# CSC165H1 Problem Set 2

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## Q1.AND vs. IMPLIES

(a) **Translate and Using Prove**: $\forall$ x,y,z $\in \mathbb{Z}$, (x|y)$\wedge$(y|z) $\Rightarrow$ x|z
$\equiv \forall$ x,y,z $\in \mathbb{Z}$, ($\exists$ $k_1 \in \mathbb{Z}$, y = $k_1$x) $\wedge$ ($\exists$ $k_2 \in \mathbb{Z}$, z = $k_2$ y) $\Rightarrow$ ($\exists$ $k_3 \in \mathbb{Z}$, z = $k_3$ x)

*Proof.* Let x,y,z $\in \mathbb{Z}$, Assume $\exists$ $k_1 \in \mathbb{Z}$, y = $k_1$x and $\exists$ $k_2 \in \mathbb{Z}$, z = $k_2$y. Want to prove $\exists$ $k_3 \in \mathbb{Z}$, z = $k_3$x. Let $k_3 = k_1 k_2$, $k_3 \in \mathbb{Z}$.

$$\begin{aligned} k_3x &= k_1k_2x \\ k_3x &= k_2y \quad (Since,\ y = k_1x) \\ k_3x &= z \quad (Since,\ z = k_2y) \end{aligned}$$

$\forall x, y, z \in \mathbb{Z}, (x \mid y) \wedge (y \mid z) \Rightarrow x \mid z \ is$ True $\square$

(b) **Using Disprove**: $\neg$ ($\forall$ x,y,z $\in \mathbb{Z}$, (x|y)$\wedge$(y|z)$\wedge$(x|z))
$\equiv \exists$ x,y,z $\in \mathbb{Z}$, (x$\nmid$y)$\vee$(y$\nmid$z)$\vee$(x$\nmid$z)

*Proof.* Let x = 3, y = 4, z = 5, x,y,z $\in \mathbb{Z}$.
$3 \nmid 4 \vee 4 \nmid 5 \vee 3 \nmid 5$
$x \nmid y \vee y \nmid z \vee x \nmid z$
$\forall x, y, z \in \mathbb{Z}, (x \mid y) \wedge (y \mid z) \wedge (x \mid z) \ is$ False $\square$

## Q2.Alternating quantifiers and the floor function.

(a) **Using Prove**: $\forall$ n $\in \mathbb{N}$, $\exists$ k $\in$ N, $\forall$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n\lfloor x \rfloor \le k$

*Proof.* Let n $\in \mathbb{N}$, Let k = n, k $\in$ N, Let x $\in \mathbb{R}$.

$$\begin{aligned} \lfloor nx \rfloor - n\lfloor x \rfloor &= (nx - \varepsilon_1) - n(x - \varepsilon_2) \quad (By\ property\ 1) \\ &= nx - \varepsilon_1 - nx + n\varepsilon_2 \\ &= n\varepsilon_2 - \varepsilon_1 \\ n\varepsilon_2 - \varepsilon_1 &\le n\varepsilon_2 \quad (Since\ 0 \le \varepsilon_1 \le 1) \\ n\varepsilon_2 &\le n \quad (Since\ 0 \le \varepsilon_2 \le 1) \\ n &= k \\ \lfloor nx \rfloor - n\lfloor x \rfloor &\le k \end{aligned}$$

$\forall$ n $\in \mathbb{N}$, $\exists$ k $\in$ N, $\forall$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n \lfloor x \rfloor \leq k$ is True $\square$

(b) **Using Disprove**: $\neg$($\exists$ k $\in$ N, $\forall$ n $\in \mathbb{N}$, $\forall$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n \lfloor x \rfloor \leq k$)
$\equiv$ $\forall$ k $\in$ N, $\exists$ n $\in \mathbb{N}$, $\exists$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n \lfloor x \rfloor > k$

*Proof.* Let k $\in \mathbb{N}$, Let n = 2(k+1), n $\in \mathbb{N}, x = k + 0.5,$ $x \in$ R.

$$\begin{aligned}
\lfloor nx \rfloor - n \lfloor x \rfloor &= \lfloor (n(k+0.5)) \rfloor - n \lfloor (k+0.5) \rfloor \\
&= \lfloor nk + 0.5n \rfloor - n(k + \lfloor 0.5) \rfloor) \quad (By\ property\ 2) \\
&= \lfloor nk + 0.5n \rfloor - nk \\
&= \lfloor nk + 0.5 * 2(k+1) \rfloor - nk \\
&= nk + k + 1 - nk \quad (By\ property\ 3, Since\ n, k \in \mathbb{N}) \\
&= k + 1 \\
k + 1 &> k \\
\lfloor nx \rfloor - n \lfloor x \rfloor &> k
\end{aligned}$$

$\forall$ k $\in$ N, $\exists$ n $\in \mathbb{N}$, $\exists$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n \lfloor x \rfloor > k$ is True
$\exists$ k $\in$ N, $\forall$ n $\in \mathbb{N}$, $\forall$ x $\in \mathbb{R}$, $\lfloor nx \rfloor - n \lfloor x \rfloor \leq k$ is False. $\square$

# Q3.More with primes.

(a) Composite(n): n > 1$\wedge$ ($\neg Prime(n)$)
$\equiv n > 1$ $\wedge$ ($\exists$ d $\in \mathbb{N}$, d | n $\wedge$ d $\neq$ 1 $\wedge$ d $\neq$ n)

(b) **Prove by cases**: $\forall$ x $\in \mathbb{N}$, Composite($x^2 + 5x + 4$)
*Let* $x \in \mathbb{N}$.

$Case\ 1 : x = 0$
$Let\ x = 0, x \in \mathbb{N}$
$x^2 + 5x + 4 = 4$
$Let\ d = 2, d \in \mathbb{N}$
(4 > 1) $\wedge$ (2 | 4) $\wedge$ (2 $\neq$ 1) $\wedge$ (2 $\neq$ 4)
4 is a Composite.

Case 2: x > 0
$x^2 + 5x + 4 = (x + 1)(x + 4)$
$Let\ x > 0, x \in \mathbb{N}$, Let d = (x + 4), d $\in \mathbb{N}$.
x > 0
x + 1 > 1
x + 4 > 4
(x + 1)(x + 4) > 4 > 1

$(x+4) \mid (x+1)(x+4)$
$(x+4) \neq 1$ (Since $x > 0$)
$(x+4) \neq (x+1)(x+4)$ (Since $x > 0$)
$x^2 + 5x + 4$ *is a Composite.*

$\forall x \in \mathbb{N}$, Composite$(x^2 + 5x + 4)$ *is True.* $\square$

$(c)$**Disprove** $:\neg$ ($\forall$ x, y $\in \mathbb{N}$, x $>$ y $\Rightarrow$ Composite$(x^2 - y^2)$)
$\equiv$ ($\exists$ x, y $\in \mathbb{N}$, x $>$ y $\wedge \neg$ Composite$(x^2 - y^2)$)
$\equiv \exists$ x, y $\in \mathbb{N}$, x $>$ y $\wedge$ $(x^2 - y^2) > 1 \wedge$ ($\exists$ d $\in \mathbb{N}$, d $\mid (x^2 - y^2) \wedge$ d $\neq 1 \wedge$ d $\neq (x^2 - y^2)$)

*Let* $x = 1, y = 0, x, y \in \mathbb{N}$.
$1 > 0$
$x > y$
$x^2 - y^2 = 1$
$1 \not> 1$.
1 *is not a Composite.*
$\exists$ x, y $\in \mathbb{N}$, x $>$ y $\wedge \neg$ Composite$(x^2 - y^2)$ *is True.*
$\forall$ x, y $\in \mathbb{N}$, x $>$ y $\Rightarrow$ Composite$(x^2 - y^2)$ *is False.*$\square$

# Q4.Function growth.

(a) **Prove**: $\exists n_0 \in R_{\geq 0}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow n + 165 \leqslant n^2$.
*Let* $n_0 = 165, n_0 \in \mathbb{R}_{\geq 0}$. *Let* $n \in \mathbb{N}$. *Assume* $n \geqslant 165$.

$$n \geqslant 165$$
$$n + n \geqslant 165 + n$$
$$2n \geqslant 165 + n$$

$$n \geqslant 2$$
$$n^2 \geqslant 2n$$

$$n^2 \geqslant 165 + n$$

$\exists n_0 \in R_{\geq 0}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow n + 165 \leqslant n^2$ *is True.*$\square$

(b) **Prove**: $\forall a, b \in \mathbb{R}_{\geq 0}$, $\exists n_0 \in R_{\geq 0}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow an + b \leqslant n^2$.
*Let* $a, b \in \mathbb{R}_{\geq 0}$, *Let* $n_0 = b + a + 1, n_0 \in \mathbb{R}_{\geq 0}$. *Let* $n \in \mathbb{N}$. *Assume* $n \geqslant b + a + 1$.

$$
\begin{aligned}
n &= n \\
an &= an \\
an + b &\leqslant an + n \quad (Since,\ n \geqslant b.) \\
an + b &\leqslant (a+1)n \\
an + b &\leqslant n^2 \quad (Since,\ n \geqslant a+1.)
\end{aligned}
$$

$\forall a, b \in \mathbb{R}_{\geq 0}, \exists \mathrm{n}_0 \in \mathrm{R}_{\geq 0}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow an + b \leqslant n^2\ is\ True. \square$