CSC165H1: Problem Set 1

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Q1. Propositional formulas

(a) (i) Truth Table

p	q	$(p \Leftrightarrow \neg q) \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

(ii)$(p \Leftrightarrow \neg q) \Rightarrow p$

$\equiv \neg(p \Leftrightarrow \neg q) \lor p$

$\equiv \neg((\neg p \lor \neg q) \land (q \lor p)) \lor p$

$\equiv \neg(\neg p \lor \neg q) \lor (\neg(q \lor p)) \lor p$

$\equiv (p \land q) \lor (\neg q \land \neg p) \lor p$

(b) (i)Truth Table

p	q	r	$p \land \neg r \Rightarrow q \lor r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

(ii)$p \land \neg r \Rightarrow q \lor r$

$\equiv \neg(p \land \neg r) \lor q \lor r$

$\equiv (\neg p \lor r) \lor (q \lor r)$

Q2. Generalizing $\Leftrightarrow$

$$(p \Leftrightarrow q) \wedge (p \Leftrightarrow r) \wedge (p \Leftrightarrow s)$$

Justification: If we want to get the formula that is True when all four variables have the same truth value. We can choose one of those four variables as the start point(for our answer we use 'p' as the start point) and check is it have the same truth values with other three variables by using $\Leftrightarrow$, One more important thing is to make sure they are all True at the same time, so we use $\wedge$ to combine those propositional formulas.

Q3. Translating Statements

(a) $\forall\, t \in T, Canada(t) \Rightarrow Stanley(t)$
(b) $BelongsTo\ (toronto, atlantic)$
(c) $\exists\, d \in D, \forall\, t \in T, BelongsTo(t, d) \Rightarrow \neg Canada(t)$
(d) $\forall\, d \in D, \exists\, t \in T, BelongsTo(t, d) \wedge Stanley(t)$
(e) $\forall\, t \in T, \exists\, d \in D, BelongsTo(t, d) \wedge (\forall\, x \in D, BelongsTo(t, x) \Rightarrow d = x)$

Q4. One-to-one functions

(a) $\forall\, x, y \in \mathbb{Z}, f(x) = f(y) \Rightarrow x = y$

Justification: Since we know OneToOne function means there are no two distinct inputs are mapped to the same output by f. In order to get the answer, we can think this is sentence in another way: if there are two inputs gives us the same outputs, then those two inputs must be the same. And we can rewrite this sentence as the propositional formula form which is our answer above.

(b) $\forall\, x, y, z \in \mathbb{Z}, f(x) = f(y) = f(z) \Rightarrow (x = y) \vee (y = z) \vee (x = z)$

Justification: Similar with part a), TwoToOne function means there are no three distinct inputs are mapped to the same output by f. And we can consider this sentence as: there are at most two distinct inputs are mapped to the same output by f. Therefore, if there are three inputs gives us the same output values, then there are at least two input are the same. We use $\vee$ to combine the relationship between three input values, which shows this statement only can be True if there are two input values are the same or they are all be the same. If the input values are all distinct and their output values are all the same, the whole statement will be False, which is what we want.

(c) $\forall\, x, y \in \mathbb{R}. f(x) = f(y) \Rightarrow x = y,\ where\ f: \mathbb{R} \to \mathbb{N}$

(d) $\neg(\exists\, f \in F, \forall x, y \in \mathbb{R}, f(x) = f(y) \Rightarrow x = y)$

$\equiv \forall\, f \in F, \neg(\forall\, x, y \in \mathbb{R}, f(x) = f(y) \Rightarrow x = y)$

$\equiv \forall\, f \in F, \exists\, x, y \in \mathbb{R}, \neg(f(x) = f(y) \Rightarrow x = y\,)$

$\equiv \forall\, f \in F, \exists\, x, y \in \mathbb{R}, \neg(f(x) \neq f(y) \vee x = y)$

$\equiv \forall\, f \in F, \exists\, x, y \in \mathbb{R}, (f(x) = f(y) \wedge (x \neq y))$

Q5. Working with infinity

(a)$\forall\, n_0 \in \mathbb{N}, \exists\, n \in \mathbb{N}, n > n_0 \wedge Prime(n+1)$

Justification: Predicate logic of infinitely many is that for every natural n_0, there is a number greater than n_0 that satisfies the predicate, and the predicate in this question is n+1 is a prime number, so we can also find infinitely many number n that is one less than the prime number, which is what the question asked for.

(b)$\neg(\forall\, n_0 \in \mathbb{N}, \exists\, n \in \mathbb{N}, n > n_0 \wedge (\neg(\exists\, a, b \in \mathbb{N}, Prime(a) \wedge Prime(b) \wedge (a+b=n))))$

$\equiv \exists n_0 \in \mathbb{N}, \neg(\exists\, n \in \mathbb{N}, n > n_0 \wedge (\neg(\exists\, a, b \in \mathbb{N}, Prime(a) \wedge Prime(b) \wedge (a+b=n)))$

$\equiv \exists n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N}, \neg(\, n > n_0 \wedge (\neg(\exists\, a, b \in \mathbb{N}, Prime(a) \wedge Prime(b) \wedge (a+b=n)))$

$\equiv \exists n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N}, n \leq n_0 \vee \neg(\neg(\exists\, a, b \in \mathbb{N}, Prime(a) \wedge Prime(b) \wedge (a+b=n))$

$\equiv \exists n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N}, n \leq n_0 \vee (\exists a, b \in \mathbb{N}, Prime(a) \wedge Prime(b) \wedge (a+b=n))$

Justification: In order to get the statement of 'There exists finitely many numbers that cannot be written as the sum of two prime numbers', we can apply the negation rule to get the answer. Since we know the negation of 'There exists infinitely many numbers that cannot be written as the sum of two prime numbers' is 'There exists finitely many numbers that cannot be written as the sum of two prime numbers'. And, we can also express 'Numbers that cannot be written as the sum of two prime numbers' as the negation of 'Numbers that can be written as the sum of two prime numbers' so we can rewrite the statement to:

$\neg(There\ are\ infinitely\ many\ number\ that\ \neg(can\ be\ written\ as\ the\ sum\ of\ two\ primes.))$

After we have simplified this statement, we will get the final answer, which is 'There exists finitely many numbers that cannot be written as the sum of two prime numbers'.

(c)$\exists\, n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N}, n > n_0 \Rightarrow Prime(n)$