作业一

姓名 学号 日期

第一题 (a) 记 $M_i = S''(x_i), m_i = S'(x_i), i = 0, 1, ..., n.$ 仍记 $S(x)$ 在区间 $[x_i, x_{i+1}]$ 上表达式为 $S_i(x)$, 记 $h_i = x_{i+1} - x_i$, 容易得到 $S_i(x)$ 仅有一个, 且它可以表示成

$$\begin{aligned} S_i(x) = \frac{x_{i+1} - x}{h_i} f(x_i) + \frac{x - x_i}{h_i} f(x_{i+1}) + \frac{1}{6h_i}(x - x_i)(x - x_{i+1})(2x_{i+1} - x_i - x)M_i \\ + \frac{1}{6h_i}(x - x_i)(x - x_{i+1})(x + x_{i+1} - 2x_i)M_{i+1} \end{aligned} \tag{1}$$

对于内节点 x_i, 由 $S_i'(x_i) = S_{i-1}'(x_i)$ 可得到

$$f(x_i, x_{i+1}) - \frac{h_i}{3}M_i - \frac{h_i}{6}M_{i+1} = f(x_{i-1}, x_i) + \frac{h_{i-1}}{6}M_{i-1} + \frac{h_{i-1}}{3}M_i \tag{2}$$

整理上式, 可得到

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, 2, ..., n-1 \tag{3}$$

其中

$$\begin{aligned} \lambda_i = \frac{h_i}{h_i + h_{i-1}}, \quad \mu_i = 1 - \lambda_i \\ d_i = 6f(x_{i-1}, x_i, x_{i+1}) \end{aligned} \tag{4}$$

再加上边界条件即可解得 M, 下面分三种情况讨论边界条件:

1. 给定 M_0, M_n 的值, 此时 n-1 阶方程组有 n-1 个未知量 $M_i, i = 1, 2, ..., n-1.$

$$\begin{bmatrix} 2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-2} & 2 & \lambda_{n-2} \\ & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 M_0 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} M_n \end{bmatrix}$$

2. 给定 $S'(x_0) = m_0, S'(x_n) = m_n$ 的值, 带入表达式得到两个方程:

$2M_0 + M_1 = \frac{6}{h_0}[f[x_0, x_1] - m_0] = d_0$

$M_{n-1} + 2M_n = \frac{6}{h_{n-1}}[m_n - f[x_{n-1}, x_n]] = d_n$

综合起来, 有 n+1 阶方程组

$$\begin{bmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

3. 被插函数以 $x_n - x_0$ 为基本周期时, 即

$S(x_0) = S(x_n), \quad S'(x_0) = S'(x_n), \quad S''(x_0) = S''(x_n)$

代入表达式得到方程组:

$4M_0 + M_1 + M_{n-1} = \frac{6}{h_0}[f[x_0, x_1] - m_0] + \frac{6}{h_{n-1}}[m_n - f[x_{n-1}, x_n]] = d_0$

此时化为 n 阶方程组

$$\begin{bmatrix} 4 & 1 & & \cdots & & 1 \\ \mu_1 2 & \lambda_1 & & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-2} & 2 & \lambda_{n-2} \\ \lambda_{n-1} & & \cdots & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{bmatrix}$$

(b) 本题相当于第一问中的第二种边界情况:

$2M_0 + M_1 = \frac{6}{h_0}[f[x_0, x_1] - m_0] = d_0$

$M_{n-1} + 2M_n = \frac{6}{h_{n-1}}[m_n - f[x_{n-1}, x_n]] = d_n$

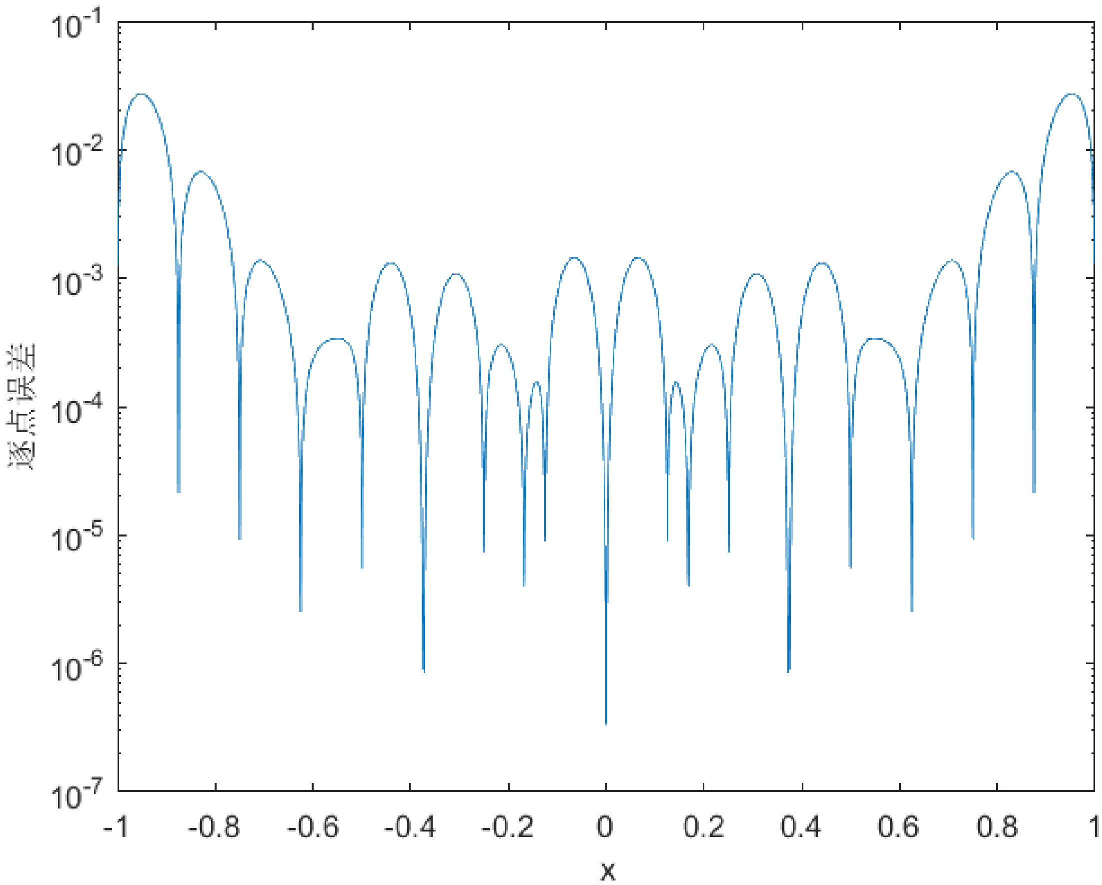
综合起来, 有 n+1 阶方程组

$$\begin{bmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

对以下函数进行三次样条插值,n=16:

$$f(x) = sin(4x^2) + sin^2(4x) \tag{5}$$

构造的三次样条插值的逐点误差的 semilogy 图:



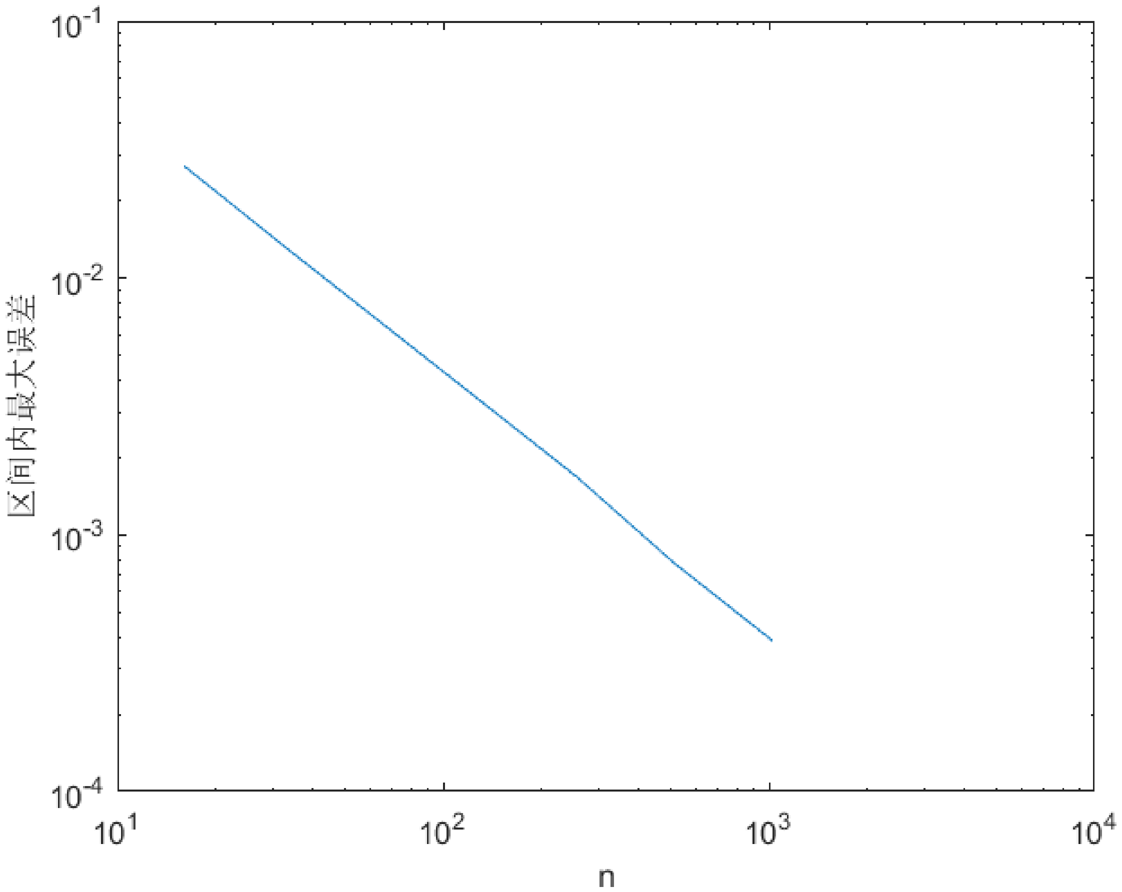
MATLAB 程序显示如下：

```
clear,clc;
f=@(x) sin(4*x^2) + (sin(4*x))^2;
h=1/8;v=1/2;u=1/2;
x=[0, 0];
d=[0; 0];
%M的运算矩阵
A=[2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v, 0;
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, u, 2, v
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2];
%计算x各点的值
x(1)=-1;i=2;
while i<=17
    x(i)=x(i-1)+h;
    i=i+1;
end
%计算对于各个x，f的d
i=1;
d(1)=48*(f(x(2))-f(x(1)))/(x(2)-x(1));
while i<16
    d(i+1)=6*chashang(f, x(i), x(i+1), x(i+2));
```

```
    i=i+1;
end
d(i+1)=-48*(f(x(17))-f(x(16)))/(x(17)-x(16));
%利用运算矩阵计算M
M=A\d;
%取2000个等距点, 记为xx
i=2;xx=[-1, 0];jianju=2/1999;
while i<2000
    xx(i)=xx(i-1)+jianju;
    i=i+1;
end
%计算绝对误差
i=1;wucha=[0, 0];
while i<2000
    wucha(i)=abs(chazhi(xx(i), x, f, M)-f(xx(i)));
    i=i+1;
end
semilogy(xx,wucha);
%差商函数
function d=chashang(f, x0, x1, x2)
    d=f(x0)/((x0-x1)*(x0-x2))+f(x1)/
    ((x1-x0)*(x1-x2))+f(x2)/((x2-x0)*(x2-x1));
end
%三次样条插值函数
function s=chazhi(xx, x, f, M)
    i=1;
    while x(i+1)<xx
        i=i+1;
        if(i>15)
            break;
        end
    end
    s1=8*(x(i+1)-xx)*f(x(i));
    s2=8*(xx-x(i))*f(x(i+1));
    s3=8/6*(xx-x(i))*(xx-x(i+1))*(2*x(i+1)-x(i)-xx)*M(i);
```

```
    s4=8/6*(xx-x(i))*(xx-x(i+1))*(xx+x(i+1)-2*x(i))*M(i+1);
    s=s1+s2+s3+s4;
end
```

(c) 插值区间上最大误差值随 n 变化的情况的 loglog 图:



MATLAB 程序显示如下:

```
clear,clc;
maxw=[0, 0];
n=[0, 0];
i=4;
while i<11
    n(i-3)=2^i;
    maxw(i-3)=test1_b(n(i-3));
    i=i+1;
```

```
end
loglog(n,maxw)
%差商函数
function d=chashang(f, x0, x1, x2)
    d=f(x0)/((x0-x1)*(x0-x2))+f(x1)/
    ((x1-x0)*(x1-x2))+f(x2)/((x2-x0)*(x2-x1));
end
%三次样条插值函数
function s=chazhi(xx, x, f, M, n, h)
    i=1;
    while x(i+1)<xx
        i=i+1;
        if(i>n-1)
            break;
        end
    end
    s1=(x(i+1)-xx)*f(x(i))/h;
    s2=(xx-x(i))*f(x(i+1))/h;
    s3=(xx-x(i))*(xx-x(i+1))*
    (2*x(i+1)-x(i)-xx)*M(i)/(6*h);
    s4=(xx-x(i))*(xx-x(i+1))*
    (xx+x(i+1)-2*x(i))*M(i+1)/(6*h);
    s=s1+s2+s3+s4;
end

function maxw=test1_b(n)
    f=@(x) sin(4*x^2) + (sin(4*x))^2;
    h=2/n;v=1/2;u=1/2;
    x=[0, 0];d=[0; 0];
    %M的运算矩阵
    A(1,1)=2;A(1,2)=1;i=2;
    while i<n+1
        A(i,i-1)=u;
        A(i,i)=2;
        A(i,i+1)=v;
```

```
        i=i+1;
    end
    A(n+1,n+1)=2;A(n+1,n)=1;
    %计算x各点的值
    x(1)=-1;i=2;
    while i<=n+1
        x(i)=x(i-1)+h;
        i=i+1;
    end
    %计算对于各个x，f的d
    i=1;
    d(1)=6/h*(f(x(2))-f(x(1)))/(x(2)-x(1));
    while i<n
        d(i+1)=6*chashang(f, x(i), x(i+1), x(i+2));
        i=i+1;
    end
    d(i+1)=-6/h*(f(x(n+1))-f(x(n)))/(x(n+1)-x(n));
    %利用运算矩阵计算M
    M=A\d;
    %取2000个等距点，记为xx
    i=2;xx=[-1, 0];jianju=2/1999;
    while i<2000
        xx(i)=xx(i-1)+jianju;
        i=i+1;
    end
    %计算绝对误差的最大值
    i=1;maxw=0;
    while i<2000
        maxw=max(maxw,abs(chazhi(xx(i), x, f, M, n, h)-f(xx(i))));
        i=i+1;
    end
end
```

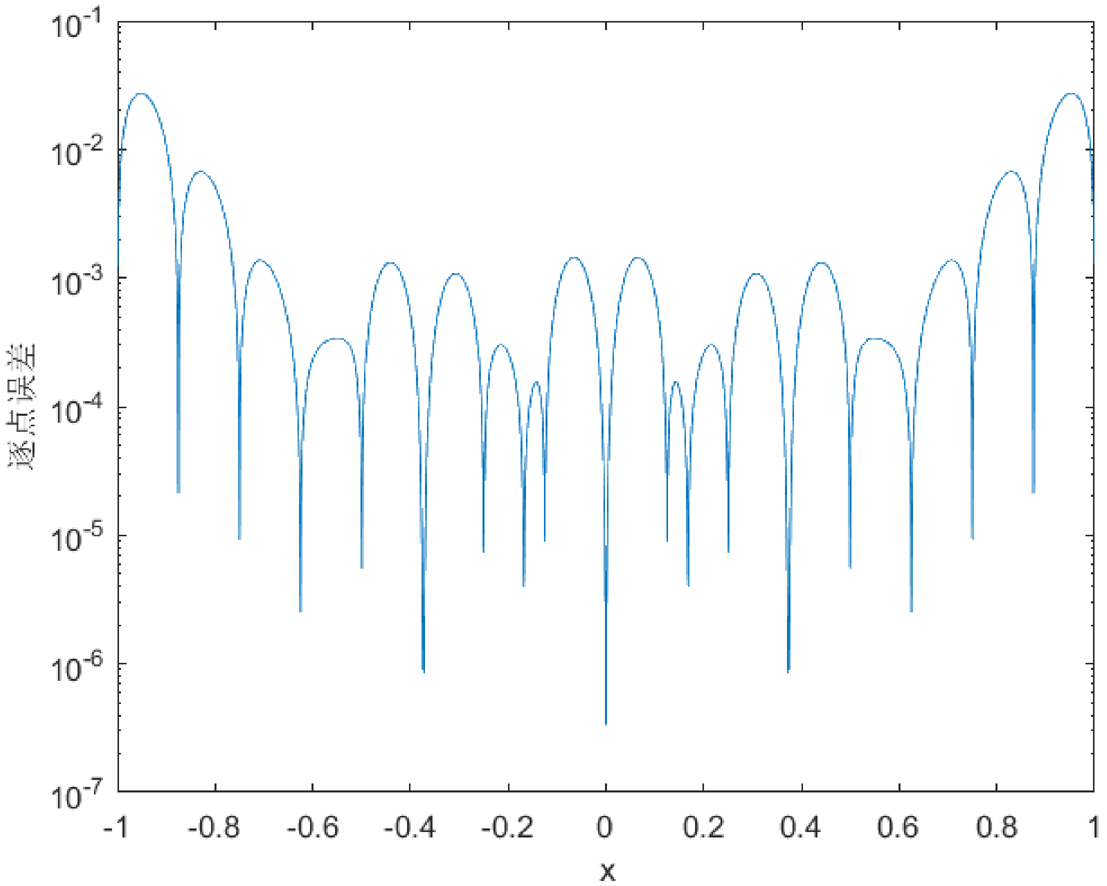
(d)a. 针对周期性边界条件, 也就是对应于第一问中的第三类边界条件:

$$4M_0 + M_1 + M_{n-1} = \frac{6}{h_0}[f[x_0,x_1] - m_0] + \frac{6}{h_{n-1}}[m_n - f[x_{n-1},x_n]] = d_0$$

此时化为 n 阶方程组

$$\begin{bmatrix} 4 & 1 & & \cdots & & 1 \\ \mu_1 2 & \lambda_1 & & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-2} & 2 & \lambda_{n-2} \\ \lambda_{n-1} & & \cdots & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{bmatrix}$$

b. 构造的三次样条插值的逐点误差的 semilogy 图:



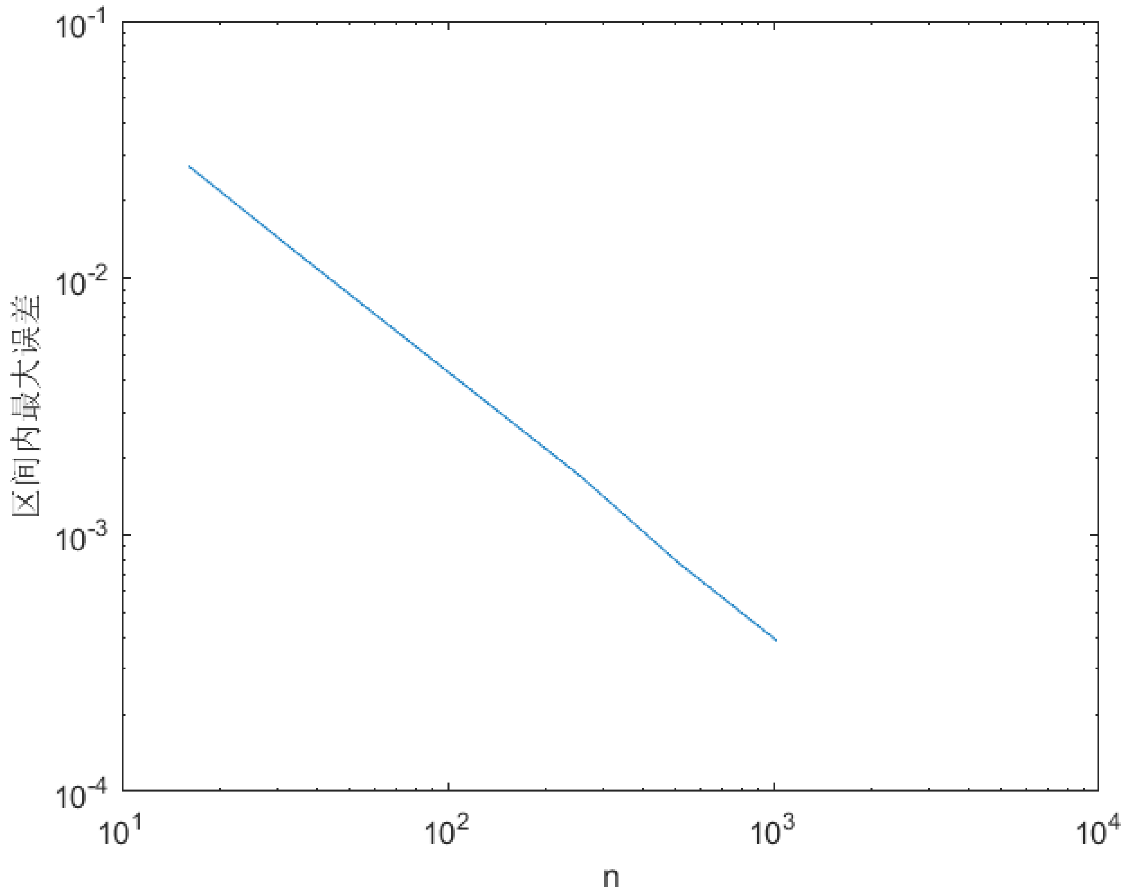
MATLAB 程序显示如下:

```
clear,clc;
    n=16;
    f=@(x) sin(4*x^2) + (sin(4*x))^2;
```

```
h=2/n;v=1/2;u=1/2;
x=[0, 0];d=[0; 0];
%M的运算矩阵，由于周期性只需要n阶，并利用M0=Mn
A(1,1)=4;A(1,2)=1;A(1,n)=1;i=2;
while i<n
    A(i,i-1)=u;
    A(i,i)=2;
    A(i,i+1)=v;
    i=i+1;
end
A(n,n-1)=u;A(n,n)=2;A(n,1)=v;
%计算x各点的值
x(1)=-1;i=2;
while i<=n+1
    x(i)=x(i-1)+h;
    i=i+1;
end
%计算对于各个x，f的d,由于周期性不需要最后一项
i=1;
d(1)=6/h*((f(x(2))-f(x(1)))/
h-(f(x(n+1))-f(x(n)))/h);
while i<n
    d(i+1)=6*chashang(f, x(i), x(i+1), x(i+2));
    i=i+1;
end
%利用运算矩阵计算M，并利用M0=Mn
M=A\d;M(n+1)=M(1);
%取2000个等距点，记为xx
i=2;xx=[-1, 0];jianju=2/1999;
while i<2000
    xx(i)=xx(i-1)+jianju;
    i=i+1;
end
%计算绝对误差
i=1;wucha=[0, 0];
```

```
    while i<2000
        wucha(i)=abs(chazhi(xx(i), x, f, M, n, h)-f(xx(i)));
        i=i+1;
    end
    semilogy(xx,wucha);
%差商函数
function d=chashang(f, x0, x1, x2)
    d=f(x0)/((x0-x1)*(x0-x2))+f(x1)/
    ((x1-x0)*(x1-x2))+f(x2)/((x2-x0)*(x2-x1));
end
%三次样条插值函数
function s=chazhi(xx, x, f, M, n, h)
    i=1;
    while x(i+1)<xx
        i=i+1;
        if(i>n-1)
            break;
        end
    end
    s1=(x(i+1)-xx)*f(x(i))/h;
    s2=(xx-x(i))*f(x(i+1))/h;
    s3=(xx-x(i))*(xx-x(i+1))*
    (2*x(i+1)-x(i)-xx)*M(i)/(6*h);
    s4=(xx-x(i))*(xx-x(i+1))*
    (xx+x(i+1)-2*x(i))*M(i+1)/(6*h);
    s=s1+s2+s3+s4;
end
```

c. 插值区间上最大误差值随 n 变化的情况的 loglog 图:



MATLAB 程序显示如下:

```
clear,clc;
maxw=[0, 0];
n=[0, 0];
i=4;
while i<11
    n(i-3)=2^i;
    maxw(i-3)=test1_b(n(i-3));
    i=i+1;
end
loglog(n,maxw)
%差商函数
function d=chashang(f, x0, x1, x2)
    d=f(x0)/((x0-x1)*(x0-x2))+f(x1)/
```

```
    ((x1-x0)*(x1-x2))+f(x2)/((x2-x0)*(x2-x1));
end
%三次样条插值函数
function s=chazhi(xx, x, f, M, n, h)
    i=1;
    while x(i+1)<xx
        i=i+1;
        if(i>n-1)
            break;
        end
    end
    s1=(x(i+1)-xx)*f(x(i))/h;
    s2=(xx-x(i))*f(x(i+1))/h;
    s3=(xx-x(i))*(xx-x(i+1))*(2*x(i+1)-x(i)-xx)*
    M(i)/(6*h);
    s4=(xx-x(i))*(xx-x(i+1))*(xx+x(i+1)-2*x(i))*
    M(i+1)/(6*h);
    s=s1+s2+s3+s4;
end

function maxw=test1_b(n)
    f=@(x) sin(4*x^2) + (sin(4*x))^2;
    h=2/n;v=1/2;u=1/2;
    x=[0, 0];d=[0; 0];
    %M的运算矩阵，由于周期性只需要n阶，并利用M0=Mn
    A(1,1)=4;A(1,2)=1;A(1,n)=1;i=2;
    while i<n
        A(i,i-1)=u;
        A(i,i)=2;
        A(i,i+1)=v;
        i=i+1;
    end
    A(n,n-1)=u;A(n,n)=2;A(n,1)=v;
    %计算x各点的值
    x(1)=-1;i=2;
```

```
    while i<=n+1
        x(i)=x(i-1)+h;
        i=i+1;
    end
    %计算对于各个x, f的d,由于周期性不需要最后一项
    i=1;
    d(1)=6/h*((f(x(2))-f(x(1)))/h-(f(x(n+1))-f(x(n)))/h);
    while i<n
        d(i+1)=6*chashang(f, x(i), x(i+1), x(i+2));
        i=i+1;
    end
    %利用运算矩阵计算M, 并利用M0=Mn
    M=A\d;M(n+1)=M(1);
    %取2000个等距点, 记为xx
    i=2;xx=[-1, 0];jianju=2/1999;
    while i<2000
        xx(i)=xx(i-1)+jianju;
        i=i+1;
    end
    %计算绝对误差的最大值
    i=1;maxw=0;
    while i<2000
        maxw=max(maxw,abs(chazhi(xx(i), x, f, M, n,
        h)-f(xx(i))));
        i=i+1;
    end
end
```

第二题 (a) 首先证明一下书上的差商性质 1.1:

$$f[x_0,x_1,...,x_k]=\sum_{i=0}^{k}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_k)} \tag{6}$$

1.k=1 时, 有 $f[x_0,x_1]=\frac{f(x_0)}{x_0-x_1}+\frac{f(x_1)}{x_1-x_0}$

2. 若存在 m>1, 使得 $f[x_0,x_1,...,x_m]=\sum_{i=0}^{m}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_m)}$

那么对于 k=m+1, 有

$f[x_0,x_1,...,x_{m+1}]=\frac{f[x_1,...,x_{m+1}]-f[x_0,x_1,...,x_m]}{x_{m+1}-x_0}=\sum_{i=0}^{m+1}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{m+1})}$

3. 由此归纳得对于任意 k>=1, 有 $f[x_0,x_1,...,x_k]=\sum_{i=0}^{k}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_k)}$

由性质 1.1 可知, 差商是一个自 0 到 k 的求和过程, 打乱 x_i 的顺序并不影响求和结果, 所以: 对于一个光滑函数 f(x), 若 $i_0,i_1,...,i_k$ 是 $0,1,...,k$ 的任意一个排列, 则 $f[x_0,x_1,...,x_k]=f[x_{i_0},x_{i_1},...,x_{i_k}]$

(b)n 阶牛顿插值多项式:

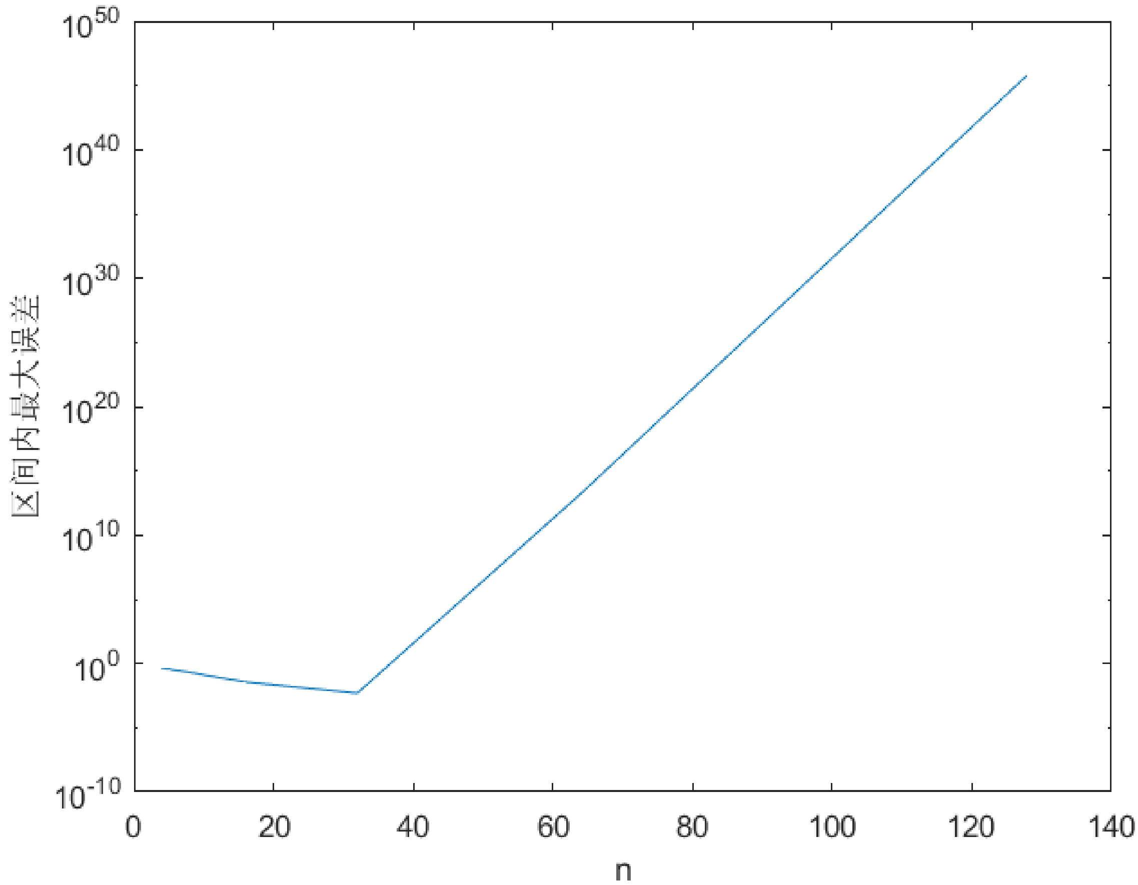
$$N(x)=f[x_0]+\sum_{k=1}^{n}f[x_0,x_1,...,x_k](x-x_0)(x-x_1)...(x-x_{k-1}) \tag{7}$$

Chebyshev 点:$x_j=cos(j/n)\quad j=0,1,...,n$ 利用 n+1 个 Chebyshev 点, 牛顿插值 [-1,1] 上的 Runge 函数:

$$f(x)=\frac{1}{1+25x^2} \tag{8}$$

由插值区间上最大误差值随 n 变化的情况的 semilogy 图可知, 虽然使用了 Chebyshev 点进行插值, 在 n=64 和 n=128 时，仍然出现了龙格现象。

插值区间上最大误差值随 n 变化的情况的 semilogy 图:



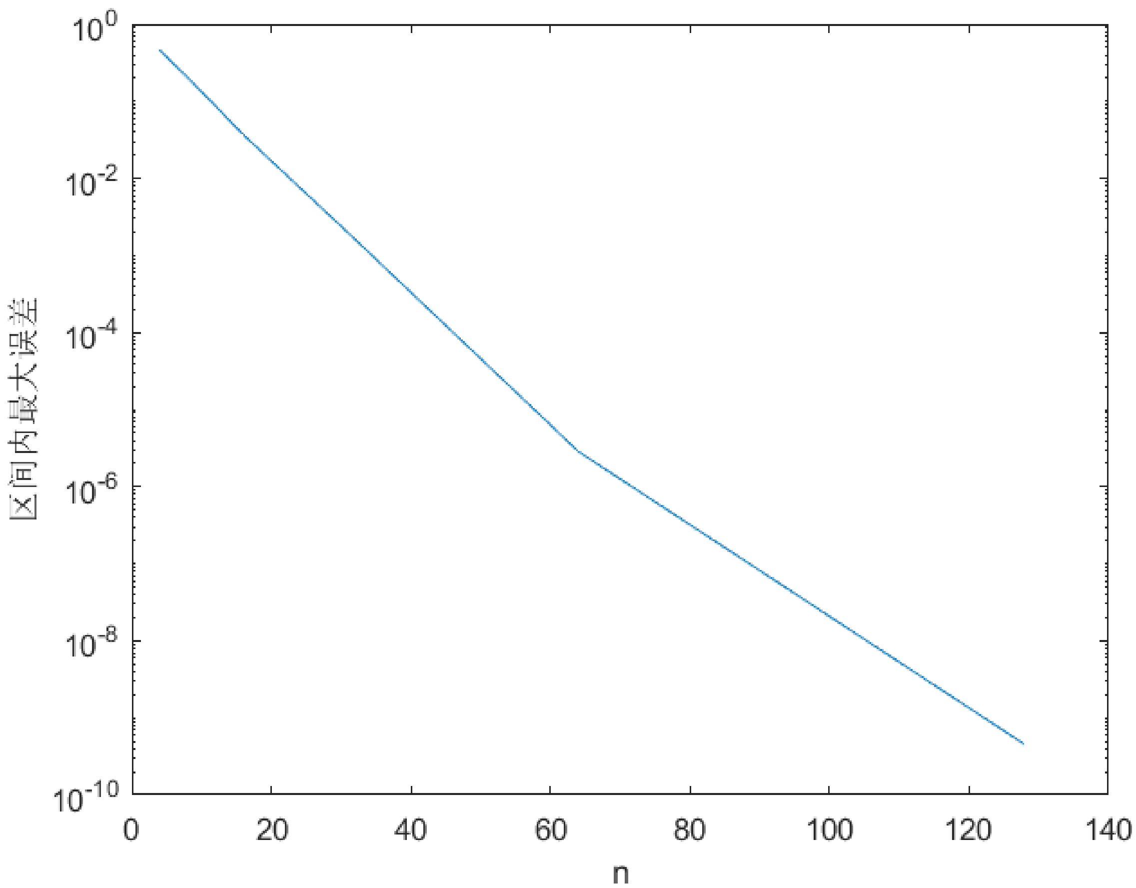
MATLAB 程序显示如下:

```
clear,clc;
maxw=[0, 0];
n=[0, 0];
i=2;
while i<8
    n(i-1)=2^i;
    maxw(i-1)=newtonwucha(n(i-1));
    i=i+1;
end
semilogy(n,maxw)
%差商函数
function d=chashang(f, x, n)
    i=1;d=0;
```

```
    while i<=n+1
        j=1;mu=1;
        while j<=n+1
            if(j~=i)
                mu=mu*(x(i)-x(j));
            end
            j=j+1;
        end
        d=d+f(x(i))/mu;
        i=i+1;
    end
end
%牛顿插值
function N=newton(f, x, xx, n)
    N=f(x(1));k=1;
    while k<=n
        zzz=chashang(f,x,k);i=0;
        while i<k
            zzz=zzz*(xx-x(i+1));
            i=i+1;
        end
        N=N+zzz;
        k=k+1;
    end
end
%主函数
function maxw=newtonwucha(n)
    f=@(x) 1/(1+25*x^2);
    x=[0, 0];i=1;
    while i<=n+1
        x(i) = cos((i-1)*pi/n);
        i=i+1;
    end
    %取2000个等距点, 记为xx
    i=2;xx=[-1, 0];jianju=2/1999;
```

```
    while i<2000
        xx(i)=xx(i-1)+jianju;
        i=i+1;
    end
   %计算绝对误差的最大值
    i=2;maxw=0;
    while i<2000
        maxw=max(maxw,abs(newton(f,x,xx(i),n)-f(xx(i))));
        i=i+1;
    end
end
```

(c) 本题没有出现龙格现象。插值区间上最大误差值随 n 变化的情况的 semilogy 图:



MATLAB 程序显示如下：

```
clear,clc;
rng(22)
maxw=[0, 0];
n=[0, 0];
i=2;
while i<8
    n(i-1)=2^i;
    maxw(i-1)=newtonwucha(n(i-1));
    i=i+1;
end
semilogy(n,maxw)
%差商函数
function d=chashang(f, x, n)
    i=1;d=0;
    while i<=n+1
        j=1;mu=1;
        while j<=n+1
            if(j~=i)
                mu=mu*(x(i)-x(j));
            end
            j=j+1;
        end
        d=d+f(x(i))/mu;
        i=i+1;
    end
end
%牛顿插值
function N=newton(f, x, xx, n)
    N=f(x(1));k=1;
    while k<=n
        zzz=chashang(f,x,k);i=0;
        while i<k
            zzz=zzz*(xx-x(i+1));
            i=i+1;
```

```
        end
        N=N+zzz;
        k=k+1;
    end
end
%主函数
function maxw=newtonwucha(n)
    f=@(x) 1/(1+25*x^2);
    ii=randperm(n+1);
    x=[0, 0];i=1;
    while i<=n+1
        x(ii(i)) = cos((i-1)*pi/n);
        i=i+1;
    end
    %取2000个等距点, 记为xx
    i=2;xx=[-1, 0];jianju=2/1999;
    while i<2000
        xx(i)=xx(i-1)+jianju;
        i=i+1;
    end
   %计算绝对误差的最大值
    i=2;maxw=0;
    while i<2000
        maxw=max(maxw,abs(newton(f,x,xx(i),n)-f(xx(i))));
        i=i+1;
    end
end
```

(d) 我在调试 2.b 程序时针对牛顿插值过程步进 debug, 发现在 n 较大时, 每个循环内牛顿插值多项式累加的值, 也就是差商与 h 累乘:$f[x_0, x_1, ..., x_k](x-x_0)(x-x_1)...(x-x_{k-1})$, 这个值会变得非常大。可能正是这个过大的值导致 matlab 运算出现龙格现象。而 2.c 打乱了 x_i 的顺序后, 这个值明显变小了。

第三题 (a)

1.n 为奇数时

$$\ell_k(x_j)=\ell_k(\frac{j}{n})=\frac{(-1)^k}{n}sin(j\pi)csc(\pi(\frac{j-k}{n})) \tag{9}$$

当 $j\neq k$ 时, 显然 $\ell_k(x_j)=0$.

当 $j=k$ 时, 分子分母为 0, 有极限, 由洛必达法则

$$\begin{aligned}\lim_{j=k}\ell_k(x)&=\lim_{j=k}\frac{(-1)^k}{n}sin(n\pi x)csc(\pi(x-x_k))=\lim_{j=k}\frac{(-1)^k}{n}\frac{sin'(n\pi x)}{sin'(\pi(x-x_k))}\\&=\lim_{j=k}\frac{(-1)^k}{n}\frac{n\pi cos(n\pi x)}{\pi cos(\pi(x-x_k))}=\frac{(-1)^k}{n}\frac{n\pi cos(j\pi)}{\pi cos(\pi(\frac{j-k}{n}))}=(-1)^k(-1)^j=1\end{aligned} \tag{10}$$

2.n 为偶数时

$$\ell_k(x_j)=\ell_k(\frac{j}{n})=\frac{(-1)^k}{n}sin(j\pi)cot(\pi(\frac{j-k}{n})) \tag{11}$$

当 $j\neq k$ 时, 显然 $\ell_k(x_j)=0$.

当 $j=k$ 时, 分子分母为 0, 有极限, 由洛必达法则

$$\begin{aligned}\lim_{j=k}\ell_k(x)&=\lim_{j=k}\frac{(-1)^k}{n}sin(n\pi x)cot(\pi(x-x_k))=\lim_{j=k}\frac{(-1)^k}{n}\frac{sin'(n\pi x)}{tan'(\pi(x-x_k))}\\&=\lim_{j=k}\frac{(-1)^k}{n}\frac{n\pi cos(n\pi x)}{\pi sec^2(\pi(x-x_k))}=\frac{(-1)^k}{n}\frac{n\pi cos(j\pi)}{\pi sec^2(\pi(\frac{j-k}{n}))}=(-1)^k(-1)^j=1\end{aligned} \tag{12}$$

综上可得,n 无论为奇数还是偶数, 有:

$$\ell_k(x_j)=\begin{cases}1 & k=j\\0 & k\neq j\end{cases}$$

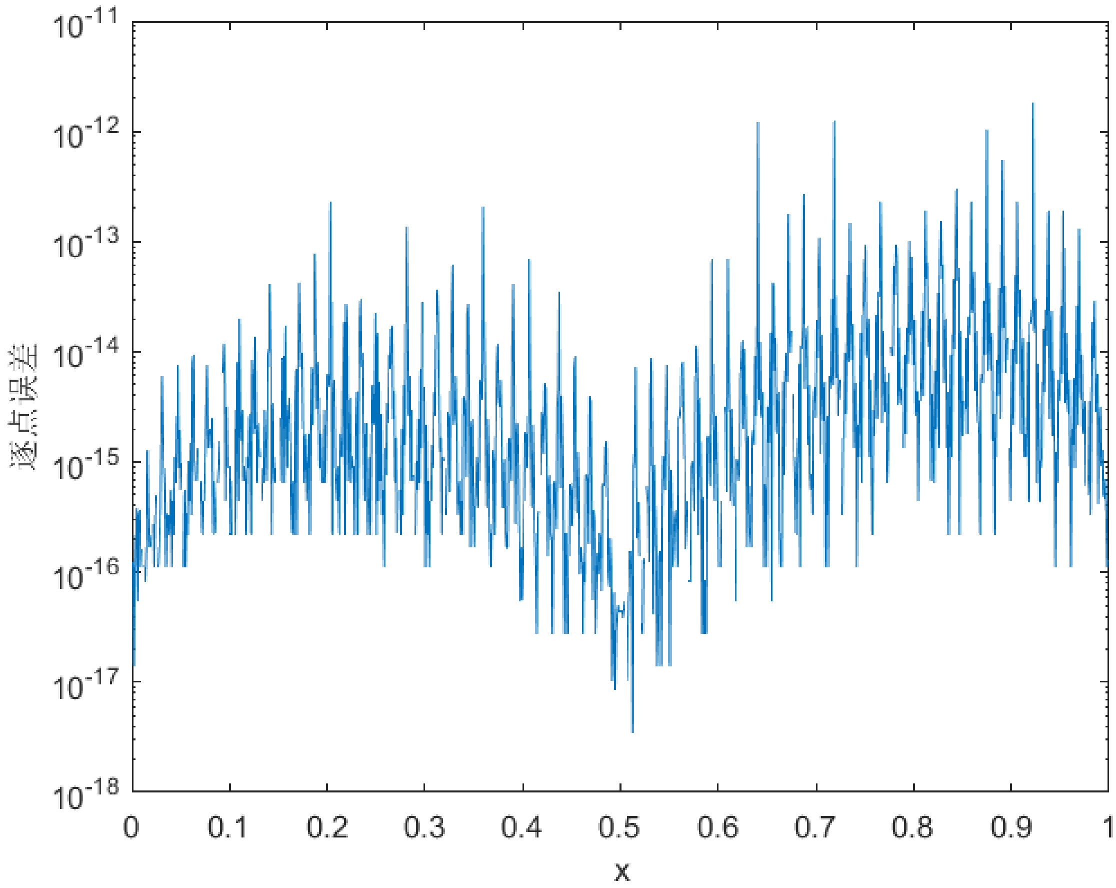
(b) n 次 Lagrange 插值多项式, 其中 $\ell_i(x)$:

$$L(x)=\sum_{i=0}^{n}\ell_i(x)f(x_i) \tag{13}$$

用第一问 n 为偶数时的基函数对以下周期函数进行 Lagrange 插值:

$$f(x)=sin(2\pi x)e^{cos(2\pi x)} \tag{14}$$

插值区间上误差值随 x 变化的情况的 semilogy 图:



MATLAB 程序显示如下:

```
clear,clc;
f=@(x) sin(2*pi*x)*exp(cos(2*pi*x));
x=[0, 0];i=2;
while i<=64
     x(i) = x(i-1)+1/64;
     i=i+1;
end
%取2000个等距点, 记为xx
 i=2;xx=[0, 0];jianju=1/999;
 while i<1000
    xx(i)=xx(i-1)+jianju;
    i=i+1;
 end
```

```
 %计算绝对误差
i=1;wucha=[0, 0];
while i<1000
    wucha(i)=abs(chazhi(f, xx(i), x)-f(xx(i)));
    i=i+1;
end
semilogy(xx,wucha);
%基函数
function l=ji(k, xx, x)
    l=(-1)^(k-1)/64*sin(64*pi*xx)*cot(pi*(xx-x(k)));
end
%插值
function ll=chazhi(f, xx, x)
    k=1;ll=0;
    while k<=64
        ll=ll+ji(k,xx,x)*f(x(k));
        k=k+1;
    end
end
```

第四题 设 $z(x) = 1/f(x) = \frac{a}{x} + b$, 用最小二乘法作形如 $z(x) = \frac{a}{x} + b$ 的拟合函数, 然后再令 $f(x) = 1/z(x)$. 那么

$$Q(a,b) = \sum_{i=1}^{m}(\frac{a}{x_i} + b - z_i)^2 \tag{15}$$

令 Q(a,b) 达到极小,a、b 应该满足:

$$\begin{cases} \dfrac{\partial Q(a,b)}{\partial a} = 2\sum_{i=1}^{m}\dfrac{\frac{a}{x_i} + b - z_i}{x_i} = 0 \\ \dfrac{\partial Q(a,b)}{\partial b} = 2\sum_{i=1}^{m}(\dfrac{a}{x_i} + b - z_i) = 0 \end{cases}$$

整理得到拟合曲线满足:

$$\begin{pmatrix} m & \sum_{i=1}^m \frac{1}{x_i} \\ \sum_{i=1}^m \frac{1}{x_i} & \sum_{i=1}^m \frac{1}{x_i^2} \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m \frac{1}{y_i} \\ \sum_{i=1}^m \frac{1}{x_i y_i} \end{pmatrix}$$

我的拟合函数对比所给数据点的误差的 2-范数:0.005978241829151

MATLAB 程序显示如下:

```
clear,clc;
A=[0, 0;0, 0];
x=[2.1, 2.5, 2.8, 3.2];
y=[0.6087, 0.6849, 0.7368, 0.8111];
d=[0;0];
answer=[0;0];
yy=[0, 0, 0, 0];
%最小二乘法
A(1,1)=4;
A(1,2)=1/x(1)+1/x(2)+1/x(3)+1/x(4);
A(2,1)=A(1,2);
A(2,2)=1/(x(1))^2+1/(x(2))^2+1/(x(3))^2+1/(x(4))^2;
d(1)=1/y(1)+1/y(2)+1/y(3)+1/y(4);
d(2)=1/y(1)/x(1)+1/y(2)/x(2)+1/y(3)/x(3)+1/y(4)/x(4);
answer=A\d;%(b;a)
b=answer(1);a=answer(2);
yy(1)=1/(a/x(1)+b);
yy(2)=1/(a/x(2)+b);
yy(3)=1/(a/x(3)+b);
yy(4)=1/(a/x(4)+b);
fprintf('%.15f', norm(y-yy,2));
```