MCA I SEMESTER
Mathematical Foundations of Computer Applications (MFCA): 20BM3101
Unit – 2: Relations and Partially Ordered Set

Cartesian product: The ***Cartesian product*** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$; that is, $A \times B = \{(a,b) \mid a \in A, b \in B\}$

Example:

Let $A = \{a,b,c\}$ and $B = \{1,2\}$. Then

i) $A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

ii) $A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

iii) $B \times B = \{(1,1), (1,2), (2,1), (2,2)\}$

iv) $B \times A = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

Note:

i) Elements of $A \times B$ are called *ordered pairs*

ii) $A \times B \neq B \times A$

iii) $A \times \phi = \phi$

iv) If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$

Relation: Let A, B be two non empty sets. Then a subset of $A \times B$ is called a *relation* from A to B.

Binary relation: Let A be a non empty set. Then a subset of $A \times A$ is called a *binary relation* (or *relation*) on A.

Note: If R is a relation from A to B, then $R \subseteq A \times B$ and for $(a, b) \in R$ we write aRb.

Example:

Let $A = \{a,b,c\}$ and $B = \{1,2\}$. Then

i) $R_1 = \{(a,1),(a,2),(b,1),(c,2)\}$

ii) $R_2 = \{(a,2), (b,1), (c,1)\}$

iii) $R_3 = \{(a,2), (b,2), (c,1), (c,2)\}$

iv) $R_4 = A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Here $R_1 \subseteq A \times B, R_2 \subseteq A \times B, R_3 \subseteq A \times B$ and $R_4 \subseteq A \times B$ hence R_1, R_2, R_3 and R_4 are relations from A to B.

1. Let $A = \{a,b,c\}$. Then

i) $R_1 = \{(a,a), (a,b), (b,b), (b,c), (c,a)\}$

ii) $R_2 = \{(a,b), (a,c), (b,a), (b,c), (c,c)\}$

iii) $R_3 = \{(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)\}$

iv) $R_4 = A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

v) $R_5 = \Delta_A = \{(a,a), (b,b), (c,c)\}$

Here $R_1 \subseteq A \times A$, $R_2 \subseteq A \times A$, $R_3 \subseteq A \times A$, $R_4 \subseteq A \times A$ and $R_5 \subseteq A \times A$ hence R_1, R_2, R_3, R_4 and R_5 are binary relations (or relations) on A.

Note:

1. If $|A| = m$ and $|B| = n$ then $|A\times B| = mn$ and the number of subsets of $A\times B$ is 2^{mn}, that is, the number of relations from A to B is 2^{mn}.
2. If $|A| = n$ then $|A\times A| = n^2$ and the number of subsets of $A\times A$ is 2^{n^2}, that is, the number of relations on A is 2^{n^2}.

Universal & Diagonal relations: Let A be a non empty set. Then

i) $A\times A$ is itself a subset of $A\times A$ and hence $A\times A$ is a relation on A. This relation is called the *universal relation* on A.

ii) $\{(a,a)|\ a\in A\}$ is a subset of $A\times A$ and hence it is a relation on A. This relation is called the *diagonal relation* on A and it is denoted by Δ_A. That is, $\Delta_A = \{(a,a)|\ a\in A\}$

Example:

1. Let $A = \{1,2,3\}$. Then

   i) $A\times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ is the universal relation on A

   ii) $\Delta_A = \{(1,1), (2,2), (3,3)\}$ is the diagonal relation on A

2. Let $A = \{1,2,3,4\}$. Then

   i) $A\times A$ is the universal relation on A

   ii) $\Delta_A = \{(1,1), (2,2), (3,3), (4,4)\}$ is the diagonal relation on A

3. Let $A = \{a,b\}$. Then

   i) $A\times A = \{(a,a), (a,b), (b,a), (b,b)\}$ is the universal relation on A

   ii) $\Delta_A = \{(a,a), (b,b)\}$ is the diagonal relation on A

Domain & Range of a relation: Let R be a relation on a non empty set A. Then

i) $\mathrm{D}(R) = \{x\in A | (x,y)\in R \text{ for some } y\in A\}$ is called the *domain* of the relation R

ii) $\mathrm{R}(R) = \{y\in A | (x,y)\in R \text{ for some } x\in A\}$ is called the *range* of the relation R

Example:

Let $A = \{1,2,3,4\}$. Then

i) $R_1 = \{(1,1), (1,2), (3,1)\}$ is a relation on A

Here the domain of R_1, $\mathrm{D}(R_1) = \{1,3\}$ and the range of R_1, $\mathrm{R}(R_1) = \{1,2\}$

ii) $R_2 = \{(1,1), (1,3), (2,1), (2,4), (3,1)\}$ is a relation on A

Here the domain of R_2, $\mathrm{D}(R_2) = \{1,2,3\}$ and the range of R_2, $\mathrm{R}(R_2) = \{1,3,4\}$

Problems:

1. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A \mid x-y \text{ is even}\}$ and $S=\{(x,y)\in A\times A \mid x+y=5\}$
   i) Write the elements of R and S
   ii) Write the domain and range of R and S
   iii) Find $R\cap S$
   iv) Find $R\cup S$

   Solution:
   i) $R=\{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}$
   $S=\{(1,4),(2,3),(3,2),(4,1)\}$
   ii) $\mathrm{D}(R)=\{1,2,3,4\}$, $\mathrm{D}(S)=\{1,2,3,4\}$, $\mathrm{R}(R)=\{1,2,3,4\}$, $\mathrm{R}(S)=\{1,2,3,4\}$
   iii) $R\cap S=\phi$
   iv) $R\cup S=\{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4),(1,4),(2,3),(3,2),(4,1)\}$

2. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A \mid x-y \text{ is divisible by } 3\}$ and $S=\{(x,y)\in A\times A \mid x+y \text{ is even}\}$
   i) Write the elements of R and S
   ii) Find $R\cap S$
   iii) Find $R\cup S$
   iv) Find $R-S$
   v) Find $S-R$

   Solution:
   i) $R=\{(1,1),(1,4),(2,2),(3,3),(4,1),(4,4)\}$
   $S=\{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}$
   ii) $R\cap S=\{(1,1),(2,2),(3,3),(4,4)\}$
   iii) $R\cup S=\{(1,1),(1,3),(1,4),(2,2),(2,4),(3,1),(3,3),(4,1),(4,2),(4,4)\}$
   iv) $R-S=\{(1,4),(4,1)\}$
   v) $S-R=\{(1,3),(2,4),(3,1),(4,2)\}$

3. Let $A=\{1,2,3,4,5\}, R=\{(x,y)\in A\times A \mid x-y \text{ is divisible by } 3\}$ and $S=\{(x,y)\in A\times A \mid x+y \text{ is even}\}$
   i) Write the elements of R and S
   ii) Find $R\cap S$
   iii) Find $R\cup S$

   Solution:
   i) $R=\{(1,1),(1,4),(2,2),(2,5),(3,3),(4,1),(4,4),(5,2),(5,5)\}$
   $S=\{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}$
   ii) $R\cap S=\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$
   iii) $R\cup S=\{(1,1),(1,3),(1,4),(1,5),(2,2),(2,4),(2,5),(3,1),(3,3),(3,5),$
   $(4,1),(4,2),(4,4),(5,1),(5,2)(5,3),(5,5)\}$

4. If $\mathrm{N}=\{0,1,2,3,\cdots\}$, write the ranges of the relations $S=\{(x,x^2)|\, x\in\mathrm{N}\}$ and $T=\{(x,2x)|\, x\in\mathrm{N}\}$. Also find $R\cup S$ and $R\cap S$

Solution: Here $S=\{(x,x^2)|\, x\in\mathrm{N}\}=\{(0,0),(1,1),(2,4),(3,9),(4,16),\cdots\}$ and
$T=\{(x,2x)|\, x\in\mathrm{N}\}=\{(0,0),(1,2),(2,4),(3,6),(4,8),\cdots\}$
Range of S, $\mathrm{R}(S)=\{0,1,4,9,\cdots\}$
Range of T, $\mathrm{R}(T)=\{0,2,4,6,\cdots\}$
Now $R\cup S=\{(0,0),(1,1),(1,2),(2,4),(3,6),(3,9),(4,8),(4,16),(5,10),(5,25),\cdots\}$
$=\{(x,y)|\, x\in\mathrm{N},\ y=2x \text{ or } y=x^2\}$
and $R\cap S=\{(0,0),\ (2,4)\}$
$=\{(x,y)|\, x\in\mathrm{N},\ y=2x \text{ and } y=x^2\}$

5. Let L denotes the relation 'less than or equal to' and D denotes the relation 'divides', where xDy means 'x divides y'. Both L and D are defined on the set $\{1,2,3,6\}$. Write L and D as sets, and find $L\cup D, L\cap D$

Solution: Let $A=\{1,2,3,6\}$.

Then $L=\{(x,y)\in A\times A|\, x \text{ is less than or equal to } y\}$
$=\{(1,1),(1,2),(1,3),(1,6),(2,2),(2,3),(2,6),(3,3),(3,6),(6,6)\}$
and $D=\{(x,y)\in A\times A|\, x \text{ divides } y\}$
$=\{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}$
Now $L\cup D=\{(1,1),(1,2),(1,3),(1,6),(2,2),(2,3),(2,6),(3,3),(3,6),(6,6)\}$
and $L\cap D=\{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}$

Exercise:

1. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A|\ x-y=1\}$ and $S=\{(x,y)\in A\times A|\ x+y=5\}$
   i) Write the elements of R and S
   ii) Write the domain and range of R and S
   iii) Find $R\cap S$
   iv) Find $R\cup S$
   v) Find $R-S$
   vi) Find $S-R$
2. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A|\ x-y \text{ is divisible by } 3\}$ and $S=\{(x,y)\in A\times A|\ x+y\le 4\}$
   i) Write the elements of R and S

ii) Write the domain and range of R and S

iii) Find $R \cap S$

iv) Find $R \cup S$

v) Find $R - S$

vi) Find $S - R$

3. Let $A = \{1, 2, 4, 6\}, R = \{(x, y) \in A \times A \mid x \text{ divides } y\}$ and $S = \{(x, y) \in A \times A \mid x - y \text{ is divisible by } 4\}$

   i) Write the elements of R and S

   ii) Write the domain and range of R and S

   iii) Find $R \cap S$

   iv) Find $R \cup S$

   v) Find $R - S$

   vi) Find $S - R$

Matrix of a relation: Let $X = \{x_1, x_2, x_3, \cdots x_m\}$ and $Y = \{y_1, y_2, y_3, \cdots y_n\}$ be two finite sets and R be a relation from X to Y. Then the matrix of the relation R is denoted by M_R and is defined as the $m \times n$ matrix given below.

$$M_R = (r_{ij})_{m \times n}, \text{ where } r_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in R \\ 0 & \text{if } (x_i, y_j) \notin R \end{cases}$$

Note:

i) If $|A| = m,\ |B| = n$ and R is a relation A to B, then M_R is $m \times n$ matrix

ii) If $|A| = n$, and R is a relation on A, then M_R is $n \times n$ square matrix

iii) If $|A| = n$, then matrix the universal relation on A, is the $n \times n$ matrix with all 1's

iv) If $|A| = n$, then matrix the diagonal relation on A, is the $n \times n$ unit matrix

Example:

1. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$.

   i) If $R = \{(a,1), (a,2), (b,1), (c,2)\}$, then $M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3 \times 2} \end{array}$

   ii) If $S = \{(a,1), (a,2), (b,1), (b,2)\}$, then $M_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}_{3 \times 2} \end{array}$

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iii) If $T=\{\ (a,2),\ (b,2),\ (c,1),\ (c,2)\ \}$, then $M_T = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$

2. Let $A=\{a,b,c\}$.

i) If $R=\{(a,a),(a,b),(b,a),(b,c),(c,c)\}$, then $M_R = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 0 & 1 \end{array}_{3\times 3}$

ii) If $S=\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,a)\}$, then $M_S = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 1 & 1 & 1 \\ c & 1 & 0 & 0 \end{array}_{3\times 3}$

iii) If $T=\{(a,b),(a,c),(b,b),(b,c),(c,b),(c,c)\}$, then $M_T = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 0 & 1 & 1 \end{array}_{3\times 3}$

iv) If $R=\{(a,a),(b,b),(c,c)\}$, then $M_R = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}_{3\times 3}$

3. Let $A=\{1,2,3,4\}$.

i) If $R=\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,1),(4,3),(4,4)\}$, then $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 4}$

ii) If $R=\{(1,2),(1,4),(2,1),(2,3),(2,4),(3,1),(3,3),(4,2),(4,3)\}$, then $M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{4\times 4}$

iii) If $R=\{(1,1),(2,2),(3,3),(4,4)\}$, then $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$

iv) If $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 4}$, then $R = \{(1,1),(1,2),(2,2),(3,3),(3,4),(4,2),(4,4)\}$

Graph of a relation: Let $X = \{x_1, x_2, x_3, \cdots x_n\}$ be a finite set and R be a relation on X. Then the graph of the relation R is denoted by G_R and is described as follows.

i) Every point in X is represented by a dot (or a small circle) known as a vertex or node.

ii) If $(x_i, x_j) \in R, i \neq j$ then the circle of x_i is connected to the circle of x_j with a directed arc in the direction from x_i to x_j.

iii) If $(x_i, x_i) \in R$ then there is a loop at the circle of x_i.

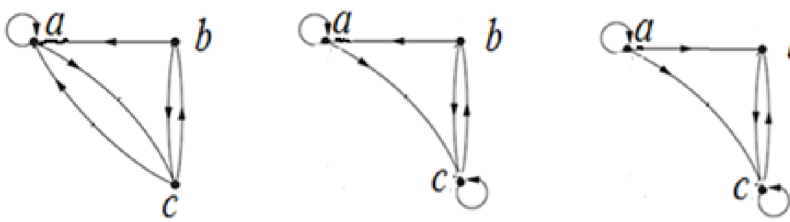
Example:

1. Let $A = \{a, b, c\}$.

i) If $R = \{(a,a),(a,c),(b,a),(b,c),(c,a),(c,b)\}$, then the graph G_R is given as follows.

ii) If $R = \{(a,a),(a,c),(b,a),(b,c),(c,b),(c,c)\}$, then the graph G_R is given as follows.

iii) If $R = \{(a,a),(a,b),(a,c),(b,c),(c,b),(c,c)\}$, then the graph G_R is given as follows.



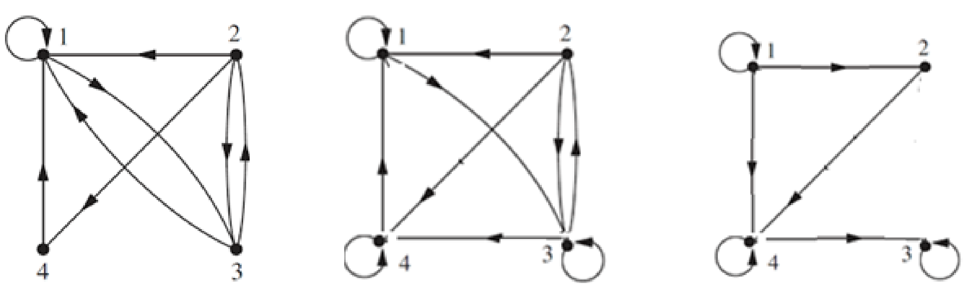
(i) (ii) (iii)

2. Let $A = \{1, 2, 3, 4\}$.

i) If $R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$, then the graph G_R is given as follows.

ii) If $R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,2),(3,3),(3,4),(4,1),(4,4)\}$, then the graph G_R is given as follows.

iii) If $R = \{(1,1),(1,2),(1,4),(2,4),(3,3),(4,3),(4,4)\}$, then the graph G_R is given as follows



(i) **(ii)** **(iii)**

Exercise:

1. Let $A=\{a,b,c\}, R=\{(a,b),(a,c),(b,a),(b,c),(c,c)\}$. Write the matrix and draw the graph of the relation R.
2. Let $A=\{1,2,3,4,5\}, R=\{(1,1),(1,4),(2,2),(2,5),(3,3),(4,1),(4,4),(5,2),(5,5)\}$. Write the matrix and draw the graph of the relation R.
3. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A \mid x>y\}$. Write the matrix and draw the graph of the relation R.
4. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A \mid x-y \text{ is even}\}$ and $S=\{(x,y)\in A\times A \mid x+y=5\}$.
   Write the matrices and draw the graphs of the relations R and S.
5. Let $A=\{1,2,3,4\}, R=\{(x,y)\in A\times A \mid x-y \text{ is divisible by } 3\}$ and $S=\{(x,y)\in A\times A \mid x+y \text{ is even}\}$
   Write the matrices and draw the graphs of the relations R and S.
6. Let $A=\{1,2,4,6\}, R=\{(x,y)\in A\times A \mid x \text{ divides } y\}$ and $S=\{(x,y)\in A\times A \mid x-y \text{ is divisible by } 4\}$
   Write the matrices and draw the graphs of the relations R and S.

Converse of a relation: Let A and B be two sets. Let R be a relation from A to B. Then the converse of R is denoted by $\tilde{R}$ and defined as the relation B to A given below.

$$\tilde{R}=\{(x,y) \mid (y,x)\in R\}$$

Note:

i) If R is a relations on A, then $\tilde{R}$ is also a relation on A.

ii) If R is a relations on A and $\tilde{R}$ is its converse, then $M_{\tilde{R}}=(M_R)^T$

Example:

1. Let $A=\{a,b,c\}$ and $B=\{1,2\}$.

   i) If $R=\{(a,1),(a,2),(b,1),(c,2)\}$, then $\tilde{R}=\{(1,a),(2,a),(1,b),(2,c)\}$

   Here $M_R=\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}_{3\times2}$ and $M_{\tilde{R}}=\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}=(M_R)^T$

   ii) If $S=\{(a,1),(a,2),(b,1),(b,2)\}$, then $\tilde{S}=\{(1,a),(2,a),(1,b),(2,b)\}$

   Here $M_S=\begin{bmatrix}1&1\\1&1\\0&0\end{bmatrix}_{3\times2}$ and $M_{\tilde{S}}=\begin{bmatrix}1&1&0\\1&1&0\end{bmatrix}=(M_S)^T$

2. Let $A=\{1,2,3\}$.

   i) If $R=\{(1,1),(1,2),(2,1),(2,2),(3,2)\}$, then $\tilde{R}=\{(1,1),(2,1),(1,2),(2,2),(2,3)\}$

   Here $M_R=\begin{bmatrix}1&1&0\\1&1&0\\0&1&0\end{bmatrix}_{3\times3}$ and $M_{\tilde{R}}=\begin{bmatrix}1&1&0\\1&1&1\\0&0&0\end{bmatrix}=(M_R)^T$

   ii) If $S=\{(1,1),(2,1),(3,1),(3,2)\}$, then $\tilde{S}=\{(1,1),(1,2),(1,3),(2,3)\}$

Here $M_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$ and $M_{\tilde{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = (M_S)^T$

Composition of relations: Let A, B and C be three sets. Let R be a relation from A to B, and S be a relation from B to C. Then the composition R and S is denoted by $R \circ S$ and is the relation from A to C defined as follows.

$$R \circ S = \{(a,c) \in A \times C \,|\, (a,b) \in R \text{ and } (b,c) \in S \text{ for some } b \in B\}$$

Note:

i) If R and S are relations on A, then $R \circ S$ is also a relation on A.

ii) If R and S are relations on A, then $R \tilde{\circ} S = \tilde{S} \circ \tilde{R}$

iii) If R is a relations on A then $R \circ R$, $R \circ R \circ R$, $\cdots$ are respectively denoted by R^2, R^3, $\cdots$

Transitive closure of a relation: If R is a relations on A then the transitive closure of R is defined as $R^+ = R \cup R^2 \cup R^3 \cup \cdots$

Problems:

1. Let $A = \{a,b,c\}$, $B = \{1,2\}$ and $C = \{x,y\}$. Let $R = \{(a,1),(a,2),(b,1),(c,2)\}$ be a relation from A to B, and $S = \{(1,x),(2,x),(2,y)\}$ be a relation from B to C. Find $R \circ S$

Solution: $R \circ S = \{(a,x),(a,y),(b,x),(c,x),(c,y)\}$

	Elements of R	Elements of S	Elements of $R \circ S$
1	$(a,1)$	$(1,x)$	(a,x)
2	$(a,2)$	$(2,x),(2,y)$	$(a,x),(a,y)$
3	$(b,1)$	$(1,x)$	(b,x)
4	$(c,2)$	$(2,x),(2,y)$	$(c,x),(c,y)$

2. Let $R = \{(1,1),(1,2),(2,1),(3,2),(3,3)\}$ and $S = \{(1,2),(2,1),(2,2)\}$ be relations on the set $A = \{1,2,3\}$. Find (i) $R \circ S$ (ii) $S \circ R$ (iii) R^2 (iv) S^2

Solution:

i) $R \circ S = \{(1,1),(1,2),(2,2),(3,1),(3,2)\}$

$$R = \{(1,1),(1,2),(2,1),(3,2),(3,3)\},\ S = \{(1,2),(2,1),(2,2)\}$$

	Elements of R	Elements of S	Elements of $R \circ S$
1	$(1,1)$	$(1,2)$	$(1,2)$
2	$(1,2)$	$(2,1),(2,2)$	$(1,1),(1,2)$
3	$(2,1)$	$(1,2)$	$(2,2)$
4	$(3,2)$	$(2,1),(2,2)$	$(3,1),(3,2)$
5	$(3,3)$	-	-

ii) $S \circ R = \{ (1,1), (2,1), (2,2) \}$

$$S = \{ (1,2), (2,1), (2,2) \}, \ R = \{ (1,1), (1,2), (2,1), (3,2), (3,3) \}$$
$$(1,2) : (2,1) \to (1,1)$$
$$(2,1) : (1,1), (1,2) \to (2,1), (2,2)$$
$$(2,2) : (2,1) \to (2,1)$$

iii) $R \circ R = \{ (1,1), (1,2), (2,1), (2,2), (3,1), (3,2) (3,3) \}$

$$R = \{ (1,1), (1,2), (2,1), (3,2), (3,3) \}$$
$$(1,1) : (1,1), (1,2) \to (1,1), (1,2)$$
$$(1,2) : (2,1) \to (1,1)$$
$$(2,1) : (1,1), (1,2) \to (2,1), (2,2)$$
$$(3,2) : (2,1) \to (3,1)$$
$$(3,3) : (3,2), (3,3) \to (3,2), (3,3)$$

iv) $S \circ S = \{ (1,1), (1,2), (2,1), (2,2) \}$

$$S = \{ (1,2), (2,1), (2,2) \}$$
$$(1,2) : (2,1), (2,2) \to (1,1), (1,2)$$
$$(2,1) : (1,2) \to (2,2)$$
$$(2,2) : (2,1), (2,2) \to (2,1), (2,2)$$

3. Let $A = \{ 1, 2, 3, 4, 5 \}$ and $R = \{ (1,1), (1,2), (2,2), (2,3) \}$ be a relation on A. Find (i) R^2 (ii) R^3 (iii) R^4 (iv) The transitive closure R^+

Solution:

(i) $R^2 = R \circ R = \{ (1,1), (1,2), (1,3), (2,2), (2,3) \}$

$$(1,1) : (1,1), (1,2) \to (1,1), (1,2)$$
$$(1,2) : (2,2), (2,3) \to (1,2), (1,3)$$
$$(2,2) : (2,2), (2,3) \to (2,2), (2,3)$$
$$(2,3) : -$$

(ii) $R^3 = R^2 \circ R = \{ (1,1), (1,2), (1,3), (2,2), (2,3) \} = R^2$

Therefore, $R^3 = R^2$

$$(1,1) : (1,1), (1,2) \to (1,1), (1,2)$$
$$(1,2) : (2,2), (2,3) \to (1,2), (1,3)$$
$$(1,3) : -$$
$$(2,2) : (2,2), (2,3) \to (2,2), (2,3)$$
$$(2,3) : -$$

(iii) $R^4 = R^3 \circ R$
$= R^2 \circ R$
$= R^3$
$= R^2$

(iv) $R^+ = R \cup R^2 \cup R^3 \cup \cdots$
$= R \cup R^2 \cup R^2 \cup \cdots$
$= R \cup R^2$
$= \{(1,1),(1,2),(1,3),(2,2),(2,3)\}$

4. Let $A = \{a,b,c\}$ and $R = \{(a,b),(a,c),(c,b)\}$ be a relation on A. Find the transitive closure of R

Solution:

$R^2 = R \circ R = \{(a,b)\}$

$R^3 = R^2 \circ R = \phi$

$R^4 = R^3 \circ R = \phi \circ R = \phi,\ R^5 = \phi,\ \cdots$

Therefore, $R^+ = R \cup R^2 \cup R^3 \cup \cdots = R \cup R^2 = \{(a,b),(a,c),(c,b)\}$

5. Let $R = \{(1,1),(1,2),(2,1),(3,2),(3,3)\}$ and $S = \{(1,2),(2,1),(2,2)\}$ be relations on the set $A = \{1,2,3\}$. Find (i) M_R (ii) M_S (iii) $M_R \circ M_S$ (iv) $R \circ S$ (v) $M_{R \circ S}$ and verify $M_{R \circ S} = M_R \circ M_S$ (vi) $M_S \circ M_R$ (vii) $S \circ R$ (viii) $M_{S \circ R}$ and verify $M_{S \circ R} = M_S \circ M_R$

Solution:

(i) $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(ii) $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $M_R \circ M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(iv) $R \circ S = \{(1,1),(1,2),(2,2),(3,1),(3,2)\}$

(v) $M_{R \circ S} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and clearly $M_{R \circ S} = M_R \circ M_S$

(vi) $M_S \circ M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(vii) $S \circ R = \{(1,1),(2,1),(2,2)\}$

(viii) $M_{S\circ R}=\begin{bmatrix}1&0&0\\1&1&0\\0&0&0\end{bmatrix}$ and clearly $M_{S\circ R}=M_S\circ M_R$

6. Let $A=\{a,b,c\}$, $B=\{1,2\}$ and $C=\{x,y\}$. Let $R=\{(a,1),(a,2),(b,1),(c,2)\}$ be a relation from A to B, and $S=\{(1,x),(2,x),(2,y)\}$ be a relation from B to C.

   Find (i) $R\circ S$ (ii) $R\,\tilde{\circ}\,S$ (iii) $\tilde{R}$ (iv) $\tilde{S}$ (v) $\tilde{S}\circ\tilde{R}$ and verify $R\,\tilde{\circ}\,S=\tilde{S}\circ\tilde{R}$

   Solution:

   (i) $R\circ S=\{(a,x),(a,y),(b,x),(c,x),(c,y)\}$

   (ii) $R\,\tilde{\circ}\,S=\{(x,a),(y,a),(x,b),(x,c),(y,c)\}$

   (iii) $\tilde{R}=\{(1,a),(2,a),(1,b),(2,c)\}$

   (iv) $\tilde{S}=\{(x,1),(x,2),(y,2)\}$

   (v) $\tilde{S}\circ\tilde{R}=\{(x,a),(x,b),(x,c),(y,a),(y,c)\}$

   And therefore, $R\,\tilde{\circ}\,S=\tilde{S}\circ\tilde{R}$

Exercise:

1. Let $R=\{(1,1),(1,2),(2,1),(3,2),(3,3)\}$ and $S=\{(1,2),(1,3),(2,1),(2,2),(3,1)\}$ be the relations on $A=\{1,2,3\}$. Find (i) $R\circ S$ (ii) $S\circ R$ (iii) $R\circ R$ (iv) R^3 (v) $S\circ S$

2. Let $R=\{(1,2),(2,2),(3,4)\}$ and $S=\{(1,3),(2,5),(3,1),(4,2)\}$ be the relations on $A=\{1,2,3,4,5\}$. Find (i) $R\circ S$ (ii) $S\circ R$ (iii) $R\circ R$ (iv) R^3 (v) $S\circ S$ (vi) $R\circ(S\circ R)$ (vii) $(R\circ S)\circ R$

3. Let $A=\{a,b,c\}$ and $R=\{(a,b),(b,c),(c,c)\}$ be a relation on A. Find the transitive closure of R

4. Let $A=\{a,b,c\}$ and $R=\{(a,b),(b,c),(c,a)\}$ be a relation on A. Find the transitive closure of R

5. Let $A=\{1,2,3,4\}$ and $R=\{(1,2),(2,1),(3,3)\}$ be a relation on A. Find the transitive closure of R

6. Let $R=\{(1,2),(2,2),(3,4)\}$ and $S=\{(1,3),(2,5),(3,1),(4,2)\}$ be relations on the set $A=\{1,2,3,4,5\}$. Find (i) M_R (ii) M_S (iii) $M_R\circ M_S$ (iv) $R\circ S$ (v) $M_{R\circ S}$ and verify $M_{R\circ S}=M_R\circ M_S$ (vi) $M_S\circ M_R$ (vii) $S\circ R$ (viii) $M_{S\circ R}$ and verify $M_{S\circ R}=M_S\circ M_R$

7. Let $R=\{(x,y)\in A\times A\mid x-y \text{ is divisible by } 3\}$ and $S=\{(x,y)\in A\times A\mid x+y \text{ is even}\}$ be relations on $A=\{1,3,4\}$. Find (i) $R\circ S$ (ii) $R\,\tilde{\circ}\,S$ (iii) $\tilde{R}$ (iv) $\tilde{S}$ (v) $\tilde{S}\circ\tilde{R}$ and verify $R\,\tilde{\circ}\,S=\tilde{S}\circ\tilde{R}$

8. Given the relation matrices $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$,

find (i) $M_{R\circ S}$ (ii) $M_{\tilde{R}}$ (iii) $M_{\tilde{S}}$ (iv) $M_{\widetilde{R\circ S}}$ and show that (v) $M_{\widetilde{R\circ S}} = M_{\tilde{S}\circ\tilde{R}}$

(i) $M_{R\circ S} = M_R \circ M_S =$

(ii) $M_{\tilde{R}} = (M_R)^T =$

(iii) $M_{\tilde{S}} = (M_S)^T =$

(iv) $M_{\widetilde{R\circ S}} = (M_{R\circ S})^T =$

(v) $M_{\tilde{S}\circ\tilde{R}} = M_{\tilde{S}} \circ M_{\tilde{R}} =$

Properties of relations (or Types of relations): Let R be a relation on a non empty set A. Then R is called

i) *Reflexive* if $(x,x) \in R$ for all $x \in A$ or xRx for all $x \in A$

ii) *Irreflexive* if $(x,x) \notin R$ for all $x \in A$ or $x\not{R}x$ for all $x \in A$

iii) *Symmetric* if for $x, y \in A$ and $(x,y) \in R$ *implies* $(y,x) \in R$

iv) *Anti symmetric* if for $x, y \in A$ and $(x,y) \in R$, $(y,x) \in R$ *implies* $x = y$

v) *Transitive* if for $x, y, z \in A$ and $(x,y) \in R$, $(y,z) \in R$ *implies* $(x,z) \in R$

vi) *Compatible relation* if R is reflexive and symmetric

vii) *Equivalence relation* if R is reflexive, symmetric and transitive

viii) *Partial Order relation* if R is reflexive, anti symmetric and transitive

Note:

1. The diagonal relation Δ_A is reflexive, symmetric, anti symmetric and transitive. Hence Δ_A is both equivalence relation and partial order relation
2. Δ_A is the smallest equivalence relation on A; that is, every equivalence relation on A contains Δ_A
3. Δ_A is the smallest partial order relation on A; that is, every partial order relation on A contains Δ_A
4. The universal relation $A \times A$ is reflexive, symmetric and transitive. Hence $A \times A$ is an equivalence relation and it is the largest.
5. If R is a relations on A then the transitive closure R^+ is the smallest transitive relation containing R
6. Let R be a relation on a finite set A and M_R be the matrix of R.

   i) R is reflexive iff all the diagonal elements of M_R are equal to 1

   ii) R is irreflexive iff all the diagonal elements of M_R are equal to 0

   iii) R is symmetric iff M_R is symmetric

   iv) R is anti symmetric iff for $i \neq j$, $r_{ij} = 1$ *implies* $r_{ji} = 0$, where $M_R = (r_{ij})_{n\times n}$

   v) R is transitive iff for i, j, k, $r_{ij} = 1, r_{jk} = 1$ *implies* $r_{ik} = 1$, where $M_R = (r_{ij})_{n\times n}$

7. Let R be a relation on a finite set A and G_R be the graph of R.
   i) R is reflexive iff there is a loop at every vertex
   ii) R is irreflexive iff there is no loop at every vertex
   iii) R is symmetric iff all arrows (other than loops) come in pairs with reverse direction
   iv) R is anti symmetric iff there are no pairs of arrows with reverse direction
8. Let R be a relation on a finite set A and $\tilde{R}$ be its converse.
   i) R is reflexive iff $\tilde{R}$ is reflexive
   ii) R is irreflexive iff $\tilde{R}$ is irreflexive
   iii) R is symmetric iff $\tilde{R}$ is symmetric
   iv) R is anti symmetric iff $\tilde{R}$ is anti symmetric
   v) R is transitive iff $\tilde{R}$ is transitive

Problems:

1. Let $A=\{1,2,3\}$ and $R=\{(1,1),(1,2),(2,1),(2,3)\}$. Determine the properties of R
   i) *Reflexive*: R is not reflexive, since $(2,2)\notin R$
   ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$
   iii) *Symmetric*: R is not symmetric, since $(2,3)\in R$ but $(3,2)\notin R$
   iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   v) *Transitive*: R is not transitive, since $(1,2),(2,3)\in R$ but $(1,3)\notin R$
2. Let $A=\{1,2,3\}$ and $R=\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3)\}$. Determine the properties of R
   i) *Reflexive*: R is reflexive, since $(1,1),(2,2),(3,3)\in R$
   ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$
   iii) *Symmetric*: R is not symmetric, since $(2,3)\in R$ but $(3,2)\notin R$
   iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   v) *Transitive*: R is not transitive, since $(1,2),(2,3)\in R$ but $(1,3)\notin R$
3. Let $A=\{1,2,3\}$ and $R=\{(1,2),(2,1),(2,3)\}$. Determine the properties of R
   i) *Reflexive*: R is not reflexive, since $(1,1)\notin R$
   ii) *Irreflexive*: R is irreflexive, since $(1,1),(2,2),(3,3)\notin R$
   iii) *Symmetric*: R is not symmetric, since $(2,3)\in R$ but $(3,2)\notin R$
   iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   v) *Transitive*: R is not transitive, since $(1,2),(2,3)\in R$ but $(1,3)\notin R$
4. Let $A=\{1,2,3\}$ and $R=\{(1,1),(1,2),(2,1),(2,3),(3,2)\}$. Determine the properties of R
   i) *Reflexive*: R is not reflexive, since $(2,2)\notin R$
   ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$
   iii) *Symmetric*: R is symmetric, since for $x,y\in A$ and $(x,y)\in R$ *implies* $(y,x)\in R$
   iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   v) *Transitive*: R is not transitive, since $(1,2),(2,3)\in R$ but $(1,3)\notin R$

Verification:		Anti symmetric		Transitive	
	Element of R: (x,y)	$(y,x)\in R$	$x=y$	$(y,z)\in R$	$(x,z)\in R$
1	$(1,1)$	$(1,1)$	Yes	$(1,1),(1,2)$	Yes
2	$(1,2)$	$(2,1)$	No	$(2,1),(2,3)$	No
3	$(2,1)$	$(1,2)$	No	$(1,1),(1,2)$	No
4	$(2,3)$	$(3,2)$	No	$(3,2)$	No
5	$(3,2)$	$(2,3)$	No	$(2,1),(2,3)$	No
	Result	No		No	

5. Let $A=\{1,2,3\}$ and $R=\{(1,1),(1,2),(2,3)\}$. Determine the properties of R
   - i) *Reflexive*: R is not reflexive, since $(2,2)\notin R$
   - ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$
   - iii) *Symmetric*: R is not symmetric, since $(2,3)\in R$ but $(3,2)\notin R$
   - iv) *Anti symmetric*: R is anti symmetric, since for $x,y\in A$ and $(x,y)\in R$, $(y,x)\in R$ *implies* $x=y$
   - v) *Transitive*: R is not transitive, since $(1,2),(2,3)\in R$ but $(1,3)\notin R$

Verification:		Anti symmetric		Transitive	
	Element of R: (x,y)	$(y,x)\in R$	$x=y$	$(y,z)\in R$	$(x,z)\in R$
1	$(1,1)$	$(1,1)$	Yes	$(1,1),(1,2)$	Yes
2	$(1,2)$	-	-	$(2,3)$	No
3	$(2,3)$	-	-	-	-
	Result	Yes		No	

6. Let $A=\{1,2,3\}$ and $R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$. Determine the properties of R
   - i) *Reflexive*: R is not reflexive, since $(3,3)\notin R$
   - ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$
   - iii) *Symmetric*: R is not symmetric, since $(1,3)\in R$ but $(3,1)\notin R$
   - iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   - v) *Transitive*: R is transitive, since for $x,y,z\in A$ and $(x,y)\in R$, $(y,z)\in R$ *implies* $(x,z)\in R$

Verification:		Anti symmetric		Transitive	
	Element of R: (x,y)	$(y,x)\in R$	$x=y$	$(y,z)\in R$	$(x,z)\in R$
1	$(1,1)$	$(1,1)$	Yes	$(1,1),(1,2),(1,3)$	Yes
2	$(1,2)$	$(2,1)$	No	$(2,1),(2,2),(2,3)$	Yes
3	$(1,3)$	-	-	-	-
4	$(2,1)$	$(1,2)$	No	$(1,1),(1,2),(1,3)$	Yes
5	$(2,2)$	$(2,2)$	Yes	$(2,1),(2,2),(2,3)$	Yes
6	$(2,3)$	-	-	-	-
	Result	No		Yes	

7. Let $A=\{1,2,3\}$ and $\Delta_A=\{(1,1),(2,2),(3,3)\}$ the diagonal relation. Determine the properties of Δ_A
   i) *Reflexive*: Δ_A is reflexive, since $(1,1),(2,2),(3,3)\in\Delta_A$
   ii) *Irreflexive*: Δ_A is not irreflexive, since $(1,1)\in\Delta_A$
   iii) *Symmetric*: Δ_A is symmetric, since for $x,y\in A$ and $(x,y)\in\Delta_A$ *implies* $(y,x)\in\Delta_A$
   iv) *Anti symmetric*: Δ_A is anti symmetric, since for $x,y\in A$ and $(x,y)\in\Delta_A$, $(y,x)\in\Delta_A$ *implies* $x=y$
   v) *Transitive*: Δ_A is transitive, since for $x,y,z\in A$ and $(x,y)\in\Delta_A$, $(y,z)\in\Delta_A$ *implies* $(x,z)\in\Delta_A$
   vi) Δ_A is both equivalence relation and partial order relation
8. Let $A=\{1,2,3\}$ and $R=\{(1,2),(1,3),(2,1),(2,3)\}$. Determine the properties of R
   i) *Reflexive*: R is not reflexive, since $(1,1)\notin R$
   ii) *Irreflexive*: R is irreflexive, since $(1,1),(2,2),(3,3)\notin R$
   iii) *Symmetric*: R is not symmetric, since $(1,3)\in R$ but $(3,1)\notin R$
   iv) *Anti symmetric*: R is not anti symmetric, since $(1,2),(2,1)\in R$ but $1\neq 2$
   v) *Transitive*: R is not transitive, since $(1,2),(2,1)\in R$ but $(1,1)\notin R$

Verification:		Anti symmetric		Transitive	
	Element of R: (x,y)	$(y,x)\in R$	$x=y$	$(y,z)\in R$	$(x,z)\in R$
1	$(1,2)$	$(2,1)$	No	$(2,1),(2,3)$	No
2	$(1,3)$	-	-	-	-
3	$(2,1)$	$(1,2)$	No	$(1,2),(1,3)$	No
4	$(2,3)$	-	-	-	-
	Result	No		No	

9. Let $A=\{1,2,\cdots 10\}$ and $R=\{(x,y)\mid x+y=10\}$. Determine the properties of R
   i) *Reflexive*: R is not reflexive, since $(1,1)\notin R$, $1+1\neq 10$
   ii) *Irreflexive*: R is not irreflexive, since $(5,5)\in R$, $5+5=10$
   iii) *Symmetric*: R is symmetric, since for $x,y\in A$

$$\begin{aligned}(x,y)\in R &\Rightarrow x+y=10\\ &\Rightarrow y+x=10\\ &\Rightarrow (y,x)\in R\end{aligned}$$

   iv) *Anti symmetric*: R is not anti symmetric, since $(2,8),(8,2)\in R$ but $2\neq 8$
   v) *Transitive*: R is not transitive, since $(2,8),(8,2)\in R$ but $(2,2)\notin R$
10. Let $A=\{1,2,\cdots 100\}$ and $R=\{(x,y)\mid x-y \text{ is even}\}$. Determine the properties of R
   i) *Reflexive*: For $x\in A$, $x-x=0$ is even. So $(x,x)\in R$ $\therefore R$ is reflexive
   ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$, $1-1=0$ is even
   iii) *Symmetric*: For $x,y\in A$,

$$\begin{aligned}(x,y)\in R &\Rightarrow x-y \text{ is even}\\ &\Rightarrow y-x \text{ is even}\\ &\Rightarrow (y,x)\in R\end{aligned}$$

$\therefore R$ is symmetric

iv) *Anti symmetric*: R is not anti symmetric, since $(1,3),(3,1)\in R$ but $1\neq 3$

v) *Transitive*: For $x,y,z\in A$,

$(x,y),(y,z)\in R \Rightarrow x-y$ is even, $y-z$ is even

$\Rightarrow x-z=(x-y)+(y-z)$ is also even

$\Rightarrow (x,z)\in R$

$\therefore R$ is transitive. Hence R is an equivalence relation

11. Let $A=\{1,2,\cdots 25\}$ and $R=\{(x,y)\,|\,x-y \text{ is divisible by } 3\}$. Determine the properties of R

i) *Reflexive*: For $x\in A$, $x-x=0$ is divisible by 3. So $(x,x)\in R$

$\therefore R$ is reflexive

ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$, $1-1=0$ is divisible by 3

iii) *Symmetric*: For $x,y\in A$,

$(x,y)\in R \Rightarrow x-y$ is divisible by 3

$\Rightarrow y-x$ is also divisible by 3

$\Rightarrow (y,x)\in R$

$\therefore R$ is symmetric

iv) *Anti symmetric*: R is not anti symmetric, since $(1,4),(4,1)\in R$ but $1\neq 4$

v) *Transitive*: For $x,y,z\in A$,

$(x,y),(y,z)\in R \Rightarrow x-y$ is divisible by 3, $y-z$ is divisible by 3

$\Rightarrow x-z=(x-y)+(y-z)$ is also divisible by 3

$\Rightarrow (x,z)\in R$

$\therefore R$ is transitive

Hence R is an equivalence relation but not a partial order relation

12. Let $A=\{1,2,\cdots\}$ and $R=\{(x,y)\,|\,x\leq y\}$. Determine the properties of R.

(This relation is called '*less than or equal to*' relation)

i) *Reflexive*: For $x\in A$, $x\leq x$. So $(x,x)\in R$

$\therefore R$ is reflexive

ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$, $1\leq 1$

iii) *Symmetric*: R is not symmetric, since $(1,2)\in R$ but $(2,1)\notin R$, $1\leq 2$ but $2\not\leq 1$

iv) *Anti symmetric*: For $x,y\in A$,

$(x,y),(y,x)\in R \Rightarrow x\leq y,\ y\leq x$

$\Rightarrow x=y$

$\therefore R$ is anti symmetric

v) *Transitive*: For $x,y,z\in A$,

$(x,y),(y,z)\in R \Rightarrow x\leq y,\ y\leq z$

$\Rightarrow x\leq z$

$\Rightarrow (x,z)\in R$

$\therefore R$ is transitive

Hence R is a partial order relation but not an equivalence relation

13. Let $A=\{1,2,\cdots\}$ and $R=\{(x,y)\,|\,x \text{ divides } y\}$. Determine the properties of R

(This relation is called '*divides*' relation)

i) *Reflexive*: For $x\in A$, x divides x. So $(x,x)\in R$ $\therefore R$ is reflexive

ii) *Irreflexive*: R is not irreflexive, since $(1,1)\in R$, 1 divides 1

iii) *Symmetric*: R is not symmetric, since $(1,2)\in R$ but $(2,1)\notin R$,

1 divides 2, but 2 does not divide 1

iv) *Anti symmetric*: For $x,y\in A$,

$$\begin{aligned}(x,y),(y,x)\in R &\Rightarrow x \text{ divides } y,\ y \text{ divides } x\\ &\Rightarrow x\le y,\quad y\le x\\ &\Rightarrow x=y\end{aligned}$$

$\therefore R$ is anti symmetric

v) *Transitive*: For $x,y,z\in A$,

$$\begin{aligned}(x,y),(y,z)\in R &\Rightarrow x \text{ divides } y,\ y \text{ divides } z\\ &\Rightarrow x \text{ divides } z\\ &\Rightarrow (x,z)\in R\end{aligned}$$

$\therefore R$ is transitive Hence R is a partial order relation but not an equivalence relation

Exercise:

1. Let $A=\{1,2,3\}$ and $R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3)\}$. Determine the properties of R
2. Let $A=\{1,2,3,4\}$ and $R=\{(1,1),(1,2),(1,4),(2,2)\}$. Determine the properties of R
3. Give an example of a relation which is neither reflexive nor irreflexive
4. Give an example of a relation which is both symmetric and anti symmetric
5. Let $A=\{1,2,\cdots 10\}$ and $R=\{(x,y)\,|\,x+y=12\}$. Determine the properties of R
6. Let $A=\{1,2,\cdots 10\}$ and $R=\{(x,y)\,|\,xy=1\}$. Determine the properties of R
7. Let $A=\{1,2,\cdots 100\}$ and $R=\{(x,y)\,|\,xy \text{ is even}\}$. Determine the properties of R
8. Let $A=\{1,2,4,6\}$, $R=\{(x,y)\in A\times A\,|\,x \text{ divides } y\}$. Determine the properties of R
9. If R and S are both reflexive, show that $R\cap S$ and $R\cup S$ are also reflexive

Partition and Covering of a Set: Let A be non empty set and $A_1,A_2,A_3,\cdots A_n$ be subsets of A. Then we say that

1. $X=\{A_1,A_2,A_3,\cdots A_n\}$ is a *covering* of A if $A_1\cup A_2\cup A_3\cup\cdots\cup A_n=A$
2. $X=\{A_1,A_2,A_3,\cdots A_n\}$ is a *partition* of A if

   (i) $A_1\cup A_2\cup A_3\cup\cdots\cup A_n=A$

   (ii) $A_1,A_2,A_3,\cdots A_n$ are mutually disjoint (ie., $A_i\cap A_j=\phi$ for $i\neq j$)

   In this case $A_1,A_2,A_3,\cdots A_n$ are called the *blocks* of the partition.

Equivalence relation corresponding to a partition: If $X=\{A_1,A_2,A_3,\cdots A_n\}$ is a partition of A, then $R_X=(A_1\times A_1)\cup(A_2\times A_2)\cup\cdots(A_n\times A_n)$ becomes an equivalence relation on A corresponding to X.

Example: Let $A=\{1,2,3,4,5,6,7,8\}$.

1. If $X=\{A_1,A_2,A_3\}$, where $A_1=\{2,3,5,8\}$, $A_2=\{1,3,6\}$, $A_3=\{4,6,7\}$ then
   (i) $A_1\cup A_2\cup A_3=A$, therefore X is a covering of A
   (ii) $A_1\cap A_2=\{3\}\neq\phi$, therefore X is not a partition of A
2. If $X=\{A_1,A_2,A_3\}$, where $A_1=\{5,8\}$, $A_2=\{1,3,6\}$, $A_3=\{4,7\}$ then
   (i) $A_1\cup A_2\cup A_3\neq A$, therefore X is not a covering of A and hence X is not a partition of A
3. If $X=\{A_1,A_2,A_3,A_4\}$, where $A_1=\{1,2,3\}$, $A_2=\{3,4,5\}$, $A_3=\{5,6,7\}$, $A_4=\{7,8\}$ then
   (i) $A_1\cup A_2\cup A_3\cup A_4=A$, therefore X is a covering of A
   (ii) $A_1\cap A_2=\{3\}\neq\phi$, therefore X is not a partition of A
4. If $X=\{A_1,A_2,A_3\}$, where $A_1=\{2,3,5,8\}$, $A_2=\{1,6\}$, $A_3=\{4,7\}$ then
   (i) $A_1\cup A_2\cup A_3=A$, therefore X is a covering of A
   (ii) $A_1\cap A_2=\phi$, $A_1\cap A_3=\phi$, $A_2\cap A_3=\phi$, therefore X is a partition of A
   (iii) The equivalence relation corresponding to this partition $X=\{A_1,A_2,A_3\}$, is given by
   $$R_X=(A_1\times A_1)\cup(A_2\times A_2)\cup(A_3\times A_3)=\{(2,2),(2,3),(2,5),(2,8),(3,2),(3,3),(3,5),(3,8),$$
   $$(5,2),(5,3),(5,5),(5,8),(8,2),(8,3),(8,5),(8,8),\ (1,1),(1,6),(6,1),(6,6),\ (4,4),(4,7),(7,4),(7,7)\}$$

5. If $X=\{A_1,A_2,A_3,A_4\}$, where $A_1=\{1,2,3\}$, $A_2=\{4,5\}$, $A_3=\{6,7\}$, $A_4=\{8\}$ then
   (i) $A_1\cup A_2\cup A_3\cup A_4=A$, therefore X is a covering of A
   (ii) $A_1\cap A_2=\phi$, $A_1\cap A_3=\phi$, $A_1\cap A_4=\phi$, $A_2\cap A_3=\phi$, $A_2\cap A_4=\phi$, $A_3\cap A_4=\phi$,
   Therefore, X is a partition of A
   (iii) The equivalence relation corresponding to this partition $X=\{A_1,A_2,A_3,A_4\}$, is given by
   $$R_X=(A_1\times A_1)\cup(A_2\times A_2)\cup(A_3\times A_3)\cup(A_4\times A_4)=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),$$
   $$(3,1),(3,2),(3,3),\ (4,4),(4,5),(5,4),(5,5),\ (6,6),(6,7),(7,6),(7,7),\ (8,8)\}$$

4. $A_1=\{2,3,5,8\}$, $A_2=\{1,6\}$, $A_3=\{4,7\}$
No. blocks: 3

5. $A_1=\{1,2,3\}$, $A_2=\{4,5\}$, $A_3=\{6,7\}$, $A_4=\{8\}$
No. blocks: 4

Exercise:

1. Check which of the following are coverings or partitions of the set $S=\{a,b,c\}$. In the case of partition, write the corresponding equivalence relation.

   (i) $A=\{\{a,b\},\{b,c\}\}$ (ii) $B=\{\{a\},\{a,c\}\}$ (iii) $C=\{\{a\},\{b,c\}\}$ (iv) $D=\{\{a,b,c\}\}$

   (v) $E=\{\{a\},\{b\},\{c\}\}$ (vi) $F=\{\{a\},\{a,b\},\{a,c\}\}$

2. Check which of the following are coverings or partitions of the set $S=\{1,2,3,4\}$. In the case of partition, write the corresponding equivalence relation.

   (i) $A=\{\{2,4\},\{1,3\}\}$ (ii) $B=\{\{1\},\{2,4\}\}$ (iii) $C=\{\{1,2\},\{2,3\},\{3,4\}\}$ (iv) $D=\{\{1,2,3,4\}\}$

   (v) $E=\{\{1\},\{2\},\{3\},\{4\}\}$ (vi) $F=\{\{1,2\},\{3\},\{4\}\}$

Equivalence relation: A relation R on a non empty set A is called an *equivalence relation* if R is

(i) Reflexive; that is, $(x,x)\in R$ for all $x\in A$

(ii) Symmetric; that is, for $x,y\in A$ and $(x,y)\in R$ *implies* $(y,x)\in R$

(iii) Transitive; that is, for $x,y,z\in A$ and $(x,y)\in R,\ (y,z)\in R$ *implies* $(x,z)\in R$

Note:

i) If A is a non empty set, then the diagonal relation Δ_A is always an equivalence relation. Also Δ_A is the smallest equivalence relation on A; that is, every equivalence relation on A contains Δ_A.

ii) If A is a non empty set, then the universal relation $A\times A$ is always an equivalence relation. Also $A\times A$ is the largest equivalence relation on A; that is, every equivalence relation on A contained in $A\times A$.

Equivalence class: Let R be an equivalence relation on a non empty set A and $a\in A$.

Then '***R*-equivalence class** of a' is denoted by $[a]_R$ *or* $[a]$ *or* $\bar{a}$ and is defined as follows.

$$[a]_R \ \ or\ \ \bar{a}=\{x\in A|(a,x)\in R\}$$

That is, $[a]_R$ is the set of all elements of A, which are related to a.

Note:

i) $a\in[a]_R$

ii) $[a]_R$ is always non empty

iii) $(a,b)\in R \iff b\in[a]_R$ and $a\in[b]_R \iff [a]_R=[b]_R \iff [a]_R\cap[b]_R\neq\phi$

iv) Union of all equivalence classes (in A) is equal to A; that is $[a]_R\cup[b]_R\cup[c]_R\cup\cdots=A$

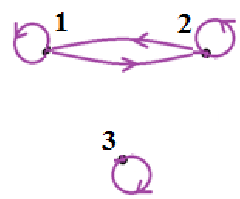
In other words, the set of all equivalence classes is covering of A

v) Intersection of any two equivalence classes is either disjoint or equal

vi) The set $\dfrac{A}{R}$ of all distinct equivalence classes is partition of A

Problems:

1. Prove that $R=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ is an equivalence relation on the set $A=\{1,2,3\}$. Write the matrix and draw the graph of R. Also write all the equivalence classes.
   i) *Reflexive*: Here $(1,1),(2,2),(3,3)\in R$ $\therefore R$ is reflexive
   ii) *Symmetric*: For $x,y\in A$ and $(x,y)\in R$ *implies* $(y,x)\in R$ $\therefore R$ is symmetric
   iii) *Transitive*: For $x,y,z\in A$ and $(x,y)\in R,\ (y,z)\in R$ *implies* $(x,z)\in R$
   $\therefore R$ is transitive. Hence R is an equivalence relation
   iv) *Matrix*: The matrix of R is given by $M_R=\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$
   v) *Graph*: The graph of R is given by G_R

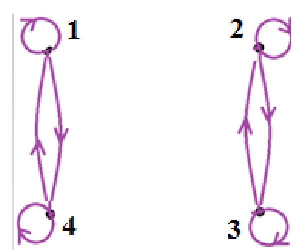


   vi) *Equivalence classes:*

   $[1]_R=\{1,2\},\quad [2]_R=\{1,2\}=[1]_R$ and $[3]_R=\{3\}$

   Therefore, the distinct equivalence classes, $\dfrac{A}{R}=\{[1],\ [3]\}$ and it is a partition of A

2. Prove that $R=\{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2),(1,4),(4,1)\}$ is an equivalence relation on the set $A=\{1,2,3,4\}$. Write the matrix and draw the graph of R. Also write all the equivalence classes.
   i) *Reflexive*: Here $(1,1),(2,2),(3,3),(4,4)\in R$ $\therefore R$ is reflexive
   ii) *Symmetric*: For $x,y\in A$ and $(x,y)\in R$ *implies* $(y,x)\in R$ $\therefore R$ is symmetric
   iii) *Transitive*: For $x,y,z\in A$ and $(x,y)\in R,\ (y,z)\in R$ *implies* $(x,z)\in R$
   $\therefore R$ is transitive. Hence R is an equivalence relation
   iv) *Matrix*: The matrix of R is given by $M_R=\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{bmatrix}$
   v) *Graph*: The graph of R is given by G_R



   vi) *Equivalence classes:*

   $[1]_R=\{1,4\},\quad [2]_R=\{2,3\},\quad [3]_R=\{2,3\}$ and $[4]_R=\{1,4\}$

   Therefore, the distinct equivalence classes, $\dfrac{A}{R}=\{[1],\ [2]\}$ and it is a partition of A

3. Prove that $R=\{(x,y)\in N\times N \mid x=y\}$ is an equivalence relation on the set $N=\{1,2,3\cdots\}$ of all natural numbers. Write the matrix and draw the graph of R. Also write all the equivalence classes.

i) *Reflexive*: For $x\in N$, $x=x$. So $(x,x)\in R$

$\therefore R$ is reflexive

ii) *Symmetric*: For $x,y\in N$,

$$\begin{aligned}(x,y)\in R &\Rightarrow x=y\\ &\Rightarrow y=x\\ &\Rightarrow (y,x)\in R\end{aligned}$$

$\therefore R$ is symmetric

iii) *Transitive*: For $x,y,z\in N$,

$$\begin{aligned}(x,y),(y,z)\in R &\Rightarrow x=y,\quad y=z\\ &\Rightarrow x=z\\ &\Rightarrow (x,z)\in R\end{aligned}$$

$\therefore R$ is transitive. Hence R is an equivalence relation

iv) *Matrix*: The matrix of R is given by $M_R=\begin{bmatrix}1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\cdots\\\cdots&\cdots&\cdots&\cdots\end{bmatrix}$

v) *Graph*: The graph of R is given by G_R

vi) *Equivalence classes:*

$[1]_R=\{1\},\quad [2]_R=\{2\},\quad [3]_R=\{3\}_R,\ [4]_R=\{4\}\ \cdots$

Therefore, the distinct equivalence classes, $\frac{N}{R}=\{[1],[2],\ [3]\cdots\}$ and it is a partition of N

4. Prove that $R=\{(x,y)\mid x-y \text{ is divisible by } 3\}$ is an equivalence relation on the set $A=\{1,2,3,4\}$. Write the matrix and draw the graph of R. Also write all the equivalence classes.

i) *Reflexive*: For $x\in A$, $x-x=0$ is divisible by 3. So $(x,x)\in R$

$\therefore R$ is reflexive

ii) *Symmetric*: For $x,y\in A$,

$$\begin{aligned}(x,y)\in R &\Rightarrow x-y \text{ is divisible by } 3\\ &\Rightarrow y-x \text{ is also divisible by } 3\\ &\Rightarrow (y,x)\in R\end{aligned}$$

$\therefore R$ is symmetric

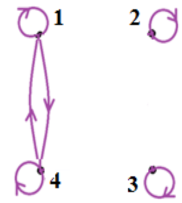
iii) *Transitive*: For $x,y,z\in A$,

$$\begin{aligned}(x,y),(y,z)\in R &\Rightarrow x-y \text{ is divisible by } 3,\ y-z \text{ is divisible by } 3\\ &\Rightarrow x-z=(x-y)+(y-z) \text{ is also divisible by } 3\\ &\Rightarrow (x,z)\in R\end{aligned}$$

$\therefore R$ is transitive. Hence R is an equivalence relation

iv) *Matrix*: The matrix of R is given by $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

v) *Graph*: The graph of R is given by G_R



vi) *Equivalence classes*: $[1]_R = \{1,4\}$, $[2]_R = \{2\}$, $[3]_R = \{3\}$ and $[4]_R = \{1,4\}$

Therefore, the distinct equivalence classes, $\dfrac{A}{R} = \{\,[1],\ [2],\ [3]\,\}$ and it is a partition of A

5. Prove that $R = \{(x,y) \mid x-y \text{ is divisible by } 4\}$, is an equivalence relation on the set $Z = \{\cdots -2, -1, 0, 1, 2, 3 \cdots\}$ of all integers. Write all the equivalence classes.

(This relation is called ‘**congruence modulo 4 relation**’ or ‘**congruence relation**’. Here R is denoted by the symbol $\equiv$ and we write $x \equiv y \pmod 4$ for $x-y$ is divisible by 4. The equivalence classes are called ‘**residue classes modulo 4’** and the distinct equivalence classes is denoted by Z_4)

i) *Reflexive*: For $x \in Z$, $x - x = 0$ is divisible by 4. So $(x,x) \in R$

$\therefore R$ is reflexive

ii) *Symmetric*: For $x, y \in Z$,

$(x,y) \in R \Rightarrow x - y$ is divisible by 4

$\Rightarrow y - x$ is also divisible by 4

$\Rightarrow (y,x) \in R$

$\therefore R$ is symmetric

iii) *Transitive*: For $x, y, z \in Z$,

$(x,y),(y,z) \in R \Rightarrow x-y$ is divisible by 4, $y-z$ is divisible by 4

$\Rightarrow x - z = (x-y)+(y-z)$ is also divisible by 4

$\Rightarrow (x,z) \in R$

$\therefore R$ is transitive. Hence R is an equivalence relation

iv) *Equivalence classes*:

$[0]_R = \{\cdots -8, -4, 0, 4, 8, \cdots\}$, $[1]_R = \{\cdots -7, -3, 1, 5, 9, \cdots\}$,

$[2]_R = \{\cdots -6, -2, 2, 6, 10, \cdots\}$, $[3]_R = \{\cdots -5, -1, 3, 7, 11, \cdots\}$,

$[4]_R = \{\cdots -8, -4, 0, 4, 8, \cdots\} = [0]_R$,

$[5]_R = [1]_R$, $[6]_R = [2]_R$, $[7]_R = [3]_R$, $\cdots$

Hence the distinct equivalence classes, $\dfrac{Z}{R}$ *or* $Z_4 = \{\,[0],\ [1],\ [2],\ [3]\,\}$ and it is a partition of Z

6. **Congruence relation:** If m is a positive integer and $Z=\{\cdots-2,-1,0,1,2,3\cdots\}$ is the set of all integers, then the equivalence relation given by $R=\{(x,y)\in Z\times Z \mid x-y \text{ is divisible by } m\}$ is called '**congruence modulo m relation**' or '**congruence relation**'. Here R is denoted by the symbol $\equiv$ and we write $x\equiv y \pmod{m}$ for $x-y$ is divisible by m. The equivalence classes are called '**residue classes modulo m**' or '**congruence classes modulo m**' and the set of all distinct equivalence classes is denoted by Z_m. That is, $Z_m=\{[0],[1],[2],\cdots[m-1]\}$

i) *Reflexive*: For $x\in Z$, $x-x=0$ is divisible by m. So $(x,x)\in R$

$\therefore R$ is reflexive

ii) *Symmetric*: For $x,y\in Z$,

$(x,y)\in R \Rightarrow x-y$ is divisible by m

$\Rightarrow y-x$ is also divisible by m

$\Rightarrow (y,x)\in R$

$\therefore R$ is symmetric

iii) *Transitive*: For $x,y,z\in Z$,

$(x,y),(y,z)\in R \Rightarrow x-y$ is divisible by m, $y-z$ is divisible by m

$\Rightarrow x-z=(x-y)+(y-z)$ is also divisible by m

$\Rightarrow (x,z)\in R$

$\therefore R$ is transitive. Hence R is an equivalence relation

7. Let R denote a relation on the set of ordered pairs of positive integers such that $(a,b)R(x,y)$ iff $ay=bx$. Show that R is an equivalence relation

Here $R=\{((a,b),(x,y)) \mid a,b,x,y\in Z^+,\ ay=bx\}$

i) *Reflexive*: For $a,b\in Z^+$ we have $ab=ba$, so $((a,b),(a,b))\in R$ $\quad p=(a,b)$

$\therefore R$ is reflexive $\quad (p,p)\in R$

ii) *Symmetric*: For $a,b,c,d\in Z^+$, $\quad p=(a,b),\ q=(c,d)$

$((a,b),(c,d))\in R \Rightarrow ad=bc$ $\quad (p,q)\in R\Rightarrow(q,p)\in R$

$\Rightarrow cb=da$

$\Rightarrow ((c,d),(a,b))\in R$

$\therefore R$ is symmetric

iii) *Transitive*: For $a,b,c,d,e,f\in Z^+$, $\quad p=(a,b),\ q=(c,d),\ r=(e,f)$

$((a,b),(c,d)),\ ((c,d),(e,f))\in R \Rightarrow ad=bc,\ cf=de$ $\quad (p,q),(q,r)\in R\Rightarrow(p,r)\in R$

$$\Rightarrow \frac{a}{b}=\frac{c}{d},\ \frac{c}{d}=\frac{e}{f}$$

$$\Rightarrow \frac{a}{b}=\frac{e}{f}$$

$\Rightarrow af=be$

$\Rightarrow ((a,b),(e,f))\in R$

$\therefore R$ is transitive. Hence R is an equivalence relation

8. If R and S are both equivalence relations, show that $R \cap S$ is also equivalence relation

Reflexive: Since R and S are both equivalence relations, we have R and S are both reflexive.
So, $(x,x) \in R$ and $(x,x) \in S$ for all x
Therefore, $(x,x) \in R \cap S$ for all x
Hence $R \cap S$ is reflexive

Symmetric: Since R and S are both equivalence relations, we have R and S are both symmetric.
Now for $x, y \in A$, $(x,y) \in R \cap S \Rightarrow (x,y) \in R, \quad (x,y) \in S$

$$\Rightarrow (y,x) \in R, \quad (y,x) \in S$$
$$\Rightarrow (y,x) \in R \cap S$$

Therefore, $R \cap S$ is symmetric

Transitive: Since R and S are both equivalence relations, we have R and S are both transitive.
Now for $x, y, z \in A$, $(x,y),(y,z) \in R \cap S \Rightarrow (x,y),(y,z) \in R, \quad (x,y),(y,z) \in S$

$$\Rightarrow (x,z) \in R, \quad (x,z) \in S$$
$$\Rightarrow (x,z) \in R \cap S$$

Therefore, $R \cap S$ is transitive. Hence $R \cap S$ is an equivalence relation.

Exercise:

1. If R and S are both equivalence relations, show that $R \cap S$ is also equivalence relation
2. Prove that $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,5),(5,2),(1,4),(4,1)\}$ is an equivalence relation on the set $A = \{1,2,3,4,5\}$. Write the matrix and draw the graph of R. Also write all the equivalence classes.
3. Prove that $R = \{(x,y) \in A \times A \mid x - y \text{ is divisible by } 3\}$ is an equivalence relation on the set $A = \{1,2,\cdots 6\}$. Write the matrix and draw the graph of R. Also write all the equivalence classes.
4. Prove that $R = \{(x,y) \mid x - y \text{ is divisible by } 6\}$, is an equivalence relation on the set $Z = \{\cdots -2,-1,0,1,2,3\cdots\}$ of all integers. Write all the equivalence classes.
5. Prove that the relation '**congruence modulo** m'given by $\equiv = \{(x,y) \in Z \times Z \mid x - y \text{ is divisible by } m\}$ over the set of positive integers is an equivalence relation. Show also that if $x_1 \equiv y_1$ and $x_2 \equiv y_2$, then $(x_1 + x_2) \equiv (y_1 + y_2)$

Partial order relation: A relation R on a non empty set A is called a *partial order relation* or a *partial ordering* in A if R is

(i) Reflexive; that is, $(x,x) \in R$ for all $x \in A$

(ii) Anti symmetric; that is, for $x, y \in A$ and $(x,y) \in R$, $(y,x) \in R$ *implies* $x = y$

Or for $x, y \in A$ and $x \neq y \Rightarrow$ *at least one of* $(x,y), (y,x) \notin R$

(iii) Transitive; that is, for $x, y, z \in A$ and $(x,y) \in R$, $(y,z) \in R$ *implies* $(x,z) \in R$

Note:

i) Generally partial order relations are denoted by the symbols like $\leq$, $\geq$, $\subseteq$, $\supseteq$

ii) If $\leq$ is a partial order relation on a non empty set A, then $\leq$ is a subset of $A \times A$ and for $(a, b) \in \leq$ we write $a \leq b$; that is, $a \leq b$ *if and only if* $(a,b) \in \leq$

iii) If A is a non empty set, then the diagonal relation Δ_A is always a partial order relation. Also Δ_A is the smallest partial order relation on A; that is, every partial order relation on A contains Δ_A.

Partially Ordered Set (or Poset): If $\leq$ is a partial order relation on a non empty set P, then the ordered pair $(P, \leq)$ is called a *partially ordered set* or a *poset*.

Comparable and Incomparable elements: Let $(P, \leq)$ be a poset and $a, b \in P$. Then we say that

(i) a, b are comparable if $a \leq b$ *or* $b \leq a$ ie., $(a, b) \in \leq$ *or* $(b, a) \in \leq$

(ii) a, b are incomparable if *neither* $a \leq b$ *nor* $b \leq a$ ie., $(a, b) \notin \leq$ *and* $(b, a) \notin \leq$

Totally Ordered Set (or Chain or Simply Ordered Set): A poset $(P, \leq)$ is called a *chain* or *totally ordered set* or *simply ordered set* if every pair of elements of P are comparable; that is $a, b \in P$ *implies* $a \leq b$ *or* $b \leq a$. In this case $\leq$ is called a total order relation.

Problems:

1. Prove that $\leq = \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$ is a partial order relation on the set $A = \{1,2,3\}$. Also prove that $\leq$ is a total order relation

   i) *Reflexive*: Here $(1,1),(2,2),(3,3) \in \leq$ $\quad \therefore \leq$ is reflexive

   ii) *Anti symmetric*: For $x, y \in A$ and $(x, y) \in \leq$, $(y, x) \in \leq$ *implies* $x = y$
   $\therefore \leq$ is anti symmetric

   iii) *Transitive*: For $x, y, z \in A$ and $(x, y) \in \leq$, $(y, z) \in \leq$ *implies* $(x, z) \in \leq$
   $\therefore \leq$ is transitive

   Therefore, $\leq$ is a partial order relation and hence $(A, \leq)$ is a partially ordered set or a poset.

   iv) Here observe that $1 \leq 2$, $1 \leq 3$, $2 \leq 3$; that is, $(1,2)$, $(1,3)$, $(2,3) \in \leq$
   Therefore, every pair of elements of A are comparable; that is,
   for $x, y \in A$ *we have* $x \leq y$ *or* $y \leq x$
   Thus, $\leq$ is a total order relation and hence $(A, \leq)$ is a totally ordered set or a chain.

2. Prove that $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4)\}$ is a partial order relation on the set $A = \{1,2,3,4\}$. But not a chain.

   i) *Reflexive*: Here $(1,1),(2,2),(3,3),(4,4) \in R$ $\quad \therefore R$ is reflexive

   ii) *Anti symmetric*: For $x, y \in A$ and $(x, y) \in R$, $(y, x) \in R$ *implies* $x = y$ $\quad \therefore R$ is anti symmetric

   iii) *Transitive*: For $x, y, z \in A$ and $(x, y) \in R$, $(y, z) \in R$ *implies* $(x, z) \in R \therefore R$ is transitive

   Therefore, R is a partial order relation and hence (A, R) is a partially ordered set or a poset.

   iv) Here observe that $1 \leq 2$, $1 \leq 3$, $1 \leq 4$, $2 \leq 4$; that is, $(1,2)$, $(1,3)$, $(1,4)$, $(2,4) \in R$

Therefore, the pairs 1 *and* 2, 1 *and* 3, 1 *and* 4, 2 *and* 4 are comparable

But we have $2 \nleq 3$ *and* $3 \nleq 2$; that is, $(2,3) \notin R$ and $(3,2) \notin R$

Therefore, the pair 2 *and* 3 is incomparable.

Similarly the pair 3 *and* 4 is also incomparable, since $3 \nleq 4$ *and* $4 \nleq 3$.

Therefore, R is not a total order relation and hence (A,R) is a poset but not a chain.

3. Prove that $R = \{(x,y) \mid x \leq y\}$, where $\leq$ is the usual 'less than or equal to', is a partial order relation on the set $A = \{1,2,3,\cdots\}$ of all integers. Also prove that R is a total order relation. (This relation is called '*less than or equal to*' relation)

   i) *Reflexive*: For $x \in A$, we have $x \leq x$. So $(x,x) \in R$ $\therefore R$ is reflexive

   ii) *Anti symmetric*: For $x, y \in A$,

   $$(x,y) \in R, (y,x) \in R \Rightarrow x \leq y, \ y \leq x$$
   $$\Rightarrow x = y \qquad \therefore R \text{ is anti symmetric}$$

   iii) *Transitive*: For $x, y, z \in A$,

   $$(x,y) \in R, (y,z) \in R \Rightarrow x \leq y, \ y \leq z$$
   $$\Rightarrow x \leq z$$
   $$\Rightarrow (x,z) \in R \qquad \therefore R \text{ is transitive}$$

   Therefore, R is a partial order relation and hence (A,R) is a partially ordered set or a poset.

   iv) Here observe that every pair of elements of A are comparable; that is,

   for $x, y \in A$ *we have* $x \leq y$ *or* $y \leq x$

   Thus, R is a total order relation and hence (A,R) is a totally ordered set or a chain.

4. Prove that $\leq = \{(x,y) \mid x \text{ divides } y\}$ is a partial order relation on the set $P = \{1,2,3,6\}$. Also prove that $\leq$ is a total order relation. (This relation is called '*divides*' relation on P)

   Or

   Prove that the set $P = \{1,2,3,6\}$ is a chain with respect to the '*divides*' relation.

   The '*divides*' relation is given by $\leq = \{(x,y) \mid x \text{ divides } y\}$

   i) *Reflexive*: For $x \in P$, x divides x. So $(x,x) \in \leq$ $\therefore \leq$ is reflexive

   ii) *Anti symmetric*: For $x, y \in P$,

   $$(x,y), (y,x) \in \leq \ \Rightarrow x \text{ divides } y, \ y \text{ divides } x$$
   $$\Rightarrow x = y \qquad \therefore \leq \text{ is anti symmetric}$$

   iii) *Transitive*: For $x, y, z \in P$,

   $$(x,y),(y,z) \in \leq \ \Rightarrow x \text{ divides } y, \ y \text{ divides } z$$
   $$\Rightarrow x \text{ divides } z$$
   $$\Rightarrow (x,z) \in \leq \qquad \therefore \leq \text{ is transitive}$$

   Therefore, $\leq$ is a partial order relation and hence $(P, \leq)$ is a partially ordered set or a poset.

   iv) Here observe that every pair of elements of P are comparable; that is,

   for $x, y \in P$ *we have* $x \leq y$ *or* $y \leq x$

Thus, $\leq$ is a total order relation and hence $(P, \leq)$ is a totally ordered set or a chain.

5. Let $X = \{a, b, c\}$ and P be the set of all subsets (power set) of X. Prove that the '*subset relation*' given by $\leq = \{(A, B) \in P \times P \mid A \subseteq B\}$ is a partial order relation on P. Also prove that $\leq$ is not a total order relation.

   Or

   Prove that the power set P of the set $X = \{a, b, c\}$ is a poset with respect to the '*subset*' relation.

   The '*subset*' relation is given by $\leq = \{(A, B) \in P \times P \mid A \subseteq B\}$

   i) *Reflexive*: For $A \in P$, we have $A \subseteq A$. So $(A, A) \in \leq$ $\therefore \leq$ is reflexive

   ii) *Anti symmetric*: For $A, B \in P$,

   $(A, B), (B, A) \in \leq \Rightarrow A \subseteq B,\ B \subseteq A$

   $\Rightarrow A = B$

   $\therefore \leq$ is anti symmetric

   iii) *Transitive*: For $A, B, C \in P$,

   $(A, B), (B, C) \in \leq \Rightarrow A \subseteq B,\ B \subseteq C$

   $\Rightarrow A \subseteq C$

   $\Rightarrow (A, C) \in \leq$

   $\therefore \leq$ is transitive

   Therefore, $\leq$ is a partial order relation and hence $(P, \leq)$ is a partially ordered set or a poset.

   iv) Here observe that $\{a\}, \{b\} \in P$ and $\{a\} \not\subseteq \{b\},\ \{b\} \not\subseteq \{a\}$

   Thus, $\leq$ is not a total order relation and hence $(P, \leq)$ is a poset but not a chain.

Exercise:

1. If R and S are both partial order relation, show that $R \cap S$ is also partial order relation
2. Prove that $R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$ is a partial order relation on the set $A = \{1, 2, 3\}$
3. Prove that the '*divides relation*' given by $\leq = \{(x, y) \in A \times A \mid x \text{ divides } y\}$ is a partial order relation on the set $A = \{1, 2, 3, \cdots 100\}$
4. Prove that the '*less than or equal to relation*' given by $R = \{(x, y) \in N \times N \mid x \leq y\}$ is a partial order relation on the set $N = \{1, 2, 3, \cdots 20\}$ of all natural numbers.
5. Prove that $R = \{(x, y) \in N \times N \mid x \leq y\}$ is a partial order relation on the set $N = \{1, 2, 3, \cdots\}$ of all natural numbers.
6. Let $X = \{a, b, c, d\}$ and P be the set of all subsets (power set) of X. Prove that the '*subset relation*' given by $\leq = \{(A, B) \in P \times P \mid A \subseteq B\}$ is a partial order relation on P.

Covering elements: Let $(P, \leq)$ be a poset and $a, b \in P$. Then we say that

(i) $a<b$ *if and only if* $a\leq b$ and $a \neq b$

(ii) b covers a *if and only if* $a<b$ and there is no element $x\in P$ such that $a\leq x \leq b$

(Also we say that a is covered by b). In this case we write $a \prec b$

Hasse Diagram: Every poset can be represented by means of a diagram known as a 'Hasse diagram' or 'poset diagram' of the poset. In this diagram, we have the following.

(i) Every element of the poset represented by a small circle or a dot.

(ii) If $a < b$, then the circle for a is below the circle for b.

(iii) If b covers a, then a straight line is drawn between the circles of a and b.

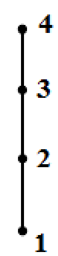
(iv) If $a < b$ and b does not cover a, then the circles of a and b are not connected directly by a single line. However they are connected through one or more elements of the poset.

Problems:

1. Let $P = \{1,2,3,4\}$ and $\leq$ be the 'less than or equal to' relation on P. Draw the Hasse diagram for the poset $(P, \leq)$.

   Here $1 \prec 2 \prec 3 \prec 4$ that is, 1 is covered by 2, 2 is covered by 3, 3 is covered by 4

   The Hasse diagram is given as follows.

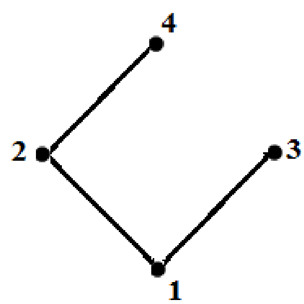


2. Let $P = \{1,2,3,4\}$ and $\leq$ be the 'divides' relation on P. Draw the Hasse diagram for the poset $(P, \leq)$.

   Here $1 \prec 2 \prec 4$ that is, 1 is covered by 2, 2 is covered by 4

   and $1 \prec 3$ that is, 1 is covered by 3

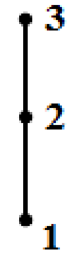
   The Hasse diagram is given as follows.



3. Let $P=\{1,2,3\}$ and $\leq=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$ be the partial order relation on P. Draw the Hasse diagram for the poset $(P,\leq)$.

   Here $1 \prec 2 \prec 3$ that is, 1 is covered by 2, 2 is covered by 3

   The Hasse diagram is given as follows.

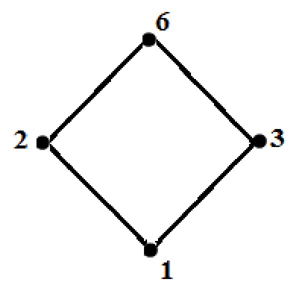


4. Let $D_6=\{1,2,3,6\}$ the divisors of 6, and $\leq$ be the 'divides' relation on P. Draw the Hasse diagram for the poset $(P,\leq)$.

   Here $1 \prec 2 \prec 6$ that is, 1 is covered by 2, 2 is covered by 6

   And $1 \prec 3 \prec 6$ that is, 1 is covered by 3, 3 is covered by 6

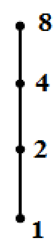
   The Hasse diagram is given as follows.



5. Let $D_8=\{1,2,4,8\}$ the divisors of 8, and $\leq$ be the 'divides' relation on P. Draw the Hasse diagram for the poset $(P,\leq)$.

   Here $1 \prec 2 \prec 4 \prec 8$ that is, 1 is covered by 2, 2 is covered by 4, 4 is covered by 8

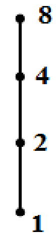
   The Hasse diagram is given as follows.



6. Let $D_8 = \{1, 2, 4, 8\}$ the divisors of 8, and $\leq$ be the 'less than or equal to' relation on P. Draw the Hasse diagram for the poset $(P, \leq)$.

Here $1 \prec 2 \prec 4 \prec 8$ that is, 1 is covered by 2, 2 is covered by 4, 4 is covered by 8

The Hasse diagram is given as follows.



7. Let $P = \{1, 2, 3, 4, 6\}$ and $\leq$ be the relation on P such that $x \leq y$ if and only if x divides y.

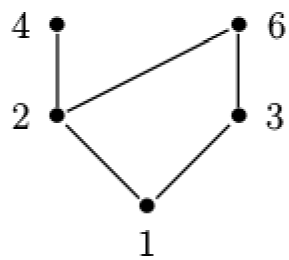
Draw the Hasse diagram for the poset $(P, \leq)$

Here we have $1 \prec 2 \prec 4$,

$1 \prec 2 \prec 6$

$1 \prec 3 \prec 6$

Now the Hasse diagram is given as follows.



8. Let $P = \{1, 2, 3, 4, 6, 12\}$ and $\leq$ be the relation on P such that $x \leq y$ if and only if x divides y.

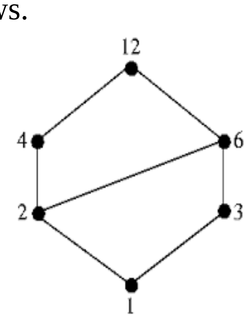
Draw the Hasse diagram for the poset $(P, \leq)$

Here we have $1 \prec 2 \prec 4 \prec 12$,

$1 \prec 2 \prec 6 \prec 12$

$1 \prec 3 \prec 6 \prec 12$

Now the Hasse diagram is given as follows.

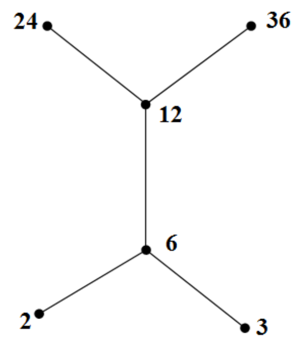


9. Let $P=\{2,3,6,12,24,36\}$ and $\leq$ be the relation on P such that $x \leq y$ if and only if x divides y. Draw the Hasse diagram for the poset $(P, \leq)$

Here we have $2 \prec 6 \prec 12 \prec 24, \quad 2 \prec 6 \prec 12 \prec 36,$

$3 \prec 6 \prec 12 \prec 24, \quad 3 \prec 6 \prec 12 \prec 36$

Now the Hasse diagram is given as follows.



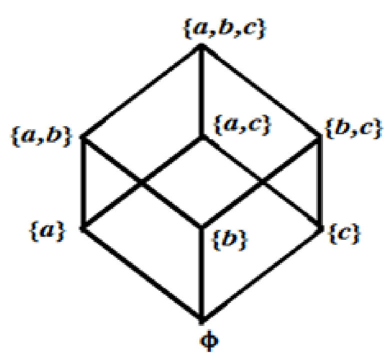
10. Let $X=\{a,b,c\}$ and P be the set of all subsets (power set) of X. Prove that the '*subset relation*' given by $\leq=\{(A,B)\in P\times P \mid A\subseteq B\}$ is a partial order relation on P. Also draw the Hasse diagram for the poset $(P, \leq)$.

Here we have $\phi \prec \{a\} \prec \{a,b\} \prec \{a,b,c\}, \quad \phi \prec \{a\} \prec \{a,c\} \prec \{a,b,c\},$

$\phi \prec \{b\} \prec \{a,b\} \prec \{a,b,c\}, \quad \phi \prec \{b\} \prec \{b,c\} \prec \{a,b,c\},$

$\phi \prec \{c\} \prec \{a,c\} \prec \{a,b,c\}, \quad \phi \prec \{c\} \prec \{b,c\} \prec \{a,b,c\}$

The Hasse diagram is given as follows.



Exercise:

1. Let $P = \{1, 2, 5, 10\}$. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on P.
2. Let $D_{10} = \{1, 2, 5, 10\}$, the divisors of 10. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on D_{10}.
3. Let $D_{12} = \{1, 2, 3, 4, 6, 12\}$, the divisors of 12. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on D_{12}.
4. Let $P = \{2, 3, 4, 6\}$. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on P.
5. Let $P = \{1, 2, 3, 6, 12\}$. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on P.
6. Let $P = \{1, 2, 3, 12, 18, 36\}$. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on P.
7. Let $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, the divisors of 30. Draw the Hasse diagram for the poset with respect to the (i) 'less than or equal to' relation (ii) 'divides' relation on D_{30}.
8. Let $P = \{2, 4, 5, 10, 12, 20, 25\}$ and $\leq$ be the relation on P such that $x \leq y$ if and only if x divides y. Draw the Hasse diagram for the poset $(P, \leq)$.
9. Let $P = \{\ \phi, \{1\}, \{1,2\}, \{1,2,3\}\ \}$ and $\leq$ be the subset relation on P. Draw the Hasse diagram for the poset $(P, \leq)$.
10. Let $P = \{\ \phi, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\ \}$ and $\leq$ be the subset relation on P. Draw the Hasse diagram for the poset $(P, \leq)$.