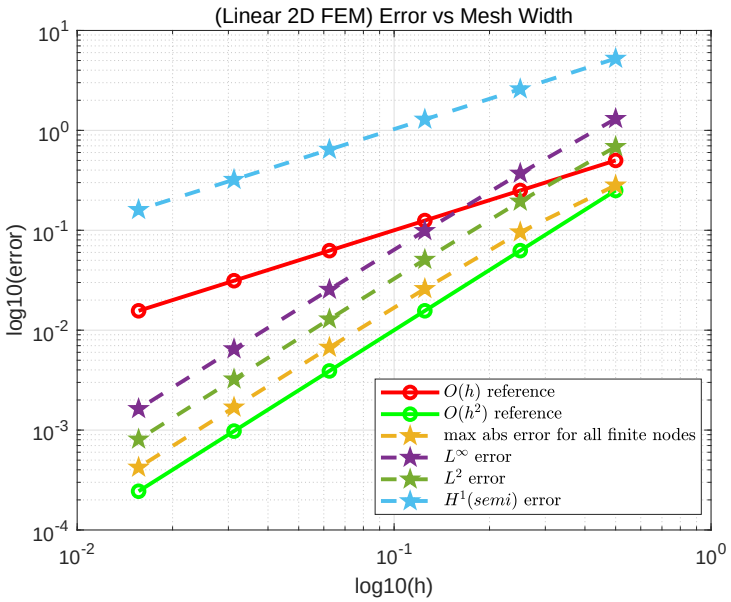


**图 37:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.12670550 s, Total: 6
Completed task num: 2, time passed: 0.36313200 s, Total: 6
Completed task num: 3, time passed: 0.78502040 s, Total: 6
Completed task num: 4, time passed: 1.56430120 s, Total: 6
Completed task num: 5, time passed: 4.59556850 s, Total: 6
Completed task num: 6, time passed: 43.94506580 s, Total: 6
```

```
     hx       hy       ht       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    ----     ----     ------    ------------------    -------------    ----------    ---------------

    1/2      1/2      1                0.28183             1.3065         0.68523         5.2336
    1/4      1/4      1/4             0.095719            0.37039         0.19449         2.5875
    1/8      1/8      1/16            0.025937           0.098704        0.050853         1.2865
    1/16     1/16     1/64           0.0067094           0.025483        0.012871        0.64214
    1/32     1/32     1/256          0.0016886          0.0064745       0.0032279        0.32092
    1/64     1/64     1/1024        0.00042269          0.0016318      0.00080763        0.16044

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          1.5579              1.8186          1.8169          1.0162
          1.8838              1.9079          1.9353          1.0081
          1.9508              1.9536          1.9822          1.0025
          1.9904              1.9767          1.9954          1.0007
          1.9982              1.9883          1.9988          1.0002
```

### 4.2 二次元求解二维二阶含时抛物方程,Backward Euler 格式

$$\begin{cases} u_t - \nabla \cdot (2\nabla u) = -3e^{x+y+t}, x \in \Omega \times T, \\ u(\partial\Omega, t = 0) = e^{x+y}, \\ u = e^{y+t}, x = 0, \\ u = e^{2+y+t}, x = 2, \\ u = e^{x+t}, y = 0, \\ u = e^{x+1+y}, y = 1, \\ \Omega = [0, 2] \times [0, 1], T = [0, 1]. \end{cases}$$

取差分格式为:

$$\left[\frac{M}{\Delta t} + \theta A\right] X^{m+1} = \theta b(t_{m+1}) + (1-\theta)b(t_m) + \left[\frac{M}{\Delta t}X^m - (1-\theta)A\right] X^m.$$

其中 $\theta = 1, \Delta t = 8h^3$, 该问题的解析解为:$u(x, y) = e^{x+y+t}$.

### 4.2.1 收敛阶及各误差计算

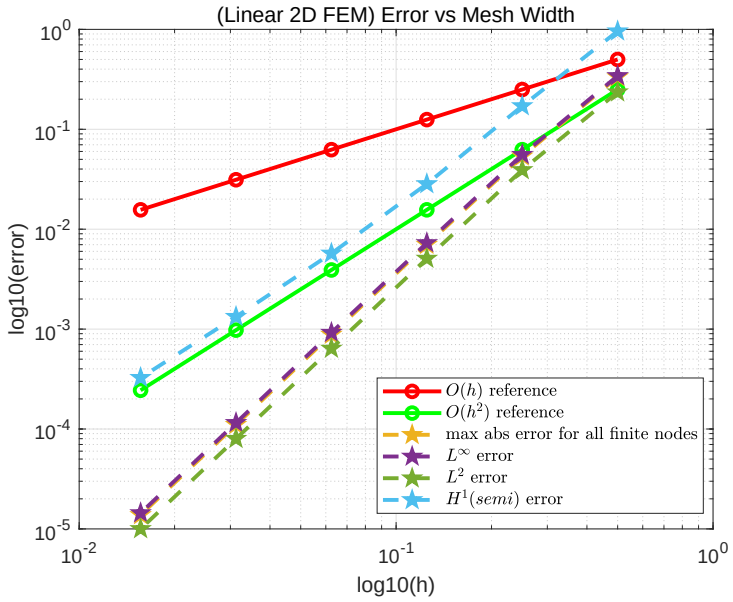


**图 38:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.14833690 s, Total: 6
Completed task num: 2, time passed: 0.43047710 s, Total: 6
Completed task num: 3, time passed: 1.49390740 s, Total: 6
Completed task num: 4, time passed: 6.59329080 s, Total: 6
Completed task num: 5, time passed: 169.17579220 s, Total: 6
Completed task num: 6, time passed: 6228.39032600 s, Total: 6
     hx       hy       ht       Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    ----     ----     -------    ------------------    -------------    ----------    ---------------

    1/2      1/2      1                 0.33348            0.34233         0.23517         0.95886
    1/4      1/4      1/8              0.053672           0.055658         0.03918         0.17096
    1/8      1/8      1/64             0.0070092          0.0072844        0.005084        0.028364
    1/16     1/16     1/512          0.00088166         0.00092169       0.00063901       0.0057212
    1/32     1/32     1/4096         0.00011034         0.00011532       7.9966e-05       0.0013298
    1/64     1/64     1/32768        1.3791e-05         1.4425e-05       9.9993e-06      0.00032586

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          2.6354              2.6208          2.5855          2.4876
          2.9368              2.9337          2.9461          2.5915
          2.9909              2.9825           2.992          2.3097
          2.9983              2.9986          2.9984          2.1051
          3.0001              2.9991          2.9995          2.0289
```

## 4.3 线性元求解二维二阶含时抛物方程,Crank-Nicolson 格式

$$\begin{cases} u_t - \nabla \cdot (2\nabla u) = -3e^{x+y+t}, x \in \Omega \times T, \\ u(\partial\Omega, t=0) = e^{x+y}, \\ u = e^{y+t}, x = 0, \\ u = e^{2+y+t}, x = 2, \\ u = e^{x+t}, y = 0, \\ u = e^{x+1+y}, y = 1, \\ \Omega = [0,2] \times [0,1], T = [0,1]. \end{cases}$$

取差分格式为:

$$\left[\frac{M}{\Delta t} + \theta A\right] X^{m+1} = \theta b(t_{m+1}) + (1-\theta)b(t_m) + \left[\frac{M}{\Delta t} X^m - (1-\theta)A\right] X^m.$$

其中 $\theta = \frac{1}{2}, \Delta t = h$, 该问题的解析解为:$u(x,y) = e^{x+y+t}$.

### 4.3.1 收敛阶及各误差计算

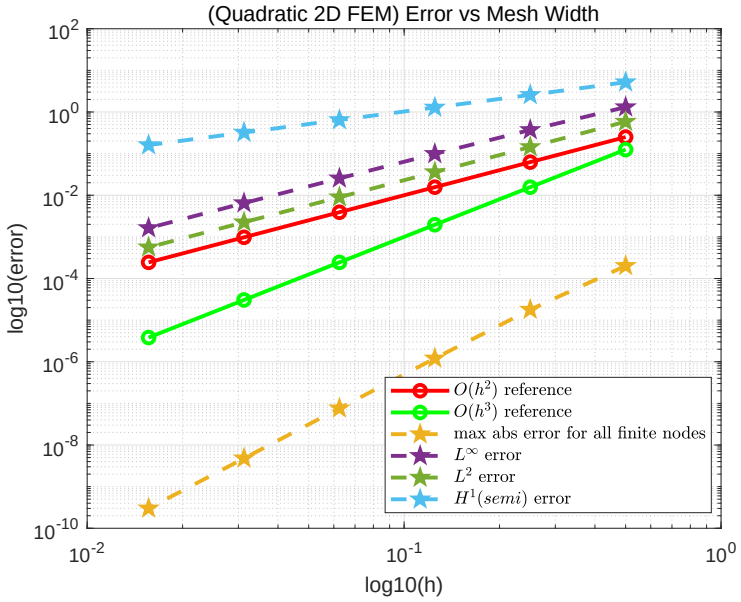


**图 39:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.15244930 s, Total: 6
Completed task num: 2, time passed: 0.44037240 s, Total: 6
Completed task num: 3, time passed: 0.87612820 s, Total: 6
Completed task num: 4, time passed: 1.50573120 s, Total: 6
Completed task num: 5, time passed: 2.57012760 s, Total: 6
Completed task num: 6, time passed: 6.75857230 s, Total: 6
     hx      hy      ht     Max Absolute Error    L^infty Error    L^2 Error     H^1(semi) Error
    ----    ----    ----    ------------------    -------------    ----------    ---------------

    1/2     1/2     1/2          0.00020131             1.3065         0.58511           5.193
    1/4     1/4     1/4          1.7863e-05            0.37039         0.14423          2.5748
    1/8     1/8     1/8          1.2042e-06           0.098704        0.035921          1.2845
```

```
    1/16     1/16     1/16          7.6275e-08            0.025483         0.0089715          0.64187
    1/32     1/32     1/32          4.7846e-09           0.0064745         0.0022423          0.32089
    1/64     1/64     1/64          3.0079e-10           0.0016318        0.00056055          0.16044

Convergence Order:
    Max Absolute Error    L^infty Error    L^2 Error    H^1(semi) Error
    ------------------    -------------    ---------    ---------------

          3.4944               1.8186          2.0204           1.0121
          3.8909               1.9079          2.0055           1.0033
          3.9807               1.9536          2.0014           1.0008
          3.9947               1.9767          2.0003           1.0002
          3.9916               1.9883          2.0001           1.0001
```

### 4.4 二次元求解二维二阶含时抛物方程,Crank-Nicolson 格式

$$\begin{cases} u_t - \nabla \cdot (2\nabla u) = -3e^{x+y+t}, x \in \Omega \times T, \\ u(\partial\Omega, t = 0) = e^{x+y}, \\ u = e^{y+t}, x = 0, \\ u = e^{2+y+t}, x = 2, \\ u = e^{x+t}, y = 0, \\ u = e^{x+1+y}, y = 1, \\ \Omega = [0, 2] \times [0, 1], T = [0, 1]. \end{cases}$$

取差分格式为:

$$\left[\frac{M}{\Delta t} + \theta A\right] X^{m+1} = \theta b(t_{m+1}) + (1-\theta)b(t_m) + \left[\frac{M}{\Delta t} X^m - (1-\theta)A\right] X^m.$$

其中 $\theta = \frac{1}{2}, \Delta t^2 \approx h^3$, 该问题的解析解为:$u(x, y) = e^{x+y+t}$.

#### 4.4.1 收敛阶及各误差计算

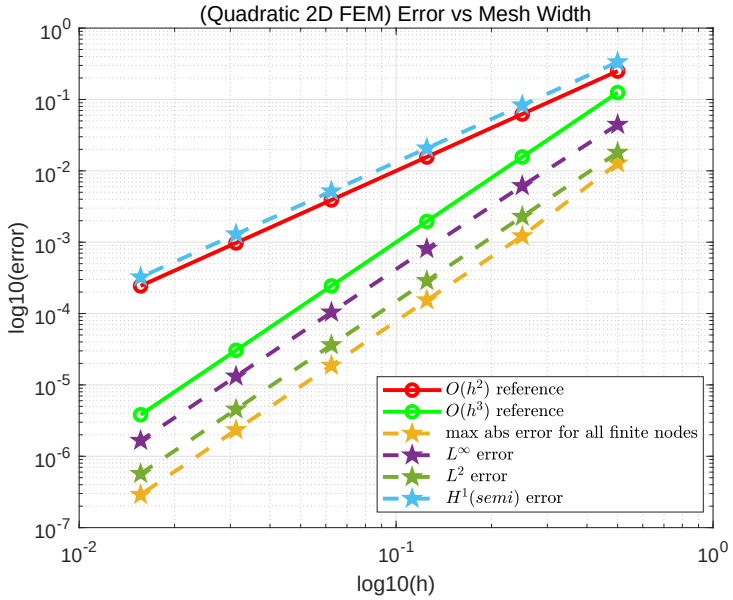


**图 40:** 收敛阶可视化

```
Now starting...
Completed task num: 1, time passed: 0.15499190 s, Total: 6
Completed task num: 2, time passed: 0.42463920 s, Total: 6
Completed task num: 3, time passed: 0.88384030 s, Total: 6
Completed task num: 4, time passed: 2.15469710 s, Total: 6
Completed task num: 5, time passed: 11.65146850 s, Total: 6
Completed task num: 6, time passed: 124.09508500 s, Total: 6
```

| hx | hy | ht | Max Absolute Error | L^infty Error | L^2 Error | H^1(semi) Error |
|---|---|---|---|---|---|---|
| 1/2 | 1/2 | 1/3 | 0.012758 | 0.044395 | 0.018065 | 0.33455 |
| 1/4 | 1/4 | 1/8 | 0.0012194 | 0.0061549 | 0.002283 | 0.083065 |
| 1/8 | 1/8 | 1/23 | 0.00015408 | 0.00081024 | 0.00028702 | 0.020725 |
| 1/16 | 1/16 | 1/64 | 1.8672e-05 | 0.00010403 | 3.6236e-05 | 0.0051789 |
| 1/32 | 1/32 | 1/181 | 2.3399e-06 | 1.3179e-05 | 4.5451e-06 | 0.0012946 |
| 1/64 | 1/64 | 1/512 | 2.8912e-07 | 1.6587e-06 | 5.6913e-07 | 0.00032363 |

Convergence Order:

| Max Absolute Error | L^infty Error | L^2 Error | H^1(semi) Error |
|---|---|---|---|
| 3.3872 | 2.8506 | 2.9842 | 2.0099 |
| 2.9844 | 2.9253 | 2.9917 | 2.0028 |
| 3.0447 | 2.9613 | 2.9857 | 2.0007 |
| 2.9964 | 2.9807 | 2.9951 | 2.0002 |
| 3.0167 | 2.9902 | 2.9975 | 2 |