MCA I SEMESTER
Mathematical Foundations of Computer Applications (MFCA): 20BM3101
Unit – 1: Mathematical Logic

Logic: A conclusion about an argument or a hypothesis in terms of truth values is called Logic

Two Valued Logic: Logic with two truth values TRUE and FALSE is called two valued logic. Here the truth values TRUE and FALSE are respectively denoted by the letters T and F (or 1 and 0).

Statement: A declarative sentence which is either TRUE or FALSE but not both is called a statement. Usually statements are denoted by the letters P, Q, R … (or p, q, r …)

Example: The following are statements.

1. The Sun rises in the east (true)
2. Delhi is the capital of INDIA (true)
3. Paris is in France (true)
4. 6 is a Prime number (false)
5. $2 < 4$ (true)
6. $4 = 7$ (false)

However the following are not statements.

1. What is your name? (this is a question)
2. Do your homework (this is a command)
3. This sentence is false (neither true nor false)
4. x is an even number (it depends on what x represents)
5. Socrates (it is not even a sentence)

Connectives: In the theory of logic, the words NOT, AND, OR, IF…THEN, IF AND ONLY IF are called connectives. These are respectively denoted by the following symbols.

NOT	$\neg$ or $\sim$	(Not)
AND	$\wedge$	(Meet or cap)
OR	$\vee$	(Join or cup)
IF…THEN	$\rightarrow$	(Implies)
IF AND ONLY IF	$\leftrightarrow$	(Double implies)

Other connectives: In addition to the above five connectives we have two more connectives namely NAND and NOR. These are respectively denoted by the following symbols.

NAND	$\uparrow$	(Nand)
NOR	$\downarrow$	(Nor)

Negation of a Statement: Let P be a statement. Then the statement whose truth value is FALSE only when the truth value of P is TRUE, is called the negation of P. It is denoted by $\neg$P or ~P. (That is, the negation of P is the statement obtained by introducing 'not' at a proper place in the statement P).

The truth table for **$\neg$P** is given as follows.

P	**$\neg$P**
T	F
F	T

Example: Consider the following statement

1. P: London is a city. Then $\neg$P: London is not a city
2. P: 2 is an Odd number. Then $\neg$P: 2 is not an Odd number

Conjunction: Let P, Q be two statements. Then the statement whose truth value is TRUE only when the truth values of P and Q are TRUE is called the conjunction of P and Q. It is denoted by '**P$\wedge$Q**' and we read this as 'P and Q' or 'P meet Q'. The truth table for **P$\wedge$Q** is given as follows.

P	**Q**	**P$\wedge$Q**
T	T	T
T	F	F
F	T	F
F	F	F

Example: Consider the following statements

P: It is raining today. Q: Roses are red

Then the conjunction of P and Q, P$\wedge$Q: It is raining today and roses are red.

Disjunction: Let P, Q be two statements. Then the statement whose truth value is FALSE only when the truth values of P and Q are FALSE is called the disjunction of P and Q. It is denoted by '**P$\vee$Q**' and we read this as 'P or Q' or 'P join Q'. The truth table for **P $\vee$ Q** is given as follows.

P	**Q**	**P $\vee$ Q**
T	T	T
T	F	T
F	T	T
F	F	F

Example: Consider the following statements

P: It is raining today. Q: Roses are red

Then the disjunction of P and Q, $P \vee Q$: It is raining today or roses are red.

Conditional: Let P, Q be two statements. Then the statement whose truth value is FALSE only when the truth value of P is TRUE and the truth value of Q is FALSE is called the conditional statement of P and Q. It is denoted by '$\mathbf{P \rightarrow Q}$' and we read this as 'if P then Q' or 'P implies Q'.

The truth table for $\mathbf{P \rightarrow Q}$ is given as follows.

P	**Q**	$\mathbf{P \rightarrow Q}$
T	T	T
T	F	F
F	T	T
F	F	T

Example: Consider the following statements

P: It is raining today Q: Roses are red

Then the conditional of P and Q, $P \rightarrow Q$: If it is raining today then roses are red

Example: Consider the following statements

P: Ram is intelligent Q: $2 > 6$

Then the conditional of P and Q, $P \rightarrow Q$: If Ram is intelligent then $2 > 6$

Note: 1. In the conditional statement $\mathbf{P \rightarrow Q}$, P is called antecedent and Q is called consequent.

2. We read the conditional statement $\mathbf{P \rightarrow Q}$ also as follows.

(i) Q is necessary for P (ii) P is sufficient for Q (iii) P only if Q (iv) P implies Q

Biconditional: Let P, Q be two statements. Then the statement whose truth value is TRUE only when the truth values of P and Q are equal, is called the biconditional statement of P and Q. It is denoted by '$\mathbf{P \leftrightarrow Q}$' and we read this as 'P if and only if Q' or 'P is necessary and sufficient for Q'

The truth table for $\mathbf{P \leftrightarrow Q}$ is given as follows.

P	**Q**	$\mathbf{P \leftrightarrow Q}$
T	T	T
T	F	F
F	T	F
F	F	T

Example: Consider the following statements

P: It is raining today Q: Roses are red

Then the biconditional of P and Q, $P \leftrightarrow Q$: It is raining today if and only if roses are red

Example: Consider the following statements

P: Ram is intelligent Q: $2 > 6$

Then the biconditional of P and Q, $P \leftrightarrow Q$: Ram is intelligent if and only if $2 > 6$

Negation of conjunction: Let P, Q be two statements. Then the statement whose truth value is FALSE only when the truth values of P and Q are TRUE, is called the negation of conjunction of P and Q. It is denoted by '$\mathbf{P \uparrow Q}$' and we read this as 'P nand Q'. In other words, $\mathbf{P \uparrow Q}$ is equivalent to $\neg$ $\mathbf{(P \wedge Q)}$.

The truth table for $\mathbf{P \uparrow Q}$ is given as follows.

P	**Q**	$\mathbf{P \uparrow Q}$
T	T	F
T	F	T
F	T	T
F	F	T

Negation of disjunction: Let P, Q be two statements. Then the statement whose truth value is TRUE only when the truth values of P and Q are FALSE, is called the negation of disjunction of P and Q. It is denoted by '$\mathbf{P \downarrow Q}$' and we read this as 'P nor Q'. In other words, $\mathbf{P \downarrow Q}$ is equivalent to $\neg$ $\mathbf{(P \vee Q)}$.

The truth table for $\mathbf{P \downarrow Q}$ is given as follows.

P	**Q**	$\mathbf{P \downarrow Q}$
T	T	F
T	F	F
F	T	F
F	F	T

Primary and Compound statements: A statement which does not contain any connectives is called an Atomic or Primary or Simple statement. A variable which denote a primary statement is called a primary variable. A statement which is not primary is called Compound or Composite statement.

Example:

1. The Sun rises in the east (Primary)
2. Delhi is the capital of INDIA (Primary)
3. 6 is a Prime number (Primary)
4. 3 and 4 are Prime numbers (Compound)
5. Ram and Laxman are brothers (Primary)

Statement Formula: A string consisting of primary variables, parentheses and connective symbols is called a statement formula.

Well formed formula (Wff): A well formed formula is defined inductively as follows.

i. Every primary variable is a well formed formula (wff)

ii. If A is a wff then $\neg$A is a wff.

iii. If A and B are wffs then $A \wedge B$, $A \vee B$, $A \rightarrow B$ and $A \leftrightarrow B$ are wffs.

iv. A statement formula is a wff if and only if it is obtained by finite number of steps of (i), (ii) and (iii).

Example:

1. $P \vee Q \rightarrow R$ (Not Wff)
2. $(P \vee Q) \rightarrow R$ (Wff)
3. $P \vee (Q \rightarrow R)$ (Wff)
4. $P \vee Q \wedge R$ (Not Wff)
5. $(P \vee Q) \wedge R$ (Wff)
6. $P \vee (Q \wedge R)$ (Wff)

Problems:

1. Let P: Mark is rich   Q: Mark is happy

   Write each of the following in symbolic form

   a) Mark is poor but happy
   b) Mark is neither rich nor happy
   c) Mark is either rich or happy
   d) If Mark is rich then Mark is not happy

Solution:

a) Mark is poor but happy **$\neg P \wedge Q$**

b) Mark is neither rich nor happy **$\neg P \wedge \neg Q$**

c) Mark is either rich or happy **$P \vee Q$**

d) If Mark is rich then Mark is not happy **$P \rightarrow \neg Q$**

2. Construct the truth table for the formula $\neg P \vee Q$

P	**Q**	**$\neg P$**	**$\neg P \vee Q$**
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

3. Construct the truth table for the formula $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$

P	**Q**	$P \leftrightarrow Q$	$P \wedge (P \leftrightarrow Q)$	$[P \wedge (P \leftrightarrow Q)] \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

4. Construct the truth table for the formula [Q ∧ (P → Q)] → P

P	**Q**	**P → Q**	**Q ∧ (P → Q)**	**[Q ∧ (P → Q)] → P**
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

5. Construct the truth table for the formula $P \rightarrow (Q \rightarrow R)$

P	**Q**	**R**	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

6. Construct the truth table for the formula $(P \leftrightarrow R) \wedge (Q \rightarrow P)$

P	**Q**	**R**	$P \leftrightarrow R$	$Q \rightarrow P$	$(P \leftrightarrow R) \wedge (Q \rightarrow P)$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	F
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	T	T	T

Exercise:

1. Let P: It is cold   Q: It is raining; Write each of the following in symbolic form
   a) It is not cold
   b) It is cold and raining
   c) It is cold but not raining
   d) It is raining but not cold
   e) If it is raining then it is cold
   f) If it is not raining then it is not cold
2. Construct the truth table for the formula $(P \rightarrow Q) \leftrightarrow Q$
3. Construct the truth table for the formula $[\neg(P \vee Q)] \wedge (P \rightarrow Q)$
4. Construct the truth table for the formula $(\neg P) \rightarrow (\neg Q)$
5. Construct the truth table for the formula $Q \vee (P \leftrightarrow R)$
6. Construct the truth table for the formula $(P \vee Q) \leftrightarrow (Q \rightarrow R)$

Tautology: A statement formula, whose truth value is TRUE irrespective of the truth values of the primary variables, is called a **Tautology** or a **universally valid formula** or a **Logical truth**.

Contradiction: A statement formula, whose truth value is FALSE irrespective of the truth values of the primary variables, is called a **Contradiction.**

Contingency: A statement formula which is neither a Tautology nor a contradiction is called a Contingency.

Note: Negation of contradiction is a tautology and vice versa.

Problems:

1. Verify the formula $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$ for tautology

   Solution:

P	Q	$P \leftrightarrow Q$	$P \wedge (P \leftrightarrow Q)$	$[P \wedge (P \leftrightarrow Q)] \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

   Therefore, $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$ is a tautology

2. Verify the formula $[\neg P \wedge (P \leftrightarrow Q)] \wedge Q$ for tautology

   Solution:

P	Q	$\neg P$	$P \leftrightarrow Q$	$\neg P \wedge (P \leftrightarrow Q)$	$[\neg P \wedge (P \leftrightarrow Q)] \wedge Q$
T	T	F	T	F	F
T	F	F	F	F	F
F	T	T	F	F	F
F	F	T	T	T	F

   Therefore, $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$ is a contradiction

3. Determine whether $\left[\neg P \wedge (P \to Q)\right] \to \neg Q$ is a tautology

P	Q	$\neg P$	$P \to Q$	$\neg P \wedge (P \to Q)$	$\neg Q$	$\left[\neg P \wedge (P \to Q)\right] \to \neg Q$
T	T	F	T	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Therefore, $\left[\neg P \wedge (P \to Q)\right] \to \neg Q$ is not a tautology, it is a contingency

4. Show that $\left[P \to (Q \to R)\right] \to \left[(P \to Q) \to (P \to R)\right]$ is a tautology

P	Q	R	$P \to Q$	$P \to R$	$Q \to R$	$P \to (Q \to R)$ A	$(P \to Q) \to (P \to R)$ B	$A \to B$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Therefore, $\left[P \to (Q \to R)\right] \to \left[(P \to Q) \to (P \to R)\right]$ is a tautology

Exercise:

1. Determine whether $(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology
2. Show that $\left[P \wedge (P \to Q)\right] \wedge \neg Q$ is a contradiction
3. Determine whether $\left[\neg P \wedge (\neg Q \wedge R)\right] \vee (Q \wedge R) \vee (P \wedge R)$ is a tautology
4. Show that $\left((P \vee Q) \wedge \neg\left[\neg P \wedge (\neg Q \vee \neg R)\right] \right) \vee \left(\neg P \wedge \neg Q\right) \vee \left(\neg P \wedge \neg R\right)$ is a tautology

Equivalence of Formulas: Two formulas A and B are said to be equivalent if $A \leftrightarrow B$ is a tautology or A and B have same truth values. In this case we write $A \Leftrightarrow B$ and we read this as 'A is equivalent B'

Problems:

1. Show that $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

Solution:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

Therefore, the truth values of $P \to Q$ and $\neg P \vee Q$ are same

Hence $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

2. Show that $(P \to Q) \Leftrightarrow (\neg Q \to \neg P)$

Solution:

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Therefore, the truth values of $P \to Q$ and $\neg Q \to \neg P$ are same

Hence $(P \to Q) \Leftrightarrow (\neg Q \to \neg P)$

3. Show that $[P \to (Q \to R)] \Leftrightarrow [(P \to Q) \to (P \to R)]$

P	Q	R	$Q \to R$	$P \to (Q \to R)$	$P \to Q$	$P \to R$	$(P \to Q) \to (P \to R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**F**	T	F	**F**
T	F	T	T	**T**	F	T	**T**
T	F	F	T	**T**	F	F	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**T**	T	T	**T**
F	F	T	T	**T**	T	T	**T**
F	F	F	T	**T**	T	T	**T**

Therefore, the truth values of $P \to (Q \to R)$ and $(P \to Q) \to (P \to R)$ are same

Hence $[P \to (Q \to R)] \Leftrightarrow [(P \to Q) \to (P \to R)]$

5. Show that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

p	**q**	**r**	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Therefore, the truth values of $p \to (q \to r)$ and $(p \wedge q) \to r$ are same

Hence $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

Basic equivalent formulas: The following table contains some basic equivalent formulas.

S. No	Equivalent Laws of Logic		
1	$(P \vee P) \Leftrightarrow P$	$(P \wedge P) \Leftrightarrow P$	Idempotent law
2	$(P \vee Q) \Leftrightarrow (Q \vee P)$	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$	Commutative law
3	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative law
4	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive law
5	$P \vee F \Leftrightarrow P$	$P \wedge T \Leftrightarrow P$	Identity law
6	$P \vee T \Leftrightarrow T$	$P \wedge F \Leftrightarrow F$	Zero law
7	$P \vee \neg P \Leftrightarrow T$	$P \wedge \neg P \Leftrightarrow F$	Complement law
8	$P \vee (P \wedge Q) \Leftrightarrow P$	$P \wedge (P \vee Q) \Leftrightarrow P$	Absorption Law
9	$\neg (P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	$\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	De Morgan's Law
10	$\neg(\neg P) \Leftrightarrow P$		Double complement Law
	Other equivalent Laws of Logic		
1	$P \to Q \Leftrightarrow \neg P \vee Q$		
2	$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P) \Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$		
3	$P \to Q \Leftrightarrow \neg Q \to \neg P$		
4	$P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$		

Tautological implications: Let A, B be two formulas. We say that 'A *is tautologically imply* B' (or 'A *is logically implies to* B' or 'B follows logically from A') if and only if $A \to B$ is a tautology. In this case we write $A \Rightarrow B$

Note: Let A, B be two formulas. Then

1. $A \to B$ read as A *is implies to* B
2. $A \leftrightarrow B$ read as A *is double implies to* B
3. $A \Rightarrow B$ read as A *is logically implies to* B
4. $A \Leftrightarrow B$ read as A *is equivalent to* B
5. $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$
6. $A \Leftrightarrow B$ if and only if $A \leftrightarrow B$ is a Tautology (Definition)
7. $A \Rightarrow B$ if and only if $A \to B$ is a Tautology (Definition)

Problems:

1. Show that (i) $P \to Q \Rightarrow \neg P \vee Q$

   (ii) $\neg P \vee Q \Rightarrow P \to Q$

   Solution: (i) Consider the truth table for $(P \to Q) \to (\neg P \vee Q)$

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \to (\neg P \vee Q)$
T	T	**T**	F	**T**	**T**
T	F	**F**	F	**F**	**T**
F	T	**T**	T	**T**	**T**
F	F	**T**	T	**T**	**T**

Therefore, $(P \to Q) \to (\neg P \vee Q)$ is a tautology and hence $P \to Q \Rightarrow \neg P \vee Q$

(ii) Consider the truth table for $(\neg P \vee Q) \to (P \to Q)$

P	Q	$\neg P$	$\neg P \vee Q$	$P \to Q$	$(\neg P \vee Q) \to (P \to Q)$
T	T	F	**T**	**T**	**T**
T	F	F	**F**	**F**	**T**
F	T	T	**T**	**T**	**T**
F	F	T	**T**	**T**	**T**

Therefore, $(\neg P \vee Q) \to (P \to Q)$ is a tautology and hence $\neg P \vee Q \Rightarrow P \to Q$

2. Show that (i) $P \wedge Q \Rightarrow P$

   (ii) $P \wedge Q \Rightarrow Q$

   Solution: (i) Consider the truth table for $(P \wedge Q) \to P$

P	Q	$P \wedge Q$	$(P \wedge Q) \to P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Therefore, $(P \wedge Q) \to P$ is a tautology and hence $P \wedge Q \Rightarrow P$

(ii) Consider the truth table for $(P \wedge Q) \rightarrow Q$

P	Q	$P \wedge Q$	$(P \wedge Q) \rightarrow Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Therefore, $(P \wedge Q) \rightarrow Q$ is a tautology and hence $P \wedge Q \Rightarrow Q$

3. Check (i) $P \Rightarrow P \wedge Q$

   (ii) $Q \Rightarrow P \wedge Q$

Solution: (i) Consider the truth table for $P \rightarrow (P \wedge Q)$

P	Q	$P \wedge Q$	$P \rightarrow (P \wedge Q)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

Therefore, $P \rightarrow (P \wedge Q)$ is not a tautology and hence $P \Rightarrow P \wedge Q$ is not true

(ii) Consider the truth table for $Q \rightarrow (P \wedge Q)$

P	Q	$P \wedge Q$	$Q \rightarrow (P \wedge Q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	T

Therefore, $Q \rightarrow (P \wedge Q)$ is not a tautology and hence $Q \Rightarrow P \wedge Q$ is not true

Exercise:

1. Show that (i) $P \Rightarrow P \vee Q$

   (ii) $Q \Rightarrow P \vee Q$

2. Show that $P \wedge (P \rightarrow Q) \Rightarrow Q$

3. Check $Q \Rightarrow P \wedge (P \rightarrow Q)$

Basic logical implications: The following table contains some logical implications.

S. No	Logical implication	S. No	Logical implication
1	$P \wedge Q \Rightarrow P$ $P \wedge Q \Rightarrow Q$ (Simplification)	5	$\neg(P \rightarrow Q) \Rightarrow P$ $\neg(P \rightarrow Q) \Rightarrow \neg Q$
2	$P \Rightarrow P \vee Q$ $Q \Rightarrow P \vee Q$ (Addition)	6	$\neg P \Rightarrow P \rightarrow Q$ $Q \Rightarrow P \rightarrow Q$
3	$P \wedge (P \rightarrow Q) \Rightarrow Q$ or $P, P \rightarrow Q \Rightarrow Q$ (Modus ponens)	7	$\neg P \wedge (P \vee Q) \Rightarrow Q$ or $\neg P, P \vee Q \Rightarrow Q$
4	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$ or $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (Modus tollens)	8	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R$ or $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

Exercise:

1. Show that $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$
2. Show that $P \rightarrow Q \Rightarrow P \rightarrow (P \wedge Q)$
3. Show that $(P \rightarrow Q) \rightarrow Q \Rightarrow P \vee Q$
4. Show that $[P \rightarrow (Q \rightarrow R)] \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$
5. Show that $[P \rightarrow (Q \vee R)] \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$
6. Show that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$
7. Show that $\neg(P \leftrightarrow Q) \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$
8. Show that $\neg(P \leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$
9. Show that $(\neg P \wedge [\neg Q \wedge R]) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$
10. Show that $(P \vee Q) \wedge (\neg P \wedge [\neg P \wedge Q]) \Leftrightarrow \neg P \wedge Q$

Converse, Inverse and the Contrapositive of an implication:

For the conditional statement formula $A \rightarrow B$,

i) $B \rightarrow A$ is called Converse

ii) $\neg A \rightarrow \neg B$ is called Inverse

iii) $\neg B \rightarrow \neg A$ is called Contrapositive

Note:

i) $A \rightarrow B \Leftrightarrow \neg B \rightarrow \neg A$

ii) $B \rightarrow A \Leftrightarrow \neg A \rightarrow \neg B$

Problems:

1. Write the converse, inverse and contrapositive of the following statement

   'If today is Thursday, then I have a test today'

   Solution: Let P: Today is Thursday

   Q: I have a test today

   Then the given statement in symbolic form is $P \to Q$

   Converse: $Q \to P$, that is, 'If I have a test today, then today is Thursday'

   Inverse: $\neg P \to \neg Q$, that is, 'If today is not Thursday, then I do not have a test today'

   Contrapositive: $\neg Q \to \neg P$, that is, 'If I do not have a test today, then today is not Thursday'

2. Write the converse, inverse and contrapositive of the following statement

   'If 3 is an even number, then 4 is a prime number'

   Solution: Let P: 3 is an even number Q: 4 is a prime number

   Then the given statement in symbolic form is $P \to Q$

   Converse: $Q \to P$, that is, 'If 4 is a prime number, then 3 is an even number'

   Inverse: $\neg P \to \neg Q$, that is, 'If 3 is not an even number, then 4 is not a prime number'

   Contrapositive: $\neg Q \to \neg P$, that is, 'If 4 is not a prime number, then 3 is not an even number'

Dual formulas: Let A be a formula containing the connectives $\vee$, $\wedge$ and $\neg$ only. Then the formula obtained from A by interchanging $\wedge$ and $\vee$ also F and T, is called the dual of A. Dual of A is denoted by A*.

Problems:

1. Write the duals of (i) $P \vee (Q \wedge \neg P)$ (ii) $\neg P \wedge (R \vee Q)$ (iii) $P \vee (\neg Q \wedge R)$ (iv) $P \vee (\neg Q \wedge F)$

   (i) $P \vee (Q \wedge \neg P)$, the dual is $P \wedge (Q \vee \neg P)$

   (ii) $\neg P \wedge (R \vee Q)$, the dual is $\neg P \vee (R \wedge Q)$

   (iii) $P \vee (\neg Q \wedge R)$, the dual is $P \wedge (\neg Q \vee R)$

   (iv) $P \vee (\neg Q \wedge F)$, the dual is $P \wedge (\neg Q \vee T)$

2. Write the duals of (i) $(P \wedge T) \vee (Q \wedge \neg R)$ (ii) $\neg(P \vee Q) \wedge [P \vee \neg(Q \wedge \neg S)]$

   (i) $(P \wedge T) \vee (Q \wedge \neg R)$, the dual is $(P \vee F) \wedge (Q \vee \neg R)$

   (ii) $\neg(P \vee Q) \wedge [P \vee \neg(Q \wedge \neg S)]$, the dual is $\neg(P \wedge Q) \vee [P \wedge \neg(Q \vee \neg S)]$

3. Write the dual of $P \to Q$

   Solution: First express $P \to Q$ in terms of the connectives $\vee$, $\wedge$ and $\neg$ only.

   $P \to Q \Leftrightarrow \neg P \vee Q$

   Therefore the dual is $\neg P \wedge Q$

4. Write the dual of $P \leftrightarrow Q$

   Solution: First express $P \leftrightarrow Q$ in terms of the connectives $\vee$, $\wedge$ and $\neg$ only.

   $$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$$
   $$\Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$$

   Therefore the dual is $(\neg P \wedge Q) \vee (\neg Q \wedge P)$

5. Write the duals (i) 2 and 3 are prime numbers

   (ii) If 4 is a prime number then 6 is an Odd number

   (iii) 3 is an even number if and only if 2+3 = 8

Duality Law: Let $A(P_1, P_2, \cdots P_n)$ be a formula containing the primary variables $P_1, P_2, \cdots P_n$ and connectives $\vee$, $\wedge$ and $\neg$. Let $A^*(P_1, P_2, \cdots P_n)$ be the dual of $A(P_1, P_2, \cdots P_n)$. Then the duality law is given by

$\neg A(P_1, P_2, \cdots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \cdots \neg P_n)$ or $A(P_1, P_2, \cdots P_n) \Leftrightarrow \neg A^*(\neg P_1, \neg P_2, \cdots \neg P_n)$

Problems:

1. Using duality law prove that $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

   Solution: Let $A(P,Q) = P \vee Q$.

   Then $A^*(P,Q) = P \wedge Q$ and $A^*(\neg P, \neg Q) = \neg P \wedge \neg Q$

   Therefore by the duality law, $\neg A(P,Q) \Leftrightarrow A^*(\neg P, \neg Q)$

   That is, $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

2. Using duality law prove that $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

   Solution: Let $A(P,Q) = P \wedge Q$.

   Then $A^*(P,Q) = P \vee Q$ and $A^*(\neg P, \neg Q) = \neg P \vee \neg Q$

   Therefore by the duality law, $\neg A(P,Q) \Leftrightarrow A^*(\neg P, \neg Q)$

   That is, $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

3. Using duality law prove that $\neg(P \wedge Q \wedge R) \Leftrightarrow \neg P \vee \neg Q \vee \neg R$

   Solution: Let $A(P,Q,R) = P \wedge Q \wedge R$.

   Then $A^*(P,Q,R) = P \vee Q \vee R$ and $A^*(\neg P, \neg Q, \neg R) = \neg P \vee \neg Q \vee \neg R$

   Therefore by the duality law, $\neg A(P,Q,R) \Leftrightarrow A^*(\neg P, \neg Q, \neg R)$

   That is, $\neg(P \wedge Q \wedge R) \Leftrightarrow \neg P \vee \neg Q \vee \neg R$

Exercise:

1. Write the converse, inverse and contrapositive of the following statement

   'If numbers are not equal, then their squares are not equal'

2. Write the converse, inverse and contrapositive of the following
   i) $p \to \neg q$
   ii) $p \to (q \to r)$
   iii) $[p \wedge (p \to q)] \to q$
3. Write the duals of (i) $(\neg P \wedge Q) \vee (\neg Q \wedge \neg P)$ (ii) $(P \vee \neg Q) \wedge (\neg R \vee Q \vee P)$ (iii) $T \vee (P \wedge \neg Q \wedge R)$
4. Write the duals: (i) 2 and 3 are prime numbers ii) If 4 is a prime number then 6 is an Odd number (iii) 3 is an even number if and only if 2+3 = 8
5. Write the dual of $(P \to Q) \wedge (Q \leftrightarrow R)$
6. Using duality law prove that $\neg(P \wedge \neg Q) \Leftrightarrow \neg P \vee Q$
7. Using duality law prove that $\neg(P \vee Q \vee R) \Leftrightarrow \neg P \wedge \neg Q \wedge \neg R$

Normal Forms: In this section we discuss the following

i) Disjunctive Normal Form (DNF)
ii) Principal Disjunctive Normal Form (PDNF)
iii) Conjunctive Normal Form (CNF)
iv) Principal Conjunctive Normal Form (PCNF)

Sum and Product: In this topic, for convenience the connectives $\vee$ and $\wedge$ are respectively known as Sum and Product.

Elementary Product: A product of primary variables and their negations is called an elementary product

Elementary Sum: A sum of primary variables and their negations is called an elementary sum

Example:

i) P, Q, $P \wedge Q$, $P \wedge \neg P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge R \wedge \neg Q$ are some elementary products

ii) P, Q, $P \vee Q$, $\neg P \vee P \vee \neg Q$, $P \vee \neg Q$, $P \vee Q \vee \neg R$, $P \vee \neg Q \vee R \vee Q$ are some elementary sums

iii) $(P \vee Q) \wedge \neg Q$ is neither ‘an elementary product’ nor ‘an elementary sum’

Disjunctive Normal Form (DNF): A formula which is in the form of a ‘sum of elementary products’ is called a Disjunctive Normal Form (DNF)

Conjunctive Normal Form (CNF): A formula which is in the form of a ‘product of elementary sums’ is called a Conjunctive Normal Form (CNF)

Note: Every formula can be expressed as both DNF and CNF

Example:

i) $(P \wedge Q) \vee (P \wedge \neg P \wedge Q)$ is a DNF

(since it is the sum of two elementary products $P \wedge Q$ and $P \wedge \neg P \wedge Q$)

ii) $(\neg P \wedge Q) \vee (Q \wedge \neg P \wedge Q) \vee (P \wedge \neg Q)$ is a DNF

(since it is the sum of three elementary products $\neg P \wedge Q$, $Q \wedge \neg P \wedge Q$ and $P \wedge \neg Q$)

iii) $P \wedge Q$ is a DNF

(since it is the sum of one elementary product $P \wedge Q$)

iv) P is a DNF

(since it is the sum of one elementary product P)

v) $(P \vee \neg Q) \wedge (P \vee \neg P \vee Q)$ is a CNF

(since it is the product of two elementary sums $P \vee \neg Q$ and $P \vee \neg P \vee Q$)

vi) $(P \vee Q) \wedge (Q \vee P \vee \neg Q) \wedge (P \vee \neg Q)$ is a CNF

(since it is the product of three elementary sums $P \vee Q$, $Q \vee P \vee \neg Q$ and $P \vee \neg Q$)

vii) $P \wedge Q$ is a CNF

(since it is the product of two elementary sums P and Q)

viii) P is a CNF

(since it is the product of one elementary sum P)

Minterm: A product of given primary variables in which every variable or its negation, but not both, appears only once, is called a Minterm

Maxterm: A sum of given primary variables in which every variable or its negation, but not both, appears only once, is called a Maxterm

Example:

i) Corresponding to one primary variable P,

Minterms: P, $\neg P$

Maxterms: P, $\neg P$

ii) Corresponding to two primary variables P, Q

Minterms: $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

Maxterms: $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$

iii) Corresponding to three primary variables P, Q, R

Minterms: $P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$,

$\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$, $\neg P \wedge \neg Q \wedge \neg R$

Maxterms: $P \vee Q \vee R$, $P \vee Q \vee \neg R$, $P \vee \neg Q \vee R$, $P \vee \neg Q \vee \neg R$,

$\neg P \vee Q \vee R$, $\neg P \vee Q \vee \neg R$, $\neg P \vee \neg Q \vee R$, $\neg P \vee \neg Q \vee \neg R$

Note:

i) Corresponding to n primary variables, there are exactly 2^n minterms and 2^n maxterms

ii) Every minterm is an elementary product but an elementary product need not be a minterm

iii) Every maxterm is an elementary sum but an elementary sum need not be a maxterm

iv) Sum of all minterms (corresponding to a given number of primary variables) is a tautology

v) Product of all maxterms (corresponding to a given number of primary variables) is a contradiction

vi) Each minterm has the truth value T for exactly one combination of the truth values of the primary variables. Similarly, each maxterm has the truth value F for exactly one combination of the truth values of the primary variables.

Example:

i) Corresponding to two primary variables P, Q

Consider the truth table for the minterms $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

				Minterms			
P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	**T**	F	F	**T**	F	F	F
T	**F**	F	T	F	**T**	F	F
F	**T**	T	F	F	F	**T**	F
F	**F**	T	T	F	F	F	**T**

Consider the truth table for the maxterms $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$

				Maxterms			
P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$P \vee \neg Q$	$\neg P \vee Q$	$\neg P \vee \neg Q$
T	**T**	F	F	T	T	T	**F**
T	**F**	F	T	T	T	**F**	T
F	**T**	T	F	T	**F**	T	T
F	**F**	T	T	**F**	T	T	T

Note:

i) For each truth value T of a formula (in the truth table), a minterm exists and it can be written as follows. If the truth value of a primary variable is T, then the variable appears otherwise the negation of the variable appears in the minterm.

ii) For each truth value F of a formula (in the truth table), a maxterm exists and it can be written as follows. If the truth value of a primary variable is F, then the variable appears otherwise the negation of the variable appears in the maxterm.

Principal Disjunctive Normal Form (PDNF): A formula which is in the form of a 'sum of minterms' is called a 'Principal Disjunctive Normal Form (PDNF)' or 'sum of products canonical form'.

Principal Conjunctive Normal Form (PCNF): A formula which is in the form of a 'product of maxterms' is called a 'Principal Conjunctive Normal Form (PCNF)' or 'product of sums canonical form'.

Note:

i) Every formula other than a contradiction can be expressed as PDNF

ii) Every formula other than a tautology can be expressed as PCNF

iii) PDNF (PCNF) of a formula (if exists) is unique except for the rearrangement of minterms (maxterms)

Problems:

1) Determine the PDNF and PCNF of the formula $P \to Q$

Solution: Consider the truth table for $P \to Q$

P	Q	$P \to Q$		
T	T	T	minterm	$P \wedge Q$
T	F	F	maxterm	$\neg P \vee Q$
F	T	T	minterm	$\neg P \wedge Q$
F	F	T	minterm	$\neg P \wedge \neg Q$

PDNF: Here the no. of Ts in the last column is 3 and the minterms corresponding to each T are given by $P \wedge Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$

Therefore, PDNF is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF: The no. of Fs in the last column is 1 and the maxterm corresponding to F is given by $\neg P \vee Q$

Therefore, PCNF is $\neg P \vee Q$ (or negations of the remaining minters)

2) Determine the PDNF and PCNF of the formula $P \leftrightarrow Q$

Solution: Consider the truth table for $P \leftrightarrow Q$

P	Q	$P \leftrightarrow Q$		
T	T	T	minterm	$P \wedge Q$
T	F	F	maxterm	$\neg P \vee Q$
F	T	F	maxterm	$P \vee \neg Q$
F	F	T	minterm	$\neg P \wedge \neg Q$

PDNF: Here the no. of Ts in the last column is 2 and the minterms corresponding to each T are given by $P \wedge Q$ and $\neg P \wedge \neg Q$

Therefore, PDNF is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF: Here the no. of Fs in the last column is 2 and the maxterms corresponding to each F are given by $P \vee \neg Q$ and $\neg P \vee Q$

Therefore, PCNF is $(P \vee \neg Q) \wedge (\neg P \vee Q)$

3) Determine the PDNF and PCNF of the formula $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$

Solution: Consider the truth table for $[P \wedge (P \leftrightarrow Q)] \rightarrow Q$

P	Q	$P \leftrightarrow Q$	$P \wedge (P \leftrightarrow Q)$	$[P \wedge (P \leftrightarrow Q)] \rightarrow Q$		
T	T	T	T	T	minterm	$P \wedge Q$
T	F	F	F	T	minterm	$P \wedge \neg Q$
F	T	F	F	T	minterm	$\neg P \wedge Q$
F	F	T	F	T	minterm	$\neg P \wedge \neg Q$

PDNF: Here the no. of Ts in the last column is 4 and the minterms corresponding to each T are given by $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$

Therefore, PDNF is $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF: Here there is no F in the last column. Therefore, PCNF does not exists

4) Determine the PDNF and PCNF of the formula $[\neg P \wedge (P \leftrightarrow Q)] \wedge Q$

Solution: Consider the truth table for $[\neg P \wedge (P \leftrightarrow Q)] \wedge Q$

P	Q	$\neg P$	$P \leftrightarrow Q$	$\neg P \wedge (P \leftrightarrow Q)$	$[\neg P \wedge (P \leftrightarrow Q)] \wedge Q$		
T	T	F	T	F	F	maxterm	$\neg P \vee \neg Q$
T	F	F	F	F	F	maxterm	$\neg P \vee Q$
F	T	T	F	F	F	maxterm	$P \vee \neg Q$
F	F	T	T	T	F	maxterm	$P \vee Q$

PDNF: Here there is no T in the last column. Therefore, PDNF does not exists

PCNF: Here the no. of Fs in the last column is 4 and the maxterms corresponding to each F are given by $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$ and $\neg P \vee \neg Q$

Therefore, PDNF is $(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$

5) Determine the PDNF and PCNF of the formula $[\neg P \wedge (P \to Q)] \to \neg Q$

Solution: Consider the truth table for $[\neg P \wedge (P \to Q)] \to \neg Q$

P	Q	$\neg P$	$P \to Q$	$\neg P \wedge (P \to Q)$	$\neg Q$	$[\neg P \wedge (P \to Q)] \to \neg Q$		
T	T	F	T	F	F	T	minterm	$P \wedge Q$
T	F	F	F	F	T	T	minterm	$P \wedge \neg Q$
F	T	T	T	T	F	F	maxterm	$P \vee \neg Q$
F	F	T	T	T	T	T	minterm	$\neg P \wedge \neg Q$

PDNF: Here the no. of Ts in the last column is 3 and the minterms corresponding to each T are given by $P \wedge Q$, $P \wedge \neg Q$ and $\neg P \wedge \neg Q$
Therefore, PDNF is $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

PCNF: Here the no. of Fs in the last column is 1 and the maxterm corresponding F is given by $P \vee \neg Q$
Therefore, PCNF is $P \vee \neg Q$

6) Determine the PDNF and PCNF of the formula $P \to (Q \to R)$

P	**Q**	**R**	$Q \to R$	$P \to (Q \to R)$		
T	T	T	T	T	minterm	$P \wedge Q \wedge R$
T	T	F	F	F	maxterm	$\neg P \vee \neg Q \vee R$
T	F	T	T	T	minterm	$P \wedge \neg Q \wedge R$
T	F	F	T	T	minterm	$P \wedge \neg Q \wedge \neg R$
F	T	T	T	T	minterm	$\neg P \wedge Q \wedge R$
F	T	F	F	T	minterm	$\neg P \wedge Q \wedge \neg R$
F	F	T	T	T	minterm	$\neg P \wedge \neg Q \wedge R$
F	F	F	T	T	minterm	$\neg P \wedge \neg Q \wedge \neg R$

PDNF: Here the no. of Ts in the last column is 7 and the minterms corresponding to each T are given by $P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$,

$\neg P \wedge Q \wedge R, \neg P \wedge Q \wedge \neg R, \neg P \wedge \neg Q \wedge R$, $\neg P \wedge \neg Q \wedge \neg R$

Therefore, PDNF is

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

PCNF: Here the no. of Fs in the last column is 1 and the maxterm corresponding F is given by $\neg P \vee \neg Q \vee R$
Therefore, PCNF is $\neg P \vee \neg Q \vee R$

7) Determine the PDNF and PCNF of the formula $(P \leftrightarrow R) \wedge (Q \rightarrow P)$

P	**Q**	**R**	$P \leftrightarrow R$	$Q \rightarrow P$	$(P \leftrightarrow R) \wedge (Q \rightarrow P)$	
T	T	T	T	T	T	minterm
T	T	F	F	T	F	maxterm
T	F	T	T	T	T	minterm
T	F	F	F	T	F	maxterm
F	T	T	F	F	F	maxterm
F	T	F	T	F	F	maxterm
F	F	T	F	T	F	maxterm
F	F	F	T	T	T	minterm

PDNF: Here the no. of Ts in the last column is 3 and the minterms corresponding to each T are given by $P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$ and $\neg P \wedge \neg Q \wedge \neg R$

Therefore, PDNF is $(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

PCNF: Here the no. of Fs in the last column is 5 and the maxterm corresponding F is given by
$\neg P \vee \neg Q \vee R,\ \neg P \vee Q \vee R,\ P \vee \neg Q \vee \neg R,\ P \vee \neg Q \vee R,\ P \vee Q \vee \neg R,$
Therefore, PCNF is

$$(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$$

Exercise:

1. Determine the PDNF and PCNF of the formula $(Q \rightarrow P) \wedge (\neg P \wedge Q)$
2. Determine the PDNF and PCNF of the formula $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$
3. Determine the PDNF and PCNF of the formula $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$
4. Determine the PDNF and PCNF of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
5. Determine the PDNF and PCNF of the formula $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
6. Determine the PDNF and PCNF of the formula $[\neg P \wedge (\neg Q \wedge R)] \vee (Q \wedge R) \vee (P \wedge R)$

Normal Forms			
1.	Elementary Product	Product of primary variables and their negations	
2.	Elementary Sum	Sum of primary variables and their negations	
3.	Minterm	Product of given primary variables in which every variable or its negation, but not both, appears only once	For each truth value T of a formula (in the truth table), a minterm exists
4.	Maxterm	Sum of given primary variables in which every variable or its negation, but not both, appears only once	For each truth value F of a formula (in the truth table), a maxterm exists
5.	DNF	Sum of elementary products	Every formula can be expressed as DNF
6.	CNF	Product of elementary sums	Every formula can be expressed as CNF
7.	PDNF	Sum of minterms	Other than a contradiction can be expressed as PDNF
8.	PCNF	Product of maxterms	Other than a tautology can be expressed as PCNF

Theory of Inference: The main function of logic is to provide rules of inference. The theory associated with such rules is known as inference theory because it is concerned with the inferring of a conclusion from certain premises. When a conclusion is derived from a set of premises using accepted rules of reasoning, then such process of derivation is called a *formal proof*.

The rules of inference are criteria for determining the validity of an argument. In any argument, a conclusion is admitted to be true provided that the premises (assumptions, axioms, hypotheses) are accepted as true and the reasoning used in deriving the conclusion from the premises follows certain accepted rules of logical inference. Any conclusion which is arrived at by following inference rules is called *valid conclusion*, and the argument is called *valid argument*.

Premise: In this topic, a statement formula is also known as a Premise.

Valid conclusion: Let $H_1, H_2, H_3, \cdots H_n$ and C be premises. We say that

C follows logically from the premises $H_1, H_2, H_3, \cdots H_n$ or

C is a valid conclusion from the premises $H_1, H_2, H_3, \cdots H_n$

if $(H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n) \to C$ is a tautology; that is, $(H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n) \Rightarrow C$

Note: If $(H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n) \to C$ is not a tautology then we said that C is an **invalid conclusion**

Problems:

1. Show that $C : Q$ follows logically from the premises $H_1 : P \to Q$ and $H_2 : P$

   Solution:

   Given Premises: $P \to Q$ and P

   Conclusion: Q

   Consider the truth table for $(H_1 \wedge H_2) \to C$; that is, $[(P \to Q) \wedge P] \to Q$

P	Q	$P \to Q$	$(P \to Q) \wedge P$	$[(P \to Q) \wedge P] \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

   Therefore, $[(P \to Q) \wedge P] \to Q$ is a tautology and hence Q follows logically from the premises $P \to Q$ and P

2. Show that $\neg P$ follows logically from the premises $P \to Q$ and $\neg(P \wedge Q)$

Or Show that $\neg P$ is a valid conclusion from the premises $P \to Q$ and $\neg(P \wedge Q)$

Solution:

Given Premises: $P \to Q$ and $\neg(P \wedge Q)$

Conclusion: $\neg P$

Consider the truth table for $[(P \to Q) \wedge \neg(P \wedge Q)] \to \neg P$

P	Q	$P \to Q$ H_1	$P \wedge Q$	$\neg(P \wedge Q)$ H_2	$H_1 \wedge H_2$	$\neg P$ C	$(H_1 \wedge H_2) \to C$
T	T	T	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	F	T	F	T	T	T	T

Therefore, $[(P \to Q) \wedge \neg(P \wedge Q)] \to \neg P$ is a tautology and hence $\neg P$ follows logically from the premises $P \to Q$ and $\neg(P \wedge Q)$

3. Show that Q does not follow logically from the premises $P \to Q$ and $\neg P$

Or Show that Q is not a valid conclusion from the premises $P \to Q$ and $\neg P$

Solution:

Given Premises: $P \to Q$ and $\neg P$

Conclusion: Q

Consider the truth table for $[(P \to Q) \wedge \neg P] \to Q$

P	Q	$P \to Q$	$\neg P$	$(P \to Q) \wedge \neg P$	$[(P \to Q) \wedge \neg P] \to Q$
T	T	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	F

Therefore, $[(P \to Q) \wedge \neg P] \to Q$ is not a tautology and hence Q does not follow logically from the premises $P \to Q$ and $\neg P$

4. Show that R is a valid conclusion from the premises $P \vee Q,\ P \rightarrow R$ and $Q \rightarrow R$

Solution:

Given Premises: $P \vee Q,\ P \rightarrow R$ and $Q \rightarrow R$

Conclusion: R

Consider the truth table for $[(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)] \rightarrow R$

P	**Q**	**R**	$P \vee Q$ H_1	$P \rightarrow R$ H_2	$Q \rightarrow R$ H_3	$H_1 \wedge H_2 \wedge H_3$	$(H_1 \wedge H_2 \wedge H_3) \rightarrow R$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Therefore, $[(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)] \rightarrow R$ is a tautology and hence R is a valid conclusion from the premises $P \vee Q,\ P \rightarrow R$ and $Q \rightarrow R$

Exercise:

1. Show that $C : Q$ follows logically from the premises $H_1 : P \vee Q$ and $H_2 : \neg P$
2. Show that $\neg(P \wedge Q)$ follows logically from the premises $\neg P$ and $P \leftrightarrow Q$
3. Show that P is not a valid conclusion from the premises $P \rightarrow Q$ and Q
4. Show that $P \rightarrow R$ is a valid conclusion from the premises $P \rightarrow Q$ and $Q \rightarrow R$
5. Show that $\neg P$ follows logically from the premises $\neg P \vee Q,\ \neg(Q \wedge \neg R)$ and $\neg R$

Rules of inference: A valid conclusion can be derived by step by step process from a given set of premises, using the following rules of inference.

Rule P: A given premise may be introduced at any step in the derivation

Rule T: A formula may be introduced at any step in the derivation if it follows logically from one or more preceding steps in the derivation.

Rule CP: If we can derive S from R and a set of premises, then we can derive R$\rightarrow$S from the set of premises alone.

Note: Rule CP is also called as the **Deduction theorem** and it is generally used if the conclusion is in the form of $R \rightarrow S$. In such cases, **R is taken as an additional premise** and S is derived from the given premises and R.

Useful results:

S.No.	Result	S.No.	Result
1.	$P \to Q \Leftrightarrow \neg P \vee Q$	6.	$P, Q \Rightarrow P \wedge Q$
2.	$P \to Q \Leftrightarrow \neg Q \to \neg P$	7.	$P \wedge Q \Rightarrow P$ $P \wedge Q \Rightarrow Q$ (Simplification)
3.	$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$	8.	$P \Rightarrow P \vee Q$ $Q \Rightarrow P \vee Q$ (Addition)
4.	$P \vee \neg P \Leftrightarrow T$ $P \wedge \neg P \Leftrightarrow F$	9.	$P \wedge (P \to Q) \Rightarrow Q$ or $P, P \to Q \Rightarrow Q$ (Modus ponens)
5.	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	10.	$\neg P \wedge (P \vee Q) \Rightarrow Q$ or $\neg P, P \vee Q \Rightarrow Q$

Problems:

1. Show that R is a valid conclusion from the premises $P \to Q,\ Q \to R$ and P

Given premises: $P \to Q,\ Q \to R,\ P$

Conclusion: R

(1) $P \to Q$	rule P	
(2) $Q \to R$	rule P	
(3) P	rule P	
(4) Q	rule T	from (1) and (3)
(5) R	rule T	from (2) and (4)

Therefore, R a valid conclusion from the given premises

Note: (i) If we continue the above derivation as follows,

(6) $P \wedge R$	rule T	from (3) and (5)

Then it becomes $P \wedge R$ a valid conclusion from the same premises

(ii) If we continue the above derivation as follows,

(6) $P \wedge Q \wedge R$	rule T	from (3), (4) and (5)

Then it becomes $P \wedge Q \wedge R$ a valid conclusion from the same premises

(iii) If we stop the above derivation at the step (4),

Then it becomes Q a valid conclusion from the same premises

2. Show that $\neg P$ is a valid conclusion from the premises $\neg Q,\ P \rightarrow Q$
Or Show that $\neg Q,\ P \rightarrow Q \Rightarrow \neg P$

Given premises: $\neg Q,\ P \rightarrow Q$
Conclusion: $\neg P$

(1)	$\neg Q$	rule P	
(2)	$P \rightarrow Q$	rule P	
(3)	$\neg Q \rightarrow \neg P$	rule T	from (2)
(4)	$\neg P$	rule T	from (1) and (3)

Therefore, $\neg P$ is a valid conclusion from the given premises

3. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q, Q \rightarrow R, P \rightarrow M$ and $\neg M$

Given premises: $P \vee Q, Q \rightarrow R, P \rightarrow M, \neg M$
Conclusion: $R \wedge (P \vee Q)$

(1)	$\neg M$	rule P	
(2)	$P \rightarrow M$	rule P	
(3)	$\neg M \rightarrow \neg P$	rule T	from (2)
(4)	$\neg P$	rule T	from (1) and (3)
(5)	$P \vee Q$	rule P	
(6)	Q	rule T	from (4) and (5)
(7)	$Q \rightarrow R$	rule P	
(8)	R	rule T	from (6) and (7)
(9)	$R \wedge (P \vee Q)$	rule T	from (5) and (8)

Therefore, $R \wedge (P \vee Q)$ is a valid conclusion from the given premises

4. Show that $R \vee S$ follows logically from the premises $C \vee D, (C \vee D) \rightarrow \neg H, \neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$

Given premises: $C \vee D, (C \vee D) \rightarrow \neg H, \neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$
Conclusion: $R \vee S$

(1)	$C \vee D$	rule P	
(2)	$(C \vee D) \rightarrow \neg H$	rule P	
(3)	$\neg H$	rule T	from (1) and (2)
(4)	$\neg H \rightarrow (A \wedge \neg B)$	rule P	
(5)	$A \wedge \neg B$	rule T	from (3) and (4)
(6)	$(A \wedge \neg B) \rightarrow (R \vee S)$	rule P	
(7)	$R \vee S$	rule T	from (5) and (6)

Therefore, $R \vee S$ is a valid conclusion from the given premises

5. Show that $S \vee R$ follows logically from the premises $P \vee Q$, $P \rightarrow R$, and $Q \rightarrow S$

Given premises: $P \vee Q$, $P \rightarrow R$, and $Q \rightarrow S$

Conclusion: $S \vee R$

(1) $P \vee Q$	rule P	
(2) $\neg P \rightarrow Q$	rule T	from (1)
(3) $Q \rightarrow S$	rule P	
(4) $\neg P \rightarrow S$	rule T	from (2) and (3)
(5) $P \rightarrow R$	rule P	
(6) $\neg R \rightarrow \neg P$	rule T	from (5)
(7) $\neg R \rightarrow S$	rule T	from (4) and (6)
(8) $\neg(\neg R) \vee S$	rule T	from (7)
(9) $R \vee S$	rule T	from (8)

Therefore, $R \vee S$ is a valid conclusion from the given premises

6. Show that $R \rightarrow S$ is a valid conclusion from the premises $P \rightarrow (Q \rightarrow S), \neg R \vee P$ and Q

Given premises: $P \rightarrow (Q \rightarrow S), \neg R \vee P$ and Q

Conclusion: $R \rightarrow S$

(1) R	conditional premise (or additional premise)	
(2) $\neg R \vee P$	rule P	
(3) P	rule T	from (1) and (2)
(4) $P \rightarrow (Q \rightarrow S)$	rule P	
(5) $Q \rightarrow S$	rule T	from (3) and (4)
(6) Q	rule P	
(7) S	rule T	from (5) and (6)
(8) $R \rightarrow S$	rule CP	from (1) and (7)

Therefore, $R \rightarrow S$ is a valid conclusion

7. Show that $Q \rightarrow S$ is a valid conclusion from the premises $P, P \rightarrow (Q \rightarrow [R \wedge S])$

Or $P, P \rightarrow (Q \rightarrow [R \wedge S]) \Rightarrow Q \rightarrow S$

Given premises: $P, P \rightarrow (Q \rightarrow [R \wedge S])$

Conclusion: $Q \rightarrow S$

(1) Q	conditional premise (or additional premise)	
(2) P	rule P	
(3) $P \rightarrow (Q \rightarrow [R \wedge S])$	rule P	
(4) $Q \rightarrow [R \wedge S]$	rule T	from (2) and (3)
(5) $R \wedge S$	rule T	from (1) and (4)
(6) S	rule T	from (5)
(7) $Q \rightarrow S$	rule CP	from (1) and (6)

Therefore, $Q \rightarrow S$ is a valid conclusion from the premises $P, P \rightarrow (Q \rightarrow [R \wedge S])$

Consistency and Inconsistency of premises: A set of premises $H_1, H_2, H_3, \cdots H_n$ is said to be Inconsistent if $H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n$ is a contradiction. Otherwise we say that Consistent.

Indirect method of proof (or Proof by contradiction): The notion of inconsistency is used in a procedure called indirect method of proof or proof by contradiction.

In order to show that C is a valid conclusion from the premises $H_1, H_2, H_3, \cdots H_n$, we assume that $\neg C$ as an additional premise and we show that the new set of premises $H_1, H_2, H_3, \cdots H_n$ and $\neg C$ is inconsistent; that is $H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n \wedge \neg C$ is a contradiction.

Problems:

1. Show that the premises $P \to Q, P \to R, Q \to \neg R, P$ are inconsistent

Given premises: $P \to Q, P \to R, Q \to \neg R, P$;

Conclusion: F

(1) P	Rule P	
(2) $P \to Q$	Rule P	
(3) Q	Rule T	from (1) and (2)
(4) $P \to R$	Rule P	
(5) R	Rule T	from (1) and (4)
(6) $Q \to \neg R$	Rule P	
(7) $\neg R$	Rule T	from (3) and (6)
(8) F	Rule T	from (5) and (7)

Therefore, the given premises are inconsistent

2. Show that the following premises are inconsistent

(i) If Jack misses many classes through illness, then he fails high school.
(ii) If Jack fails high school, then he is uneducated.
(iii) If Jack reads a lot of books, then he is not uneducated.
(iv) Jack misses many classes through illness and reads a lot of books.

Solution: First express the statements in symbolic form

Let P: Jack misses many classes through illness
Q: Jack fails high school
R: Jack is uneducated
S: Jack reads a lot of books

Then the given premises: $P \to Q, Q \to R, S \to \neg R, P \wedge S$;

Conclusion: F

(1) $P \to Q$	Rule P	
(2) $Q \to R$	Rule P	
(3) $P \to R$	Rule T	from (1) and (2)
(4) $P \wedge S$	Rule P	
(5) P	Rule T	from (4)
(6) R	Rule T	from (3) and (5)
(7) S	Rule T	from (4)
(8) $S \to \neg R$	Rule P	
(9) $\neg R$	Rule T	from (7) and (8)
(10) F	Rule T	from (6) and (9)

Therefore, the given premises are inconsistent

3. Using indirect method, show that $\neg P$ is a valid conclusion from the premises $R \to \neg Q, R \vee S, S \to \neg Q$ and $P \to Q$

Given premises: $R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q$

Conclusion: $\neg P$

According to indirect method, consider P as an additional premise and proceed to prove that the given premises along with the additional premise are inconsistent.

(1) P	Additional premise	
(2) $P \to Q$	Rule P	
(3) Q	Rule T	from (1) and (2)
(4) $R \to \neg Q$	Rule P	
(5) $Q \to \neg R$	Rule T	from (4)
(6) $\neg R$	Rule T	from (3) and (5)
(7) $R \vee S$	Rule P	
(8) S	Rule T	from (6) and (7)
(9) $S \to \neg Q$	Rule P	
(10) $\neg Q$	Rule T	from (8) and (9)
(11) F	Rule T	from (3) and (10)

Therefore, the given premises along with the additional premise are inconsistent.

Hence $\neg P$ is a valid conclusion from the premises $R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q$

Exercise:

1. Show that $\neg P$ is a valid conclusion from the premises $\neg(P \wedge \neg Q), \neg Q \vee R,$ and $\neg R$
2. Show that $J \wedge S$ follows logically from the premises $P \to Q,\ Q \to \neg R,\ R$ and $P \vee (J \wedge S)$
3. Show that $\neg P \vee \neg Q$ follows logically from the premises $(P \wedge Q) \to R,\ \neg R \vee S$ and $\neg S$
4. Show that $M \vee N$ follows logically from the premises $\neg J \to (M \vee N), (H \vee G) \to \neg J$ and $H \vee G$
5. Show that $P \to S$ is a valid conclusion from the premises $\neg P \vee Q,\ \neg Q \vee R$ and $R \to S$
6. Show that the premises $P \to (Q \to R),\ S \to (Q \wedge \neg R),\ P \wedge S$ are inconsistent
7. Using indirect method, show that $\neg P$ is a valid conclusion from the premises $P \to Q, Q \to R, S \to \neg R$ and S

Predicate Logic:

Predicate: A **Predicate** (or propositional function or statement function) in a variable x is an expression of the form $P(x)$ which gives a statement for a definite value of $x.$ The set of all values in which x can take, is called the **Universe of discourse**.

Usually, we write the predicates in the variable x, as $P(x), Q(x), \cdots$ and the predicates in two variables x, y, as $P(x, y), Q(x, y), \cdots$

Example: Consider the following statements

2 is a prime number

3 is a prime number

4 is a prime number

5 is a prime number

The above statements can be written in the predicate form as follows.

$$P(x): x \text{ is a prime number, for } x = 2,3,4,5$$

Here $P(2), P(3), P(4)$ and $P(5)$ are given statements and the Universe of discourse is $\{2,3,4,5\}$

Example: Consider the following predicate

$$P(x, y): x \text{ is greater than } y, \text{ for } x = 1,3,4 \text{ and } y = 0, 5$$

Then $P(1,0): 1$ is greater than 0 $\quad$ $P(1,5): 1$ is greater than 5

$P(3,0): 3$ is greater than 0 $\quad$ $P(3,5): 3$ is greater than 5

$P(4,0): 4$ is greater than 0 $\quad$ $P(4,5): 4$ is greater than 5 are the statements.

Here the universe of discourse of x is $\{1,3,4\}$ and the universe of discourse of y is $\{0,5\}$

Predicate logic: The logic based upon the analysis of predicates is called predicate logic.

Predicate formula: An expression containing predicates, statements, connectives and quantifiers is called a predicate formula.

Quantifiers: There are two types of quantifiers (i) Universal quantifier (ii) Existential quantifier

Universal quantifier: The word 'all' is called the Universal quantifier and it is denoted by $\forall$ or ( )

$\forall x$, (x), *for all* x, *for every* x, *for each* x are same

Existential quantifier: The word 'some' is called the Existential quantifier and it is denoted by $\exists$

$\exists x$, *for some* x, *there exists* x, *there is* x are same

Negation of the quantifiers: The negation of the universal quantifier is the existential quantifier and vice versa. In fact, we have the following.

(1) $\neg[(x)P(x)] \Leftrightarrow (\exists x)[\neg P(x)]$

(2) $\neg[(\exists x)P(x)] \Leftrightarrow (x)[\neg P(x)]$

Problems:

1. If the universe of discourse is $\{a,b,c\}$, eliminate the quantifiers in the following formulas.

i)	$(x)P(x)$	$P(a)\wedge P(b)\wedge P(c)$
ii)	$(\exists x)P(x)$	$P(a)\vee P(b)\vee P(c)$
iii)	$(x)P(x)\wedge(x)Q(x)$	$[P(a)\wedge P(b)\wedge P(c)]\wedge[Q(a)\wedge Q(b)\wedge Q(c)]$
iv)	$(x)[P(x)\wedge Q(x)]$	$[P(a)\wedge Q(a)]\wedge[P(b)\wedge Q(b)]\wedge[P(c)\wedge Q(c)]$
v)	$(\exists x)P(x)\vee(\exists x)Q(x)$	$[P(a)\vee P(b)\vee P(c)]\vee[Q(a)\vee Q(b)\vee Q(c)]$
vi)	$(\exists x)[P(x)\vee Q(x)]$	$[P(a)\vee Q(a)]\vee[P(b)\vee Q(b)]\vee[P(c)\vee Q(c)]$
vii)	$\neg[(x)P(x)]$	$\neg[P(a)\wedge P(b)\wedge P(c)]$
viii)	$(\exists x)[\neg P(x)]$	$\neg P(a)\vee\neg P(b)\vee\neg P(c)$
ix)	$\neg[(\exists x)P(x)]$	$\neg[P(a)\vee P(b)\vee P(c)]$
x)	$(x)[\neg P(x)]$	$\neg P(a)\wedge\neg P(b)\wedge\neg P(c)$

Note: (1) From (iii) and (iv) we have $(x)[P(x)\wedge Q(x)] \Leftrightarrow (x)P(x)\wedge(x)Q(x)$

(2) From (v) and (vi) we have $(\exists x)[P(x)\vee Q(x)] \Leftrightarrow (\exists x)P(x)\vee(\exists x)Q(x)$

(3) From (vii) and (viii) we have $\neg[(x)P(x)] \Leftrightarrow (\exists x)[\neg P(x)]$

(4) From (ix) and (x) we have $\neg[(\exists x)P(x)] \Leftrightarrow (x)[\neg P(x)]$

2. Write the statement in predicate form: 'every apple is red'

   Let $P(x): x$ is an apple, and $Q(x): x$ is red

   Then the given statement can written as $(x)\,(P(x)\rightarrow Q(x))$

3. Write the statement in predicate form: 'every integer is even or odd'

   Let $P(x): x$ is an integer, $Q(x): x$ is even and $R(x): x$ is odd

   Then the given statement can written as $(x)[\,P(x)\rightarrow(Q(x)\vee R(x))\,]$

4. Write the statement in predicate form: 'there exists a prime integer'

   Let $P(x): x$ is an integer, $Q(x): x$ is prime

   Then the given statement can written as $(\exists x)\,(P(x)\wedge Q(x))$

5. Write the statement in predicate form: 'there exists an even or odd integer'

   Let $P(x): x$ is an integer, $Q(x): x$ is an even, $R(x): x$ is an odd

   Then the given statement can written as $(\exists x)\,[P(x)\wedge(Q(x)\vee R(x))]$

6. Find the truth values

   a) $(x)\,(P(x)\vee Q(x))$, where $P(x): x=1,\quad Q(x): x=2$ and the universe of discourse is $\{1,2\}$.

      Here $(x)\,(P(x)\vee Q(x)) \Leftrightarrow (P(1)\vee Q(1))\wedge(P(2)\vee Q(2)) \Leftrightarrow (T\vee F)\wedge(F\vee T) \Leftrightarrow T\vee T \Leftrightarrow T$

   b) $(x)\,[P\to Q(x)]\vee R(a)$, where $P: 2>1,\quad Q(x): x\le 3,\ R(x): x>5,\ a:5$ and the universe of discourse is $\{-2,3,6\}$.

      Here $(x)\,[P\to Q(x)]\vee R(a) \Leftrightarrow (\,[P\to Q(-2)]\wedge[P\to Q(3)]\wedge[P\to Q(6)]\,)\vee R(5)$

$$\begin{aligned}
&\Leftrightarrow (\,[T\to T]\wedge[T\to T]\wedge[T\to F]\,)\vee F\\
&\Leftrightarrow (T\wedge T\wedge F\,)\vee F\\
&\Leftrightarrow (T\wedge T\wedge F\,)\vee F\\
&\Leftrightarrow F\vee F\\
&\Leftrightarrow F
\end{aligned}$$

7. Write the following statements in predicate form

   i) All men are good

   ii) No men are good

   iii) Some men are good

   iv) Some men are not good

   Let $P(x): x$ is a man, $\quad Q(x): x$ is good

   Then

i) All men are good	$(x)[\,P(x)\to Q(x)\,]$
ii) No men are good	$(x)[\,P(x)\to\neg\,Q(x)\,]$
iii) Some men are good	$(\exists x)\,(P(x)\wedge Q(x))$
iv) Some men are not good	$(\exists x)\,(P(x)\wedge\neg Q(x))$

8. Write the negations of the following

i) All tigers are white	Some tigers are not white
ii) No tigers are white	Some tigers are white
iii) Some tigers are white	No tigers are white
iv) Some tigers are not white	All tigers are white

Bound and Free Variables: In a predicate formula, a part of the form $(x)(\cdots)$ or $(\exists x)(\cdots)$ is called $x-bound\ part$ of the formula.

The variable x in $x-bound\ part$ is called bound variable or bound occurrence. Otherwise the variable x is called free variable or free occurrence.

The part immediately followed by a quantifier (x) or $(\exists x)$ is called the scope of the corresponding quantifier.

Example: Consider the predicate formula $(x)\left[P(x)\rightarrow Q(x)\right]\wedge\left[R(x)\rightarrow P(y)\right]$

Here

i) There is only one universal quantifier

ii) $(x)\left[P(x)\rightarrow Q(x)\right]$ is the $x-bound\ part$ and all the 3 occurrences of x are bound occurrences or bound variables.

iii) $\left[P(x)\rightarrow Q(x)\right]$ is the scope of the universal quantifier

iv) $R(x)\rightarrow P(y)$ is the non bound part and the occurrences of both x and y are free occurrences or free variables.

Example: Consider the predicate formula $(x)\left[P(x)\rightarrow(\exists y)Q(x,\ y)\right]\wedge(x)R(x,\ y)$

Here

i) There are two universal quantifiers and one existential quantifier

ii) $(x)\left[P(x)\rightarrow(\exists y)Q(x,\ y)\right]$, $(x)R(x,\ y)$ are the $x-bound\ parts$ and all the occurrences of x are bound occurrences or bound variables.

iii) $\left[P(x)\rightarrow(\exists y)Q(x,\ y)\right]$ is the scope of the 1^{st} universal quantifier

iv) $R(x,\ y)$ is the scope of the 2^{nd} universal quantifier

v) $(\exists y)Q(x,\ y)$ is the $y-bound\ part$ and all the occurrences of y are bound occurrences or bound variables.

vi) $Q(x,\ y)$ is the scope of the existential quantifier

vii) Therefore in this formula, all the occurrences of x are bound occurrences, 1^{st} and 2^{nd} occurrences of y are bound occurrences and the 3^{rd} occurrence of y is free occurrence.

Valid Formulas in the Predicated Calculus: In order to prove that a conclusion is valid from a given set of premises in the predicate calculus, we the following inference rules in addition to the rules P, T and CP.

Rule US (Universal Specification): From $(x)P(x)$, we can conclude $P(a)$ (a can be any variable)

Rule ES (Existential Specification): From $(\exists x)P(x)$, we can conclude $P(a)$ (a can be new variable)

Rule EG (Existential Generalization): From $P(a)$, we can conclude $(\exists x)P(x)$

Rule UG (Universal Generalization): From $P(a)$, we can conclude $(x)P(x)$ (provided a is not introduced by the rule ES in the preceding steps, a is not in the premises)

Problems:

1. Prove that $(\exists x)[P(x)\wedge Q(x)] \Rightarrow (\exists x)P(x)\wedge(\exists x)Q(x)$

Given premise: $(\exists x)[P(x)\wedge Q(x)]$
Conclusion: $(\exists x)P(x)\wedge(\exists x)Q(x)$

(1) $(\exists x)(P(x)\wedge Q(x))$	Rule P	
(2) $P(a)\wedge Q(a)$	Rule ES	from (1)
(3) $P(a)$	Rule T	from (2)
(4) $Q(a)$	Rule T	from (2)
(5) $(\exists x)P(x)$	Rule EG	from (3)
(6) $(\exists x)Q(x)$	Rule EG	from (4)
(7) $(\exists x)P(x)\wedge(\exists x)Q(x)$	Rule T	from (5) and (6)

Therefore, $(\exists x)P(x)\wedge(\exists x)Q(x)$ is a valid conclusion from the premise $(\exists x)[P(x)\wedge Q(x)]$

Or $(\exists x)[P(x)\wedge Q(x)] \Rightarrow (\exists x)P(x)\wedge(\exists x)Q(x)$

2. Prove that $(x)[H(x)\rightarrow M(x)],\ H(s) \Rightarrow M(s)$

Given premises: $(x)[H(x)\rightarrow M(x)],\ H(s)$
Conclusion: $M(s)$

(1) $(x)[H(x)\rightarrow M(x)]$	Rule P	
(2) $H(s)\rightarrow M(s)$	Rule US	from (1)
(3) $H(s)$	Rule P	
(4) $M(s)$	Rule T	from (2) and (3)

Therefore, $M(s)$ is a valid conclusion from the premises $(x)[H(x)\rightarrow M(x)],\ H(s)$

Or $(x)[H(x)\rightarrow M(x)],\ H(s) \Rightarrow M(s)$

3. Prove that $(\exists x)M(x)$ follows logically from the premises $(\exists x)H(x)$ and $(x)[H(x)\rightarrow M(x)]$

Given premises: $(\exists x)H(x)$ and $(x)[H(x)\rightarrow M(x)]$
Conclusion: $(\exists x)M(x)$

(1) $(\exists x)H(x)$	Rule P	
(2) $H(a)$	Rule ES	from (1)
(3) $(x)[H(x)\rightarrow M(x)]$	Rule P	
(4) $H(a)\rightarrow M(a)$	Rule US	from (3)
(5) $M(a)$	Rule T	from (2) and (4)
(6) $(\exists x)M(x)$	Rule EG	from (5)

Therefore, $(\exists x)M(x)$ is a valid conclusion from the premises $(\exists x)H(x)$ and $(x)[H(x)\rightarrow M(x)]$

Or $(\exists x)H(x)\ \wedge\ (x)[H(x)\rightarrow M(x)] \Rightarrow (\exists x)M(x)$

4. Verify the validity of the following argument
Tigers are dangerous animals.
There are Tigers.
Therefore there are dangerous animals

Solution:
Let $P(x): x$ is Tiger, $Q(x): x$ is dangerous animal. Then
the given premises: $(x)[P(x) \to Q(x)]$ and $(\exists x)P(x)$
the conclusion: $(\exists x)Q(x)$

(1) $(\exists x)P(x)$	rule P	
(2) $P(a)$	rule ES	from (1)
(3) $(x)[P(x) \to Q(x)]$	rule P	
(4) $P(a) \to Q(a)$	rule US	from (3)
(5) $Q(a)$	rule T	from (2) and (4)
(6) $(\exists x)Q(x)$	rule EG	from (5)

Therefore, the given argument is valid

5. Verify the validity of the following argument
Every integer is a rational number.
5 is an integer.
Therefore there are rational numbers

Solution:
Let $P(x): x$ is an integer, $Q(x): x$ is a rational number. Then
Given premises: $(x)[P(x) \to Q(x)]$ and $P(5)$
Conclusion: $(\exists x)Q(x)$

(1) $P(5)$	rule P	
(2) $(x)[P(x) \to Q(x)]$	rule P	
(3) $P(5) \to Q(5)$	rule US	from (2)
(4) $Q(5)$	rule T	from (2) and (3)
(5) $(\exists x)Q(x)$	rule EG	from (5)

Therefore, the given argument is valid