CSC236 2017 Summer Assignment 2

Yuwei Yang
Make Zhang
Jingyi Chen

Due Date: July 15th, 2017

Q1.Part1:Find closed form for T(n).

Since $S(n) = (S(n-1))^2 + 2S(n-1)$
Since $T(n) = S(n) + 1$, Then
$T(n) = (T(n-1) - 1)^2 + 2(T(n-1) - 1) + 1$
$T(n) = (T(n-1) + 1)^2$
$T(n) = (T(n-1))^2$

T(n)= $\begin{cases} 2, & \text{if } n = 0 \\ T(n) = (T(n-1))^2, & \text{if } n > 4 \end{cases}$

k=1, $T(n) = (T(n-1))^2$
k=2, $T(n) = (T(T(n-2))^2)^2 = (T(n-2))^4$
k=3, $T(n) = (T(T(n-3))^2)^4 = (T(n-3))^8$
k=4, $T(n) = (T(T(n-4))^2)^8 = (T(n-4))^{16}$
...
k=k, $T(n) = (T(n-k))^{2^k}$
let n-k = 0, Then n = k.
k=n, T(n)=$(T(n-n))^{2^n} = 2^{2^n}$(Since T(0) = 2)
Then S(n) = 2^{2^n}-1

Part2:Proof by Simple induction.
Define P(n) = S(n) = $2^{2^n} - 1$.
We want to prove $\forall n \in N$, P(n)
Base case: Let n = 0, Prove P(0).
S(0) = 1, and $2^{2^0} - 1 = 1$
Then $S(0) = 2^{2^0} - 1 = 1$
Induction step:Let $k \in N$.Assume P(k), i.e, $S(k) = 2^{2^k} - 1$
We want to prove : S(k+1) = $2^{2^{(k+1)}} - 1$
$S(k+1) = (S(k))^2 + 2S(k)$
=$(2^{2^k} - 1)^2 + 2(2^{2^k} - 1)$(By Induction Hypothesis)
= $2^{2^{k+1}} - 1$
Hence, $\forall n \in N$, P(n)

Part3:Prove S(n) $\in \theta(2^{2^n})$

Let g(n) = 2^{2^n}

Then we calculate $\lim_{x\to\infty} \frac{S(n)}{g(n)} = \frac{2^{2^n}-1}{2^{2^n}} = 1$

Then S(n) $\in \theta(2^{2^n})$

(By the theorem, let f,g $\in R^+$,then f(n) = $\theta(g(n))$ if $\lim_{x\to\infty} \frac{f(n)}{g(n)}$ = c, where$0 < c < \infty$)

(b)Part1: Find the close form and prove it.

Case1:when n is even:

let k = 2, S(n) = S(n-2)+2n-1

let k = 4, S(n) = S(n-4)+2(n-2)-1+2n-1

S(n) = S(n-4)+4n-6

let k = 6, S(n) = S(n-6)+2(n-4)-1+4n-6

S(n) = S(n-6)+6n-15

let k = 8, S(n) = S(n-8)+2(n-6)-1+6n-15

S(n) = S(n-8)+8n-28

...

Let k = k, S(n) = $S(n-k) + kn - (\frac{k^2-k}{2})$

let n-k = 0, Then n = k.

let k = n,S(n) = $\frac{n^2+n}{2}$

Part2:Prove $\forall n \in N$, n is even $\Rightarrow$ S(n) = $\frac{n^2+n}{2}$

Base case:Let n = 0, Prove S(0)

Since $\frac{0^2+0}{2} = 0$

Then S(0) is ture

Induction step: Let $0 \leq i \leq n$,Assume S(i),i.e, S(i) = $\frac{i^2+i}{2}$.

We want to prove : S(n), i.e, S(n) = $\frac{n^2+n}{2}$. Let n $\geq$ 2.

Then S(n) = S(n-2)+2n-1 = $\frac{(n-2)^2+(n-2)}{2} + 2n - 1$(By Induction Hypothesis)

$=\frac{(n-2)^2+(n-2)+4n-2}{2}$

$=\frac{n^2-4n+4+n-2+4n-2}{2}$

$=\frac{n^2+n}{2}$

Hence, $\forall n \in N$,n is even $\Rightarrow$ S(n) = $\frac{n^2+n}{2}$

Case2:When n is odd:

let k = 3,S(n) = S(n-2)+3n

let k = 5,S(n) = S(n-4)+3(n-2)+3n = S(n-4)+6n-6

let k = 7,S(n) = S(n-6)+3(n-4)+6n-6 = S(n-6)+9n-18

let k = 9,S(n) = S(n-8)+3(n-6)+9n-18 = S(n-8)+12n-36

...

let k = k, S(n) = $S(n-k+1) + (k-1)/2 * 3 * n - (\frac{k-1}{2})(\frac{k-3}{2}) * 3$

let k = n, $S(1) = S(1) + (n-1)/2 * 3 * n - (\frac{n-1}{2})(\frac{n-3}{2}) * 3$

$=\frac{4+6n^2-6n-3n^2+3n+9n-9}{4}$

$=\frac{3n^2+6n-5}{4}$

Proof by Complete Induction:

Define P(n) : S(n) = $\frac{3n^2+6n-5}{4}$

Prove $\forall n \in N$, n is odd $\Rightarrow$ S(n) $= \frac{3n^2+6n-5}{4}$
Base case:Let n =1, Prove P(1)
Then we have $S(1) = 1$
And $\frac{3*1^2+6*1-5}{4}$=1
Then S(1)=$\frac{3*1^2+6*1-5}{4}$=1;
Induction step:Let $0 \leq i < k$, Assume P(i), i.e, S(i) $= \frac{3i^2+6i-5}{4}$
We want to prove P(n), i.e, S(n) = S(n-2)+3n
S(n) = S(n-2)+3n
$=\frac{3(n-2)^2+6(n-2)-5}{4} + 3n$(since $k-2<k$,by I.H)
$=\frac{3n^2-12n+12+6n-12-5+12n}{4}$
$=\frac{3n^2+6n-5}{4}$
Hence $\forall n \in N$, n is odd $\Rightarrow$ S(n) $= \frac{3n^2+6n-5}{4}$

Part2: Prove S(n) $\in \theta(n^2)$.
Case1:when n is even:
Let g(n) $= n^2$
Then we calculate $\lim_{x\to\infty} \frac{S(n)}{g(n)} = \frac{\frac{n^2+n}{2}}{n^2} = \frac{1}{2}$
Then S(n) $\in \theta(n^2)$
(By the theorem, let f,g $\in R^+$,then f(n) $= \theta(g(n))$ if $\lim_{x\to\infty} \frac{f(n)}{g(n)}$ = c, where$0 < c < \infty$)
Case2:when n is odd: Let g(n) $= n^2$
Then we calculate $\lim_{x\to\infty} \frac{S(n)}{g(n)} = \frac{\frac{3n^2+6n-5}{4}}{n^2} = \frac{3}{4}$
(By the theorem, let f,g $\in R^+$,then f(n) $= \theta(g(n))$ if $\lim_{x\to\infty} \frac{f(n)}{g(n)}$ = c, where$0 < c < \infty$)
Combine the two cases, Hence S(n) $\in \theta(n^2)$.

Q2.We want to prove $\forall n \in N$, $S(n) \in \theta(nlogn)$.
Since $2^{k-1} \leq n \leq 2^k$
We have $S(2^{k-1}) \leq S(n) \leq S(2^k)$ (Since hint(a) and S(n) is monotone-non deceasing).
Then $2^{k-1}*(k-1)+3*2^{k-1}-5 \leq S(n) \leq 2^k*k+3*2^k-5$
(Since S(n) = nlogn+3n-5).
Let f(n) = S(n).
Let g(n) = nlogn.
Since $2^{k-1} \leq n \leq 2^k$
We have $2^{k-1}*(k-1) \leq g(n) \leq 2^k*k$ (By hint (a) and g(n) is monotone-non decreasing)
We want to calculate $\lim_{x\to\infty} \frac{f(n)}{g(n)} = \frac{S(n)}{nlogn}$
Then $\lim_{x\to\infty} \frac{2^{k-1}*(k-1)+3*2^{k-1}-5}{2^k*(k)} \leq \lim_{x\to\infty} \frac{f(n)}{g(n)} = \frac{S(n)}{nlogn} \leq \lim_{x\to\infty} \frac{2^k*(k)+3*2^k-5}{2^{k-1}*(k-1)}$.
Then $\frac{1}{2} \leq \lim_{x\to\infty} \frac{S(n)}{g(n)} \leq 2$.
We have $\lim_{x\to\infty} \frac{S(n)}{g(n)} = c$, where $\frac{1}{2} \leq c \leq 2$.
Then $\lim_{x\to\infty} \frac{S(n)}{cg(n)} = 1$.
Then $S(n) \in \theta(cg(n))$.(By hint(b), since $\lim_{x\to\infty} \frac{S(n)}{cg(n)} = 1$.)

Then $S(n) \in \theta(g(n))$.
Then $\forall n \in N, S(n) \in \theta(nlogn))$.

Q3.
3.(a)

def max_sum(A)
  if len(A) == 0:
    return 0
  sum = A[0];
  for i in range (len(A)):
    local_sum = 0
    for j in range (i, len(A)):
      local_sum += A[j];
      if local_sum > sum:
        sum = local_sum;
  return sum;

For a fixed iteration of the outer loop, the inner loop takes exactly n - i iterations, with each iteration costing a single step. Note that this is the same for each iteration of the outer loop.There are n iterations of the outer loop. For each iteration costing a single step.We multiply with this by the cost of each outer loop $\sum_{i=0}^{n-1} n = n^2$, to get a total cost of n^2, which is $\theta(n^2)$.

(b)Divide and conquer:

$$\text{T(n)} = \begin{cases} c1, & \text{if } n = 1 \\ 2(\lceil \frac{n}{2} \rceil) + n + c, & \text{if } n > 1 \end{cases}$$

From T(n), we know that b = 2,a = 2, so it can apply the Master theorem.

(c)

```
def max_sum(A):
    if len(A) == 0:
        return 0
    elif len(A) == 1:
        return A[0]
    m = len(A)//2
    l_sum = 0
    l_max = A[m-1]
    r_sum = 0
    r_max = A[m]
    for i in range(m-1,-1,-1):
        l_sum += A[i]
```

```
        if l_sum > l_max:
            l_max = l_sum
    for j in range(m,len(A)):
        r_sum += A[j]
        if r_sum > r_max:
            r_max = r_sum
    sum = l_max+r_max
    sum2 = max_sum(A[:m])
    sum3 = max_sum(A[m:])

    return max(sum,sum2,sum3)
```

(d)From the above DC algorithms, we can apply the Master Theorem.
And a = 2, b = 2, l=1
Since a = b^l = 2
Then $T(n) \in \theta(nlogn)$ (By Master Theorem)

Q4.(a)
Case1: If postage made by at least one 4 cents stamp.
Then we have T1(n) = T(n-4).
Case2: If postage made by all 6 cents stamp.
Then we have

$$T2(n) = \begin{cases} 1, & \text{if } 6|n \\ 0, & \text{if } 6 \not| n \end{cases}$$

Then T(n) = T1(n) + T2(n).

$$T(n) = \begin{cases} 0, & \text{if } n < 4 \\ 1, & \text{if } n = 4 \\ T(n-4)+1, & \text{if } n \geq 4 \wedge 6|n \\ T(n-4), & \text{if } n \geq 4 \wedge 6 \not| n \end{cases}$$

```
def foo(n):
    if n == 4:
        return 1
    elif n < 4:
        return 0
    else:
        if n % 6 == 0:
            return foo(n-4) + 1
        else:
```

return foo(n-4)

(b)

Assume the postage of n made by at least one 10-cent stamps and other made by 4-cents and 6-cents.

Then S(n) = S(n-10) + T(n)

(c)

Proof by complete induction:

Define $P(n) = S(n) - S(n-2) \geq 0$

We want to prove $\forall n \in N, P(n)$.

Base Case: For convenient, we create tuples (number of 4-cents, number of 6-cents, number of 10-cents) here to represents how many ways can form n.

let n = 4,Then we have: (1,0,0) and S(n) = 1

let n = 6,Then we have: (0,1,0) and S(n) = 1

let n = 8,Then we have: (2,0,0) and S(n) = 1

let n = 10,Then we have: (1,1,0),(0,0,1) and S(n) = 2

let n = 12,Then we have: (3,0,0),(0,2,1) and S(n) = 2

let n = 14,Then we have: (1,0,1),(2,1,0) and S(n) = 2

let n = 16,Then we have: (4,0,0),(1,2,0),(,1,1) and S(n) = 3

let n = 18,Then we have: (3,1,0),(0,3,0),(2,0,1) and S(n) = 3

let n = 20,Then we have: (5,0,0),(0,0,2),(2,2,0),(1,1,2) and S(n) = 4

let n = 22,Then we have: (4,1,0),(1,6,0),(3,0,1),(0,2,1) and S(n) = 4

let n = 24,Then we have: (6,0,0),(3,2,0),(0,4,0),(2,1,1),(1,0,5) and S(n) = 5

Then it is easy to say that$\forall n \in N, 4 \leq n \leq 26, S(n) - S(n-2) \geq 0$

Induction Step:Let $i \in N, 4 \leq i <$ n Assume P(i), i.e ,$S(i) - S(i-2) \geq 0$.

We want to prove P(n),i.e ,$S(n) - S(n-2) \geq 0$. Let $n \geq 26$.

Let X(n) = $\begin{cases} 1, & \text{if } 6|n \\ 0, & \text{if } 6 \not| n \end{cases}$

Let Y(n) = $\begin{cases} 1, & \text{if } 4|n \\ 0, & \text{if } 4 \not| n \end{cases}$

Then we have S(n) - S(n-2) = S(n-10) + T(n) -S(n-12)-T(n-2)

= S(n-20)+T(n-10)+T(n)-S(n-22)-T(n-12)-T(n-2)

= (S(n-20)-S(n-22))+(T(n-10)+T(n)-T(n-12)-T(n-2))

$\geq T(n-10) + T(n) - T(n-12) - T(n-2)$

(By IH,since $4 \leq n-20 \leq n$ and $4 \leq n-22 \leq n$)

=T(n-16)+Y(n-10)+T(n-6)+Y(n)-(T(n-16)+X(n-12)-T(n-6)+X(n-2))

= Y(n-10)+Y(n)-(X(n-12)+X(n-2))

Then we just need to prove Y(n-10)+Y(n)-(X(n-12)+X(n-2)) $\geq$ 0.

Case1:if $n\%12 = 0$:

Then Y(n-10) =0(Since 4 $\not| n$),Y(n) = 1 (Since 4$|n$),

X(n-12)=1(Since 6$|n$),X(n-2)=0(Since 6 $\not| n$)

Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 0

Case2:if $n\%12 = 2$:

Then Y(n-10) =1(Since $4|n$),Y(n) = 0 (Since $4 \nmid n$),
X(n-12)=0(Since $6 \nmid n$),X(n-2)=1(Since $6|n$)
Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 0

Case3:if $n\%12 = 4$:
Then Y(n-10) =0(Since $4 \nmid n$),Y(n) = 1 (Since $4|n$),
X(n-12)=0(Since $6 \nmid n$),X(n-2)=0(Since $6 \nmid n$)
Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 1

Case4:if $n\%12 = 6$:
Then Y(n-10) =1(Since $4|n$),Y(n) = 0 (Since $4 \nmid n$),
X(n-12)=1(Since $6|n$),X(n-2)=0(Since $6 \nmid n$)
Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 0

Case5:if $n\%12 = 8$:
Then Y(n-10) =0(Since $4 \nmid n$),Y(n) = 1 (Since $4|n$),
X(n-12)=0(Since $6 \nmid n$),X(n-2)=1(Since $6|n$)
Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 0

Case5:if $n\%12 = 10$:
Then Y(n-10) =1(Since $4|n$),Y(n) = 0 (Since $4 \nmid n$),
X(n-12)=0(Since $6 \nmid n$),X(n-2)=0(Since $6 \nmid n$)
Then Y(n-10)+Y(n)-(X(n-12)+X(n-2)) = 1
Conclusion all cases:Y(n-10)+Y(n)-(X(n-12)+X(n-2)) ≥ 0.
Then $\forall n \in N, P(n)$.

Q5.(a)

```
def closest_pair(lst):
    i = 0
    global_min = (lst[0][0] - lst[1][0]) ** 2 +
    (lst[0][1] - lst[1][1]) ** 2
    global_pair = [lst[0], lst[1]]
    while i < len(lst) - 1:
        j = i + 1
        local_min = (lst[i][0] - lst[j][0]) ** 2 +
        (lst[i][1] - lst[j][1]) ** 2
        local_pair = [lst[i], [lst[j]]]
        while i + 1 <= j < len(lst):
            temp_dis = (lst[i][0] - lst[j][0]) ** 2 +
            (lst[i][1] - lst[j][1]) ** 2
            temp_pair = [lst[i], [lst[j]]]
            if temp_dis < local_min:
                local_min = temp_dis
                local_pair = temp_pair
            j += 1
```

```
        i += 1
        if local_min < global_min:
            global_min = local_min
            global_pair = local_pair
    return global_pair
```

(b) Let F(n) be the function from part(a).
From the code, we could know that:
For the first outer loop, it will work for i = 0, j = 1 ... (n - 1)
and the run time will be (n - 1) for the first outer loop.
For the second outer loop, it will work for i = 1, j = 2 ... (n - 1) and the run time will be (n - 2)
By such Analogy,
the total run time for F(n) should be (n - 1) + (n - 2) + ... + 1 = $\frac{((n-1)+1)(n-1)}{2} = \frac{n^2-n}{2}$
Therefore, F(n) $\in \theta(n^2)$.

(c) For proving the correctness, we should consider both inner and outer loop.
First prove for inner loop, define the loop invariant(LI): i + 1 $\leq$ j $\leq$ len(lst) $\wedge$ local_min, local_pair will become the closet distance and corresponding points in lst[i, i+1,...j].
Define the predicate P(n): LI holds after n-th iteration.
WTS P(n), Proof by simple induction.
Base Case: Let n = 0, so i = 0, j = 1.
i + 1 = 1
len(lst) $\geq$ 2
i + 1 $\leq$ j $\leq$ len(lst)
And for local_min, local_pair, they will satisfy the condition since they are the only points in the range of lst[0] and lst[1]
P(0) holds.
Induction Step: $\forall$ k $\in \mathbb{N}$, Assume P(k), WTS P(k + 1)
Case 1: if there is no (k + 1)-th iteration
Then $j_{k+1} = j_k$,
local_min$_{k+1}$, local_pair$_{k+1}$ = local_min$_k$, local_pair$_k$
Therefore, P(k + 1) is true.
Case 2: if there is (k+1)-th iteration
From loop guard, we know that $j_k <$ len(lst) - 1
Then i + 1 $\leq j_k + 1 \leq$ len(lst) (By IH)
Then i + 1 $\leq j_{k+1} \leq$ len(lst)
And after k-th iteration, local_min$_k$, local_pair$_k$ will satisfy the condition, and we need to compare it with the last point.
If temp_dis$_{k+1}$ < local_min$_k$, we will replace the local_pair$_{k+1}$ to temp_pair$_{k+1}$ and local_min$_{k+1}$ with temp_dis$_{k+1}$, in order to store the closet points.
Else, the previous points are already satisfied the condition. And local_min$_{k+1}$, local_pair$_{k+1}$ will be the same as local_min$_k$, local_pair$_k$, which is satisfied the condition

P(k + 1) holds
Hence P(n) is true.
After proving the LI, we also need to prove the inner loop will terminate.
Let m = len(lst) - j
then, len(lst), j ∈ N, and j ≤ len(lst)
then m ∈ N, and m ≥ 0
then m strictly decreases on each iteration.
Therefore the inner loop terminates.
Finally, proving the postcondition
Assume LI and termination together
Then j = len(lst)
Then local_min$_{k+1}$, local_pair$_{k+1}$ will be the smallest distance and closest pair in lst[i,i+1,...n - 1].
Then postcondition is satisfied in inner loop.

And then, prove for outer loop, define the loop invariant(LI): 0 ≤ i ≤ len(lst) - 1 ∧ global_min, global_pair will become the closet distance and corresponding points in lst[0,...,i-1].
Define the predicate P(n): LI holds after n-th iteration.
WTS P(n), Proof by simple induction.
Base Case: Let n = 0, so i = 0
len(lst) ≥ 2
0≤ i ≤ len(lst) - 1
And for global_min, global_pair, they will satisfy the condition since there are no other points to compare.
P(0) holds.
Induction Step: ∀ k ∈ ℕ, Assume P(k), WTS P(k + 1)
Case 1: if there is no (k + 1)-th iteration
Then $i_{k+1} = i_k$,
global_min$_{k+1}$, global_pair$_{k+1}$ = global_min$_k$, global_pair$_k$
Therefore, P(k + 1) is true.
Case 2: if there is (k+1)-th iteration
From loop guard, we know that i_k < len(lst) - 1
Then 0 ≤ i_k + 1 ≤ len(lst) - 1 (By IH)
Then 0 ≤ i_{k+1} ≤ len(lst) - 1
And after comparing all the points in A[0,...,i], global_min$_k$, global_pair$_k$ will satisfy the condition, and the last outer loop will compare them with the last point.
If the new distance is smaller than previous one, we will replace the global_pair$_{k+1}$ with the new point and global_min$_{k+1}$ with the new smallest distance, in order to get the closet points for the whole list.
Else, the previous points are already satisfied the condition. And global_min$_{k+1}$, global_pair$_{k+1}$ will be the same as global_min$_k$, global_pair$_k$, which is satisfied the condition
P(k + 1) holds
Hence P(n) is true.
After proving the LI, we also need to prove the inner loop will terminate.

Let m = len(lst) - i
then, len(lst), j $\in$ N, and i < len(lst)
then m $\in$ N, and m $\geq$ 0
then m strictly decreases on each iteration.
Therefore the outer loop terminates.
Finally, prove the postcondition
Assume LI and termination together
Then i = len(lst)
Then global_min, global_pair will be the smallest distance and closest pair in the outer loop which is also in the whole list.
Then postcondition is satisfied.
We have proved both inner and outer loop are satisfied the postcondition
Hence the program is correct.