MCA I SEMESTER

Mathematical Foundations of Computer Applications (MFCA): 20BM3101

Unit – 4: Recurrence relations

Recurrence relations: Generating Functions of sequences, calculating coefficients of generating functions, recurrence relations, solving recurrence relations by substitution, generating functions and the method of characteristic roots (Sections 3.1 - 3.5 of the text book 2)

Generating function: Let $\{a_n \mid n = 0,1,2,3,\cdots\}$ be a given sequence. Then the expression $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ or $\sum_{n=0}^{\infty} a_n x^n$ is called the *generating function* of the sequence.

Example:

(i) $1 + 2x + x^2 = (1+x)^2$ is the generating function of the sequence $\{a_n\}$, where

$$a_n = \begin{cases} 1 & \text{for } n = 0,2 \\ 2 & \text{for } n = 1 \\ 0 & \text{for } n = 3,4,5,\cdots \end{cases}$$

(ii) $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ is the generating function of the sequence $\{a_n\}$, where $a_n = 1$ for $n = 0,1,2,3,\cdots$

(iii) $1 + ax + a^2x^2 + a^3x^3 + \cdots = \dfrac{1}{1-ax}$ is the generating function of the sequence $\{a_n\}$, where $a_n = a^n$ for $n = 0,1,2,3,\cdots$

(iv) $1 - ax + a^2x^2 - a^3x^3 + \cdots = \dfrac{1}{1+ax}$ is the generating function of the sequence $\{a_n\}$, where

$$a_n = \begin{cases} a^n & \text{for } n = 0,2,4,\cdots \\ -a^n & \text{for } n = 1,3,5,\cdots \end{cases}$$

(v) $$\frac{1}{(1-x)^n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^n + \cdots \quad , n > 0;$$

that is, $\dfrac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r, n > 0$ is the generating function of the sequence $\{a_r\}$, where $a_r = {}^{n+r-1}C_r$ for $r = 0,1,2,3,\cdots$

(vi) $1 + 2x + 3x^2 + 4x^3 + \cdots = \dfrac{1}{(1-x)^2}$ is the generating function of the sequence $\{a_n\}$, where $a_n = n+1$ for $n = 0,1,2,3,\cdots$

(vii) $x + 2x^2 + 3x^3 + 4x^4 + \cdots = \dfrac{x}{(1-x)^2}$ is the generating function of the sequence $\{a_n\}$, where $a_n = n$ for $n = 0,1,2,3,\cdots$

(viii) $x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots = \dfrac{x^2}{(1-x)^2}$ is the generating function of the sequence $\{a_n\}$, where $a_n = n-1$ for $n = 0,1,2,3,\cdots$

Sum, Difference and Product of two generating functions:

$$Let\ A(x)=\sum_{r=0}^{\infty}a_r x^r\ and\ B(x)=\sum_{r=0}^{\infty}b_r x^r\ be\ two\ generating\ functions\ then$$

$$sum\ is\ defined\ as\ A(x)+B(x)=\sum_{r=0}^{\infty}(a_r+b_r)x^r$$

$$subtraction\ is\ defined\ as\ A(x)-B(x)=\sum_{r=0}^{\infty}(a_r-b_r)x^r$$

$$product\ of\ two\ generating\ functions\ are\ defined\ as\ A(x)B(x)=\sum_{r=0}^{\infty}P_r x^r;\ where\ P_r=\sum_{j+k=r}a_j b_k$$

$a_0b_0,\quad a_0b_1+a_1b_0,\quad a_0b_2+a_1b_1+a_2b_0$

Problem: if A(X)=$a_0+a_3X^3+a_4X^4+a_8X^8\ and\ B(X)=b_0+b_4X^4+b_5X^5+b_8X^8$ then find coefficient of X^8 in the product of $A(X)B(X)$?

Solution: the coefficient of X^8 can be obtained by using
X^0 in the first and X^8 in the second;
X^3 in the first and X^5 in the second;
X^4 in the first and X^4 in the second;
X^8 in the first and X^0 in the second

Thus the coefficient of X^8 in the product of $A(X)B(X)$ is $P_8=a_0b_8+a_3b_5+a_4b_4+a_8b_0$ because (0,8),(3,5),(4,4) and (8,0) are the only pairs of exponents of A(X) and B(X) whose sum is 8.

In particular, in the case, the coefficient of X^8 in the product is just the number of pairs of exponents whose sum is 8, that is, the coefficient of X^8 in the product $(1+X^3+X^4+X^8)(1+X^4+X^5+X^8)$ is just the number of integral solutions to the equation $e_1+e_2=8$, where e_1 and e_2 represent the exponents of A(X) and B(X), respectively. Hence e_1 can only be 0,3,4, or 8 and e_2 =0,4,5, or 8.

Thus if we want to count the number of integral solutions to an equation $e_1+e_2=r$ with certain constraints on e_1 and e_2 here , we need to find only the coefficient of X^r in the product of generating functions $A(X)B(X)$ where the exponents of A(x) reflect the constraints on e_1 and the exponents of B(x) reflect the constraints on e_2.

Generating function Models:

1. The generating function for the sequence a_r given by the number of integer solutions to an equation $e_1+e_2+e_3+\cdots+e_n=r$ with certain conditions on each e_i is given by $A_1(x)A_2(x)A_3(x)\cdots A_n(x)$ and each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$.

2. The generating function for the sequence a_r given by the number ways of distributing r similar balls into n boxes **is same as** a generating function for the sequence a_r given by the number of integer solutions to an equation $e_1+e_2+e_3+\cdots+e_n=r$ with each $e_i \geq 0$.
3. The generating function for the sequence a_r given by the number ways of distributing r similar balls into n boxes, where there are certain conditions on the number of balls in each box, **is same as** a generating function for the sequence a_r given by the number of integer solutions to an equation $e_1+e_2+e_3+\cdots+e_n=r$ with the condition on e_i same as the condition on the number of balls in i^{th} box.

Note:

1. ${}^nC_r = \dfrac{n!}{r!\,(n-r)!} = \dfrac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{r!}$, ${}^nC_r = {}^nC_{n-r}$
2. The number of binary numbers with $n-1$ 1s and r 0s is given by ${}^{n+r-1}C_r$
3. The number of r combinations with repetitions taken from the set $\{a_1, a_2, \cdots a_n\}$ of n objects is given by ${}^{n+r-1}C_r$
4. The number of ways of placing r similar balls in n boxes is given by ${}^{n+r-1}C_r$
5. The number of non negative integer solutions of the equation $x_1+x_2+x_3+\cdots+x_n=r$ is given by ${}^{n+r-1}C_r$

Problems:

1. Determine a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2=r$ with $e_1=0,3,4,8$ and $e_2=0,4,5,8$.

   Solution: Here each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$
   Since $e_1=0,3,4,8$, we have $A_1(x)=1+x^3+x^4+x^8$
   Since $e_2=0,4,5,8$, we have $A_2(x)=1+x^4+x^5+x^8$
   Therefore, the generating function for the sequence a_r is given by
   $A_1(x)A_2(x)=(1+x^3+x^4+x^8)(1+x^4+x^5+x^8)$
   That is, $a_r=$ coefficient of x^r in the product $(1+x^3+x^4+x^8)(1+x^4+x^5+x^8)$

2. Determine a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $0\leq e_1\leq 6$, $2<e_2\leq 7$, $5\leq e_3\leq 7$, e_1 is even and e_2 is odd.

   Solution: Here each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$
   Since $0\leq e_1\leq 6$ and e_1 is even, we have $A_1(x)=1+x^2+x^4+x^6$
   Since $2<e_2\leq 7$ and e_2 is odd, we have $A_2(x)=x^3+x^5+x^7$
   Since $5\leq e_3\leq 7$, we have $A_3(x)=x^5+x^6+x^7$

Therefore, the generating function for the sequence a_r is given by

$$A_1(x)A_2(x)A_3(x) = (1+x^2+x^4+x^6)(x^3+x^5+x^7)(x^5+x^6+x^7)$$
$$= x^8(1+x^2+x^4+x^6)(1+x^2+x^4)(1+x+x^2)$$

That is, $a_r =$ coefficient of x^r in the product $x^8(1+x^2+x^4+x^6)(1+x^2+x^4)(1+x+x^2)$

3. Build a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $0 \le e_i \le 3$ for each i.

   Solution: Here each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$

   Since $0 \le e_1, e_2, e_3 \le 3$, we have $A_1(x) = A_2(x) = A_3(x) = 1+x+x^2+x^3$

   Therefore, the generating function for the sequence a_r is given by

   $A_1(x)A_2(x)A_3(x) = (1+x+x^2+x^3)^3$

   That is, $a_r =$ coefficient of x^r in the expansion of $(1+x+x^2+x^3)^3$

4. Build a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $2 \le e_i \le 5$ for each i.

   Solution: Here each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$

   Since $2 \le e_1, e_2, e_3 \le 5$, we have $A_1(x) = A_2(x) = A_3(x) = x^2+x^3+x^4+x^5$

   Therefore, the generating function for the sequence a_r is given by

   $A_1(x)A_2(x)A_3(x) = (x^2+x^3+x^4+x^5)^3$

   That is, $a_r =$ coefficient of x^r in the expansion of $(x^2+x^3+x^4+x^5)^3$

5. Determine a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $e_i > 0$ for each i.

   Solution: Here each e_i is determined by the exponents (powers of x) of the i^{th} factor $A_i(x)$

   Since $e_1, e_2, e_3 > 0$, we have $A_1(x) = A_2(x) = A_3(x) = x+x^2+x^3+x^4+\cdots$

   Therefore, the generating function for the sequence a_r is given by

   $A_1(x)A_2(x)A_3(x) = (x+x^2+x^3+x^4+\cdots)^3$

   That is, $a_r =$ coefficient of x^r in the expansion of $(x+x^2+x^3+x^4+\cdots)^3$

6. Find the generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3+e_4+e_5=r$ with $0 \le e_1 \le 3,\ 0 \le e_2 \le 3,\ 2 \le e_3 \le 6,\ 2 \le e_4 \le 6,\ 0 \le e_5 \le 9$ and e_5 is odd.

   Solution: Here each e_i is determined by the exponents of the i^{th} factor $A_i(x)$

   Since $0 \le e_1 \le 3$ and $0 \le e_2 \le 3$, we have $A_1(x) = 1+x+x^2+x^3$ and $A_2(x) = 1+x+x^2+x^3$

   Since $2 \le e_3 \le 6$ and $2 \le e_4 \le 6$, we have $A_3(x) = A_4(x) = x^2+x^3+x^4+x^5+x^6$

Since $0 \le e_5 \le 9$ and e_5 is odd, we have $A_5(x) = x + x^3 + x^5 + x^7 + x^9$

Therefore, the generating function for the sequence a_r is given by $A_1(x)A_2(x)A_3(x)A_4(x)A_5(x)$
$= (1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$

That is, $a_r =$ coefficient of x^r in the product
$(1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$

7. Determine a generating function for a_r the number of ways to distributing r similar balls into 3 boxes where each box contains at least 2 balls.

Solution: Let e_1, e_2, e_3 be the number balls placed in 1st, 2nd and 3rd boxes respectively. Then the number of ways to distributing r balls into 3 boxes where each box contains at least 2 balls **is same as** the number of integer solutions to the equation $e_1 + e_2 + e_3 = r$ with $e_i \ge 2$ for each i.

Since $e_1, e_2, e_3 \ge 2$, we have $A_1(x) = A_2(x) = A_3(x) = x^2 + x^3 + x^4 + \cdots$

Therefore, the generating function for the sequence a_r is given by
$A_1(x)A_2(x)A_3(x) = (x^2 + x^3 + x^4 + \cdots)^3$

That is, $a_r =$ coefficient of x^r in the expansion of $(x^2 + x^3 + x^4 + \cdots)^3$

8. Determine a generating function for a_r the number of ways to distributing r similar balls into 4 boxes where each box contains at most 3 balls.

Solution: Let e_1, e_2, e_3, e_4 be the number balls placed in 1st, 2nd, 3rd and 4th boxes respectively. Then the number of ways to distributing r balls into 4 boxes where each box contains at most 3 balls **is same as** the number of integer solutions to the equation $e_1 + e_2 + e_3 + e_4 = r$ with $e_i \le 3$ for each i.

Since $e_1, e_2, e_3, e_4 \le 3$, we have $A_1(x) = A_2(x) = A_3(x) = A_4(x) = 1 + x + x^2 + x^3$

Therefore, the generating function for the sequence a_r is given by
$A_1(x)A_2(x)A_3(x)A_4(x) = (1 + x + x^2 + x^3)^4$

That is, $a_r =$ coefficient of x^r in the expansion of $(1 + x + x^2 + x^3)^4$

9. Determine a generating function for the number of ways to distribute r balls into 5 boxes where each box contains at least 3 balls and at most 7 balls

Solution: Let e_1, e_2, e_3, e_4, e_5 be the number balls placed in 1st, 2nd, 3rd, 4th and 5th boxes respectively. Then the number of ways to distributing r balls into 5 boxes where each box contains at least 3 balls and at most 7 balls **is same as** the number of integer solutions to the equation $e_1 + e_2 + e_3 + e_4 + e_5 = r$ with $3 \le e_i \le 7$ for each i.

Since $3 \le e_1, e_2, e_3, e_4, e_5 \le 7$, we have $A_1(x) = A_2(x) = A_3(x) = A_4(x) = A_5(x) = x^3 + x^4 + x^5 + x^6 + x^7$

Therefore, the generating function for the sequence a_r is given by
$A_1(x)A_2(x)A_3(x)A_4(x)A_5(x) = (x^3 + x^4 + x^5 + x^6 + x^7)^5$

That is, $a_r =$ coefficient of x^r in the expansion of $(x^3 + x^4 + x^5 + x^6 + x^7)^5$

10. Determine a generating function for the number of ways of selecting r balls from 3 red balls, 5 blue balls, 7 white balls.

Solution: Let e_1, e_2, e_3 be the number of red, blue and white balls selected respectively. Then the number of ways of selecting r balls from 3 red, 5 blue, 7 white balls **is same as** the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $0\le e_1\le 3, 0\le e_2\le 5, 0\le e_1\le 7$.

Since $0\le e_1\le 3$, we have $A_1(x)=1+x+x^2+x^3$

Since $0\le e_1\le 5$, we have $A_2(x)=1+x+x^2+x^3+x^4+x^5$

Since $0\le e_3\le 7$, we have $A_3(x)=1+x+x^2+x^3+x^4+x^5+x^6+x^7$

Therefore, the generating function for the sequence a_r is given by

$A_1(x)A_2(x)A_3(x) = (1+x+x^2+x^3)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5+x^6+x^7)$

That is, $a_r =$ coefficient of x^r in the expansion of $(1+x+x^2+x^3)(1+x+\cdots+x^5)(1+x+x^2\cdots+x^7)$

Exercise:

1. Determine a generating function for the sequence a_r given by the of integer solutions to the equation $e_1+e_2=r$ with $e_1=0,1,6,$ and $e_2=1,2,4,7$.
2. Determine a generating function for the sequence a_r given by the of integer solutions to the equation $e_1+e_2=r$ with $e_1=1,2,4,5$ and $e_2=1,3,6,8$.
3. Determine a generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3=r$ with $1\le e_1\le 5,\ 2<e_2\le 8,\ 3\le e_3\le 8,\ e_2$ is odd and e_3 is even.
4. Find the generating function for the sequence a_r given by the number of integer solutions to the equation $e_1+e_2+e_3+e_4+e_5=r$ with $1\le e_1\le 5,\ 2\le e_2\le 4,\ 0\le e_3\le 6, 3\le e_4\le 5, 1\le e_5\le 10$, e_3 is odd and e_5 is prime.
5. Determine a generating function for a_r the number of ways to distributing r similar balls into 4 boxes where each box contains at least 2 balls.
6. Determine a generating function for a_r the number of ways to distributing r similar balls into 7 boxes where second, third, fourth and fifth boxes are non empty.

Calculating coefficients of generating functions: The following results are useful in calculating the coefficients of certain generating functions.

1. $\frac{1-x^{n+1}}{1-x}=1+x+x^2+\cdots+x^n$ **or** $\frac{1-x^{n+1}}{1-x}=\sum_{r=0}^{n}x^r$

2. $\frac{1}{1-x}=1+x+x^2+x^3+\cdots$ **or** $\frac{1}{1-x}=\sum_{r=0}^{\infty}x^r$

3. $\frac{1}{(1-x)^n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots;\ n>0$

Or $\frac{1}{(1-x)^n}=\sum_{r=0}^{\infty}{}^{n+r-1}C_r\ x^r,\quad n>0$

Problems:

1. Determine the coefficient of x^r in the expansion of $\dfrac{1}{2-5x}$

Consider $\dfrac{1}{2-5x} = \dfrac{1}{2}\left[\dfrac{1}{1-\frac{5}{2}x}\right] = \dfrac{1}{2}\sum_{r=0}^{\infty}\left(\dfrac{5}{2}\right)^r x^r \qquad \left(\because \dfrac{1}{1-x} = \sum_{r=0}^{\infty} x^r\right)$

Therefore, the coefficient of $x^r = \dfrac{1}{2}\left(\dfrac{5}{2}\right)^r$

2. Determine the coefficient of x^r in the expansion of $\dfrac{1}{x^2-5x+6}$

Consider $\dfrac{1}{x^2-5x+6} = \dfrac{1}{(x-2)(x-3)}$

$$= \frac{1}{x-3} - \frac{1}{x-2}$$

$$= \left(-\frac{1}{3}\right)\left[\frac{1}{1-\frac{x}{3}}\right] - \left(-\frac{1}{2}\right)\left[\frac{1}{1-\frac{x}{2}}\right]$$

$$= \left(-\frac{1}{3}\right)\left[\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r x^r\right] + \left(\frac{1}{2}\right)\left[\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r x^r\right] \qquad \left(\because \frac{1}{1-x} = \sum_{r=0}^{\infty} x^r\right)$$

Therefore, the coefficient of $x^r = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^r - \dfrac{1}{3}\left(\dfrac{1}{3}\right)^r = \dfrac{1}{2^{r+1}} - \dfrac{1}{3^{r+1}}$

3. Determine the coefficient of x^r in the expansion of $\dfrac{5x-12}{x^2-5x+6}$

Suppose that $\dfrac{5x-12}{x^2-5x+6} = \dfrac{5x-12}{(x-2)(x-3)} = \dfrac{A}{x-3} + \dfrac{B}{x-2}$

Then (by partial fractions) $A=3$ and $B=2$

Now $\dfrac{5x-12}{x^2-5x+6} = \dfrac{3}{x-3} + \dfrac{2}{x-2}$

$$= \left(-\frac{1}{3}\right)\left[\frac{3}{1-\frac{x}{3}}\right] + \left(-\frac{1}{2}\right)\left[\frac{2}{1-\frac{x}{2}}\right]$$

$$= -\left[\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r x^r\right] - \left[\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r x^r\right] \qquad \left(\because \frac{1}{1-x} = \sum_{r=0}^{\infty} x^r\right)$$

Therefore, the coefficient of $x^r = -\left(\dfrac{1}{2}\right)^r - \left(\dfrac{1}{3}\right)^r = -\dfrac{1}{2^r} - \dfrac{1}{3^r}$

4. Determine the coefficient of (i) x^{15} (ii) x^{18} (iii) x^{20} in the expansion of $(x^3+x^4+x^5+\cdots)^5$

Consider $(x^3+x^4+x^5+\cdots)^5 = x^{15}(1+x+x^2+x^3+\cdots)^5$

$$= x^{15}\left[\frac{1}{(1-x)^5}\right]$$

$$= x^{15}\left[\sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right] \quad \left(\because \frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right)$$

$$= x^{15}\left[\sum_{r=0}^{\infty} {}^{r+4}C_r\ x^r\right] \text{ or } x^{15}\left[\sum_{r=0}^{\infty} {}^{r+4}C_4\ x^r\right]$$

Therefore,

(i) the coefficient of $x^{15} = {}^4C_0 = 1$

(ii) the coefficient of $x^{18} = {}^7C_3$

(iii) the coefficient of $x^{20} = {}^9C_5$

5. Determine the coefficient of (i) x^6 (ii) x^{10} (iii) x^{12} (iv) x^{14} in the expansion of $(1+x+x^2+x^3)^{10}$

Consider $(1+x+x^2+x^3)^{10} = \left(\dfrac{1-x^4}{1-x}\right)^{10} = \dfrac{(1-x^4)^{10}}{(1-x)^{10}}$

$$= \left[1-{}^{10}C_1x^4+{}^{10}C_2x^8-{}^{10}C_3x^{12}+{}^{10}C_4x^{16}-\cdots+x^{40}\right]\left[\sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right] \quad \left(\because \frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right)$$

$$= \left[1-{}^{10}C_1x^4+{}^{10}C_2x^8-{}^{10}C_3x^{12}+{}^{10}C_4x^{16}-\cdots+x^{40}\right]\left[\sum_{r=0}^{\infty} {}^{r+9}C_r\ x^r\right]$$

Therefore,

(i) Coefficient of $x^6 = {}^{15}C_6-{}^{10}C_1\,{}^{11}C_2$

(ii) Coefficient of $x^{10} = {}^{19}C_{10}-{}^{10}C_1\,{}^{15}C_6+{}^{10}C_2\,{}^{11}C_2$

(iii) Coefficient of $x^{12} = {}^{21}C_{12}-{}^{10}C_1\,{}^{17}C_8+{}^{10}C_2\,{}^{13}C_4-{}^{10}C_3\,{}^{9}C_0$

(iv) Coefficient of $x^{14} = {}^{23}C_{14}-{}^{10}C_1\,{}^{19}C_{10}+{}^{10}C_2\,{}^{15}C_6-{}^{10}C_3\,{}^{11}C_2$

6. Find the coefficient of (i) x^{16} (ii) x^{18} (iii) x^{20} in the product $(x+x^2+x^3+x^4+x^5)(x^2+x^3+\cdots)^5$

Consider $(x+x^2+x^3+x^4+x^5)(x^2+x^3+\cdots)^5 = x^{11}(1+x+x^2+x^3+x^4)(1+x+x^2+\cdots)^5$

$$= x^{11}\left(\frac{1-x^5}{1-x}\right)\frac{1}{(1-x)^5} = x^{11}(1-x^5)\frac{1}{(1-x)^6}$$

$$= (x^{11}-x^{16})\left[\sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right] \quad \left(\because \frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r\ x^r\right)$$

$$= (x^{11}-x^{16})\left[\sum_{r=0}^{\infty} {}^{r+5}C_r\ x^r\right]$$

Therefore,

(i) Coefficient of $x^{16} = {}^{10}C_5 - {}^5C_0 = {}^{10}C_5 - 1$

(ii) Coefficient of $x^{18} = {}^{12}C_7 - {}^{7}C_2$

(iii) Coefficient of $x^{20} = {}^{14}C_9 - {}^{9}C_4$

7. Determine the coefficient of x^{15} in the product of $(x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x+x^2+\cdots+x^{15})$

Consider $(x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x+x^2+\cdots+x^{15})$

$$= x^2(1+x+x^2+x^3)x(1+x+x^2+x^3+x^4+x^5+x^6)(1+x+x^2+\cdots+x^{15})$$

$$= x^3\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^7}{1-x}\right)\left(\frac{1-x^{16}}{1-x}\right) = x^3\,\frac{(1-x^4)(1-x^7)(1-x^{16})}{(1-x)^3}$$

$$= (x^3-x^7)(1-x^7-x^{16}+x^{23})\left[\sum_{r=0}^{\infty} {}^{n+r-1}C_r\, a^r\, x^r\right] \qquad \left(\because \frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r\, x^r\right)$$

$$= (x^3-x^{10}-x^{19}+x^{26}-x^7+x^{14}+x^{23}-x^{30})\left[\sum_{r=0}^{\infty} {}^{r+2}C_r\, x^r\right]$$

$$= (x^3-x^7-x^{10}+x^{14}-x^{19}+x^{23}+x^{26}-x^{30})\left[\sum_{r=0}^{\infty} {}^{r+2}C_r\, x^r\right]$$

Therefore, the coefficient of $x^{15} = {}^{14}C_{12} - {}^{10}C_8 - {}^{7}C_5 + {}^{3}C_1$

Exercise:

1. Determine the coefficient of x^r in the expansion of $\frac{1}{2+5x}$
2. Determine the coefficient of x^r in the expansion of $\frac{1}{x^2-6x+8}$
3. Determine the coefficient of x^r in the expansion of $\frac{1}{1-x} + \frac{5}{1+2x} + \frac{7}{(1-x)^5}$
4. Determine the coefficient of x^r in the expansion of $\frac{9x-31}{x^2-8x+15}$
5. Determine the coefficient of x^{10} in the expansion of the following

   (i) $(1+x+x^2+x^3\cdots)^2$ (ii) $\frac{1}{(1-x)^3}$ (iii) $\frac{1}{(1+2x)^5}$ (iv) $\frac{1}{(1-x)^3}$
6. Determine the coefficient of (i) x^{25} (ii) x^{22} (iii) x^{20} in the expansion of $(x^2+x^3+x^4+x^5+x^6)^7$
7. Determine the coefficient of x^{12} in the expansion of the following

   (i) $(x^3+x^4+x^5+\cdots)^2$ (ii) $(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+\cdots+x^{12})$
8. Determine the coefficient of x^{14} in the expansion of the following

   (i) $(1+x+x^2+x^3)^{10}$ (ii) $(1+x+x^2+\cdots+x^8)^{10}$ (iii) $(x^2+x^3+x^4+x^5+x^6+x^7)^4$

Recurrence relation: A recurrence relation for a sequence $\{a_n\}$ or $\{a_n \mid n = 0,1,2,3,\cdots\}$ is an equation in which a_n gives in terms of one or more of the preceding terms of the sequence.

Example:

(i) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$

(ii) $a_n = a_{n-1}^2 + 2a_{n-2}^2$ for $n \geq 2$

(iii) $a_n = a_{n-1} + n$ for $n \geq 1$

(iv) $a_n = na_{n-1} + a_{n-3}$ for $n \geq 3$

(v) $a_n = na_{n-1} + 5a_{n-2} + 3$ for $n \geq 2$

Linear recurrence relation: Let n, k be non negative integers. Then a relation of the form $c_0(n)a_n + c_1(n)a_{n-1} + c_2(n)a_{n-2} + \cdots c_k(n)a_{n-k} = f(n)$ for $n - k \geq 0$ where, $c_0, c_1, c_2, \cdots c_k$ and f are functions of n, is called a *linear recurrence relation*.

Note:

(i) In the above definition, if $c_0, c_1, c_2, \cdots c_k$ are constants then it is called a *linear recurrence relation with constant coefficients* or a *linear relation.*

(ii) In the above definition, if $f(n) = 0$ then it is called *homogeneous* and if $f(n) \neq 0$ then it is called *inhomogeneous* or *non homogeneous*.

(iii) In the above definition, if $c_0 \neq 0$ and $c_k \neq 0$ then it is called a linear recurrence relation of *degree* k

(iv) In the above definition, if $c_0, c_1, c_2, \cdots c_k$ are constants, $c_0 \neq 0$ and $c_k \neq 0$, then $c_0 r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k = 0$ is called the *characteristic equation* and the roots are called the *characteristic roots* of the given recurrence relation.

Solution of a recurrence relation: A sequence $\{a_n\}$ is called a solution of a recurrence relation if a_n satisfy the recurrence relation.

Example:

(i) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$

It is a homogeneous linear recurrence relation with constant coefficients of degree 2

(ii) $a_n = a_{n-1} + 5$ for $n \geq 1$

It is a non homogeneous linear recurrence relation with constant coefficients of degree 1

(iii) $a_n = a_{n-1} + n$ for $n \geq 1$

It is a non homogeneous linear recurrence relation with constant coefficients of degree 1

(iv) $a_n - 4a_{n-1} + 6a_{n-2} + 4a_{n-3} = 0$ for $n \geq 3$

It is a homogeneous linear recurrence relation with constant coefficients of degree 3

(v) $a_n^2 + 2a_{n-1}^2 = 0$ for $n \geq 2$

It is a homogeneous non linear recurrence relation with constant coefficients of degree 1

(vi) $a_n^2 + 2a_{n-1}^2 = n^2$ for $n \geq 2$

It is a non homogeneous non linear recurrence relation with constant coefficients of degree 1

Problems:

1. Find the first five terms of the recurrence relation given by $a_n = a_{n-1} + 5$ for $n \geq 1$, $a_0 = 2$

   Solution: Given that $a_0 = 2$

   Now,

   $a_1 = a_0 + 5 = 2 + 5 = 7$

   $a_2 = a_1 + 5 = 7 + 5 = 12$

   $a_3 = a_2 + 5 = 12 + 5 = 17$

   $a_4 = a_3 + 5 = 17 + 5 = 22$

2. Find the first six terms of the recurrence relation given by $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

   Solution: Given that $a_0 = 0$, $a_1 = 1$

   Now,

   $a_2 = a_1 + a_0 = 1 + 0 = 1$

   $a_3 = a_2 + a_1 = 1 + 1 = 2$

   $a_4 = a_3 + a_2 = 2 + 1 = 3$

   $a_5 = a_4 + a_3 = 3 + 2 = 5$

3. Find the first six terms of the recurrence relation $a_n - 2a_{n-1} + 4a_{n-2} = 0$ for $n \geq 2$, $a_0 = 1$, $a_1 = -1$

   Solution: $a_n - 2a_{n-1} + 4a_{n-2} = 0 \Rightarrow a_n = 2a_{n-1} - 4a_{n-2}$

   Given that $a_0 = 1$, $a_1 = -1$

   Now, $a_2 = 2a_1 - 4a_0 = 2(-1) - 4(1) = -6$

   $a_3 = 2a_2 - 4a_1 = 2(-6) - 4(-1) = -8$

   $a_4 = 2a_3 - 4a_2 = 2(-8) - 4(-6) = 8$

   $a_5 = 2a_4 - 4a_3 = 2(8) - 4(-8) = 48$

4. Show that $a_n = 2^n$ is a solution of the recurrence relation $a_n = 2a_{n-1}$ for $n \geq 1$

   Solution: Given that $a_n = 2^n$. Then $a_{n-1} = 2^{n-1}$

   Consider $a_n - 2a_{n-1} = 2^n - 2(2^{n-1}) = 0$

   Therefore, a_n is a solution of the given recurrence relation

5. Show that $a_n = C_1 2^n + C_2 5^n$ is a solution of the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$

   Solution: Given that $a_n = C_1 2^n + C_2 5^n$

   Then $a_{n-1} = C_1 2^{n-1} + C_2 5^{n-1}$ and $a_{n-2} = C_1 2^{n-2} + C_2 5^{n-2}$

   Consider
$$\begin{aligned} a_n - 7a_{n-1} + 10a_{n-2} &= \left[C_1 2^n + C_2 5^n\right] - 7\left[C_1 2^{n-1} + C_2 5^{n-1}\right] + 10\left[C_1 2^{n-2} + C_2 5^{n-2}\right] \\ &= C_1\left[2^n - 7(2^{n-1}) + 10(2^{n-2})\right] + C_2\left[5^n - 7(5^{n-1}) + 10(5^{n-2})\right] \\ &= C_1(2^{n-2})\left[4 - 14 + 10\right] + C_2(5^{n-2})\left[25 - 35 + 10\right] \\ &= C_1(2^{n-2})\left[0\right] + C_2(5^{n-2})\left[0\right] = 0 \end{aligned}$$

   Therefore, a_n is a solution of the given recurrence relation

6. Show that $a_n = C_1 3^n + C_2 4^n$ is a solution of the recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 0$ for $n \geq 2$

   Solution: Given that $a_n = C_1 3^n + C_2 4^n$

   Then $a_{n-1} = C_1 3^{n-1} + C_2 4^{n-1}$ and $a_{n-2} = C_1 3^{n-2} + C_2 4^{n-2}$

   Consider $a_n - 7a_{n-1} + 12a_{n-2} = \left[C_1 3^n + C_2 4^n\right] - 7\left[C_1 3^{n-1} + C_2 4^{n-1}\right] + 12\left[C_1 3^{n-2} + C_2 4^{n-2}\right]$

   $= C_1\left[3^n - 7(3^{n-1}) + 12(3^{n-2})\right] + C_2\left[4^n - 7(4^{n-1}) + 12(4^{n-2})\right]$

   $= C_1(3^{n-2})\left[9 - 21 + 12\right] + C_2(4^{n-2})\left[16 - 28 + 12\right]$

   $= C_1(3^{n-2})\left[0\right] + C_2(4^{n-2})\left[0\right] = 0$

   Therefore, a_n is a solution of the given recurrence relation

7. Find the characteristic equation and the roots of the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$

   Solution: The characteristic equation is given by $r^2 - 7r + 10 = 0$

   Now $r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow r = 2, 5$

   Therefore, the characteristic roots are given by $r = 2, 5$

8. Find the characteristic equation and the roots of the recurrence relation $a_n - 10a_{n-1} + 25a_{n-2} = 0$ for $n \geq 2$

   Solution: The characteristic equation is given by $r^2 - 10r + 25 = 0$

   Now $r^2 - 10r + 25 = 0 \Rightarrow (r-5)(r-5) = 0 \Rightarrow r = 5, 5$

   Therefore, the characteristic roots are given by $r = 5, 5$

9. Find the characteristic equation and the roots of the recurrence relation $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$

   Solution: The characteristic equation is given by $r^3 - 7r^2 + 16r - 12 = 0$

   Now $r^3 - 7r^2 + 16r - 12 = 0 \Rightarrow (r-2)(r-2)(r-3) = 0 \Rightarrow r = 2, 2, 3$

   Therefore, the characteristic roots are given by $r = 2, 2, 3$

Exercise:

1. Find the first six terms of the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2,\ a_0 = 1,\ a_1 = 2$
2. Find the first six terms of the recurrence relation $a_n + 5a_{n-1} - 8a_{n-2} = 0$ for $n \geq 2,\ a_0 = 0,\ a_1 = 2$
3. Show that $a_n = C_1(-2)^n + C_2 3^n$ is a solution of the recurrence relation $a_n - a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$
4. Show that $a_n = K_1 5^n + K_2 n 5^n$ is a solution of the recurrence relation $a_n - 10a_{n-1} + 25a_{n-2} = 0$ for $n \geq 2$
5. Show that $a_n = C_1 + C_2 2^n + C_3 3^n$ is a solution of the recurrence relation $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ for $n \geq 3$
6. Find the characteristic equation and the roots of the recurrence relation $a_n - 6a_{n-1} + 5a_{n-2} = 0$ for $n \geq 2$
7. Find the characteristic equation and the roots of the recurrence relation $a_n + a_{n-1} + a_{n-2} = 0$ for $n \geq 2$
8. Find the characteristic equation and the roots of the recurrence relation $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ for $n \geq 3$

Fibonacci recurrence relation: The recurrence relation given by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = F_1 = 1$, is called the Fibonacci recurrence relation. Here $F_0, F_1, F_2, \cdots$ are called Fibonacci numbers and $\{F_n\}$ is called Fibonacci sequence.

Note: The Fibonacci numbers are given by

$F_0 = 1$

$F_1 = 1$

$F_2 = F_1 + F_0 = 2$

$F_3 = F_2 + F_1 = 3$

$F_4 = F_3 + F_2 = 5$ and so on.

Solution of Fibonacci recurrence relation:

(i) The general solution of the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, is given by

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(ii) The solution of the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = F_1 = 1$, is given by $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

Properties of Fibonacci numbers:

(i) $F_0 + F_1 + F_2 + \cdots F_n = F_{n+2} - 1$

(ii) $F_0 + F_2 + F_4 + \cdots F_{2n} = F_{2n+1}$

(iii) $F_1 + F_3 + F_5 + \cdots F_{2n-1} = F_{2n} - 1$

(iv) $F_0^2 + F_1^2 + F_2^2 + \cdots F_n^2 = F_n F_{n+1}$

(v) $F_n^2 = F_{n-1} F_{n+1} + (-1)^n$ for $n \geq 1$

Solving a recurrence relation by Substitution Method: In the substitution method, the recurrence relation for a_n is used repeatedly to solve for a general expression for a_n in terms of n.

Problems:

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by the substitution method.

Solution: Consider

$a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

Therefore, $a_n = a_0 + \sum_1^n f(n)$ is the general solution of the given recurrence relation.

2. Solve the recurrence relation $a_n = a_{n-1} + n$ for $n \ge 1$, $a_0 = 2$ by the substitution method.

Solution: Let $f(n) = n$. Then we have $a_n = a_{n-1} + f(n)$ for $n \ge 1$, $a_0 = 2$

Consider

$a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

That is, $a_n = a_0 + \sum_1^n f(n) = 2 + \sum_1^n n = 2 + \frac{n(n+1)}{2}$

Therefore, the solution of the given recurrence relation is $a_n = 2 + \frac{n(n+1)}{2}$

3. Solve the recurrence relation $a_n = a_{n-1} + n^2 + 2$ for $n \ge 1$, $a_0 = 1$ by the substitution method.

Solution: Let $f(n) = n^2 + 2$. Then we have $a_n = a_{n-1} + f(n)$ for $n \ge 1$, $a_0 = 1$

Consider

$a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

That is, $a_n = a_0 + \sum_1^n f(n) = 1 + \sum_1^n (n^2 + 2) = 1 + \sum_1^n n^2 + \sum_1^n 2 = 1 + \frac{n(n+1)(2n+1)}{6} + 2n$

Therefore, the solution of the given recurrence relation is $a_n = 1 + \frac{n(n+1)(2n+1)}{6} + 2n$

4. Solve the recurrence relation $a_n = a_{n-1} + n(n-2)$ for $n \ge 1$, $a_0 = 2$ by the substitution method.

Solution: Let $f(n) = n(n-2)$. Then we have $a_n = a_{n-1} + f(n)$ for $n \ge 1$, $a_0 = 2$

Consider $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

That is, $a_n = a_0 + \sum_1^n f(n) = 2 + \sum_1^n n(n-2) = 2 + \sum_1^n n^2 - 2\sum_1^n n = 2 + \frac{n(n+1)(2n+1)}{6} - (2)\frac{n(n+1)}{2}$

Therefore, the solution of the given recurrence relation is $a_n = 2 + \frac{n(n+1)(2n+1)}{6} - n(n+1)$

5. Solve the recurrence relation $a_n = a_{n-1} + 3^n$ for $n \geq 1$, $a_0 = 5$ by the substitution method.

Solution: Let $f(n) = 3^n$. Then we have $a_n = a_{n-1} + f(n)$ for $n \geq 1$, $a_0 = 5$

Consider, $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

That is, $a_n = a_0 + \sum_1^n f(n) = 5 + \sum_1^n 3^n = 5 + \frac{1-3^n}{1-3} = 5 + \frac{1}{2}(3^n - 1)$

Therefore, the general solution of the given recurrence relation is $a_n = 5 + \frac{1}{2}(3^n - 1)$

6. Solve the recurrence relation $a_n = a_{n-1} + \frac{1}{n(n+1)}$ for $n \geq 1$, $a_0 = 3$ by the substitution method.

Solution: Let $f(n) = \frac{1}{n(n+1)}$. Then we have $a_n = a_{n-1} + f(n)$ for $n \geq 1$, $a_0 = 3$

Consider $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3), \cdots$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n) = a_0 + \sum_1^n f(n)$$

That is, $a_n = a_0 + \sum_1^n f(n) = 3 + \sum_1^n \frac{1}{n(n+1)} = 3 + \sum_1^n \left[\frac{1}{n} - \frac{1}{n+1}\right]$

$$= 3 + \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n} - \frac{1}{n+1}\right] = 3 + \left(1 - \frac{1}{n+1}\right) = 4 - \frac{1}{n+1}$$

Therefore, the solution of the given recurrence relation is $a_n = 4 - \frac{1}{(n+1)}$

Exercise:

1. Solve the recurrence relation $a_n = a_{n-1} + 3 + n + n^2$ for $n \geq 1$, $a_0 = 2$ by the substitution method.
2. Solve the recurrence relation $a_n = a_{n-1} + n(n^2 - 1)$ for $n \geq 1$, $a_0 = -1$ by the substitution method.
3. Solve the recurrence relation $a_n = a_{n-1} + (n+1)(n+2)$ for $n \geq 1$, $a_0 = 3$ by the substitution method.
4. Solve the recurrence relation $a_n = a_{n-1} + 3^n + 2n^3$ for $n \geq 1$, $a_0 = 4$ by the substitution method.
5. Solve the recurrence relation $a_n = a_{n-1} + \frac{1}{(n+1)(n+2)}$ for $n \geq 1$, $a_0 = 1$ by the substitution method.
6. Solve the recurrence relation $a_n = a_{n-1} + \frac{1}{(2n-1)(2n+1)}$ for $n \geq 1$, $a_0 = 5$ by the substitution method.

Solving a recurrence relation by Characteristic roots Method: Let $a_n + k_1 a_{n-1} + k_2 a_{n-2} = 0$ for $n \geq 2$ be the given liner recurrence relation of degree 2. Let r_1 and r_2 be the characteristic roots.

(i) If $r_1 \neq r_2$ then $a_n = C_1 r_1^n + C_2 r_2^n$ is the general solution of the given recurrence relation

(ii) If $r_1 = r_2 = r$ then $a_n = (C_1 + C_2 n) r^n$ is the general solution of the given recurrence relation

Note: Let r_1, r_2 and r_3 be the characteristic roots of a liner recurrence relation of degree 3 of the form $a_n + k_1 a_{n-1} + k_2 a_{n-2} + k_3 a_{n-3} = 0$ for $n \geq 3$.

(i) If $r_1 \neq r_2 \neq r_3 \neq r_1$ then $a_n = C_1 r_1^n + C_2 r_2^n + C_3 r_3^n$ is the general solution of the given recurrence relation

(ii) If $r_1 = r_2 = r_3 = r$ then $a_n = (C_1 + C_2 n + C_3 n^2) r^n$ is the general solution of the given recurrence relation

(iii) If $r_1 = r_2 \neq r_3$ then $a_n = (C_1 + C_2 n) r_1^n + C_3 r_3^n$ is the general solution of the given recurrence relation

Problems:

1. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ by the method of characteristic roots.

   Solution: The characteristic equation is given by $r^2 - 7r + 10 = 0$
   Now $r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow r = 2, 5$
   Therefore, the characteristic roots are given by $r = 2, 5$
   Therefore, the general solution is given by $a_n = C_1 2^n + C_2 5^n$

2. Solve the recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 0$ for $n \geq 2$, $a_0 = 2$, $a_1 = 5$ by using the method of characteristic roots.

   Solution: The characteristic equation is given by $r^2 - 7r + 12 = 0$
   Now $r^2 - 7r + 12 = 0 \Rightarrow (r-3)(r-4) = 0 \Rightarrow r = 3, 4$
   Therefore, the characteristic roots are given by $r = 3, 4$
   Therefore, the general solution is given by $a_n = C_1 3^n + C_2 4^n$
   Now $a_0 = 2 \Rightarrow C_1 + C_2 = 2 \quad \cdots\cdots\cdots (i)$
   And $a_1 = 5 \Rightarrow 3C_1 + 4C_2 = 5 \quad \cdots\cdots\cdots (ii)$
   Solving (i) and (ii), we have $C_1 = 3$ and $C_2 = -1$
   Hence, the solution is given by $a_n = 3(3^n) - 4^n$ or $a_n = 3^{n+1} - 4^n$

3. Solve the recurrence relation $a(n) - 7a(n-1) + 10a(n-2) = 0$ for $n \geq 2$, $a(0) = 0$, $a(1) = 1$ by the method of characteristic roots.
   Solution: The given recurrence relation can be written as $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
   The characteristic equation is given by $r^2 - 7r + 10 = 0$
   Now $r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow r = 2, 5$
   Therefore, the characteristic roots are given by $r = 2, 5$

Therefore, the general solution is given by $a_n = C_1 2^n + C_2 5^n$

Now $a_0 = 0 \Rightarrow C_1 + C_2 = 0 \quad \cdots\cdots\cdots (i)$

And $a_1 = 1 \Rightarrow 2C_1 + 5C_2 = 1 \quad \cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$

Hence, the solution is given by $a_n = -\frac{1}{3}(2)^n + \frac{1}{3}(5)^n$

4. Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0$ for $n \geq 2$, $a_0 = \frac{5}{2}$, $a_1 = 8$ by the method of characteristic roots.

   Solution: The characteristic equation is given by $r^2 - 4r + 4 = 0$

   Now $r^2 - 4r + 4 = 0 \Rightarrow (r-2)(r-2) = 0 \Rightarrow r = 2, 2$

   Therefore, the characteristic roots are given by $r = 2, 2$

   Therefore, the general solution is given by $a_n = (C_1 + C_2 n) 2^n$

   Now $a_0 = \frac{5}{2} \Rightarrow C_1 = \frac{5}{2} \quad \cdots\cdots\cdots (i)$

   And $a_1 = 8 \Rightarrow 2C_1 + 2C_2 = 8 \quad \cdots\cdots\cdots (ii)$

   Solving (i) and (ii), we have $C_1 = \frac{5}{2}$ and $C_2 = \frac{3}{2}$

   Hence, the solution is given by $a_n = \left(\frac{5}{2} + \frac{3}{2}n\right) 2^n$ or $a_n = (5+3n) 2^{n-1}$

5. Solve the recurrence relation $a_n - 4a_{n-1} - 12a_{n-2} = 0$ for $n \geq 2$, $a_0 = 4$, $a_1 = \frac{16}{3}$ by the method of characteristic roots.

   Solution: The characteristic equation is given by $r^2 - 4r - 12 = 0$

   Now $r^2 - 4r - 12 = 0 \Rightarrow (r+2)(r-6) = 0 \Rightarrow r = -2, 6$

   Therefore, the characteristic roots are given by $r = -2, 6$

   Therefore, the general solution is given by $a_n = C_1(-2)^n + C_2 6^n$

   Now $a_0 = 4 \Rightarrow C_1 + C_2 = 4 \quad \cdots\cdots\cdots (i)$

   And $a_1 = \frac{16}{3} \Rightarrow -2C_1 + 6C_2 = \frac{16}{3} \quad \cdots\cdots\cdots (ii)$

   Solving (i) and (ii), we have $C_1 = \frac{7}{3}$ and $C_2 = \frac{5}{3}$

   Hence, the solution is given by $a_n = \left(\frac{7}{3}\right)(-2)^n + \left(\frac{5}{3}\right) 6^n$

6. Solve the recurrence relation $a_n + 5a_{n-1} + 5a_{n-2} = 0$ for $n \geq 2$, $a_0 = 0$, $a_1 = 2\sqrt{5}$ by the method of characteristic roots.

   Solution: The characteristic equation is given by $r^2 + 5r + 5 = 0$

Now $r^2+5r+5=0 \Rightarrow r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{-5\pm\sqrt{25-20}}{2} \Rightarrow r=\dfrac{-5\pm\sqrt{5}}{2}$

Therefore, the characteristic roots are given by $r=\dfrac{-5\pm\sqrt{5}}{2}$

Therefore, the general solution is given by $a_n = C_1\left(\dfrac{-5+\sqrt{5}}{2}\right)^n + C_2\left(\dfrac{-5-\sqrt{5}}{2}\right)^n$

Now $a_0=0 \Rightarrow C_1+C_2=0 \quad \cdots\cdots\cdots (i)$

And $a_1=2\sqrt{5} \Rightarrow C_1\left(\dfrac{-5+\sqrt{5}}{2}\right)+C_2\left(\dfrac{-5-\sqrt{5}}{2}\right)=2\sqrt{5} \quad \cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1=2$ and $C_2=-2$

Hence, the solution is given by $a_n = 2\left(\dfrac{-5+\sqrt{5}}{2}\right)^n - 2\left(\dfrac{-5-\sqrt{5}}{2}\right)^n$

7. Solve the recurrence relation $a_n-7a_{n-1}+16a_{n-2}-12a_{n-3}=0$ for $n\geq 3$, $a_0=1$, $a_1=4$ and $a_2=8$ by using the method of characteristic roots.

Solution: The characteristic equation is given by $r^3-7r^2+16r-12=0$

Now $r^3-7r^2+16r-12=0 \Rightarrow (r-2)(r-2)(r-3)=0 \Rightarrow r=2, 2, 3$

Therefore, the characteristic roots are given by $r=2, 2, 3$

Therefore, the general solution is given by $a_n=(C_1+C_2n)\,2^n+C_3 3^n$

Now $a_0=1 \Rightarrow C_1+C_3=1 \quad \cdots\cdots\cdots (i)$

$a_1=4 \Rightarrow 2C_1+2C_2+3C_3=4 \quad \cdots\cdots\cdots (ii)$

And $a_2=8 \Rightarrow 4C_1+8C_2+9C_3=8 \quad \cdots\cdots\cdots (iii)$

Solving (i), (ii) and (iii), we have $C_1=5$, $C_2=3$ and $C_3=-4$

Hence, the solution is given by $a_n=(5+3n)\,2^n-4(3)^n$ or $a_n=5(2)^n+3n(2)^n-4(3)^n$

8. Solve the recurrence relation $a_n-6a_{n-1}+12a_{n-2}-8a_{n-3}=0$ for $n\geq 3$, $a_0=1$, $a_1=2$ and $a_2=12$ by using the method of characteristic roots.

Solution: The characteristic equation is given by $r^3-6r^2+12r-8=0$

Now $r^3-6r^2+12r-8=0 \Rightarrow (r-2)(r-2)(r-2)=0 \Rightarrow r=2, 2, 2$

Therefore, the characteristic roots are given by $r=2, 2, 2$

Therefore, the general solution is given by $a_n=(C_1+C_2n+C_3n^2)\,2^n$

Now $a_0=1 \Rightarrow C_1=1 \quad \cdots\cdots\cdots (i)$

$a_1=2 \Rightarrow 2C_1+2C_2+2C_3=2 \quad \cdots\cdots\cdots (ii)$

And $a_2=12 \Rightarrow 4C_1+8C_2+16C_3=12 \cdots\cdots\cdots (iii)$

Solving (i), (ii) and (iii), we have $C_1=1$, $C_2=-1$ and $C_3=1$

Hence, the solution is given by $a_n=(1-n+n^2)\,2^n$

9. Solve the recurrence relation $a_n + a_{n-1} - 5a_{n-2} + 3a_{n-3} = 0$ for $n \ge 3,\ a_0 = 0,\ a_1 = 1$ and $a_2 = 2$ by using the method of characteristic roots.

Solution: The characteristic equation is given by $r^3 + r^2 - 5r + 3 = 0$

Now $r^3 + r^2 - 5r + 3 = 0 \Rightarrow (r-1)(r-1)(r+3) = 0 \Rightarrow r = 1, 1, -3$

Therefore, the characteristic roots are given by $r = 1, 1, -3$

Therefore, the general solution is given by $a_n = (C_1 + C_2 n)1^n + C_3(-3)^n$ or $a_n = C_1 + C_2 n + C_3(-3)^n$

Now $a_0 = 0 \Rightarrow C_1 + C_3 = 0 \qquad \cdots\cdots\cdots (i)$

$a_1 = 1 \Rightarrow C_1 + C_2 - 3C_3 = 1 \qquad \cdots\cdots\cdots (ii)$

And $a_2 = 2 \Rightarrow C_1 + 2C_2 + 9C_3 = 2 \qquad \cdots\cdots\cdots (iii)$

Solving (i), (ii) and (iii), we have $C_1 = 0,\ C_2 = 1$ and $C_3 = 0$

Hence, the solution is given by $a_n = n$

10. Solve the Fibonacci recurrence relation given by $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2,\ F_0 = 1,\ F_1 = 1$.

Solution: The characteristic equation is given by $r^2 - r - 1 = 0$

Now $r^2 - r - 1 = 0 = 0 \Rightarrow \ r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{1 \pm \sqrt{1+4}}{2} \Rightarrow r = \dfrac{1 \pm \sqrt{5}}{2}$

Therefore, the characteristic roots are given by $r = \dfrac{1 \pm \sqrt{5}}{2}$

Therefore, the general solution is given by $F_n = C_1\left(\dfrac{1+\sqrt{5}}{2}\right)^n + C_2\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Now $F_0 = 1 \Rightarrow C_1 + C_2 = 1 \qquad \cdots\cdots\cdots (i)$

And $F_1 = 1 \Rightarrow C_1\left(\dfrac{1+\sqrt{5}}{2}\right) + C_2\left(\dfrac{1-\sqrt{5}}{2}\right) = 1 \qquad \cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1 = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)$ and $C_2 = \dfrac{-1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)$

Hence, the solution is given by $F_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

$$\text{or} \qquad F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Exercise:

1. Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \ge 2,\ a_0 = 1,\ a_1 = -1$ by using the method of characteristic roots.
2. Solve the recurrence relation $a_n - 6a_{n-1} + 5a_{n-2} = 0$ for $n \ge 2,\ a_0 = 1,\ a_1 = 3$ by using the method of characteristic roots.
3. Use the method of characteristic roots, solve the recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ for $n \ge 2$, and given that $a_0 = 1,\ a_1 = -1$

4. Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$, $a_0 = -4$, $a_1 = -7$ by using the method of characteristic roots.
5. Solve the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 0$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$ by using the method of characteristic roots.
6. Solve the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 0$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$ by using the method of characteristic roots.
7. Solve the recurrence relation $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ and $a_2 = 14$ by using the method of characteristic roots. (Ans. $a_n = 1 + 2^n + 3^n$)
8. Solve the recurrence relation $a_n - a_{n-1} - 4a_{n-2} + 4a_{n-3} = 0$ for $n \geq 3$, $a_0 = 2$, $a_1 = 0$ and $a_2 = 8$ by using the method of characteristic roots. (Ans. $a_n = 2^n + (-2)^n$)

Solving a recurrence relation by generating functions:

The following results are useful in solving recurrence relations by generating functions.

1. $\dfrac{1}{(1-x)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$
2. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
3. $\dfrac{1}{(1-x)^2} = \sum_{n=0}^{\infty} {}^{n+1}C_n x^n = \sum_{n=0}^{\infty} {}^{n+1}C_1 x^n = \sum_{n=0}^{\infty} (n+1)x^n$
4. $\dfrac{1}{(1-x)^3} = \sum_{n=0}^{\infty} {}^{n+2}C_n x^n = \sum_{n=0}^{\infty} {}^{n+2}C_2 x^n = \sum_{n=0}^{\infty} \dfrac{(n+2)(n+1)}{2} x^n$
5. $\dfrac{1}{(1-x)^4} = \sum_{n=0}^{\infty} {}^{n+3}C_n x^n = \sum_{n=0}^{\infty} {}^{n+3}C_3 x^n = \sum_{n=0}^{\infty} \dfrac{(n+3)(n+2)(n+1)}{6} x^n$

Problems:

1. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_0 = 1$, $a_1 = 6$ using generating functions.

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 7a_{n-1} + 10a_{n-2} = 0$

$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$ (Multiplying by x^n and sum 2 to ∞)

$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 10x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$

$\Rightarrow [A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 10x^2[A(x)] = 0$ (Expressing in terms of $A(x)$)

$\Rightarrow A(x) = \dfrac{a_0 + (a_1 - 7a_0)x}{1 - 7x + 10x^2} \Rightarrow A(x) = \dfrac{a_0 + (a_1 - 7a_0)x}{(1-2x)(1-5x)}$

$\Rightarrow A(x) = \dfrac{C_1}{1-2x} + \dfrac{C_2}{1-5x}$ $\left(C_1 \text{ and } C_2 \text{ are constants}\right)$

$$\Rightarrow A(x) = C_1\sum_{n=0}^{\infty} 2^n x^n + C_2\sum_{n=0}^{\infty} 5^n x^n \qquad \left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right)$$

Comparing the coefficient of x^n, we have $a_n = C_1 2^n + C_2 5^n$

Now $a_0 = 1 \Rightarrow C_1 + C_2 = 1 \qquad \cdots\cdots\cdots (i)$

And $a_1 = 6 \Rightarrow 2C_1 + 5C_2 = 6 \qquad \cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1 = -\frac{1}{3}$ and $C_2 = \frac{4}{3}$

Therefore, the solution is given by $a_n = \left(-\frac{1}{3}\right)2^n + \left(\frac{4}{3}\right)5^n$

2. Solve the recurrence relation $a_n - 10a_{n-1} + 21a_{n-2} = 0$ for $n \geq 2$, $a_0 = \frac{10}{21}$, $a_1 = 2$ using generating functions

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 10a_{n-1} + 21a_{n-2} = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 10\sum_{n=2}^{\infty} a_{n-1} x^n + 21\sum_{n=2}^{\infty} a_{n-2} x^n = 0 \qquad \text{(Multiplying by } x^n \text{ and sum 2 to } \infty\text{)}$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 10x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 21x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x] - 10x[A(x) - a_0] + 21x^2[A(x)] = 0 \qquad \text{(Expressing in terms of } A(x)\text{)}$$

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 10a_0)x}{1 - 10x + 21x^2} \Rightarrow A(x) = \frac{a_0 + (a_1 - 10a_0)x}{(1-3x)(1-7x)}$$

$$\Rightarrow A(x) = \frac{C_1}{1-3x} + \frac{C_2}{1-7x} \qquad (C_1 \text{ and } C_2 \text{ are constants})$$

$$\Rightarrow A(x) = C_1\sum_{n=0}^{\infty} 3^n x^n + C_2\sum_{n=0}^{\infty} 7^n x^n \qquad \left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right)$$

Comparing the coefficient of x^n, we have $a_n = C_1 3^n + C_2 7^n$

Now $a_0 = \frac{10}{21} \Rightarrow C_1 + C_2 = \frac{10}{21} \qquad \cdots\cdots\cdots (i)$

And $a_1 = 2 \Rightarrow 3C_1 + 7C_2 = 2 \qquad \cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1 = \frac{1}{3}$ and $C_2 = \frac{1}{7}$

Therefore, the solution is given by $a_n = \left(\frac{1}{3}\right)3^n + \left(\frac{1}{7}\right)7^n$ or $a_n = 3^{n-1} + 7^{n-1}$

3. Solve the recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ for $n \geq 2$, $a_0 = -\frac{4}{3}$, $a_1 = \frac{16}{3}$ using generating functions

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 8a_{n-1} + 16a_{n-2} = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 8\sum_{n=2}^{\infty} a_{n-1} x^n + 16\sum_{n=2}^{\infty} a_{n-2} x^n = 0 \qquad \text{(Multiplying by } x^n \text{ and sum 2 to} \infty)$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 8x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 16x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x] - 8x[A(x) - a_0] + 16x^2[A(x)] = 0 \qquad \text{(Expressing in terms of } A(x))$$

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 8a_0)x}{1 - 8x + 16x^2} \Rightarrow A(x) = \frac{a_0 + (a_1 - 8a_0)x}{(1-4x)^2}$$

$$\Rightarrow A(x) = \frac{k_1}{1-4x} + \frac{k_2}{(1-4x)^2} \quad (k_1 \text{ and } k_2 \text{ are constants})$$

$$\Rightarrow A(x) = k_1\sum_{n=0}^{\infty} 4^n x^n + k_2\sum_{n=0}^{\infty} {}^{n+1}C_1 4^n x^n \qquad \left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} {}^{n+1}C_1 x^n\right)$$

$$\Rightarrow A(x) = k_1\sum_{n=0}^{\infty} 4^n x^n + k_2\sum_{n=0}^{\infty} (n+1) 4^n x^n$$

Comparing the coefficient of x^n, we have $a_n = [k_1 + k_2(n+1)]\,4^n$ or $a_n = (C_1 + C_2 n)\,4^n$

Now $a_0 = -\frac{4}{3} \Rightarrow C_1 = -\frac{4}{3}$ $\cdots\cdots\cdots (i)$

And $a_1 = \frac{16}{3} \Rightarrow 4C_1 + 4C_2 = \frac{16}{3}$ $\cdots\cdots\cdots (ii)$

Solving (i) and (ii), we have $C_1 = -\frac{4}{3}$ and $C_2 = \frac{8}{3}$

Therefore, the solution is given by $a_n = \left(-\frac{4}{3} + \frac{8}{3}n\right)4^n$ or $a_n = (2n-1)\,\frac{4^{n+1}}{3}$

4. Solve the recurrence relation $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ for $n \ge 3$ by using generating functions.

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6\sum_{n=3}^{\infty} a_{n-1} x^n + 11\sum_{n=3}^{\infty} a_{n-2} x^n - 6\sum_{n=3}^{\infty} a_{n-3} x^n = 0 \qquad \text{(Multiplying by } x^n \text{ and sum 3 to} \infty)$$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6x\sum_{n=3}^{\infty} a_{n-1} x^{n-1} + 11x^2\sum_{n=3}^{\infty} a_{n-2} x^{n-2} - 6x^3\sum_{n=3}^{\infty} a_{n-3} x^{n-3} = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x - a_2 x^2] - 6x[A(x) - a_0 - a_1 x] + 11x^2[A(x) - a_0] - 6x^3[A(x)] = 0$$

(Expressing in terms of $A(x)$)

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 11a_0)x^2}{1 - 6x + 11x^2 - 6x^3}$$

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 11a_0)x^2}{(1-x)(1-2x)(1-3x)}$$

$$\Rightarrow A(x) = \frac{C_1}{1-x} + \frac{C_2}{1-2x} + \frac{C_3}{1-3x} \quad (C_1, C_2 \text{ and } C_3 \text{ are constants})$$

$$\Rightarrow A(x) = C_1 \sum_{n=0}^{\infty} x^n + C_2 \sum_{n=0}^{\infty} 2^n x^n + C_3 \sum_{n=0}^{\infty} 3^n x^n \qquad \left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \right)$$

Comparing the coefficient of x^n, we have $a_n = C_1 + C_2 2^n + C_3 3^n$.

It is the required solution.

5. Solve the recurrence relation $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ for $n \geq 3$, $a_0 = 2$, $a_1 = 5$ and $a_2 = 13$ by using generating functions.

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6\sum_{n=3}^{\infty} a_{n-1} x^n + 11\sum_{n=3}^{\infty} a_{n-2} x^n - 6\sum_{n=3}^{\infty} a_{n-3} x^n = 0 \qquad \text{(Multiplying by } x^n \text{ and sum 3 to} \infty)$$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6x\sum_{n=3}^{\infty} a_{n-1} x^{n-1} + 11x^2 \sum_{n=3}^{\infty} a_{n-2} x^{n-2} - 6x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3} = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x - a_2 x^2] - 6x[A(x) - a_0 - a_1 x] + 11x^2 [A(x) - a_0] - 6x^3 [A(x)] = 0$$

(Expressing in terms of $A(x)$)

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 11a_0)x^2}{1 - 6x + 11x^2 - 6x^3}$$

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 11a_0)x^2}{(1-x)(1-2x)(1-3x)}$$

$$\Rightarrow A(x) = \frac{C_1}{1-x} + \frac{C_2}{1-2x} + \frac{C_3}{1-3x} \quad (C_1, C_2 \text{ and } C_3 \text{ are constants})$$

$$\Rightarrow A(x) = C_1 \sum_{n=0}^{\infty} x^n + C_2 \sum_{n=0}^{\infty} 2^n x^n + C_3 \sum_{n=0}^{\infty} 3^n x^n \qquad \left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \right)$$

Comparing the coefficient of x^n, we have $a_n = C_1 + C_2 2^n + C_3 3^n$

Now $a_0 = 2 \Rightarrow C_1 + C_2 + C_3 = 2 \qquad \cdots \cdots \cdots (i)$

$a_1 = 5 \Rightarrow C_1 + 2C_2 + 3C_3 = 5 \qquad \cdots \cdots \cdots (ii)$

And $a_2 = 13 \Rightarrow C_1 + 4C_2 + 9C_3 = 13 \cdots \cdots \cdots (iii)$

Solving (i), (ii) and (iii), we have $C_1 = 0$, $C_2 = 1$ and $C_3 = 1$

Hence, the solution is given by $a_n = 2^n + 3^n$

6. Solve the recurrence relation $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$ for $n \geq 3$, $a_0 = 1$, $a_1 = 2$ and $a_2 = 12$ by using generating functions.

Solution: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be a generating function of a_n

Now, $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6\sum_{n=3}^{\infty} a_{n-1} x^n + 12\sum_{n=3}^{\infty} a_{n-2} x^n - 8\sum_{n=3}^{\infty} a_{n-3} x^n = 0 \qquad \text{(Multiplying by } x^n \text{ and sum 3 to} \infty)$$

$$\Rightarrow \sum_{n=3}^{\infty} a_n x^n - 6x \sum_{n=3}^{\infty} a_{n-1} x^{n-1} + 12x^2 \sum_{n=3}^{\infty} a_{n-2} x^{n-2} - 8x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3} = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x - a_2 x^2] - 6x[A(x) - a_0 - a_1 x] + 12x^2 [A(x) - a_0] - 8x^3 [A(x)] = 0$$

(Expressing in terms of $A(x)$)

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 12a_0)x^2}{1 - 6x + 12x^2 - 8x^3}$$

$$\Rightarrow A(x) = \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 12a_0)x^2}{(1-2x)^3}$$

$$\Rightarrow A(x) = \frac{k_1}{1-2x} + \frac{k_2}{(1-2x)^2} + \frac{k_3}{(1-2x)^3} \quad (k_1, k_2 \text{ and } k_3 \text{ are constants})$$

$$\Rightarrow A(x) = k_1 \sum_{n=0}^{\infty} 2^n x^n + k_2 \sum_{n=0}^{\infty} {}^{n+1}C_1 \, 2^n x^n + k_3 \sum_{n=0}^{\infty} {}^{n+2}C_2 \, 2^n x^n$$

$$\left(\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} {}^{n+1}C_1 x^n, \ \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} {}^{n+2}C_2 x^n \right)$$

$$\Rightarrow A(x) = k_1 \sum_{n=0}^{\infty} 2^n x^n + k_2 \sum_{n=0}^{\infty} (n+1) \, 2^n x^n + k_3 \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} 2^n x^n$$

Comparing the coefficient of x^n, we have $a_n = \left[k_1 + k_2(n+1) + k_3 \frac{(n+2)(n+1)}{2} \right] 2^n$

or $\quad a_n = (C_1 + C_2 n + C_3 n^2) 2^n$

Now $a_0 = 1 \Rightarrow C_1 = 1 \qquad \cdots \cdots \cdots (i)$

$a_1 = 2 \Rightarrow 2C_1 + 2C_2 + 2C_3 = 2 \qquad \cdots \cdots \cdots (ii)$

And $a_2 = 12 \Rightarrow 4C_1 + 8C_2 + 16C_3 = 12 \cdots \cdots \cdots (iii)$

Solving (i), (ii) and (iii), we have $C_1 = 1,\ C_2 = -1$ and $C_3 = 1$

Hence, the solution is given by $a_n = (1 - n + n^2) 2^n$

Exercise:

1. Solve the recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 0$ for $n \geq 2, a_0 = 0, a_1 = 1$ using generating functions
2. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_0 = 10$, $a_1 = 41$ using generating functions
3. Solve the recurrence relation $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$, $a_0 = -3$ and $a_1 = -10$ using generating functions
4. Solve the recurrence relation $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$ for $n \geq 3$, $a_0 = 0$, $a_1 = 2$ and $a_2 = 16$ by using generating functions. (Ans. $a_n = n^2 2^n$)
5. Solve the recurrence relation $a_n + a_{n-1} - 5a_{n-2} + 3a_{n-3} = 0$ for $n \geq 3$, $a_0 = 0$, $a_1 = -2$ and $a_2 = 12$ by using generating functions. (Ans. $a_n = -1 + 2n + (-3)^n$)
6. Solve the recurrence relation $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$, $a_0 = -1$, $a_1 = 0$ and $a_2 = 6$ by using generating functions. (Ans. $a_n = 2(3^n) - 3(2^n)$)