Gaussian Elimination

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This problem, worth 10 points, is due May 22, 2023, by 11:59 pm via BBLearn. You may work on this problem in a team of up to two people. One submission per group will suffice. Please submit original work.

Consider the problem of solving a system of linear equations of the form

$$\begin{array}{lllll} a_{0,0}x_0 & + a_{0,1}x_1 & + \cdots & + a_{0,n-1}x_{n-1} & = b_0, \\ a_{1,0}x_0 & + a_{1,1}x_1 & + \cdots & + a_{1,n-1}x_{n-1} & = b_1, \\ . & . & & . & . \\ . & . & & . & . \\ a_{n-1,0}x_0 & + a_{n-1,1}x_1 & + \cdots & + a_{n-1,n-1}x_{n-1} & = b_{n-1}. \end{array}$$

In matrix notation, the above system is written as $Ax = b$ where A is a dense $n \times n$ matrix of coefficients such that $A[i,j] = a_{i,j}$, b is an $n \times 1$ vector $[b_0, b_1, \ldots, b_{n-1}]^T$, and x is the desired solution vector $[x_0, x_1, \ldots, x_{n-1}]^T$. From here on, we will denote the matrix elements $a_{i,j}$ and x_i by $A[i,j]$ and $x[i]$, respectively. A system of equations $Ax = b$ is usually solved in two stages. First, through a set of algebraic manipulations, the original system of equations is reduced to an upper triangular system of the form

$$\begin{array}{llllll} x_0 & + u_{0,1}x_1 & + u_{0,2}x_2 & + \cdots & + u_{0,n-1}x_{n-1} & = y_0, \\ & x_1 & + u_{1,2}x_2 & + \cdots & + u_{1,n-1}x_{n-1} & = y_1, \\ & & & & . & . \\ & & & & . & . \\ & & & & x_{n-1} & = y_{n-1}. \end{array}$$

We write the above system as $Ux = y$, where U is an upper-triangular matrix, that is, one where the subdiagonal entries are zero and all principal diagonal entries are equal to one. More formally, $U[i,j] = 0$ if $i > j$, otherwise $U[i,j] = u_{i,j}$, and furthermore, $U[i,i] = 1$ for $0 \leq i < n$. In the second stage of solving a system of linear equations, the upper-triangular system is solved for the variables in reverse order, from $x[n-1]$ to $x[0]$ using a procedure called back-substitution.

A serial implementation of a simple Gaussian elimination algorithm is shown below. The algorithm converts the system of linear equations $Ax = b$ into a unit upper-triangular system $Ux = y$. We assume that the matrix U shares storage with A and overwrites the upper-triangular portion of A. So, the element $A[k, j]$ computed in line 5 of the code is actually $U[k, j]$. Similarly, the element $A[k, k]$ that is equated to 1 in line 8 is $U[k, k]$. Also, we assume that $A[k, k] \neq 0$ when it is used as a divisor in lines 5 and 7. So, our implementation is numerically unstable, though it should not be a concern for this assignment. For k ranging from 0 to $n - 1$, the Gaussian elimination procedure systematically eliminates the variable $x[k]$ from equations $k + 1$ to $n - 1$ so that the matrix of coefficients becomes upper-triangular. In the k^{th} iteration of the outer loop (line 3), an appropriate multiple of the k^{th} equation is subtracted from each of the equations $k + 1$ to $n - 1$.

```
 1: procedure GAUSS_ELIMINATE($A$, $b$, $y$)
 2: int $i$, $j$, $k$;
 3: for $k$ := 0 to $n - 1$ do
 4:    for $j$ := $k + 1$ to $n - 1$ do
 5:       $A[k, j]$ := $A[k, j]/A[k, k]$;                 /* Division step */
 6:    end for
 7:    $y[k]$ := $b[k]/A[k, k]$;
 8:    $A[k, k] := 1$;
 9:    for $i$ := $k + 1$ to $n - 1$ do
10:       for $j$ := $k + 1$ to $n - 1$ do
11:          $A[i, j]$ := $A[i, j]$ - $A[i, k] \times A[k, j]$;     /* Elimination step */
12:       end for
13:       $b[i]$ := $b[i] - A[i, k] \times y[k]$;
14:       $A[i, k] := 0$;
15:    end for
16: end for
```

Develop a parallel formulation of `GAUSS_ELIMINATE` using OpenMP. You may develop additional code as needed. The program given to you accepts the width of the square matrix as the command-line parameter. The upper-diagonal matrix generated by the multi-threaded code is compared against the reference single-threaded result and if the solutions match within a certain tolerance, the application will print out "TEST PASSED" to the screen before exiting.

For maximum points, parallelize both the division and elimination steps. Do not constantly create and destroy threads, but rather create them only once and coordinate execution with the most appropriate synchronization mechanisms. Upload all source files needed to run your code on xunil as a single zip file on BBLearn. Also, provide a short report describing the parallelization process, using code or pseudocode to help the discussion, and the speedup obtained over the serial version for matrix sizes of 512×512, 1024×1024, 2048×2048, and 4096×4096, for 4, 8, 16, and 32 threads. The report can include the names of the team members on the cover page.