CSC165H1 Problem Set 3

Make Zhang, Ruijian An, Haoran Fang

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1. A summation

Use simple induction

For all $x \in \mathbb{R}^+$, Assume $x \neq 1$

Define $\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}$ as P(n)

Base Case: Let n = 0, Want to prove P(0).

LHS = $\sum_{i=0}^{0-1} x^i = x^0 = 0$

RHS $= \frac{x^0 - 1}{x-1} = 0$

LHS = RHS

P(0) holds

Inductive Step:

Let $k \in \mathbb{N}$

Assume P(k) holds, i.e., that $\sum_{i=0}^{k-1} x^i = \frac{x^k - 1}{x-1}$. We want to prove P(k+1), i.e., that $\sum_{i=0}^{k-1+1} x^i = \frac{x^{k+1} - 1}{x-1}$

$\sum_{i=0}^{k-1+1} x^i$

$= \sum_{i=0}^{k-1} x^i + x^k$

$= \frac{x^k - 1}{x-1} + x^k$ #by P(k)

$= \frac{x^k - 1 + x^{k+1} - x^k}{x-1}$

$= \frac{x^{k+1} - 1}{x-1}$

P(k+1) holds

Therefore, $\forall x \in \mathbb{R}^+, x \neq 1 \Rightarrow (\forall n \in \mathbb{N}, \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1})$

2. Fibonacci sequence

Want to prove $\forall n \in \mathbb{N}, \gcd(F_n, F_{n+1}) = 1$

Define $\gcd(F_n, F_{n+1}) = 1$ as P(n)

Base case: Let n = 0, Want to prove P(0).

$\gcd(F_0, F_{0+1}) = \gcd(F_0, F_1) = \gcd(0, 1) = 1$ # by Claim 1

P(0) holds

Inductive Step:

Let $k \in \mathbb{N}$

Assume P(k) i.e. $\gcd(F_k, F_{k+1}) = 1$ holds, $k \geq 0$

Want to show P(k+1) i.e. $\gcd(F_{k+1}, F_{k+2}) = 1$ also hold

Prove its contrapositive form:

$\forall k \in \mathbb{N}, \neg\mathrm{P}(k+1) \Rightarrow \neg\mathrm{P}(k)$

$\equiv \forall k \in \mathbb{N}, \gcd(F_{k+1}, F_{k+2}) \neq 1 \Rightarrow \gcd(F_k, F_{k+1}) \neq 1$

Let $k \in \mathbb{N}$, assume $\gcd(F_{k+1}, F_{k+2}) = d$ and $d \neq 1$

Since $1|F_{k+1}$ and $1|F_{k+2}$, so $d > 1$

By definition of gcd, $d|F_{k+1} \wedge d|F_{k+2}$

By Claim 2, $d|F_{k+1} \wedge d|F_{k+2}$, so $d|-F_{k+1} + F_{k+2}$

$d|-F_{k+1} + F_{k+2} = d|-F_{k+1} + F_k + F_{k+1} = d|F_k$

So $d|F_k \wedge d|F_{k+1}$

So $\gcd(F_k, F_{k+1}) \geq d > 1$

So $\gcd(F_k, F_{k+1}) \neq 1$

Therefore, $\forall k \in \mathbb{N}, \neg\mathrm{P}(\mathrm{k}+1) \Rightarrow \neg\mathrm{P}(\mathrm{k})$

Therefore, $\forall n \in \mathbb{N}, \gcd(F_n, F_{n+1}) = 1$

3. Counting more subsets

(a)

$DP_1 = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\{0\}, \emptyset)\}$

$DP_2 = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\{0\}, \emptyset), (\emptyset, \{1\}), (\{1\}, \emptyset), (\emptyset, \{0,1\}), (\{1,0\}, \emptyset), (\{1\}, \{0\}), (\{0\}, \{1\})\}$

(b)

$S_n = \{0,1, \dots, n-1\}$

$DP_n = \{(A, B) | A, B \subseteq S_n \text{ and } A \wedge B \subseteq \emptyset\}$

$DP_0 = \{(\emptyset, \emptyset)\}$

$DP_1 = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\{0\}, \emptyset)\}$

$S_2 = \{0,1\}$

$A = \emptyset, \{0\}, \{1\}, \{0,1\}$

$DP_2 = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\{0\}, \emptyset), (\emptyset, \{1\}), (\{1\}, \emptyset), (\emptyset, \{0,1\}), (\{1,0\}, \emptyset), (\{1\}, \{0\}), (\{0\}, \{1\})\}$

$|DP_0| = 1$

$|DP_1| = 3$

$|DP_2| = 9$

Guess

$\forall n \in \mathbb{N}, |DP_n| = 3^n$

Proof:

Let $|DP_n| = 3^n$ as P(n)

<u>Base case:</u> let n = 0. Want to prove P(0)

LHS = $|DP_0|$ = 1

RHS = 3^0= 1

LHS = RHS

P(0) holds

<u>Inductive step:</u>

Let $k \in \mathbb{N}$

Assume P(k) i.e. $|DP_k| = 3^k$ holds

Want to show P(k+1) i.e. $|DP_{k+1}| = 3^{k+1}$ also hold.

There are three different situations we need to consider:

Situation1:

$k \notin A,\ k \notin B$

$|DP_k| = 3^k$ # By Induction hypothesis

Situation2:

$k \in A,\ k \notin B$

Since we know $k \in A,$ we can obtain all the subsets of A and add k to each element in A. Therefore, we can apply our hypothesis.

$|DP_k| = 3^k$

Situation3:

$k \notin A,\ k \in B$

Similar with Situation2, but this time we can combine k to each element in the B and get the same answer with Situation2.

$|DP_k| = 3^k$

Add all the possible situation together, and we will get our final answer:

$|DP_{k+1}| = 3^k + 3^k + 3^k = 3^{k+1}$

Therefore, $\forall n \in \mathbb{N}, |DP_n| = 3^n$

4. Pigeonhole principle

Use simple induction

Base case: Let n = 2, want to prove PHP(2)

Let $n = 2, S, T \subseteq N$

Assume $|S| = 2, |T| = 1$ and let $S = \{s_1, s_2\} T = \{t\}$

Let $f: S \to T,\ f(s_1) = f(s_2) = t$

PHP(2) holds

Inductive Step:

Let $k \in \mathbb{N}$

Assume PHP(k) holds and $k \geq 2.$

Want to show PHP(k+1) also holds

Let $S = \{s_1, \dots, s_k, s_{k+1}\}$

$T = \{t_1, \dots, t_{k-1}, t_k\}$

Let $f: S \to T$

$f(s_{k+1}) = t_i$ for some $i = 1, .., k$

Case 1:

Suppose $\exists s_j \in S, j \neq k+1, f(s_j) = t_i$ or we can say there are at least two distinct elements in S, such that the image of them are t_i

Because $s_j \neq s_{k+1},\ f(s_j) = f(s_{k+1}) = t_i$

PHP(k+1) holds

Case 2:

Suppose $\neg(\exists s_j \in S, j \neq k+1, f(s_j) = t_i)$ or we can say only $f(s_{k+1}) = t_i$

If we remove s_{k+1} from S and t_i from T i.e.($S \backslash \{s_{k+1}\},\ T \backslash \{t_i\}$) at the same time, and we know the remaining k elements in S and remaining k-1 elements in T will follow the induction hypothesis we have been made.

Since $f: S \backslash \{s_{k+1}\} \to T \backslash \{t_i\}$, we can get:

$\exists s_1, s_2 \in S \backslash \{s_{k+1}\}, s_1 \neq s_2 \wedge f(s_1) = f(s_2)$ (By Induction hypothesis)

PHP(k+1) holds

Therefore, $\forall n \in \mathbb{N}, n \geq 2 \Rightarrow PHP(n)$