CSC236 2017 Summer Assignment 1

Yuwei Yang
Make Zhang
Jingyi Chen

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1. Proof by Simple Induction:

   Define the Predicate P(n): f(0) + f(2) + ... + f(2n) = f(2n+1)

   Want to prove: $\forall n \in \mathbb{N}, P(n)$
   Base Case: Let n = 0, We want to prove P(0) i.e, f(0) = f(2*0+1) = f(1)
   By the Fibonacci function, we know that f(0) = f(1) = 1. Then f(0) = f(2*0 + 1)
   Let n = 1, We want to prove P(1) i.e, f(0) + f(2) = f(2*1+1) = f(3)
   f(2) = f(0) + f(1) = 1 + 1 = 2 (By fibonacci function)
   f(3) = f(1) + f(2) = 1 + 2 = 3
   f(0) + f(2) = 1 + 2 = 3 = f(3)
   Induction Step: Let k $\in \mathbb{N}$, k $\geq$ 0. Assume P(k) i.e, f(0) + f(2) + ... + f(2k) = f(2k+1)
   Want to prove P(k + 1) i.e, f(0) + f(2) + ... + f(2k) + f(2k+2) = f(2(k+1)+1) = f(2k+3)

$$\begin{aligned} f(0) + f(2) + ... + f(2k) + f(2k+2) &= f(2k+1) + f(2k+2) \ (By\ Induction\ hypothesis) \\ &= f(2k+3) \ (By\ Definition\ of\ Fibonacci\ function) \\ &= f(2(k+1)+1) \end{aligned}$$

   Hence $\forall n \in \mathbb{N}$, f(0) + f(2) + ... + f(2n) = f(2n+1) $\square$

2. Proof by Simple Induction:

   Define the Predicate P(n): 8 | (2n+1)$^2 - 1$

   Want to prove: $\forall n \in \mathbb{N}, P(n)$
   Base Case: Let n = 0, We want to prove P(0) i.e, 8 | 0
   8*0 = 0
   Induction Step: Let k $\in \mathbb{N}$, k $\geq$ 0. Assume P(k) i.e, 8 | (2k+1)$^2 - 1$
   Want to prove P(k + 1) i.e, 8 | (2(k+1)+1)$^2 - 1$

$$\begin{aligned}(2(k+1)+1)^2-1 &= (2k+3)^2-1\\ &= 4k^2+12k+8\\ &= (4k^2+4k+1-1)+8k+8\\ &= [(2k+1)^2-1]+8(k+1)\end{aligned}$$

We know that 8 | (2k+1)$^2 - 1$ *is true by Induction hypothesis.*
And 8 | 8(k+1) is true since k $\in \mathbb{N}$.
And then 8 | [(2k+1)$^2 - 1] + 8(k+1)$ *is true.*
Hence $\forall n \in \mathbb{N}$, 8 | (2n+1)$^2 - 1$ $\square$

3.We cannot represent any amount with coins of denominations 3 and 5. But we can find a number 8 that any amount greater or equal to it we can represented it with the above coins.

Define $P(n) = \exists a, b \in \mathbb{N}, n = 3a + 5b$.

We want to prove $\forall n \in \mathbb{N}, n \geq 8 \Rightarrow P(n)$.

Proof by simple induction.

Base Case: Let $n = 8$, We want to prove $P(8)$, i.e, $8 = 3a + 5b$.

Let $a = 1, b = 5$, Then $8 = 3a + 5b$.

Induction Step: Let $k \in N, k \geq 8$.Assume $P(k)$, i.e, $\exists a, b \in N, k = 3a + 5b$.

We want to prove $P(k + 1)$, i.e, $\exists a', b' \in N, k + 1 = 3a' + 5b'$.

We break in into two cases.

Case1: b $>$ **0**: $\begin{aligned}[t] k+1 &= 3a'+5b'\\ &= 3a+5b+1(\text{By Induction Hypothesis})\\ &= 3a+5b+(2*3-1*5)\\ &= 3(a+2)+5(b-1)\\ &= 3a'+5b'\end{aligned}$

Case2: b $=$ **0**: $\begin{aligned}[t] k+1 &= 3a'+5b'\\ &= 3a+5b+(2*5-3*3)(\text{By Induction Hypothesis})\\ &= 3(a-3)+5(b+2)\\ &= 3a'+5b'\end{aligned}$

Hence,$\forall n \in \mathbb{N}, n \geq 8 \Rightarrow \exists a, b \in N, n = 3a + 5b$.

4.Proof by Well-Ordering Principle.

Let set S:$\{m \in \mathbb{Z} | \exists k \in \mathbb{N}, n = 2^k * m\}$

First we want to prove S $\neq \emptyset$

Let $k \in \mathbb{N}$, Let k = 0

Then $n = 2^0 * m = m$

Then $n \in S$(Since $m \in S$)

Then S $\neq \emptyset$

We want to prove S $\in \mathbb{N}$

We have $2^k \geq 1$.(Since $k \in \mathbb{N}$, the least natural number is 0)

And $n = 2^k * m$

Then $n \geq 1$

We have $m \in \mathbb{Z}, m \neq 0$,otherwise, m = 0, then n = 0,which is contradict with n $\geq$ 1.

Then m > 0.(Since n $\geq$ 1 $\wedge$ $2^k$$\geq$ 1).

Then $S \in \mathbb{N}$.

Then there exist a least element m' $\in$ S.(By Well-Ordering Principle).

We want to prove m' is odd, i.e, $\exists d \in \mathbb{N}, m' = 2d + 1$

Assume m' is even, i.e,$\exists d \in \mathbb{N}, m' = 2d$

Then $n = 2^k * m' = 2^{k+1} * d$

Then $d \in S \wedge d < m'$ (Contradiction with m' is the least element in S).

Hence, given any natural number n $\geq$ 1, there exists an odd integer m and a natural number k such that n = 2k m.

5.Define $P(x, y) : \exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$.

Want to prove $\forall (x, y) \in N, P(x, y)$.

Proof by structural induction.

Base Case:Want to prove $P(3, 2)$, i.e, $\exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$.

Let $k = 0$, Then $(3, 2) = (2^1 + 1, 2^0 + 1)$.

Induction Step:Assume H({x,y}), i.e, $\exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$.

We want to prove $P(3x - 2y, x)$, i.e ,$\exists k' \in \mathbb{N}, (x, y) = (2^{k'+1} + 1, 2^{k'} + 1)$.

Let $k' = k + 1$, Then $(3x - 2y, x) = (3(2^{k+1} + 1) - 2(2^k + 1), 2^{k+1} + 1)$.

$$= (2^{k+2} + 1, 2^{k+1} + 1)$$

$$= (2^{k'+1} + 1, 2^{k'} + 1)$$

Hence,$\forall (x, y) \in N, \exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$ is true for all elements of M.

6.(a)Let $n = 2$, then there are A and B are playing this game.

Then A has a unique nearest neighbour B

And B has a unique nearest neighbour A

Then A throw B, B throw A.

Then there may be no dry person.

(b) Define $P(n)$: Suppose 2n+1 people are positioned such that each person has a unique nearest neighbour. Each person has a single water balloon that they throw at their nearest neighbour. Then there is always at least one dry person.

Want to prove $\forall n \in \mathbb{N}, n > 0 \Rightarrow P(n)$.

Proof by simple induction.

Base Case: Let n = 0, want to prove P(1).

There is no unique nearest neighbour. (Since just one person are playing this game)

Then there is always at least one dry person.

Let n = 1, want to prove P(3).

Then we have A,B and C. If A throw B, then B throw A.

C will throw A or B, and no one will throw C.

Then C is the dry person.

Then there is always at least one dry person.

Induction Step: let $n = k, k \in \mathbb{N}$. Assume P(k),i.e, Suppose Suppose 2k+1 people are positioned such that each person has a unique nearest neighbour. Each person has a single water balloon that they throw at their nearest neighbour. Then there is always at least one dry person. Want to prove P(k+1),i.e,Suppose Suppose 2k+3 people are positioned such that each person has a unique nearest neighbour. Each person has a single water balloon that they throw at their nearest neighbour. Then there is always at least one dry person. We want to separate 2k + 3 people into two group. Picking arbitrary two people A and B to formed Group 1. And the 2k+1 people formed Group2.

Case1: At least one person form Group2 throw A or B

We have one of the 2k+1 people is unique nearest A or B.

Then A or B is not dry

Then there are 2k+1 people have 2k water balloon.

Then there is always at least one dry person.

Case2: Every people from Group2 do not throw A and B

Then A will throw B, and B will throw A.

Then, we just consider Group2.

Then Group2 must have one dry person (By Induction Hypothesis).

Then 2k+3 people is always have at least one dry person.

Put Case1 and Case2 together:$2k + 3$ people always have one dry person.

Hence,$\forall n \in \mathbb{N}, P(n)$.

Q7. Proof by Complete Induction

Define Predicate P(n): A convex polygon with consecutive vertices $v_1, v_2, ..., v_n$ is triangulated into $n - 2$ triangles, the $n - 2$ triangles can be numbered $1, 2, ..., n - 2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., n - 2$.

Base Case: Let n = 3, Want to prove P(3), i.e. A convex polygon with consecutive vertices v1, v2, v3 is triangulated into 1 triangle, this triangle can be numbered as 1 so that v_i is a vertex of triangle i for $i = 1$.
The convex polygon with three vertices v1, v2, v3 must be a triangle and the triangle itself can be numbered as 1 which v1 is a vertex of this triangle. Therefore, P(3) is true.

Induction Step: Let n $\in \mathbb{N}$, and n $\geq$ 3.
Assume P(k), i.e. A convex polygon with consecutive vertices $v_1, v_2, ..., v_k$ is triangulated into $k-2$ triangles, the $k-2$ triangles can be numbered $1, 2, ..., k-2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., k-2$, for $\forall$ k $\in \mathbb{N}$, and 3 $\leqslant$ k < n.
Want to prove P(n) i.e. A convex polygon with consecutive vertices $v_1, v_2, ..., v_n$ is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, ..., n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., n-2$.
There are two cases we need to consider:

$Case1 : v_n$ is connected with one of the vertices v_m
And the diagonal between v_n and v_m will separate the convex polygon into two smaller polygons, we named them A1 and A2.
Since v_m can not be the neighbour with v_1 or v_n, we know that 1 < m < n - 1.
We consider A1 first, there are m+1 vertices such as v_1, $v_2 ... v_m$, v_n in A1.
Since we need consecutive vertices, and we rename v_n to $v_m + 1$.
From the condition above: 1 < m < n - 1, we can get 3 $\leqslant$ m + 1 < n by some simple manipulations. And this condition satisfies what we assumed in induction hypothesis.
Therefore, A1 satisfy the predicate.

And then, consider A2, there are n-m+1 vertices such as v_m, $v_m + 1 ... v_n - i$, v_n in A2.
Since we need consecutive vertices, and we rename v_i to $v_i - m + 1$.
Since we know that 1 < m < n - 1, and 1-n < -m < -1.
From 1-n < -m < -1, we can get 3 $\leqslant$ n - m + 1 < n by some simple manipulations. And this condition satisfies what we assumed in induction hypothesis.
Therefore, A2 satisfy the predicate.
Combine A1 and A2, and rename the vertices in P2 as $v_i - 1 + m$ in order to fit the condition of consecutive vertices, and then P(n) is true for the whole convex polygon.

$Case2$: $v_n - 1$ is connected with one of the vertices v_m
And the diagonal between $v_n - 1$ and v_m will separate the convex polygon into two smaller polygons, we named them A1 and A2.
Since v_m can not be the neighbour with v_1 or $v_n - 1$, we know that 1 $\leqslant$ m < n - 2.
We consider A1 first, there are m+2 vertices such as v_1, $v_2 ... v_m$, $v_n - 1$, v_n in A1.
Since we need consecutive vertices, and we rename $v_n - 1$ to $v_m + 1$ and v_n to $v_m + 2$.
From the condition above: 1 $\leqslant$ m < n - 2, we can get 3 $\leqslant$ m + 2 < n by some simple manipulations. And this condition satisfies what we assumed in induction hypothesis.
Therefore, A1 satisfy the predicate.

And then, consider A2, there are n-m vertices such as v_m, $v_m + 1 ... v_n - i$ in A2.
Since we need consecutive vertices, and we rename v_i to $v_i - m + 1$.
Since we know that 1 $\leqslant$ m < n - 2, and 2-n < -m $\leqslant$ -1.

From 2-n < -m $\leqslant$ -1, we can get 2 < n - m $\leqslant$ n - 1 by some simple manipulations. And this condition satisfies what we assumed in induction hypothesis.
Therefore, A2 satisfy the predicate.
Combine A1 and A2, and rename the vertices in P2 as $v_i - 1 + m$ in order to fit the condition of consecutive vertices, and then P(n) is true for the whole convex polygon.
Finally, we have proved P(n) for both two cases. $\square$