MCA I SEMESTER
Mathematical Foundations of Computer Applications (MFCA): 20BM3101
Unit – 3: Boolean algebra

Partial order relation: A relation R on a non empty set A is called a *partial order relation* or a *partial ordering* in A if R is

(i) Reflexive; that is, $(x, x) \in R$ for all $x \in A$

(ii) Anti symmetric; that is, for $x, y \in A$ and $(x, y) \in R,\ (y, x) \in R$ *implies* $x = y$

Or for $x, y \in A$ and $x \neq y \Rightarrow$ *at least one of* $(x, y),\ (y, x) \notin R$

(iii) Transitive; that is, for $x, y, z \in A$ and $(x, y) \in R,\ (y, z) \in R$ *implies* $(x, z) \in R$

Note:

i) Generally partial order relations are denoted by the symbols like $\leq,\ \geq,\ \subseteq,\ \supseteq$

ii) If $\leq$ is a partial order relation on a non empty set A, then $\leq$ is a subset of $A \times A$ and for $(a, b) \in \leq$ we write $a \leq b$; that is, $a \leq b$ *if and only if* $(a, b) \in \leq$

iii) If A is a non empty set, then the diagonal relation Δ_A is always a partial order relation. Also Δ_A is the smallest partial order relation on A; that is, every partial order relation on A contains Δ_A.

Partially Ordered Set (or Poset): If $\leq$ is a partial order relation on a non empty set P, then the ordered pair $(P, \leq)$ is called a *partially ordered set* or a *poset*.

Comparable and Incomparable elements: Let $(P, \leq)$ be a poset and $a,\ b \in P$. Then we say that

(i) a, b are comparable if $a \leq b$ *or* $b \leq a$ ie., $(a, b) \in \leq$ *or* $(b, a) \in \leq$

(ii) a, b are incomparable if *neither* $a \leq b$ *nor* $b \leq a$ ie., $(a, b) \notin \leq$ *and* $(b, a) \notin \leq$

Totally Ordered Set (or Chain or Simply Ordered Set): A poset $(P, \leq)$ is called a *chain* or *totally ordered set* or *simply ordered set* if every pair of elements of P are comparable; that is $a, b \in P$ *implies* $a \leq b$ *or* $b \leq a$. In this case $\leq$ is called a total order relation.

Covering elements: Let $(P, \leq)$ be a poset and $a,\ b \in P$. Then we say that

(i) $a < b$ *if and only if* $a \leq b$ and $a \neq b$

(ii) b covers a *if and only if* $a < b$ and there is no element $x \in P$ such that $a \leq x \leq b$ (Also we say that a is covered by b). In this case we write $a \prec b$

Hasse Diagram: Every poset can be represented by means of a diagram known as a 'Hasse diagram' or 'poset diagram' of the poset. In this diagram, we have the following.

(i) Every element of the poset represented by a small circle or a dot.

(ii) If $a < b$, then the circle for a is below the circle for b.

(iii) If b covers a, then a straight line is drawn between the circles of a and b.

(iv) If $a < b$ and b does not cover a, then the circles of a and b are not connected directly by a single line. However they are connected through one or more elements of the poset.

Upper and Lower bounds: Let $(P, \leq)$ be a poset and $A \subseteq P$. Then an element $x \in P$ is called

(i) an *upper bound* of A if $a \leq x$ for all $a \in A$.

(ii) a *lower bound* of A if $x \leq a$ for all $a \in A$.

Least Upper Bound (or Supremum): Let $(P, \leq)$ be a poset, and $A \subseteq P$. Then an element $x \in P$ is called a '*least upper bound'* or '*LUB*' or '*supremum'* of A if

(i) x is an upper bound of A; that is $a \leq x$ for all $a \in A$.

(ii) $x \leq y$ for every upper bound y of A;
that is, $y \in P$ and y is an upper bound of $A \Rightarrow x \leq y$

Greatest Lower Bound (or Infimum): Let $(P, \leq)$ be a poset, and $A \subseteq P$. Then an element $x \in P$ is called a '*greatest lower bound'* or '*GLB*' or '*Infimum'* of A if

(i) x is a lower bound of A; that is $x \leq a$ for all $a \in A$.

(ii) $y \leq x$ for every lower bound y of A; that is, $y \in P$ and y is a lower bound of $A \Rightarrow y \leq x$

N=1,2,3,…	A=2,3,5	Usual rel.	UBs:5,6,7…	LUB:5	LBs: 1,2	GLB: 2
N=1,2,3,…	A=2,3,5	Divides rel.	UBs:30,60,90…	LUB:30	LBs: 1	GLB: 1
N=1,2,3,…	A=2,4,12	Divides rel.	UBs:12,24,36…	LUB:12	LBs: 1,2	GLB: 2
N=1,2,3,…	A=3,4,12	Divides rel.	UBs:12,24,36…	LUB:12	LBs: 1	GLB: 1
P=1,2,3,4,5	A=2,3,5	Divides rel.	UBs: no	LUB:no	LBs: 1	GLB: 1

Note: Let $(P, \leq)$ be a poset, and $A \subseteq P$.

i) Upper bounds (or Lower bounds) of A may not exist, and if exist, they may not belong to A

ii) Supremum (or Infimum) of A may not exists, and if exists, it may not belongs to A

iii) If supremum (or infimum) of A exists, then it is unique and is denoted by 'Sup A' or 'lub A' (or 'Inf A' or 'glb A')

iv) If $A = \{a,b\}$, and supremum of A exists, then it is denoted by $\text{Sup}\{a,b\}$ or $\text{lub}\{a,b\}$ or $a \vee b$ or $a \oplus b$ or some times $a+b$.

v) If $A = \{a,b\}$, and infimum of A exists, then it is denoted by $\text{Inf}\{a,b\}$ or $\text{glb}\{a,b\}$ or $a \wedge b$ or $a * b$ or some times $a \cdot b$.

Special elements: Let $(P, \leq)$ be a poset, and $A \subseteq P$. Then an element $x \in P$ is called

i) a '*greatest element*' of A if $x \in A$, and $a \leq x$ for all $a \in A$

ii) a '*least element*' of A if $x \in A$, and $x \leq a$ for all $a \in A$

iii) a '*maximal element*' of A if $x \in A$, and there is no element $y \in A$ such that $x < y$

iv) a '*minimal element*' of A if $x \in A$, and there is no element $y \in A$ such that $y < x$

N=1,2,3,…	A=3,4,6	Usual rel.	Greatest ele: 6	Least ele: 3	Maximal:6	Minimal: 3
N=1,2,3,…	A=3,4,6	Divides rel.	Greatest ele: no	Least ele: no	Maximal:4,6	Minimal:3,4
N=1,2,3,…	A=2,4,6	Divides rel.	Greatest ele: no	Least ele: 2	Maximal:4,6	Minimal:2
N=1,2,3,…	A=3,4,12	Divides rel.	Greatest ele: 12	Least ele: no	Maximal:12	Minimal:3,4

Note: Let $(P, \leq)$ be a poset, and $A \subseteq P$.

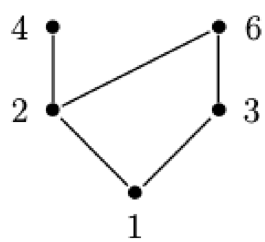
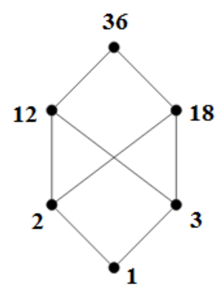
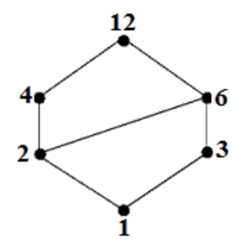
i) Greatest element (or Least element) of A may not exists, and if exists then it is unique and belongs to A

ii) Maximal element (or Minimal element) of A may not exists, and if exists then it may not be unique but belongs to A

iii) If maximal element (or minimal element) of A is exists and unique, then it becomes greatest element (or least element) of A

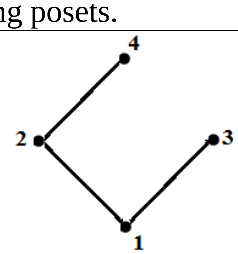
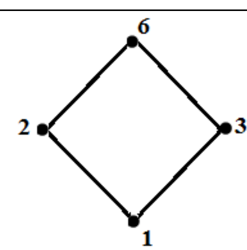
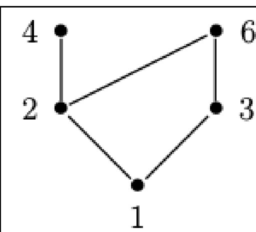
iv) If Supremum (or Infimum) of A exists, and belongs to A then it becomes greatest element (or least element) of A

v) Distinct maximal (or minimal) elements are incomparable

	Example:1	**Example:2**	**Example:3**
			
Ubs of 2,3	6	12,18,36	6,12
Lub of 2,3	6	no	6
Lbs of 2,3	1	1	1
Glb of 2,3	1	1	1
ubs	4,6: no	2,18: 18,36	3,4: 12
lub	4,6: no	2,18: 18	3,4: 12

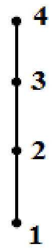
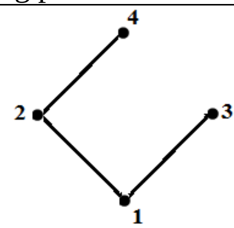
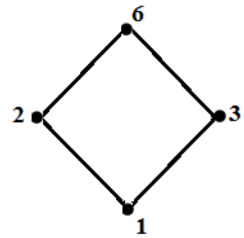
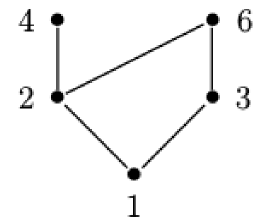
Bounded Poset: A poset $(P, \leq)$ is called a *bounded poset* if P have both least and greatest elements. The least and the greatest elements of P are respectively denoted by the symbols 0 and 1.

Example: Consider the following posets.

			
Usual relation **Bounded** $0=1$ and $1=4$	Divides relation **Not bounded**	Divides relation **Bounded** $0=1$ and $1=6$	Divides relation **Not bounded**

Lattice: A poset $(L, \le)$ is called a *lattice* if every pair of elements of L has both least upper bound (supremum) and greatest lower bound (infimum). That is, for $a, b \in L$ both $\text{lub}\{a,b\}$ and $\text{glb}\{a,b\}$ exist (or in other words, both $a \vee b$ and $a \wedge b$ exist).

Example: Consider the following posets.

			
Lattice	Not a Lattice	Lattice	Not a Lattice

Properties of a Lattice: Let $(L, \le)$ be a lattice, and $a, b, c \in L$. Then we have

i) Idempotent property: $a \wedge a = a,$ $\quad a \vee a = a$

ii) Commutative property: $a \wedge b = b \wedge a,$ $\quad a \vee b = b \vee a$

iii) Associative property: $a \wedge (b \wedge c) = (a \wedge b) \wedge c,$ $\quad a \vee (b \vee c) = (a \vee b) \vee c$

iv) Absorption property: $a \wedge (a \vee b) = a,$ $\quad a \vee (a \wedge b) = a$

Note: In the above properties, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

Bounded lattice: A lattice $(L, \le)$ is called a *bounded lattice* if L have both least and greatest elements. The least and the greatest elements of L are respectively denoted by the symbols 0 and 1.

Properties of a bounded lattice: Let $(L, \le)$ be a bounded lattice with least and greatest elements 0 and 1 respectively. Then in addition to the above properties, we have the following. For $a \in L$,

i) Identity property: $a \wedge 1 = a,$ $\quad a \vee 0 = a$

ii) Zero property: $a \wedge 0 = 0,$ $\quad a \vee 1 = 1$

Note: In the above properties, if we use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively, then we have the following.

i) Identity property: $a * 1 = a,$ $\quad a \oplus 0 = a$

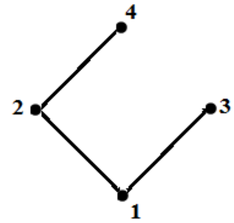
ii) Zero property: $a * 0 = 0,$ $\quad a \oplus 1 = 1$

Example:

1. The set $N = \{1, 2, 3, \cdots\}$ of all natural numbers is a lattice with respect to 'less than or equal to' relation
   If we take two natural numbers, then the bigger number becomes supremum and smaller number becomes infimum. Therefore, every pair of natural numbers has both supremum and infimum. Hence it is a lattice.

2. The set $N = \{1, 2, 3, \cdots\}$ of all natural numbers is a lattice with respect to 'divides' relation
   If we take two natural numbers, then their LCM becomes supremum and GCD becomes infimum. That is, for $a, b \in N$, we have $a \vee b = \text{LCM of } a, b$ and $a \wedge b = \text{GCD of } a, b$. Therefore, every pair of elements of N has both supremum and infimum. Hence it is a lattice.

3. Let $P=\{1,2,3,4\}$ and $\leq$ be the 'divides' relation on P.
Consider the Hasse diagram of the poset $(P,\leq)$



Here

$Sup\{1,2\}=2$ $\quad$ $Inf\{1,2\}=1$ $\quad$ $Sup\{1,1\}=1$ $\quad$ $Inf\{1,1\}=1$

$Sup\{1,3\}=3$ $\quad$ $Inf\{1,3\}=1$ $\quad$ $Sup\{2,2\}=2$ $\quad$ $Inf\{2,2\}=2$

$Sup\{1,4\}=4$ $\quad$ $Inf\{1,4\}=1$ $\quad$ $Sup\{3,3\}=3$ $\quad$ $Inf\{3,3\}=3$

$Sup\{2,3\}$ *not exists* $\quad$ $Inf\{2,3\}=1$ $\quad$ $Sup\{4,4\}=4$ $\quad$ $Inf\{4,4\}=4$

$Sup\{2,4\}=4$ $\quad$ $Inf\{2,4\}=2$

$Sup\{3,4\}$ *not exists* $\quad$ $Inf\{3,4\}=1$

We can tabulate the above as follows.
(The following tables are called composition tables for $\vee$ and $\wedge$)

$\vee$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	2	-	4
3	3	-	3	-
4	4	4	-	4

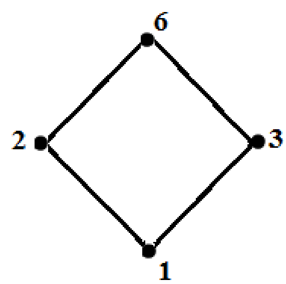
$\wedge$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	2	1	2
3	1	1	3	-
4	1	2	-	4

Since $Sup\{2,3\}$ does not exist, $(P,\leq)$ is not a lattice.

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

(ii) With respect to the 'divides' relation, $a\vee b=\text{LCM of } a, b$ and $a\wedge b=\text{GCD of } a, b$

4. Let $L=\{1,2,3,6\}$ and $\leq$ be the 'divides' relation on P.
Consider the Hasse diagram of the poset $(L,\leq)$



Here

$Sup\{1,2\}=2$ $\quad$ $Inf\{1,2\}=1$ $\quad$ $Sup\{1,1\}=1$ $\quad$ $Inf\{1,1\}=1$

$Sup\{1,3\}=3$ $\quad$ $Inf\{1,3\}=1$ $\quad$ $Sup\{2,2\}=2$ $\quad$ $Inf\{2,2\}=2$

$Sup\{1,6\}=6$ $\quad$ $Inf\{1,6\}=1$ $\quad$ $Sup\{3,3\}=3$ $\quad$ $Inf\{3,3\}=3$

$Sup\{2,3\}=6$ $\quad$ $Inf\{2,3\}=1$ $\quad$ $Sup\{6,6\}=6$ $\quad$ $Inf\{6,6\}=6$

$Sup\{2,6\}=6$ $\quad$ $Inf\{2,6\}=2$

$Sup\{3,6\}=6$ $\quad$ $Inf\{3,6\}=3$

We can tabulate the above as follows. (The composition tables for $\vee$ and $\wedge$)

$\vee$	**1**	**2**	**3**	**6**
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

$\wedge$	**1**	**2**	**3**	**6**
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Since every pair of elements of L has both least upper bound (supremum) and greatest lower bound (infimum), $(L, \leq)$ is a lattice.

Also, L has the least element 1 and greatest element 6

Therefore, $(L, \leq)$ is bounded and hence it is bounded lattice.

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

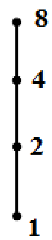
(ii) With respect to the 'divides' relation, $a \vee b = \text{LCM of } a, b$ and $a \wedge b = \text{GCD of } a, b$

5. Construct the composition tables for $\vee$ and $\wedge$ in the poset $L = \{1, 2, 4, 8\}$ with respect to the 'divides' relation and hence show that it is a lattice.

   Or

   Show that $L = \{1, 2, 4, 8\}$ is a lattice with respect to the 'divides' relation

   Consider the Hasse diagram of the poset $(L, \leq)$



The composition tables for $\vee$ and $\wedge$.

$\vee$	**1**	**2**	**4**	**8**
1	1	2	4	8
2	2	2	4	8
4	4	4	4	8
8	8	8	8	8

$\wedge$	**1**	**2**	**4**	**8**
1	1	1	1	1
2	1	2	2	2
4	1	2	4	4
8	1	2	4	8

Since every pair of elements of L has both least upper bound (supremum) and greatest lower bound (infimum), $(L, \leq)$ is a lattice.

Also, L has the least element 1 and greatest element 8

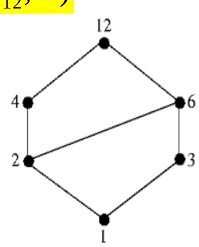
Therefore, $(L, \leq)$ is bounded and hence it is bounded lattice.

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

(ii) With respect to the 'divides' relation, $a \vee b = \text{LCM of } a, b$ and $a \wedge b = \text{GCD of } a, b$

6. Show that $D_{12} = \{1, 2, 3, 4, 6, 12\}$, the divisors of 12, is a lattice with respect to the 'divides' relation

   Consider the Hasse diagram of the poset $(D_{12}, \leq)$



The composition tables for $\vee$ and $\wedge$.

$\vee$	**1**	**2**	**3**	**4**	**6**	**12**
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

$\wedge$	**1**	**2**	**3**	**4**	**6**	**12**
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

Since every pair of elements of D_{12} has both least upper bound (supremum) and greatest lower bound (infimum), $(D_{12}, \leq)$ is a lattice.

Also, D_{12} has the least element 1 and greatest element 12

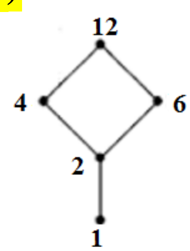
Therefore, $(D_{12}, \leq)$ is bounded and hence it is bounded lattice.

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

(ii) With respect to the 'divides' relation, $a \vee b = \text{LCM of } a, b$ and $a \wedge b = \text{GCD of } a, b$

7. Show that $L = \{1, 2, 4, 6, 12\}$ is a lattice with respect to the 'divides' relation

Consider the Hasse diagram of the poset $(L, \leq)$



The composition tables for $\vee$ and $\wedge$.

$\vee$	**1**	**2**	**4**	**6**	**12**
1	1	2	4	6	12
2	2	2	4	6	12
4	4	4	4	12	12
6	6	6	12	6	12
12	12	12	12	12	12

$\wedge$	**1**	**2**	**4**	**6**	**12**
1	1	1	1	1	1
2	1	2	2	2	2
4	1	2	4	2	4
6	1	2	2	6	6
12	1	2	4	6	12

Since every pair of elements of L has both least upper bound (supremum) and greatest lower bound (infimum), $(L, \leq)$ is a lattice.

Also, L has the least element 1 and greatest element 12

Therefore, $(L, \leq)$ is bounded and hence it is bounded lattice.

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.(ii) With respect to the 'divides' relation, $a \vee b = \text{LCM of } a, b$ and $a \wedge b = \text{GCD of } a, b$

8. **Theorem 1**: Let $(L, \leq)$ be a lattice and $a, b \in L$. Prove that the following.

(i) $a \leq b \Leftrightarrow a \wedge b = a$

(ii) $a \leq b \Leftrightarrow a \vee b = b$

(i) Suppose that $a \leq b$.

Since $a \leq b$ and $a \leq a$, we have a is a lower bound of a and b

Therefore, $a \le a \wedge b$ $\cdots\cdots\cdots$ (1) (since $a \wedge b$ is greatest lower bound of a and b)
Since $a \wedge b$ is a lower bound of a and b, we have $a \wedge b \le a$ $\cdots\cdots\cdots$ (2)
Therefore from (1) and (2), we have $a \wedge b = a$

Conversely suppose that $a \wedge b = a$.
By the definition of $a \wedge b$, we have $a \wedge b \le b$
Therefore, $a \le b$

(ii) Suppose that $a \le b$.
Since $a \le b$ and $b \le b$, we have b is an upper bound of a and b
Therefore, $a \vee b \le b$ $\cdots\cdots\cdots$ (1) (since $a \vee b$ is least upper bound of a and b)
Since $a \vee b$ is an upper bound of a and b, we have $b \le a \vee b$ $\cdots\cdots\cdots$ (2)
Therefore from (1) and (2), we have $a \vee b = b$

Conversely suppose that $a \vee b = b$.
By the definition of $a \vee b$, we have $a \le a \vee b$
Therefore, $a \le b$

9. **Theorem 2**: Let $(L, \le)$ be a lattice and $a, b, c \in L$. Prove that the *isotonic properties*.
(i) $b \le c \Rightarrow a \wedge b \le a \wedge c$ or $x \le y \Rightarrow a \wedge x \le a \wedge y$
(ii) $b \le c \Rightarrow a \vee b \le a \vee c$ or $x \le y \Rightarrow a \vee x \le a \vee y$

(i) Suppose that $b \le c$. Then $b \wedge c = b$ and $b \vee c = c$
We have to show that $a \wedge b \le a \wedge c$
Consider $(a \wedge b) \wedge (a \wedge c) = (a \wedge a) \wedge (b \wedge c)$
$= a \wedge b$
Therefore, $a \wedge b \le a \wedge c$

(ii) Suppose that $b \le c$. Then $b \wedge c = b$ and $b \vee c = c$
We have to show that $a \vee b \le a \vee c$
Consider $(a \vee b) \vee (a \vee c) = (a \vee a) \vee (b \vee c)$
$= a \vee c$
Therefore, $a \vee b \le a \vee c$

10. **Theorem 3**: Let $(L, \le)$ be a lattice and $a, b, c \in L$. Prove that the *distributive inequalities*.
(i) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$ or $a \oplus (b * c) \le (a \oplus b) * (a \oplus c)$
(ii) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$ or $a * (b \oplus c) \ge (a * b) \oplus (a * c)$

(i) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$
By the definition of $a \vee b$ and $a \vee c$, we have $a \le a \vee b$ and $a \le a \vee c$
That is, a is a lower bound of $a \vee b$ and $a \vee c$
Therefore, $a \le (a \vee b) \wedge (a \vee c)$ $\cdots\cdots\cdots$ (1)
(since $(a \vee b) \wedge (a \vee c)$ is greatest lower bound of $a \vee b$ and $a \vee c$)

By the definition of $b \wedge c$, we have $b \wedge c \le b$ and $b \wedge c \le c$
Also by the definition of $a \vee b$ and $a \vee c$, we have $b \le a \vee b$ and $c \le a \vee c$

Therefore, $b\wedge c\le a\vee b$ and $b\wedge c\le a\vee c$

That is, $b\wedge c$ is a lower bound of $a\vee b$ and $a\vee c$

Therefore, $b\wedge c\le (a\vee b)\wedge(a\vee c)$ $\cdots\cdots\cdots$ (2)

(since $(a\vee b)\wedge(a\vee c)$ is greatest lower bound of $a\vee b$ and $a\vee c$)

From (1) and (2), we have $(a\vee b)\wedge(a\vee c)$ is an upper bound of a and $b\wedge c$

Therefore, $a\vee(b\wedge c)\le(a\vee b)\wedge(a\vee c)$ (since $a\vee(b\wedge c)$ is least upper bound of a and $b\wedge c$)

(ii) $a\wedge(b\vee c)\ge(a\wedge b)\vee(a\wedge c)$

By the definition of $a\wedge b$ and $a\wedge c$, we have $a\wedge b\le a$ and $a\wedge c\le a$

That is, a is an upper bound of $a\wedge b$ and $a\wedge c$

Therefore, $(a\wedge b)\vee(a\wedge c)\le a$ $\cdots\cdots\cdots$ (1)

(since $(a\wedge b)\vee(a\wedge c)$ is least upper bound of $a\wedge b$ and $a\wedge c$)

By the definition of $b\vee c$, we have $b\le b\vee c$ and $c\le b\vee c$

Also by the definition of $a\wedge b$ and $a\wedge c$, we have $a\wedge b\le b$ and $a\wedge c\le c$

Therefore, $a\wedge b\le b\vee c$ and $a\wedge c\le b\vee c$

That is, $b\vee c$ is an upper bound of $a\wedge b$ and $a\wedge c$

Therefore, $(a\wedge b)\vee(a\wedge c)\le b\vee c$ $\cdots\cdots\cdots$ (2)

(since $(a\wedge b)\vee(a\wedge c)$ is least upper bound of $a\wedge b$ and $a\wedge c$)

From (1) and (2), we have $(a\wedge b)\vee(a\wedge c)$ is a lower bound of a and $b\vee c$

Therefore, $(a\wedge b)\vee(a\wedge c)\le a\wedge(b\vee c)$ (since $a\wedge(b\vee c)$ is greatest lower bound of a and $b\vee c$)

11. **Theorem 4**: Let $(L,\le)$ be a lattice and $a,b,c\in L$. Prove that the *Modular inequality* $a\le c\Leftrightarrow a\vee(b\wedge c)\le(a\vee b)\wedge c$

Suppose that $a\le c$. Then $a\vee c=c$ and $a\wedge c=a$

By distributive inequality, $a\vee(b\wedge c)\le(a\vee b)\wedge(a\vee c)$

$=(a\vee b)\wedge c$

Therefore, $a\vee(b\wedge c)\le(a\vee b)\wedge c$

Conversely suppose that $a\vee(b\wedge c)\le(a\vee b)\wedge c$

But we know that $a\le a\vee(b\wedge c)$ and $(a\vee b)\wedge c\le c$

Therefore, $a\le c$

12. **Theorem 5**: Show that in a lattice if $a\le b$ and $c\le d$, then (i) $a\wedge c\le b\wedge d$ (ii) $a\vee c\le b\vee d$

(i) Suppose that $a\le b$ and $c\le d$. Then $a\wedge b=a$ and $c\wedge d=c$

Consider $(a\wedge c)\wedge(b\wedge d)=(a\wedge b)\wedge(c\wedge d)$

$=a\wedge c$

Therefore, $a\wedge c\le b\wedge d$

(ii) Suppose that $a\le b$ and $c\le d$. Then $a\vee b=b$ and $c\vee d=d$

Consider $(a\vee c)\vee(b\vee d)=(a\vee b)\vee(c\vee d)$

$=b\vee d$

Therefore, $a\vee c\le b\vee d$

13. Let $(L, \leq)$ be a lattice, $*$ and $\oplus$ be two operations such that $a * b = \text{glb}\{a,b\}$ and $a \oplus b = \text{lub}\{a,b\}$. Prove that both $*$ and $\oplus$ satisfy commutative law, associative law, absorption law and idempotent law.

Commutative law: $a * b = \text{glb}\{a,b\} = \text{glb}\{b,a\} = b * a$

Associative law: $a*(b*c) = \text{glb}\{a, b*c\} = \text{glb}\{a, \text{glb}\{b,c\}\} = \text{glb}\{a,b,c\}$ and

$(a*b)*c = \text{glb}\{a*b, c\} = \text{glb}\{\text{glb}\{a,b\}, c\} = \text{glb}\{a,b,c\}$

Therefore, $a*(b*c) = (a*b)*c$

Absorption law: $a*(a \oplus b) = \text{glb}\{a, a \oplus b\} = \text{glb}\{a, \text{lub}\{a,b\}\} = a \quad (\because a \leq \text{lub}\{a,b\})$

Idempotent law: $a * a = \text{glb}\{a,a\} = a$

Similarly, we can prove the above laws with $\oplus$

Special Lattices:

Lattice (as algebraic system): An algebraic system $(L, \wedge, \vee)$ is called a *lattice* if it satisfies the following. For $a,b,c \in L$,

i) Idempotent property: $a \wedge a = a$, $a \vee a = a$

ii) Commutative property: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$

iii) Associative property: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$

iv) Absorption property: $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$

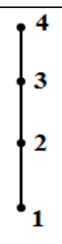
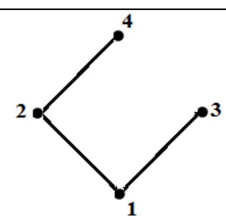
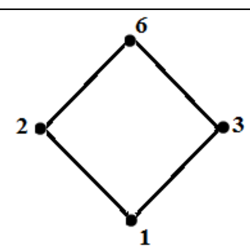
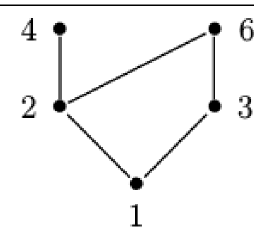
Bounded lattice: A lattice $(L, \wedge, \vee)$ is called a *bounded lattice* if there exist least and greatest elements 0 and 1 respectively. In this case, we say that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice.

Properties of a bounded lattice: Let $(L, \leq)$ be a bounded lattice with least and greatest elements 0 and 1 respectively. Then in addition to the above properties, we have the following. For $a \in L$,

i) Bounds property: $0 \leq a \leq 1$

ii) Identity property: $a \wedge 1 = a$, $a \vee 0 = a$

iii) Zero property: $a \wedge 0 = 0$, $a \vee 1 = 1$

				$N = \{1,2,3,\cdots\}$ of all natural numbers, with respect to the relations (i) less than or equal to (ii) divides
$L = \{1,2,3,4\}$ Usual order	$L = \{1,2,3,4\}$ Divides relation	$D_6 = \{1,2,3,6\}$ Divides relation	$L = \{1,2,3,4,6\}$ Divides relation	
Bounded Lattice	Not a Lattice	Bounded Lattice	Not a Lattice	Lattice but not bounded

Complete lattice: A lattice $(L, \wedge, \vee)$ is called a *complete lattice* if every non empty subset of L has both least upper bound (supremum) and greatest lower bound (infimum). That is, for $A \subseteq L$ both '$\text{lub } A$' and '$\text{glb } A$' exist.

Complement of an element: Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $a, b \in L$. Then we say that b is a *complement* of a if $a \wedge b = 0$ and $a \vee b = 1$

Note:

i) a is a complement of b $\Leftrightarrow$ b is a complement of a

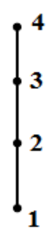
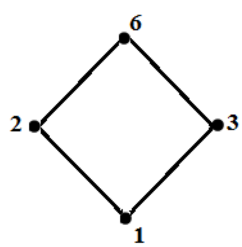
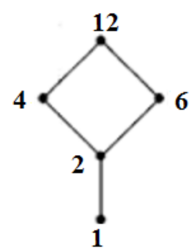
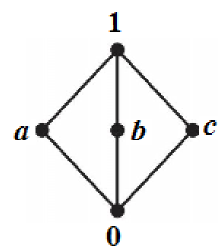
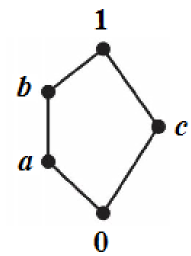
ii) In general an element need not have a complement. If an element has a complement, then it may not be unique.

iii) If complement of an element a is unique, then it is denoted by a'

iv) 0 and 1 are complements to each other (since $0 \wedge 1 = 0$ and $0 \vee 1 = 1$); that is, $0' = 1$ and $1' = 0$

Complemented lattice: A bounded lattice $(L, \wedge, \vee, 0, 1)$ is called a *complemented lattice* if every element of L has at least one complement.

Examples:

				
$L = \{1,2,3,4\}$ Usual order	$D_6 = \{1,2,3,6\}$ Divides relation	$L = \{1,2,4,6,12\}$ Divides relation	$L = \{0,1,a,b,c\}$	$L = \{0,1,a,b,c\}$
Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice
$1' = 4,\ 4' = 1$ 2 has no complement 3 has no complement	$1' = 6,\ 6' = 1,$ $2' = 3,\ 3' = 2$	$1' = 12,\ 12' = 1$ 2,4,6 have no complements	$0' = 1, 1' = 0$ $a' = b, c$ $b' = a, c$ $c' = a, b$	$0' = 1, 1' = 0$ $a' = c$ $b' = c$ $c' = a, b$
Not complemented	Complemented	Not complemented	Complemented	Complemented

Distributive lattice: A lattice $(L, \wedge, \vee)$ is called a *distributive lattice* if it satisfies the following distributive properties.

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

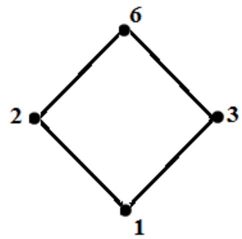
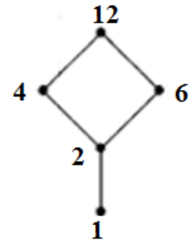
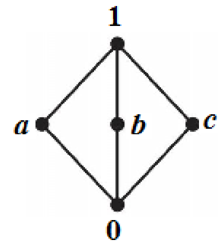
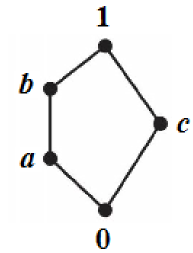
ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Note: In the above definition, (i) and (ii) are equivalent and hence one of the conditions is sufficient for verification of distributive property of a lattice.

Bounded distributive lattice: A distributive lattice $(L, \wedge, \vee)$ is called a *bounded distributive lattice* if there exist least and greatest elements 0 and 1 respectively. In this case, we say that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.

Note: Complement of an element (if exists) of a bounded distributive lattice is unique

Examples:

				
$L = \{1,2,3,4\}$ Usual order	$D_6 = \{1,2,3,6\}$ Divides relation	$L = \{1,2,4,6,12\}$ Divides relation	$L = \{0,1,a,b,c\}$	$L = \{0,1,a,b,c\}$
Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice
Distributive	Distributive	Distributive	Not Distributive	Not Distributive

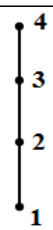
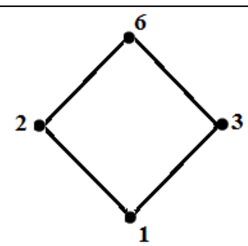
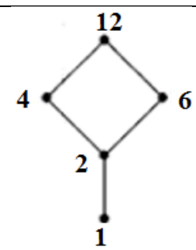
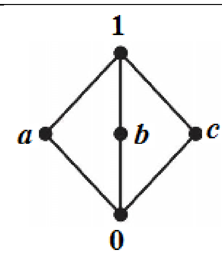
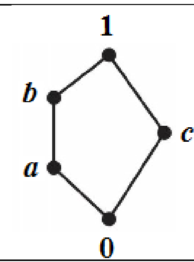
Boolean algebra: A complemented distributive lattice is called a *Boolean algebra*.
(OR) A bounded distributive lattice $(B, \wedge, \vee, 0, 1)$ is called a *Boolean algebra* if every element of B has a complement.

Note:

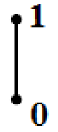
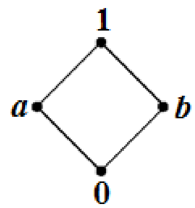
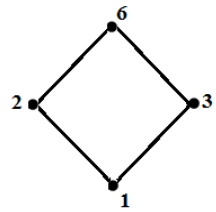
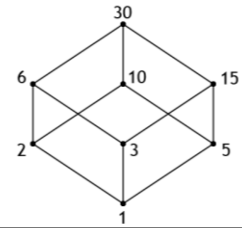
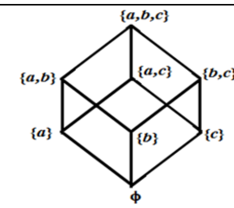
i) In a Boolean algebra $(B, \wedge, \vee, 0, 1)$, complement of every element is unique and the complement of an element a is denoted by a'. In this case we say that $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra.

ii) Every finite Boolean algebra contains 2^n elements.

Examples:

				
$L = \{1,2,3,4\}$ Usual order	$D_6 = \{1,2,3,6\}$ Divides relation	$L = \{1,2,4,6,12\}$ Divides relation	$L = \{0,1,a,b,c\}$	$L = \{0,1,a,b,c\}$
Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice
Not complemented	Complemented	Not complemented	Complemented	Complemented
Distributive	Distributive	Distributive	Not Distributive	Not Distributive
Not Boolean algebra	Boolean algebra	Not Boolean algebra	Not Boolean algebra	Not Boolean algebra

Examples: Standard Boolean algebras

2 elements	2^2 elements	2^2 elements	2^3 elements	2^3 elements
				
$B=\{0,1\}$ Any relation	$B=\{0,a,b,1\}$ Above relation	$D_6=\{1,2,3,6\}$ Divides relation	$D_{30}=\{1,2,3,5,6,10,15,30\}$ Divides relation	$B=P(X), X=\{a,b,c\}$ Subset relation
Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice	Bounded Lattice
Complemented	Complemented	Complemented	Complemented	Complemented
Distributive	Distributive	Distributive	Distributive	Distributive
Boolean algebra	Boolean algebra	Boolean algebra	Boolean algebra	Boolean algebra

Examples:

1. $B=\{0,1\}$ is a 2 element Boolean algebra with respect to usual order
2. $B=\{0,1\}$ is a 2 element Boolean algebra with respect to usual order. Then $B^n=\{\,(a_1,a_2,\cdots a_n)\,|\,a_i\in B,\ i=1,2,3,\cdots n\,\}$ is a Boolean algebra.
3. Let X be a non empty set and $P(X)$ be the power set of X. Then $(P(X),\cap,\cup,c,\phi,X)$ is a Boolean algebra.
   Here for $A,B\in P(X)$,
   $\inf\{A,B\}=A\cap B$, $\sup\{A,B\}=A\cup B$,
   least element $0=\phi$, greatest element $1=X$
   And the complement of A is A^c
4. Let S be the set of all statement formulas involving n primary variables. Then $(S,\wedge,\vee,\neg,F,T)$ is a Boolean algebra.
   Here for $P,Q\in S$,
   $\inf\{P,Q\}=P\wedge Q$, $\sup\{P,Q\}=P\vee Q$,
   least element $0=F$ contradict ion, greatest element $1=T$ tautology
   And the complement of P is $\neg P$

Properties of a Boolean algebra: If $(B,\wedge,\vee,{}',0,1)$ is a Boolean algebra, then

(1) $(B,\wedge,\vee)$ is a lattice and satisfy the following properties.

- i) Idempotent property: $a\wedge a=a,$ $a\vee a=a$
- ii) Commutative property: $a\wedge b=b\wedge a,$ $a\vee b=b\vee a$
- iii) Associative property: $a\wedge(b\wedge c)=(a\wedge b)\wedge c,$ $a\vee(b\vee c)=(a\vee b)\vee c$
- iv) Absorption property: $a\wedge(a\vee b)=a,$ $a\vee(a\wedge b)=a$

(2) $(B,\wedge,\vee)$ is a distributive lattice and satisfy the following properties.

- i) $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ $a(b+c)=ab+ac$
- ii) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$ $a+(bc)=(a+b)(a+c)$ $a\le a\vee b,\ a\wedge b\le a$
- iii) $a\wedge b=a\wedge c$ and $a\vee b=a\vee c \Rightarrow b=c$
- iv) $(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)=(a\vee b)\wedge(b\vee c)\wedge(c\vee a)$

(3) $(B, \wedge, \vee, 0, 1)$ is a bounded lattice and satisfy the following properties.

i) Bounds property: $0 \le a \le 1$

ii) Identity property: $a \wedge 1 = a$, $a \vee 0 = a$

iii) Zero property: $a \wedge 0 = 0$, $a \vee 1 = 1$

(4) $(B, \wedge, \vee, ', 0, 1)$ is a complemented lattice and satisfy the following properties.

i) Complement property: $0' = 1$, $1' = 0$

ii) Complement property: $a \wedge a' = 0$, $a \vee a' = 1$

iii) De-Morgan's property: $(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$

(5) $(B, \le)$ is a poset and satisfy the following properties.

i) $a \wedge b = \text{glb}\{a, b\}$, $a \vee b = \text{lub}\{a, b\}$

ii) $a \le b \Leftrightarrow a \wedge b = a$, $a \le b \Leftrightarrow a \vee b = b$

iii) $a \le b \Leftrightarrow b' \le a'$

iv) $a \le b \Leftrightarrow a \wedge b' = 0$ $a \le b \Leftrightarrow a' \vee b = 1$

Note: In the above properties we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively. Also we can use $\cdot$ and $+$ in place $\wedge$ and $\vee$ respectively.

Exercise:

1. Write all the properties of a Boolean algebra with respect to the operations $*$ and $\oplus$
2. Write all the properties of a Boolean algebra with respect to the operations $\cdot$ and $+$

Important Definitions		
1	Partial order $\le$	$\le$ is reflexive, anti symmetric, transitive
2	Poset $(P, \le)$	$P \ne \phi$, $\le$ is a partial order
3	Bounded Poset $(P, \le)$	$(P, \le)$ is a poset, P has least & greatest elements 0 & 1
4	Lattice $(L, \le)$	$(L, \le)$ is a poset, Every pair of elements of L has supremum & infimum
5	Complete Lattice $(L, \le)$	$(L, \le)$ is a lattice, Every non empty subset of L has supremum & infimum
6	Bounded Lattice $(L, \le)$	$(L, \le)$ is a lattice, L has least & greatest elements 0 & 1
7	Lattice $(L, \wedge, \vee)$	$(L, \wedge, \vee)$ is an algebraic system and satisfy idempotent, commutative, associative and absorption properties
8	Bounded Lattice $(L, \wedge, \vee, 0, 1)$	$(L, \wedge, \vee)$ is a lattice, L has least & greatest elements 0 & 1
9	Complemented Lattice $(L, \wedge, \vee, 0, 1)$	$(L, \wedge, \vee, 0, 1)$ is a bounded lattice, Every element of L has a complement
10	Distributive Lattice $(L, \wedge, \vee)$	$(L, \wedge, \vee)$ is a lattice, and satisfy distributive property
11	Bounded Distributive Lattice $(L, \wedge, \vee, 0, 1)$	$(L, \wedge, \vee)$ is a distributive lattice, L has least & greatest elements 0 & 1
12	Complemented Bounded Distributive Lattice or Boolean algebra $(B, \wedge, \vee, ', 0, 1)$	$(B, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, Every element of L has a complement

Problems:

1. Show that in a bounded lattice with two or more elements, no element is its own complement

   Let $(L,\leq)$ be a lattice with two or more elements. Let $a \in L$ and $a \neq 0$.
   Suppose that a has its own complement, that is $a' = a$.
   Then we have $a = a \wedge a = a \wedge a' = 0$, which is a contradiction.
   Therefore, no element has its own complement.

2. Every chain is a lattice

   Let $(L,\leq)$ be a Chain and $a, b \in L$.
   Then we have $a \leq b$ or $b \leq a$ (since L is a chain)
   If $a \leq b$, then $a \vee b = b$ and $a \wedge b = a$
   If $b \leq a$, then $a \vee b = a$ and $a \wedge b = b$
   Therefore, in either case every pair of elements of L has both supremum and infimum
   Hence $(L,\leq)$ is lattice

3. Every bounded chain with more than two elements is a lattice but not complemented

   Let $(L,\leq)$ be a bounded chain with more than two elements.
   Let $a \in L$ and $a \neq 0$, $a \neq 1$
   Suppose that a has a complement b, then $a \wedge b = 0$, $a \vee b = 1$
   Since L is a chain, we have $a \leq b$ or $b \leq a$
   If $a \leq b$, then $0 = a \wedge b = a$, a contradiction
   If $b \leq a$, then $1 = a \vee b = a$, a contradiction
   Therefore, a does not have a complement and hence L is not complemented.

4. **Theorem 6**: Every chain is a distributive lattice

   Proof: Let $(L,\leq)$ be a Chain and $a, b, c \in L$. Then we have $a \leq b$ or $b \leq a$ (since L is a chain)

   Case (i): Let $a \leq b$. Then $a \leq b \leq b \vee c$
   Now $a \wedge (b \vee c) = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee (a \wedge c) = a$
   $\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   Case (ii): Let $b \leq a$. Here we have two sub cases $a \leq c$ or $c \leq a$ (since L is a chain)

   If $a \leq c$, then $a \leq c \leq b \vee c$
   Now $a \wedge (b \vee c) = a$ and $(a \wedge b) \vee (a \wedge c) = b \vee a = a$
   $\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   If $c \leq a$, then $b \vee c \leq a$ (since $b \leq a$, $c \leq a$ and $b \vee c$ is the least upper bound b and c)
   Now $a \wedge (b \vee c) = b \vee c$ and $(a \wedge b) \vee (a \wedge c) = b \vee c$
   $\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
   Therefore, in either case we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and hence $(L,\leq)$ is a distributive lattice

5. **Theorem 7**: In a distributive lattice, prove the following. $a \wedge b = a \wedge c$ and $a \vee b = a \vee c \Rightarrow b = c$

Proof: Let $(L, \wedge, \vee)$ be a distributive lattice and $a, b, c \in L$.

Suppose that $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$

Consider $b = b \wedge (b \vee a)$ (Absorption property)

$= b \wedge (a \vee c)$ (Since, $a \vee b = a \vee c$)

$= (b \wedge a) \vee (b \wedge c)$ (Distributive property)

$= (a \wedge c) \vee (b \wedge c)$ (Since, $a \wedge b = a \wedge c$)

$= c \wedge (a \vee b)$ (Distributive property)

$= c \wedge (a \vee c)$ (Since, $a \vee b = a \vee c$)

$= c$ (Absorption property)

6. In a distributive lattice, prove that $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Solution: We prove that $ab + bc + ca = (a+b)(b+c)(c+a)$

Consider $(a+b)(b+c)(c+a) = abc + aba + acc + aca + bbc + bba + bcc + bca$

$= abc + bca + aba + acc + aca + bbc + bba + bcc$

$= abc + ab + ac + ac + bc + ba + bc$

$= abc + ab + ac + bc$

$= ab + ac + bc$

7. Prove that the complement of an element (if exists) of a bounded distributive lattice is unique.

Solution: Let $(L, \wedge, \vee)$ be a bounded distributive lattice and $a \in L$.

Suppose that $b, c \in L$ are complements of a, then we have

$a \wedge b = 0,\ a \vee b = 1$ and $a \wedge c = 0,\ a \vee c = 1$

Therefore, $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$

Consider $b = b \wedge (b \vee a)$ (Absorption property)

$= b \wedge (a \vee c)$ (Since, $a \vee b = a \vee c$)

$= (b \wedge a) \vee (b \wedge c)$ (Distributive property)

$= (a \wedge c) \vee (b \wedge c)$ (Since, $a \wedge b = a \wedge c$)

$= c \wedge (a \vee b)$ (Distributive property)

$= c \wedge (a \vee c)$ (Since, $a \vee b = a \vee c$)

$= c$ (Absorption property)

Hence the complement of an element (if exists) is unique.

8. In a complemented distributive lattice, show that the De-Morgan's laws, given by (i) $(a \wedge b)' = a' \vee b'$ (ii) $(a \vee b)' = a' \wedge b'$

(i) $(a \wedge b)' = a' \vee b'$

Consider, $(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = (0 \wedge b) \vee (a \wedge 0) = 0 \vee 0 = 0$

And $(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = (1 \vee b') \wedge (a' \vee 1) = 1 \wedge 1 = 1$

Therefore, $a' \vee b'$ is the complement of $a \wedge b$, that is $(a \wedge b)' = a' \vee b'$

(ii) $(a \vee b)' = a' \wedge b'$

Consider, $(a \vee b) \wedge (a' \wedge b') = (a \wedge a' \wedge b') \vee (b \wedge a' \wedge b') = (0 \wedge b) \vee (a' \wedge 0) = 0 \vee 0 = 0$

And $(a \vee b) \vee (a' \wedge b') = (a \vee b \vee a') \wedge (a \vee b \vee b') = (1 \vee b) \wedge (a \vee 1) = 1 \wedge 1 = 1$

Therefore, $a' \wedge b'$ is the complement of $a \vee b$, that is $(a \vee b)' = a' \wedge b'$

9. In a complemented distributive lattice, show that (i) $a \le b \Leftrightarrow b' \le a'$ (ii) $a \le b \Leftrightarrow a \wedge b' = 0$ (iii) $a \le b \Leftrightarrow a' \vee b = 1$

We know that $a \le b \Leftrightarrow a \wedge b = a$ and $a \le b \Leftrightarrow a \vee b = b$

(i) $a \le b \Leftrightarrow a \wedge b = a \quad \Leftrightarrow (a \wedge b)' = a' \quad \Leftrightarrow a' \vee b' = a' \quad \Leftrightarrow b' \le a'$

(ii) Suppose that $a \le b$

$$\begin{aligned} a \wedge b' &= a \wedge (a \vee b)' && (\text{Since, } a \vee b = b) \\ &= a \wedge a' \wedge b' \\ &= 0 \wedge b' && (\text{Since, } a \wedge a' = 0) \\ &= 0 \end{aligned}$$

Conversely suppose that $a \wedge b' = 0$

Consider, $b = b \vee 0 \quad = b \vee (a \wedge b') \quad = (b \vee a) \wedge (b \vee b') \quad = (b \vee a) \wedge 1 \quad = b \vee a$

Therefore, $a \le b$

(iii) Suppose that $a \le b$

$$\begin{aligned} a' \vee b &= (a \wedge b)' \vee b && (\text{Since, } a \wedge b = a) \\ &= a' \vee b' \vee b \\ &= a' \vee 1 && (\text{Since, } b' \vee b = 1) \\ &= 1 \end{aligned}$$

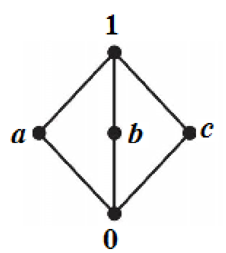
Conversely suppose that $a' \vee b = 1$

Consider, $a = a \wedge 1 \quad = a \wedge (a' \vee b) \quad = (a \wedge a') \vee (a \wedge b) \quad = 0 \vee (a \wedge b) \quad = a \wedge b$

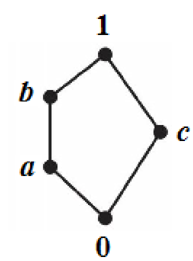
Therefore, $a \le b$

10. Show that the following lattices are not distributive

(i) L_1



(ii) L_2



Solution:

(i) Consider $a \wedge (b \vee c) = a \wedge 1 = a$ and $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

Therefore, $a \wedge (b \vee c) \ne (a \wedge b) \vee (a \wedge c)$

Hence L_1 is not distributive

(ii) Consider $b \wedge (a \vee c) = b \wedge 1 = b$ and $(b \wedge a) \vee (b \wedge c) = a \vee 0 = a$

Therefore, $b \wedge (a \vee c) \ne (b \wedge a) \vee (b \wedge c)$

Hence L_2 is not distributive

11. Prove that the following Boolean identities.

(i) $a \oplus (a' * b) = a \oplus b$ or $a + (a' \cdot b) = a + b$ or $a + a'b = a + b$

(ii) $a * (a' \oplus b) = a * b$ or $a \cdot (a' + b) = a \cdot b$ or $a(a' + b) = ab$

(iii) $(a * b) \oplus (a * b') = a$ or $ab + ab' = a$

(iv) $(a * b * c) \oplus (a * b) = a * b$ or $abc + ab = ab$

(v) $(a * b') \oplus (a' * b) = (a \oplus b) * (a' \oplus b')$ or $ab' + a'b = (a + b)(a' + b')$

Proof:

(i) $a \oplus (a' * b) = (a \oplus a') * (a \oplus b) = 1 * (a \oplus b) = a \oplus b$

(ii) $a * (a' \oplus b) = (a * a') \oplus (a * b) = 0 \oplus (a * b) = a * b$

(iii) $(a*b)\oplus(a*b')=a*(b\oplus b')=a*1=a$

(iv) $(a*b*c)\oplus(a*b)=(x*c)\oplus x=x\oplus(x*c)=x=a*b$ where $x=a*b$

(v) $(a*b')\oplus(a'*b)=(a\oplus a')*(a\oplus b)*(b'\oplus a')*(b'\oplus b)$

$$=1*(a\oplus b)*(b'\oplus a')*1$$

$$=(a\oplus b)*(a'\oplus b')$$

12. In any Boolean algebra, show that

(i) $a=b\Leftrightarrow ab'+a'b=0$

(ii) $a=0\Leftrightarrow ab'+a'b=b$

(iii) $(a+b')(b+c')(c+a')=(a'+b)(b'+c)(c'+a)$

(iv) $(a+b)(a'+c)=ac+a'b=ac+a'b+bc$

(v) $a\le b\Rightarrow a+bc=b(a+c)$

Proof:

(i) $a=b\Leftrightarrow ab'+a'b=0$

Suppose that $a=b$

Now $ab'+a'b=aa'+a'a=0+0=0$

Conversely suppose that $ab'+a'b=0$

Now $a=a+0=a+ab'+a'b=a+a'b=a+b$

and $b=b+0=b+ab'+a'b=b+a'b+ab'=b+ab'=b+a$

Therefore, $a=b$

(ii) $a=0\Leftrightarrow ab'+a'b=b$

Suppose that $a=0$

Now $ab'+a'b=0b'+1b=0+b=b$

Conversely suppose that $ab'+a'b=b$

First observe that $0=bb'=(ab'+a'b)b'=ab'b'+a'bb'=ab'$

And $ab=a(ab'+a'b)=aab'+aa'b=ab'=0$

Now $a=a1=a(b+b')=ab+ab'=0$

(iii) $(a+b')(b+c')(c+a')=(a'+b)(b'+c)(c'+a)$

Consider $(a+b')(b+c')(c+a')=(ab+ac'+b'b+b'c')(c+a')$

$$=abc+ac'c+b'c'c+aba'+ac'a'+b'c'a'$$

$$=abc+a'b'c'$$

And $(a'+b)(b'+c)(c'+a)=(a'b'+bb'+a'c+bc)(c'+a)$

$$=a'b'c'+a'cc'+bcc'+a'b'a+a'ca+bca$$

$$=a'b'c'+abc$$

Therefore, $(a+b')(b+c')(c+a')=(a'+b)(b'+c)(c'+a)$

(iv) $(a+b)(a'+c)=ac+a'b=ac+a'b+bc$

Consider $(a+b)(a'+c)=aa'+ac+ba'+bc$

$$=0+ac+ba'+bc$$

$$=ac+a'b+bc$$

Also $ac+a'b+bc=ac+a'b+bc(a+a')$ $\quad\because a+a'=1$

$$=ac+a'b+bca+bca'$$

$$=ac(1+b)+a'b(1+c)$$

$$=ac+a'b \qquad \because 1+b=1,\ 1+c=1$$

Therefore, $(a+b)(a'+c)=ac+a'b=ac+a'b+bc$

(v) $a\le b\Rightarrow a+bc=b(a+c)$

Suppose that $a\le b$, then $b(a+c)=ba+bc=a+bc$ $\quad(\because ba=a)$

Boolean Expression (or Boolean form or Boolean formula): A Boolean expression in n variables $x_1, x_2, x_3, \cdots x_n$ is defined in the following manner.

(i) 0 and 1 are Boolean expressions.

(ii) $x_1, x_2, x_3, \cdots x_n$ are Boolean expressions.

(iii) If α and β are Boolean expressions, then $\alpha \wedge \beta$ and $\alpha \vee \beta$ are also Boolean expressions.

(iv) If α is a Boolean expression, then α' is also a Boolean expression.

(v) Any expression is a Boolean expression if and only if it is obtained by finite number of steps of (i), (ii), (iii) and (iv).

Note: (i) In the above, we can use $*$ and $\oplus$ in place $\wedge$ and $\vee$ respectively.

(ii) In general Boolean expressions are denoted by $\alpha, \beta, \gamma, \cdots$

(iii) Particularly a Boolean expression α in the variables $x_1, x_2, \cdots x_n$, is denoted by $\alpha(x_1, x_2, \cdots x_n)$

Example: (i) In the variables x_1, x_2 some of the Boolean expressions are $0, 1, x_1, x_2, x_1 \wedge x_2, x_1 \vee x_2'$

(ii) $0, 1, x_1, x_2, x_1 * x_2, x_1 \oplus x_2, x_1 \oplus x_2', (x_1 * x_2) * x_1'$ are some Boolean expressions in the variables x_1, x_2.

(iii) $0, 1, x_1, x_2, x_1 \wedge x_2, x_1 \wedge x_2 \wedge x_3, x_1 \vee x_2', (x_1 \vee x_2) \wedge x_1'$ are some Boolean expressions in the variables x_1, x_2, x_3.

Value of a Boolean Expression: Let $\alpha(x_1, x_2, \cdots x_n)$ be a Boolean expression and $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Let $a_1, a_2, \cdots a_n \in B$. Then the element of B obtained from α by substituting $x_1 = a_1, x_2 = a_2, \cdots x_n = a_n$, is called a *value* of the Boolean expression $\alpha(x_1, x_2, \cdots x_n)$ and it is denoted by $\alpha(a_1, a_2, \cdots a_n)$.

Valuation process: The process of determining all the values of a Boolean expression $\alpha(x_1, x_2, \cdots x_n)$ is called a *valuation process* over a given Boolean algebra $(B, \wedge, \vee, ', 0, 1)$. In particular, if $B = \{0, 1\}$ the two element Boolean algebra, then the valuation process is called *Binary valuation process*. All the binary values can be tabulated (as in the truth table of a statement formula) known as *Binary valuation table*.

Equivalence of Boolean Expressions: Two Boolean expressions are said to be *equivalent or equal* if and only if all their binary values are equal.

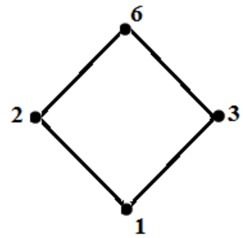
Note:

(i) If $|B| = m$ and a Boolean expression α contains n variables, then the total no. of values α is m^n.

(ii) If $B = \{0, 1\}$ and an expression α contains n variables, then the total no. of binary values α is 2^n

Example:

1. Let $B = \{1, 2, 3, 6\}$ be the Boolean algebra with respect to the divides relation.



(i) If $\alpha(x_1, x_2) = x_1 \wedge x_2$, then

$\alpha(1, 1) = 1 \wedge 1 = 1$ $\qquad$ $\alpha(1, 3) = 1 \wedge 3 = 1$ $\qquad$ $\alpha(2, 6) = 2 \wedge 6 = 2$

$\alpha(2, 3) = 2 \wedge 3 = 1$ $\qquad$ $\alpha(3, 2) = 3 \wedge 2 = 1$ $\qquad$ $\alpha(6, 3) = 6 \wedge 3 = 3$

The above are some values of the Boolean expression $\alpha(x_1, x_2) = x_1 \wedge x_2$.

Since $|B| = 4$ and α contains 2 variables, total no. of values of α is 4^2.

(ii) If $\alpha(x_1, x_2) = x_1 \oplus x_2$, then

$\alpha(1, 2) = 1 \oplus 2 = 2$ $\qquad$ $\alpha(1, 6) = 1 \oplus 6 = 6$ $\qquad$ $\alpha(2, 2) = 2 \oplus 2 = 2$

$\alpha(2, 3) = 2 \oplus 3 = 6$ $\qquad$ $\alpha(3, 6) = 3 \oplus 6 = 6$ $\qquad$ $\alpha(6, 1) = 6 \oplus 1 = 6$

The above are some values of the Boolean expression $\alpha(x_1, x_2) = x_1 \oplus x_2$.

Since $|B| = 4$ and α contains 2 variables, total no. of values of α is 4^2.

(iii) If $\alpha(x_1, x_2) = (x_1 \wedge x_2) \vee x_1'$, then

$\alpha(1, 2) = (1 \wedge 2) \vee 1' = 1 \vee 6 = 6$ $\qquad$ $\alpha(2, 3) = (2 \wedge 3) \vee 2' = 1 \vee 3 = 3$

$\alpha(3, 2) = (3 \wedge 2) \vee 3' = 1 \vee 2 = 2$ $\qquad$ $\alpha(6, 1) = (6 \wedge 1) \vee 6' = 1 \vee 1 = 1$

The above are some values of the Boolean expression $\alpha(x_1, x_2) = (x_1 \wedge x_2) \vee x_1'$.

Since $|B| = 4$ and α contains 2 variables, total no. of values of α is 4^2.

(iv) If $\alpha(x_1, x_2, x_3) = x_3 \vee (x_1 \wedge x_2')$, then

$\alpha(1, 1, 1) = 1 \vee (1 \wedge 1') = 1 \vee (1 \wedge 6) = 1 \vee 1 = 1,$ $\qquad$ $\alpha(1, 2, 3) = 3 \vee (1 \wedge 2') = 3 \vee (1 \wedge 3) = 3 \vee 1 = 3$

$\alpha(2, 3, 2) = 2 \vee (2 \wedge 3') = 2 \vee (2 \wedge 2) = 2 \vee 2 = 2,$ $\qquad$ $\alpha(3, 6, 1) = 1 \vee (3 \wedge 6') = 1 \vee (3 \wedge 1) = 1 \vee 1 = 1$

The above are some values of the Boolean expression $\alpha(x_1, x_2, x_3) = x_3 \vee (x_1 \wedge x_2')$.

Since $|B| = 4$ and α contains 3 variables, total no. of values of α is 4^3.

2. Let $B = \{0,1\}$ be the two-element Boolean algebra.

(i) If $\alpha(x_1, x_2) = x_1 \wedge x_2$, then

$\alpha(1, 1) = 1 \wedge 1 = 1,$ $\qquad$ $\alpha(1, 0) = 1 \wedge 0 = 0,$ $\qquad$ $\alpha(0, 1) = 0 \wedge 1 = 0,$ $\qquad$ $\alpha(0, 0) = 0 \wedge 0 = 0$

The above are all the binary values of the Boolean expression $\alpha(x_1, x_2) = x_1 \wedge x_2$.

Since $|B| = 2$ and α contains 2 variables, total no. of binary values of α is 2^2.

These binary values can be tabulated in the binary valuation table as follows.

x_1	x_2	$\alpha(x_1, x_2) = x_1 \wedge x_2$
1	1	1
1	0	0
0	1	0
0	0	0

(ii) If $\alpha(x_1, x_2) = (x_1' \wedge x_2) \vee x_1$, then

$\alpha(1, 1) = (1' \wedge 1) \vee 1 = (0 \wedge 1) \vee 1 = 0 \vee 1 = 1$ $\qquad$ $\alpha(1, 0) = (1' \wedge 0) \vee 1 = (0 \wedge 0) \vee 1 = 0 \vee 1 = 1$

$\alpha(0, 1) = (0' \wedge 1) \vee 0 = (1 \wedge 1) \vee 0 = 1 \vee 0 = 1$ $\qquad$ $\alpha(0, 0) = (0' \wedge 0) \vee 0 = (1 \wedge 0) \vee 0 = 0 \vee 0 = 0$

The above are all the binary values of the Boolean expression $\alpha(x_1, x_2) = (x_1' \wedge x_2) \vee x_1$.

Since $|B| = 2$ and α contains 2 variables, total no. of values of α is 2^2.

These binary values can be tabulated in the binary valuation table as follows.

x_1	x_2	x_1'	$x_1' \wedge x_2$	$\alpha(x_1, x_2) = (x_1' \wedge x_2) \vee x_1$
1	1	0	0	1
1	0	0	0	1
0	1	1	1	1
0	0	1	0	0

(iii) If $\alpha(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3')$, then

$\alpha(1, 1, 1) = 1 \wedge (1 \vee 1') = 1 \wedge (1 \vee 0) = 1 \wedge 1 = 1$, $\alpha(1, 1, 0) = 1 \wedge (1 \vee 0') = 1 \wedge (1 \vee 1) = 1 \wedge 1 = 1$

$\alpha(1, 0, 1) = 1 \wedge (0 \vee 1') = 1 \wedge (0 \vee 0) = 1 \wedge 0 = 0$, $\alpha(1, 0, 0) = 1 \wedge (0 \vee 0') = 1 \wedge (0 \vee 1) = 1 \wedge 1 = 1$

$\alpha(0, 1, 1) = 0 \wedge (1 \vee 1') = 0 \wedge (1 \vee 0) = 0 \wedge 1 = 0$, $\alpha(0, 1, 0) = 0 \wedge (1 \vee 0') = 0 \wedge (1 \vee 1) = 0 \wedge 1 = 0$

$\alpha(0, 0, 1) = 0 \wedge (0 \vee 1') = 0 \wedge (0 \vee 0) = 0 \wedge 0 = 0$, $\alpha(0, 0, 0) = 0 \wedge (0 \vee 0') = 0 \wedge (0 \vee 1) = 0 \wedge 1 = 0$

The above are all the binary values of the Boolean expression $\alpha(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3')$.

Since $|B| = 2$ and α contains 3 variables, total no. of values of α is 2^3.

These binary values can be tabulated in the binary valuation table as follows.

x_1	x_2	x_3	x_3'	$x_2 \vee x_3'$	$\alpha(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3')$
1	1	1	0	1	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	1	1	0
0	0	1	0	0	0
0	0	0	1	1	0

Binary system to Decimal system:

(i) $(0\ 0)_2 = 0 \times 2^1 + 0 \times 2^0 = 0$, $(0\ 1)_2 = 0 \times 2^1 + 1 \times 2^0 = 1$,

$(1\ 0)_2 = 1 \times 2^1 + 0 \times 2^0 = 2$, $(1\ 1)_2 = 1 \times 2^1 + 1 \times 2^0 = 3$

(ii) $(0\ 0\ 0)_2 = 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 0$, $(0\ 0\ 1)_2 = 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1$

$(0\ 1\ 0)_2 = 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 2$, $(0\ 1\ 1)_2 = 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 3$

$(1\ 0\ 0)_2 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4$, $(1\ 0\ 1)_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$

$(1\ 1\ 0)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 6$, $(1\ 1\ 1)_2 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 7$

Minterm (or Complete product or Fundamental product): A Boolean expression in n variables $x_1, x_2, x_3, \cdots x_n$ is called a *Minterm* if it is a product $(\wedge)$ of the variables in which every variable or its complement, but not both, appears only once.

Maxterm (or Complete sum or Fundamental sum): A Boolean expression in n variables $x_1, x_2, x_3, \cdots x_n$ is called a *Maxterm* if it is a sum $(\vee)$ of the variables in which every variable or its complement, but not both, appears only once.

Example:

(1) Corresponding to 2 variables x_1, x_2

Minterms: $x_1 \wedge x_2,\ x_1 \wedge x_2',\ x_1' \wedge x_2,\ x_1' \wedge x_2'$

These are respectively denoted by $\min_3, \min_2, \min_1, \min_0$

(Observe that in the binary system, $(1\ 1)_2 = 3,\ (1\ 0)_2 = 2,\ (0\ 1)_2 = 1,\ (0\ 0)_2 = 0$)

Maxterms: $x_1 \vee x_2,\ x_1 \vee x_2',\ x_1' \vee x_2,\ x_1' \vee x_2'$

These are respectively denoted by $\max_0, \max_1, \max_2, \max_3$

(Observe that in the binary system, $(0\ 0)_2 = 0,\ (0\ 1)_2 = 1,\ (1\ 0)_2 = 2,\ (1\ 1)_2 = 3$)

(2) Corresponding to 3 variables x_1, x_2, x_3

Minterms: $x_1 \wedge x_2 \wedge x_3,\ x_1 \wedge x_2 \wedge x_3',\ x_1 \wedge x_2' \wedge x_3,\ x_1 \wedge x_2' \wedge x_3',$
$x_1' \wedge x_2 \wedge x_3,\ x_1' \wedge x_2 \wedge x_3',\ x_1' \wedge x_2' \wedge x_3,\ x_1' \wedge x_2' \wedge x_3'$

These are respectively denoted by $\min_7, \min_6, \min_5, \min_4, \min_3, \min_2, \min_1, \min_0$

(Observe that in the binary system, $(1\ 1\ 1)_2 = 7,\ (1\ 1\ 0)_2 = 6,\ (1\ 0\ 1)_2 = 5,\ (1\ 0\ 0)_2 = 4,$
$(0\ 1\ 1)_2 = 3,\ (0\ 1\ 0)_2 = 2,\ (0\ 0\ 1)_2 = 1,\ (0\ 0\ 0)_2 = 0$)

Maxterms: $x_1 \vee x_2 \vee x_3,\ x_1 \vee x_2 \vee x_3',\ x_1 \vee x_2' \vee x_3,\ x_1 \vee x_2' \vee x_3',$
$x_1' \vee x_2 \vee x_3,\ x_1' \vee x_2 \vee x_3',\ x_1' \vee x_2' \vee x_3,\ x_1' \vee x_2' \vee x_3'$

These are respectively denoted by $\max_0, \max_1, \max_2, \max_3, \max_4, \max_5, \max_6, \max_7$

(Observe that in the binary system, $(0\ 0\ 0)_2 = 0,\ (0\ 0\ 1)_2 = 1,\ (0\ 1\ 0)_2 = 2,\ (0\ 1\ 1)_2 = 3,$
$(1\ 0\ 0)_2 = 4,\ (1\ 0\ 1)_2 = 5,\ (1\ 1\ 0)_2 = 6,\ (1\ 1\ 1)_2 = 7$)

Note:

(i) Corresponding to n variables $x_1, x_2, x_3, \cdots x_n$, there are exactly 2^n minterms and 2^n maxterms

(ii) Sum of all minterms (corresponding to a given variables) is the greatest element 1

(iii) Product of all maxterms (corresponding to a given variables) is the least element 0

(iv) For each binary value 1 of a Boolean expression (in the binary valuation table), a minterm exists and it can be written as follows. If the binary value of a variable is 1, then the variable appears otherwise the complement of the variable appears in the minterm

(v) For each binary value 0 of a Boolean expression (in the binary valuation table), a maxterm exists and it can be written as follows. If the binary value of a variable is 0, then the variable appears otherwise the complement of the variable appears in the maxterm

PDNF or Sum of products canonical form: A Boolean expression which is in the form of a sum of minterms is called a Principal Disjunctive Normal Form (PDNF) or sum of products canonical form.

PCNF or Product of sums canonical form: A Boolean expression which is in the form of a product of maxterms is called a Principal Conjunctive Normal Form (PCNF) or product of sums canonical form.

Note:

(i) Every Boolean expression other than 0 can be expressed as PDNF

(ii) Every Boolean expression other than 1 can be expressed as PCNF

(iii) PDNF (PCNF) of a Boolean expression (if exists) is unique except for the rearrangement of minterms (maxterms)

(iv) **Free Boolean algebra:** Consider 2^n minterms corresponding to n variables $x_1, x_2, x_3, \cdots x_n$, and also consider the set B_{2^n} of all possible sums of the minterms. Then B_{2^n} becomes a Boolean algebra known as *free Boolean algebra* generated by $x_1, x_2, x_3, \cdots x_n$.

Problems:

1. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $x_1 \wedge x_2$

Consider the Binary valuation table for $x_1 \wedge x_2$

x_1	x_2	$x_1 \wedge x_2$			
1	1	1	minterm	$x_1 \wedge x_2$	$\min_3$ or m_3
1	0	0	maxterm	$x_1' \vee x_2$	$\max_2$ or M_2
0	1	0	maxterm	$x_1 \vee x_2'$	$\max_1$ or M_1
0	0	0	maxterm	$x_1 \vee x_2$	$\max_0$ or M_0

(i) PDNF: Here the no. of 1s in the last column is 1, and the min term corresponding to 1 is $x_1 \wedge x_2$

Therefore, the sum of products canonical form is given by $x_1 \wedge x_2$ or $x_1 * x_2$ or $\min_3$

(ii) PCNF: Here the no. of 0s in the last column is 3, and the max terms corresponding to each 0 is $x_1 \vee x_2$, $x_1 \vee x_2'$, $x_1' \vee x_2$

Therefore, the product of sums canonical form is given by $(x_1 \vee x_2) \wedge (x_1 \vee x_2') \wedge (x_1' \vee x_2)$ or $(x_1 \oplus x_2) * (x_1 \oplus x_2') * (x_1' \oplus x_2)$ or $\max_0 \wedge \max_1 \wedge \max_2$

2. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $x_1 \wedge x_2$ in the variables x_1, x_2, x_3

Consider the Binary valuation table for $x_1 \wedge x_2$

x_1	x_2	x_3	$x_1 \wedge x_2$			
1	1	1	1	minterm	$x_1 \wedge x_2 \wedge x_3$	$\min_7$ or m_7
1	1	0	1	minterm	$x_1 \wedge x_2 \wedge x_3'$	$\min_6$ or m_6
1	0	1	0	maxterm	$x_1' \vee x_2 \vee x_3'$	$\max_5$ or M_5
1	0	0	0	maxterm	$x_1' \vee x_2 \vee x_3$	$\max_4$ or M_4
0	1	1	0	maxterm	$x_1 \vee x_2' \vee x_3'$	$\max_3$ or M_3
0	1	0	0	maxterm	$x_1 \vee x_2' \vee x_3$	$\max_2$ or M_2
0	0	1	0	maxterm	$x_1 \vee x_2 \vee x_3'$	$\max_1$ or M_1
0	0	0	0	maxterm	$x_1 \vee x_2 \vee x_3$	$\max_0$ or M_0

(i) PDNF: Here the no. of 1s in the last column is 2, and the min terms corresponding to each 1 is $x_1 \wedge x_2 \wedge x_3$, $x_1 \wedge x_2 \wedge x_3'$ or $\min_7, \min_6$ or m_7, m_6

Therefore, the sum of products canonical form is given by

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3')$ or $(x_1 * x_2 * x_3) \oplus (x_1 * x_2 * x_3')$ or

$\min_7 \vee \min_6$ or $m_7 \vee m_6$ or $\vee 7, 6$

(ii) PCNF: Here the no. of 0s in the last column is 6, and the max terms corresponding to each 0 is $x_1 \vee x_2 \vee x_3$, $x_1 \vee x_2 \vee x_3'$, $x_1 \vee x_2' \vee x_3$, $x_1 \vee x_2' \vee x_3'$, $x_1' \vee x_2 \vee x_3$, $x_1' \vee x_2 \vee x_3'$ or

$\max_0, \max_1, \max_2, \max_3, \max_4, \max_5$ or $M_0, M_1, M_2, M_3, M_4, M_5$

Therefore, the product of sums canonical form is given by

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3') \wedge (x_1 \vee x_2' \vee x_3) \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3')$ or

$\max_0 \wedge \max_1 \wedge \max_2 \wedge \max_3 \wedge \max_4 \wedge \max_5$ or $M_0 \wedge M_1 \wedge M_2 \wedge M_3 \wedge M_4 \wedge M_5$

or $\wedge\, 0, 1, 2, 3, 4, 5$

3. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $(x_1 * x_2') \oplus x_3$

The given Boolean expression can be written as $(x_1 \wedge x_2') \vee x_3$

Consider the Binary valuation table for $(x_1 \wedge x_2') \vee x_3$

x_1	x_2	x_3	x_2'	$x_1 \wedge x_2'$	$(x_1 \wedge x_2') \vee x_3$			
1	1	1	0	0	1	minterm	$x_1 \wedge x_2 \wedge x_3$	$\min_7$ or m_7
1	1	0	0	0	0	maxterm	$x_1' \vee x_2' \vee x_3$	$\max_6$ or M_6
1	0	1	1	1	1	minterm	$x_1 \wedge x_2' \wedge x_3$	$\min_5$ or m_5
1	0	0	1	1	1	minterm	$x_1 \wedge x_2' \wedge x_3'$	$\min_4$ or m_4
0	1	1	0	0	1	minterm	$x_1' \wedge x_2 \wedge x_3$	$\min_3$ or m_3
0	1	0	0	0	0	maxterm	$x_1 \vee x_2' \vee x_3$	$\max_2$ or M_2
0	0	1	1	0	1	minterm	$x_1' \wedge x_2' \wedge x_3$	$\min_1$ or m_1
0	0	0	1	0	0	maxterm	$x_1 \vee x_2 \vee x_3$	$\max_0$ or M_0

(i) PDNF: Here the no. of 1s in the last column is 5, and the min terms corresponding to each 1 is

$x_1 \wedge x_2 \wedge x_3$, $x_1 \wedge x_2' \wedge x_3$, $x_1 \wedge x_2' \wedge x_3'$, $x_1' \wedge x_2 \wedge x_3$, $x_1' \wedge x_2' \wedge x_3$

or $\min_7, \min_5, \min_4, \min_3, \min_1$ or m_7, m_5, m_4, m_3, m_1

Therefore, the sum of products canonical form is given by

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3)$

or $\min_7 \vee \min_5 \vee \min_4 \vee \min_3 \vee \min_1$ or $m_7 \vee m_5 \vee m_4 \vee m_3 \vee m_1$ or $\vee 7,5,4,3,1$

(ii) PCNF: Here the no. of 0s in the last column is 3, and the max terms corresponding to each 0 is

$x_1 \vee x_2 \vee x_3$, $x_1 \vee x_2' \vee x_3$, $x_1' \vee x_2' \vee x_3$ or $\max_0, \max_2, \max_6$ or M_0, M_2, M_6

Therefore, the product of sums canonical form is given by

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3)$ or $\max_0 \wedge \max_2 \wedge \max_6$ or $M_0 \wedge M_2 \wedge M_6$ or $\wedge 0,2,6$

4. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $x_1 \oplus (x_2 * x_3')$

The given Boolean expression can be written as $x_1 \vee (x_2 \wedge x_3')$

Consider the Binary valuation table for $x_1 \vee (x_2 \wedge x_3')$

x_1	x_2	x_3	x_3'	$x_2 \wedge x_3'$	$x_1 \vee (x_2 \wedge x_3')$			
1	1	1	0	0	1	minterm	$x_1 \wedge x_2 \wedge x_3$	$\min_7$ or m_7
1	1	0	1	1	1	minterm	$x_1 \wedge x_2 \wedge x_3'$	$\min_6$ or m_6
1	0	1	0	0	1	minterm	$x_1 \wedge x_2' \wedge x_3$	$\min_5$ or m_5
1	0	0	1	0	1	minterm	$x_1 \wedge x_2' \wedge x_3'$	$\min_4$ or m_4
0	1	1	0	0	0	maxterm	$x_1 \vee x_2' \vee x_3'$	$\max_3$ or M_3
0	1	0	1	1	1	minterm	$x_1' \wedge x_2 \wedge x_3'$	$\min_2$ or m_2
0	0	1	0	0	0	maxterm	$x_1 \vee x_2 \vee x_3'$	$\max_1$ or M_1
0	0	0	1	0	0	maxterm	$x_1 \vee x_2 \vee x_3$	$\max_0$ or M_0

(i) PDNF: Here the no. of 1s in the last column is 5, and the min terms corresponding to each 1 is $x_1 \wedge x_2 \wedge x_3$, $x_1 \wedge x_2 \wedge x_3'$, $x_1 \wedge x_2' \wedge x_3$, $x_1 \wedge x_2' \wedge x_3'$, $x_1' \wedge x_2 \wedge x_3'$ or m_7, m_6, m_5, m_4, m_2

Therefore, the sum of products canonical form is given by

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3')$

or $\min_7 \vee \min_6 \vee \min_5 \vee \min_4 \vee \min_2$

or $\mathrm{m}_7 \vee \mathrm{m}_6 \vee \mathrm{m}_5 \vee \mathrm{m}_4 \vee \mathrm{m}_2$

or $\vee\, 7, 6, 5, 4, 2$

(ii) PCNF: Here the no. of 0s in the last column is 3, and the max terms corresponding to each 0 is $x_1 \vee x_2 \vee x_3$, $x_1 \vee x_2 \vee x_3'$, $x_1 \vee x_2' \vee x_3'$ or M_0, M_1, M_3

Therefore, the product of sums canonical form is given by

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3') \wedge (x_1 \vee x_2' \vee x_3')$

or $\max_0 \wedge \max_1 \wedge \max_3$

or $\mathrm{M}_0 \wedge \mathrm{M}_1 \wedge \mathrm{M}_3$

or $\wedge\, 0, 1, 3$

Exercise:

1. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $x_1 \vee x_2'$
2. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $(x_1' \wedge x_2) \vee x_2'$
3. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $(x_1' \vee x_3) \wedge x_2$
4. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $(x_1' * x_2) \oplus (x_3' * x_2)$
5. Determine (i) the sum of products canonical form (ii) the product of sums canonical form of the Boolean expression $(x_1 \oplus x_2)' \oplus (x_1' * x_3)$