Lecture 1: The geometry of linear equations

$\begin{cases} -x+2y=0 \\ 2x-y=3 \end{cases}$ row picture $\underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$

column picture $x\begin{bmatrix} -1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

not any A (consists of n columns of ~~a~~ vector of length n) can be linear combination by x to ~~re~~present b.

Lecture 2: Elimination with matrices

(1) Elimination: pick the pivot one by one and eliminate all the other elements in the same column by the eliminate.
(successful: invertable)

(2) Back substitution: take the vector of the right ~~hand~~ values of the equations, and operate the same operations done to the rows in the elimination process.

(3) Matrices Multiplication:
- vector x multiply matrix A
  means: linear combination of ~~c~~ rows in A by x.
- ~~vector~~ matrix A ~~mulit~~ multiply vector x^T
  means: linear combination of columns in A by x.

(4) permutation: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & a \\ b & c \end{bmatrix}$

do row operation to the left matric   do the column operation

(5) Elimination by matrices multiplication:

$$\underbrace{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & \boxed{8} & 1 \\ 0 & 4 & \boxed{1} \end{bmatrix}}_{A} \to \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{called } E_{21}}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\text{called } E_{32}}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U}$$

$$E_{32}E_{21}A = U$$

Lecture 3: Multiplication and inverse matrices

(1) $\begin{bmatrix} A \\ m\times n \end{bmatrix} \begin{bmatrix} B \\ n\times p \end{bmatrix} = \begin{bmatrix} C \\ m\times p \end{bmatrix}$ ← rows of C are combinations of rows of B

↑ columns of C are combinations of columns of A

(2) $AB = \text{sum}((\text{colums of } A) \times (\text{rows of } B))$

(3) Singular (not invertible)

if A is singular, then you can find a vector x so that

$$Ax = 0$$

Because if A^{-1} exists, then $A^{-1}Ax = A^{-1}0 = 0$

(4) Gauss-Jordan (solve 2 equations at once)

given an invertible matric A, in order to find A^{-1}

$$[A \mid I] \xrightarrow{\text{do}} [I \mid A^{-1}]$$

elimination to A to become I, and do the same operations to I

Because elimination can be seen as a multiplication of matric E with A. a.k.a $EA = I$, E must be A^{-1}. So $EI = A^{-1}$.

Lecture 4: Factorization into A = LU

(1) $EA = U \longrightarrow A = E^{-1}U = LU$

upper triangular

~~up-tran~~ ~~up-triangu~~

lower triangular

(2) inverse for permutation matrices

$$P^{-1} = P^{T}$$

(3) if A is a matric $n\times n$. then the operations needed for elimination to U is going to be $n^2 + (n-1)^2 + (n-2)^2 + \cdots 1^2 \approx \frac{1}{3}n^3$

Lecture 5: Transposes, permutations, spaces R^n

(1) R^TR is a symmetric matric

because $(R^TR)^T = R^TR$

(2) vector space is a space that any ~~com~~ linear combinations on vectors in the space are still in the space.

R^2 : plane of x and y axes

(3) subspaces of R^2

① all of R^2

② any line through $\begin{bmatrix}0\\0\end{bmatrix}$ L

③ zero vector only Z

(4) Column Space C(A):

~~all~~ every column in A is in R^n, and all linear combinations of columns form a subspace called C(A).

Lecture 6: Column space and nullspace

(1) subspaces S and T, the intersection $S \cap T$ is a subspace

(2) $Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$ only if b is in C(A), can x be solved.

pivot columns

(3) Nullspace of A = all solutions of $Ax = 0$

Lecture 7: Solving $Ax = 0$: pivot variables, special solutions

(1) $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank of A

= # of pivots

pivots

pivot columns

free columns

we can set any values to these free columns, and we can solve $Ax=0$ based on free columns

(2) reduced row echelon form

e.g. $A \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$ so, $Ax=0 \rightarrow Rx=0$
solutions are the same

I in pivot rows and columns

(3) Algorithm to solve $Ax=0$:

a. do elimination to A

b. get the rref of A to be R

c. get the pivot variables and free variables.

d. make the free variables to be 1 one by one, and back substitude ~~to~~ R to calculate solutions,

e. linear combinations to all the solutions above is the nullspace.

Lecture 8: Solving $Ax=b$: row reduced form R

(1) find complete solution to $Ax=b$:

a. do elimination to $[A|b]$

b. find the pivot columns and free variables.

c. find X particular solution: set all free variables to 0, solve $Ax=b$ for pivot variables

d. $X_{complete} = X_{particular} + X_{\text{all vectors of nullspace of } A}$

(2) Proof: $\because$ $AX_{particular} = b$

$AX_{nullspace} = 0$

$\therefore A(X_p + X_n) = b$

(3) $m \times n$ matrix A of rank r, then $r \le m$, $r \le n$

(4) Full column rank means $r = n < m$: no free variable   0 or 1 solution

a. Nullspace of A is only the zero vector.

b. Solutions to $Ax=b$: $X = X_p$ if it exists
unique solution

(5) Full row rank means $r = m < n$ ∞ solutions

a. can solve $Ax=b$ for every b.

b. left with $n-r$ free variables.

(6) Full row and column rank means $r=m=n$ (full rank) one solution

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

a. ~~interval~~ invertible.

b. $R = I$

c. Nullspace is zero vector only

d. only one solution to $Ax=b$, and solve any b.

Lecture 9: Independence, basis, and dimension

(1) Independence

Vectors $x_1, x_2 \dots x_n$ are independent if no linear combination gives zero vector (except the zero combination)

(2) when vectors $x_1, x_2 \dots x_n$ form a matrix A $m \times n$, and $m<n$. then $x_1, x_2 \dots x_n$ are dependent.

(3) when ~~the~~ vectors $x_1, x_2 \dots x_n$, and there is zero vector, then they are dependent.

(4) Basis for a ~~vector~~ space is a sequence of vectors $v_1, v_2 \dots v_d$. with 2 properties: a. they are independent.

b. they span the space.

(5) Given a space, every basis for the space has the same number of vectors.

The number is called the Dimension of the space.

(6) Rank of a matric = # of pivot columns = dimension of $C(A)$

(7) $\dim(N(A))$ = # of free variables = n − Rank of A

dimension of the nullspace of A

Lecture 10: The four fundamental subspaces

(1) 4 subspaces:

Column Space → $C(A)$ Null space → $N(A)$ Rowspace (all combinations of rows of A) → $C(A^T)$ Nullspace of A^T → $N(A^T)$

(2) Dimension of $C(A^T) = r$

Dimension of $N(A^T) = m - r$

(3) Basis of $C(A^T)$:

a. do elimination to A

b. get rref of ~~@~~ A after elimination

c. non-zero rows are basis of $C(A^T)$

(4) Basis of $N(A^T)$.

a. do $E[A \ I] \to [R \ E]$

b. the rows in E corresponding to zero rows in R are the basis

c. all linear combinations of basis is $N(A^T)$

Lecture 11: Matrix spaces; rank 1; small world

(1) Symetric Matric S; uppertriangal Matric U

dim of $S = 6$ dim of $U = 6$

(2) ~~SU~~ $S \cap U$ = diagnal Matrix D

$S + U$ = all ~~matri~~ matrices.

(3) rank 1 matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} = uv^T$$

(4) S = all v in R^4 with $v_1 + v_2 + v_3 + v_4 = 0$ → is a subspace

= nullspace of A where $Av = 0$ $A = [1 \ 1 \ 1 \ 1]$

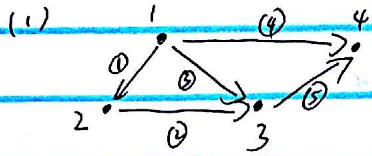
$\therefore$ dim of $S = 3$

basis of S: $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

(5) dim of a zero space is 0.

Lecture 12: Graphs, networks, incidence matrices

(1)

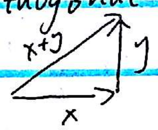


$$A = \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{edge 1} \\ \text{edge 2} \\ \text{edge 3} \\ \text{edge 4} \\ \text{edge 5} \end{array}$$

(2) Euler's formula $\quad$ #nodes − #edges + #loops = 1

Lecture 14: Orthogonal vectors and subspaces

(1) Orthogonal Vectors



$x^T y = 0$

Pythagoras $\quad \|x\|^2 + \|y\|^2 = \|x+y\|^2 \Rightarrow x^T x + y^T y = (x+y)^T (x+y)$

$$= x^T x + y^T y + x^T y + y^T x$$

$$0 = 2x^T y$$

(2) Subspaces orthogonality: if subspace S is orthogonal to subspace T,
every vector in S is orthogonal to every vector in T

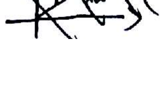
(3) row space is orthogonal to nullspace $\cancel{N(A^T)}$ $N(A)$
column space is orthogonal to nullspace $N(A^T)$

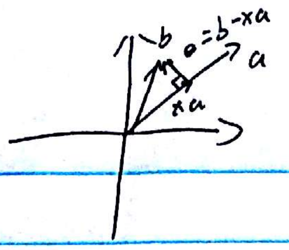
(4) $\left.\begin{array}{l}\text{dim of row space is } r \\ \text{dim of } N(A) \text{ is } n-r\end{array}\right\}$ $N(A)$ and $R(A)$ are orthogonal complements in R^n

(5) $A^T A$ is invertible exactly if A has independent columns.

Lecture 15: Projections onto subspaces

(1) $a^T(b - xa) = 0$





$a^T(b - xa) = 0$

$x a^T a = a^T b$

$x = \frac{a^T b}{a^T a}$ $P = ax \Rightarrow P = a\frac{a^T b}{a^T a}$

$\therefore$ projection matrix $M_p = \frac{aa^T}{a^T a}$

$P = M_p b$

(2) a. $C(M_p)$ = the line through a

b. $\text{rank}(M_p) = 1$

c. $M_p^T = \left(\frac{aa^T}{a^Ta}\right)^T = \frac{aa^T}{a^Ta} = M_p$ M_p is symmetric

d. $M_p^2 = M_p$ Because any b multiple M_p will be on a, any the second time multiplication won't change.

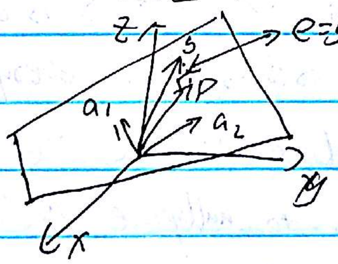
(3) Motivation of introducing Projection:

Because $Ax = b$ may do not have solution.

Solve $A\hat{x} = p$ where p is the projection of b into the ~~set~~ column spaces of A.

(4) if it is a 3D frame.

C(A) is a plane of a_1 ~~with~~ and a_2. So A can be rewrote as $\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$.



$e = b - p$ is perpendicular to the plane

p is in the plane

$\therefore P = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$= A\hat{x}$.

$\because A^T\underbrace{(b - A\hat{x})}_{e \text{ is in } N(A^T)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \hat{x} = (A^TA)^{-1}A^Tb \Rightarrow P = A\hat{x} = A(A^TA)^{-1}A^Tb$

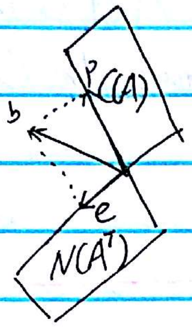
$e \perp C(A)$ because column space is orthogonal to $N(A^T)$

$A(A^TA)^{-1}A^T$ can't be extended to $AA^{-1}(A^T)^{-1}A^T$
because A is not a squre matrix, A^{-1} ~~not exists~~, doesn't exist

(5) $M_p^T = M_p \quad M_p^2 = M_p$

Lecture 16 Projection matrices and least squares

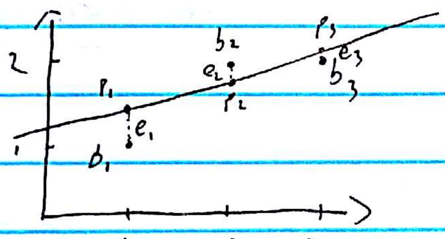
(1)



$p = M_p \cdot b$

$e = (I - M_p)\, b$

(2)



$y = c + dx$

$$\begin{matrix} c+d=1 \\ c+2d=2 \\ c+3d=2 \end{matrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

no solution. so project b to $C(A)$.

$$\min\ e_1^2 + e_2^2 + e_3^2 = (c+d-1)^2 + (c+2d-2)^2 + (c+3d-2)^2$$

$\therefore$ the original problem can be transformed to find $\hat{x} = \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix}$, ~~$M_p$~~

$$A^TA\hat{x} = A^Tb \Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\therefore \left.\begin{matrix} 3\hat{c} + 6\hat{d} = 5 \\ 6\hat{c} + 14\hat{d} = 11 \end{matrix}\right\}$$

if do partial derivative to c and d, ~~will get the same equations~~ and set to be 0, then will get the same equations

$$\left.\begin{matrix} e_1 = b_1 - p_1 = -\frac{1}{6} \\ e_2 = b_2 - p_2 = \frac{2}{6} \\ e_3 = b_3 - p_3 = -\frac{1}{6} \end{matrix}\right\} e$$

$e \perp p$, $e + p = b$, e in $N(A^T)$, p in $C(A)$.

(3) Understanding of linear least squares:
Because the perfect case is $Ax = b$, however it is ~~unrealistic~~ impossible

in real case. So $Ax=b$ can't be solved, which means b is not in $C(A)$. Then we want to minimize $e_1^2+e_2^2+e_3^2$, which is ~~exa~~ ectually is the length's square of vector e, $e = b - Ax$. So the minimum length of e is in the case that $e = b - p$, where p is the projection of b to $C(A)$.

(4) If A has independent columns, then A^TA is invertible

proof: Suppose $A^TAx = 0$, to prove x must be 0

$$x^TA^TAx = 0$$

$$(Ax)^TAx = 0 \Rightarrow \cancel{Ax = 0}\ Ax = 0 \Rightarrow x = 0$$

$\boxed{\text{Lecture 17: Orthogonal matrices and Gram-Schmidt}}$

(1) ~~Orthog.~~ Orthonormal vectors

$$q_i^Tq_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \qquad Q^TQ = I_n$$

if Q is a square matrix, then $Q^T = Q^{-1}$

(2) Q has orthonormal columns.

Project onto its column space

$$M_p = Q(Q^TQ)^{-1}Q^T = QQ^T \quad \{= I \text{ if } Q \text{ is square}\}$$

(3) Original equation: $A^TA\hat{x} = A^Tb$ normal equation

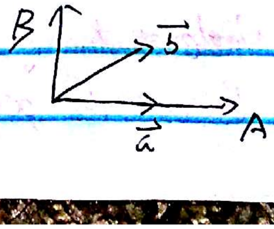
$$\Downarrow$$

$$Q^TQ\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb$$

$$\hat{x}_i = q_i^Tb$$

(4) Gram-Schmidt

a and b is independant.



$A = \cancel{\frac{a}{\|a\|}}\, a$

$B = b - \frac{A^T b}{A^T A} A$

$q_1 = \frac{A}{\|A\|}$

$q_2 = \frac{B}{\|B\|}$

if there is another vector c not in the plane of AB, then

$C = \cancel{c}\; c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$

$q_3 = \frac{C}{\|C\|}$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ the orthonormal vectors of it is $Q = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(5) $A = QR$ → is a upper ~~tran~~ triangular

e.g. $\cancel{A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}}$ $Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$

$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix}$

→ should be 0

$\cancel{q_2^T q_1 = \frac{b^T a}{\|a\|}}$

$\cancel{a_2^T q_1 =}$

$\cancel{B}\; b = B + \frac{A^T b}{A^T A} A$

Lecture 18: Properties of determinants

(1) Determinants $\det(A) = |A|$ A is a square matrix

Properties of Determinants:

① $\det I = 1$

② Exchange rows will reverse the sign of det

③ a: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

b: $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

④ two equal rows → $\det = 0$

⑤ ~~subst~~ subtract $l \times$ row i from row k det doesn't change

⑥ row of zeros $\rightarrow \det A = 0$

⑦ $\det U = \begin{vmatrix} d_1 & * & \cdots & * \\ 0 & d_2 & \cdots & * \\ 0 & \cdots & d_3 & \vdots \\ 0 & \cdots & \cdots & d_n \end{vmatrix} = \prod_{i=1}^{n} d_i$ product of pivots

⑧ $\det A = 0$ when A is singular

$\det A \neq 0$ when A is invertible

⑨ $\det AB = (\det A)(\det B)$

⑩ $\det A^{-1} = \dfrac{1}{\det A}$

$\det A^2 = (\det A)^2$

$\det 2A = 2^n (\det A)$

⑩ $\det A^T = \det A$

$\boxed{\text{Lecture 19: Determinant formulas and cofactors}}$

(1) $\det A_{n\times n} = \sum_{n!\ \text{terms}} \pm\, a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \cdots a_{n\omega}$

$(\alpha, \beta, \gamma, \dots, \omega) =$ ~~permutat~~ permutation of $(1, 2, \dots, n)$

(2) Cofactor of $a_{ij} = C_{ij}$

when $i+j$ even $\leftarrow$ $\pm \det(n-1$ matrix with row i and col j erased$)$

$\downarrow$ when $i+j$ odd

(3) cofactor formula

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \cdots a_{1n}C_{1n}$

$\boxed{\text{Lecture 20: Cramer's rule, inverse matrix, and volume}}$

(1) $A^{-1} = \dfrac{1}{\det A} C^T$

$\rightarrow$ cofactors matrix

proof : $AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 & 0 & \cdots & 0 \\ 0 & \det A & \cdots & & & 0 \\ 0 & & & & & \vdots \\ 0 & \cdots & & & & \det A \end{bmatrix} = \det A \cdot I$

(2) Cramer's Rule

$Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b$

$x_1 = \frac{\det B_1}{\det A}$ $\qquad x_j = \frac{\det B_j}{\det A}$

$x_2 = \frac{\det B_2}{\det A}$

$\vdots$

where $B_1 = \begin{bmatrix} | & | & | & & | \\ b & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$ $\qquad B_j = \begin{bmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{j-1} & b & a_{j+1} & \cdots & a_n \\ | & & | & | & | & & | \end{bmatrix}$

$\underbrace{a_2 \; a_3 \cdots a_n}_{n-1 \text{ columns of } A}$

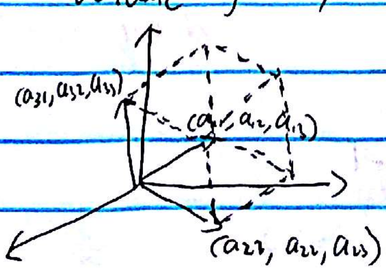
because $x = \frac{1}{\det A} \underbrace{C^T b}_{\downarrow}$

$\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

$x_1 = \underbrace{b_1 C_{11} + b_2 C_{21} + b_3 C_{31} + \cdots \; b_n C_{n1}}_{\text{is the det of}}$

$\begin{bmatrix} b_1 & | & | & & | \\ b_2 & a_2 & a_3 & \cdots & a_n \\ \vdots & & & & \\ b_n & | & | & & | \end{bmatrix}$

(3) $\underset{3\times 3}{\det A}$ = volume of box



Lecture 21: Eigenvalues and eigenvectors

(1) Ax parallel to x

$$Ax = \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{x}_{\text{eigenvector}}$$

(2) Fact: Sum of λ's = sum of ~~diagnal~~ diagonal elements

(3) to find λ and x

solve $\underbrace{(A-\lambda I)}_{\text{must be singular}} x = 0$ → $\det(A-\lambda I) = 0$

(4) $\text{trace} = \Sigma\lambda$

$\det = \Pi\lambda$

Lecture 22: Diagonalization and powers of A

(1) Suppose n linear independant eigenvectors of A, put them in columns of S

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}}_{\text{diagonal eigenvalue matrix}} = S\Lambda$$

$$S^{-1}AS = \Lambda$$

$$A = S\Lambda S^{-1}$$

(2) if $Ax = \lambda x$

then $A^2 x = \lambda A x = \lambda^2 x$

in matrix form: $A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$A^k = \cancel{S A^k S^{-1}}\ S\Lambda^k S^{-1}$

So the power of matrix ~~wont~~ will have the same S, x

(3) Thm: $A^k \to 0$ when $k \to \infty$

if all $|\lambda_i| < 1$

(4) A is sure to have n independent ~~eigene~~ eigenvectors (and be diagonalizable)

if all the λ's are different (i.e. no repeated λ's)

(if there are repeated λ's, A may also have n independent eigenvectors e.g. $A = I$)

(5) Equation: $u_{k+1} = Au_k$, starts with given vector u_0

$u_1 = Au_0,\ u_2 = A^2 u_0 \cdots\ u_k = A^k u_0$

suppose $u_0 = c_1 x_1 + c_2 x_2 + \cdots c_n x_n := Sc$

$Au_0 = c_1\lambda_1 x_1 + c_2\lambda_2 x_2 + \cdots c_n\lambda_n x_n$

$A^k u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \cdots c_n\lambda_n^k x_n = \Lambda^k S c$

Lecture 23: Differential equations and exp(At)

(1) example $\frac{du}{dt} = Au \Rightarrow \frac{du_1}{dt} = -u_1 + 2u_2$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \frac{du_2}{dt} = u_1 - 2u_2$

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\lambda_1 = 0 \quad \lambda_2 = 3$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\frac{du}{dt} = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

check: $\frac{du}{dt} = Au$

$c_1\lambda_1 e^{\lambda_1 t} x_1 + c_2\lambda_2 e^{\lambda_2 t} x_2 = c_1 e^{\lambda_1 t} Ax_1 + c_2 e^{\lambda_2 t} Ax_2$ correct take $u(0)$ into $u(t) = c_1 e^{0t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

get $u(t) = \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$

(2) Stability $u(t) \to 0 \Rightarrow$ need $e^{\lambda t} \to 0 \Rightarrow$ need real $\lambda < 0$

Steady state $\lambda_1 = 0$ and other $Re\,\lambda < 0$

Blow up if any $Re\,\lambda > 0$

(3) $\frac{du}{dt} = Au$ Set $u = Sv$

eigen vectors matrix

$S\frac{dv}{dt} = ASv$

$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$

$\therefore v(t) = e^{\Lambda} e^{\Lambda t} v(0)$

$u(t) = S e^{\Lambda t} v(0)$

$u(t) = u(0) e^{At} = S e^{\Lambda t} S^{-1} u(0)$

(4) Matrix Exponential e^{At}

$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots \frac{(At)^n}{n!} \dots$

$SS^{-1} \leftarrow$

$= I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \dots \frac{S\Lambda^n S^{-1}}{n!}t^n \dots$

$= S(I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \dots \frac{\Lambda^n t^n}{n!})S^{-1}$

$= S e^{\Lambda t} S^{-1}$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & \dots & e^{\lambda_n t} \end{bmatrix}$$

(5) calculate $y'' + by' + ky = 0$

set $u = \begin{bmatrix} y' \\ y \end{bmatrix}$ $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

$= \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

if it is a nth order differential equation

$A = \begin{bmatrix} \text{— coefficients —} \\ 1 \; 0 \cdots\cdots 0 \\ 0 \; 1 \cdots\cdots 0 \\ 0 \; 0 \; 1 \cdots 0 \\ 0 \cdots\cdots 1 \; 0 \end{bmatrix}$

Lecture 24: Markov matrices; fourier series

(1) Propeties of Markov matrices

① All entries ≥ 0

② All columns add to 1

e.g.: $A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$

key points:

① $\lambda = 1$ is an ~~eigenva~~ eigenvalue

② All other $|\lambda_i| < 1$

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots c_n \lambda_n^k x_n$

$= c_1 x_1$

$\because$ $A - \lambda I = B = A - I$

all ~~coln~~ ~~colums~~ columns of B add to zero $\rightarrow$ $A - I$ is singular

$\Downarrow$

because rows are dependent

$\Downarrow$

because $(1,1,1)$ is in ~~$N(A^T)$~~ $N((A-I)^T)$

then x_1 is in $N((A-I))$

$\because$ $(A - I)x_1 = Ax_1 - x_1 = \lambda_1 x_1 - x_1 = 0$

(2) Example

$$\begin{bmatrix} U_{cal} \\ U_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} \qquad \begin{bmatrix} U_{cal} \\ U_{mass} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} U_{cal} \\ U_{mass} \end{bmatrix}_k$$

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \quad \lambda_1 = 1 \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.7 \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

to plugin u_0 into it get $u_k = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(3) Projections with ~~orthogonal~~ orthonomal basis $q_1, q_2 \cdots q_n$

Any vector $v = x_1 q_1 + x_2 q_2 + \cdots x_n q_n$

$$v = \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$Qx = v \qquad x = \underbrace{Q^{-1} v = Q^T v}_{\text{Because } QQ^T = I}$$

$$x_1 = q_1^T v \quad \cdots \quad x_n = q_n^T v$$

(4) fourier series : periodic functions

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

inner production of two functions $g(x), f(x)$.

$\cancel{f^T g}$ $\quad f^T g = \int f(x) g(x) dx$

if $f^T g = 0$, then f and g are orthogonal

$$\therefore \int_0^{2\pi} f(x)\cos x\,dx = 0 + \int_0^{2\pi} (\cos x)^2 dx + 0 \cdots = \pi$$

$$\therefore a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

Lecture 25: symmetric matrices and positive definiteness

(1) $A = A^T$

① The eigenvalues of A are REAL

② The eigen-vectors are PERPENDICULAR

due to orthonormal columns

$$A = \underbrace{Q}_{\text{unit vectors of columns of } S} \Lambda Q^{T}$$

(2) $A = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots \lambda_n q_n q_n^T$

$\therefore$ Every symmetric matrix can be seen as a combination of perpendicular projection matrices

(3) for symmetric matrices Signs of pivots are the same as the signs of λ's

\# of ~~posiv~~ positive pivots = \# of positive λ's

(4) Symmetric Matrices Positive ~~define~~ definite matrix

all eigenvalues are positive

all pivots are positive

all sub-determinants ~~is~~ are positive

Lecture 26: Complex matrices; fast fourier transform

(1) $Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ in $\mathbb{C}^n$

$\bar{Z}^T Z$ is ~~positive~~ non-negative $= |z_1|^2 + |z_2|^2 \cdots |z_n|^2$

$\Downarrow$

$Z^H Z$ Hermitian

inner product of two complex matrices : $y^H x$

Hermitian Matrix $A^H = A \Rightarrow \bar{A}^T = A$

Perpendicular $q_1, q_2 \dots q_n$ orthonormal

$$q_i^H q_j \quad \bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \quad \bar{Q}^T Q = I \quad Q^H Q = I$$

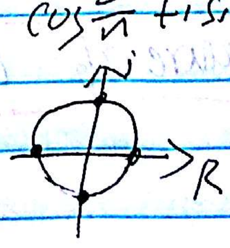
(2) n by n fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & & 1 \\ 1 & w & w^2 & w^4 & & w^{n-1} \\ \vdots & w^2 & w^4 & w^6 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2n-2} & w^{3n-3} & & w^{(n-1)^2} \end{bmatrix}$$

$(F_n)_{ij} = w^{ij} \quad i, j = 0 \dots n-1$

$w^n = 1 \quad w = e^{\frac{i2\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

when $n=4$ then $w^4 = 1 \quad w = e^{\frac{2\pi i}{4}} = i$



$\therefore w = i \quad w^2 = -1 \quad w^3 = -i \quad w^4 = 1$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{bmatrix}$$

(3)

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D_{32} \\ I & -D_{32} \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \overset{P}{\begin{bmatrix} 100\cdots0 \\ 001\cdots\cdots0 \\ 00001\cdots0 \\ 0\cdots\cdots10 \\ 010\cdots\cdots0 \\ 0001\cdots\cdots0 \\ \vdots \\ 00\cdots\cdots1 \end{bmatrix}}$$

$$D_{32} = \begin{bmatrix} 1 & & & 0 \\ & w & & \\ & & w^2 & \\ 0 & & \ddots & \\ & & & w^{31} \end{bmatrix}$$

key is $\cancel{(W^{64})^2}\ (W_{64})^2 = e^{\frac{2\pi i}{64}\times 2} = e^{\frac{2\pi i}{32}}$

$= W_{32}$

$$O(64^2) \Rightarrow O(2(32)^2+32) \Rightarrow O(2(2(16)^2+16)+32) \Rightarrow O(\boxed{6}\times\boxed{32}) \quad \boxed{6}\to\log_2 64,\ \boxed{32}\to\frac{n}{2}$$

$\therefore$ can reduce from $O(n^2)$ to $\cancel{1+2+4+\cdots 2^{n-1}}$ $O(\frac{n}{2}\log^n) = O(n\log n)$

Lecture 27: Positive definite matrices and minima

(1) Positive definite matrices

① $\lambda_i > 0$

② dets > 0

③ pivots > 0

④ $x^TAx > 0$

(2) MIN ~ MATRIX OF 2^{nd} derivs

$f(x_1, x_2 \cdots x_n)$ is Positive Definite

(3) Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ dets: 2, 3, 4

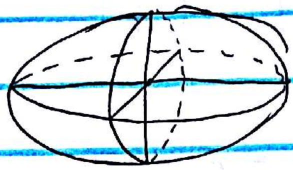
pivots: $2, \frac{3}{2}, \frac{4}{3}$

eigenvalues: $2-\sqrt{2}, 2, 2+\sqrt{2}$

$x^TAx = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 0x_1x_3 + -2x_2x_3 > 0$

$x^T A x = 1$ is an ellipsoid

$A = \cancel{Q^TAQ}\, Q\Lambda Q^T$



eigenvectors are the ~~axis~~ directions of the axes of the ellipsoid

eigenvalues are the length of ~~three~~ 3 axes.

Lecture 28: Similar matrices and jordan form

(1) A is an m by n matrix with rank $= n$

A^TA is a symmetric, square matrix

because $x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2 > 0$

(Ax): $m \times 1$ vector

because the $N(A)$ is null, so ~~$\|A$~~ Ax can't be 0.

$\therefore \|Ax\| > 0$

$\therefore A^TA$ is positive definite

(2) $\underset{n\times n}{A}$ and $\underset{n\times n}{B}$ are similar means: for some M

$$B = M^{-1}AM$$

Example: A is similar to its Λ

~~be~~ $\because S^{-1}AS = \Lambda$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

let M be $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ then $B = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$

same trace
same det

(3) Similar matrices have same eigenvalues, the same # of eigenvectors

$$Ax = \lambda x \qquad B = M^{-1}AM$$

$$AMM^{-1}x = \lambda x$$

$$M^{-1}AMM^{-1}x = \lambda M^{-1}x$$

$$BM^{-1}x = \lambda M^{-1}x \rightarrow$$ λ is also B's eigenvalue

(4) Jordan block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Every square matrix A is similar to a Jordan matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$$ # of J blocks ~~to~~ = # of λ's

$\boxed{\text{Lecture 29: Singular value decomposition}}$

(1)

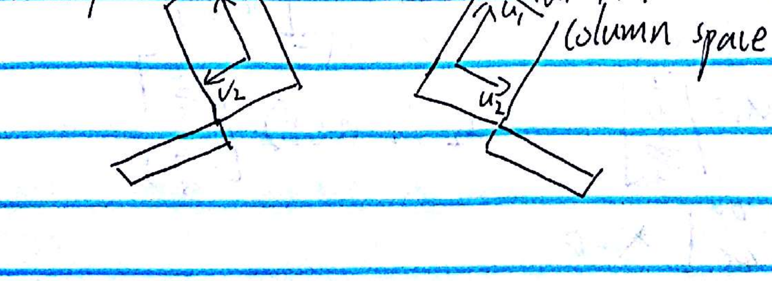
A m by n matrix.

let $v_1, v_2 \dots v_r$ are orthonormal ~~vet~~ row space basis of A

~~then~~ $u_1, u_2 \dots$ ~~a~~ u_r are orthonormal column space basis of A

$n_1 \cdots n_{n-r}$ $\qquad$ 0

$$A\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{bmatrix}\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \\ & & & & 0 \end{bmatrix}$$



(2) Example 1 $A=\begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}$

to find orthonormal $\begin{cases} v_1, v_2 \text{ in row space } \mathbb{R}^2 \\ u_1, u_2 \text{ in column space } \mathbb{R}^2 \end{cases}$

$\sigma_1, \sigma_2 > 0$

to satisfy $Av_1 = \sigma_1 u_1$
$Av_2 = \sigma_2 u_2$

$AV = U\Sigma \rightarrow A = U\Sigma V^T$

$A^TA = V\Sigma^T U^T U \Sigma V^T = \underbrace{V\begin{bmatrix}\sigma_1^2 & \\ & \sigma_2^2 \\ & & \ddots\end{bmatrix}V^T}_{\Downarrow \; Q\Lambda Q^{-1}}$ $\Big\}$ ∴ V is the eigenvectors, σ_i^2 is the eigenvalues

$AA^T = U\Sigma V^T V \Sigma^T U^T$
$= U\Sigma\Sigma^T U^T$

∴ $A^TA = \begin{bmatrix}25 & 7\\7 & 25\end{bmatrix} \Rightarrow \lambda_1 = 32 \quad x_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$\lambda_2 = 18 \quad x_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

$$\therefore A = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix} = \overset{U}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} \overset{\Sigma}{\begin{bmatrix}\sqrt{32} & 0\\0 & \sqrt{18}\end{bmatrix}} \overset{V^T}{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}}$$

$AA^T = \begin{bmatrix}32 & 0\\0 & 18\end{bmatrix} \Rightarrow \lambda_1 = 32 \quad x_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$\lambda_2 = 18 \quad x_2 = \begin{bmatrix}0\\1\end{bmatrix}$

(3) Example 2: $A=\begin{bmatrix}4 & 3\\8 & 6\end{bmatrix}$ singular matrix

row space $v_1 = \begin{bmatrix}0.8\\0.6\end{bmatrix}$ $\quad v_2$ in $N(A)$ $\begin{bmatrix}0.6\\-0.8\end{bmatrix}$

~~∴ A =~~

column space $u_1 = \begin{bmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}$ $\quad u_2$ in $N(A^T)$ $\begin{bmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{bmatrix}$

$A^TA = \begin{bmatrix}80 & 60\\60 & 45\end{bmatrix}$ $\quad \lambda_1 = 0$

$\lambda_2 = 125$

$$\therefore A = \overset{U}{\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \overset{V^T}{\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}}$$

(4) $v_1, \dots v_r$ are orthonormal basis for row space
$u_1, \dots u_r$ are orthonormal basis for column space
$v_{r+1} \dots v_n$ are orthonormal basis for $N(A)$
$u_{r+1} \dots u_m$ are orthonormal basis for $N(A^T)$

$$\underset{m\times n}{A} = \underset{n\times m}{U} \; \underset{m\times n}{\Sigma} \; \underset{m\times n}{V^T}$$

Lecture 30: Linear transformations and their matrices

(1) Linear Transformations $T(x)$

$T(v+w) = T(v) + T(w)$

$T(cv) = cT(v)$

examples: projection, ~~shift~~ rotation, matrix A.

(2) the rule to find matrix A. Given bases $v_1 \dots v_n$ Input Space, $w_1 \dots w_m$ Output Space

1st column of A: $a_{11} \cdots a_{m1}$: $T(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots a_{m1}w_m$

2nd column of A: $a_{12} \cdots a_{m2}$: $T(v_2) = a_{12}w_1 + \cdots \; a_{m2}w_m$

A (input coordinates) = (output coordinates)

Lecture 31: Change of basis; image compression

(1) Fourier Basis 8×8

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} w \\ w^2 \\ \vdots \\ w^n \end{bmatrix} \cdots$$

(2) compression

signal x

lossless $\downarrow$ change basis

coeffs c

lossy $\downarrow$ compression by discard coeffs below a threshold

$\hat{c}$ (many zeros)

$\downarrow$

$$\hat{x} = \sum \hat{c}_i v_i$$

(3) wavelet

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdots$$

(4) change basis

$P = Wc$ $\quad$ P is in standard basis, W is a new basis

$c = W^{-1}p$ $\quad$ inverse

A Good Basis: ① Fast ✓ FFT, FWT

② Few is Enough

(5) Change of basis

<u>Columns</u>: W = new basis vectors

$$\begin{bmatrix} | \\ x \\ | \end{bmatrix}_{\text{old basis}} \longrightarrow \begin{bmatrix} c \end{bmatrix}_{\text{new basis}}$$

$x = Wc$

Having a Linear Transformation T

<u>Matrix</u>: with respect to basis $v_1, v_2 \cdots v_8$, it has A

with respect to basis $w_1, \ldots w_8$, it has B

A and B are similar $\quad B = M^{-1}AM$

(6) Calculate A using basis $v_1, \ldots v_8$

Know T completely from $T(v_1), T(v_2) \ldots T(v_8)$

Because every $x = c_1v_1 + c_2v_2 + \cdots c_8v_8$

Then $T(x) = c_1T(v_1) + c_2T(v_2) + \cdots + c_8T(v_8)$

to get $Ax = T(x)$ Write $T(v_1) = a_{11}v_1 + a_{21}v_2 + \cdots a_{81}v_8$

$T(v_2) = a_{12}v_1 + \cdots \quad a_{82}v_8$

$\vdots$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{18} \\ a_{21} & \cdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{81} & \cdots & a_{88} \end{bmatrix}$$

(7) For eigenvector basis (Best Basis)

$T(v_i) = \lambda_i v_i$

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \Lambda$$

$$\boxed{\text{Lecture 33: Left and right inverses; pseudoinverse}}$$

(1) <u>2-sided inverse</u> $\quad A$ $m \times n$ with Rank $= r$

$AA^{-1} = I = A^{-1}A \qquad m = n = r$ full rank

left inverse

full column rank $r = n < m$

$$\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}}A = I$$

right inverse

full row rank $r = m < n$

$$\underbrace{A A^T (AA^T)^{-1}}_{A^{-1}_{right}} = I$$

Note: the brace in the formula above sits under $A^T (AA^T)^{-1}$ only; "A^{-1}_{right}" is written beneath it in red.

(2) $A_{m \times n}$ with Rank $r < m, n$

vector x in row space of A

then Ax is in column space of A

suppose another vector y in row space of A.

Ay is in column space of A.

define $A^+ (Ay) = y$

A^+ ↓ pseudoinverse

(3) Find the pseudoinverse A^+

① start from SVD: $A = U \Sigma V^T$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r & \\ & & & 0 \end{bmatrix} \text{ rank } r \quad m \times n$$

$$\Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_r} & \\ & & & 0 \end{bmatrix}_{n \times m}$$

$$\Sigma \Sigma^+ = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 & \\ & & & 0 \end{bmatrix}_{m \times m}$$

$$\Sigma^+ \Sigma = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 & \\ & & & 0 \end{bmatrix}_{n \times n}$$

$A^+ = V \Sigma^+ U^T$