Faster maximal clique enumeration in large real-world link streams

Alexis Baudin

January 30, 2024

IRIF – Graph seminar

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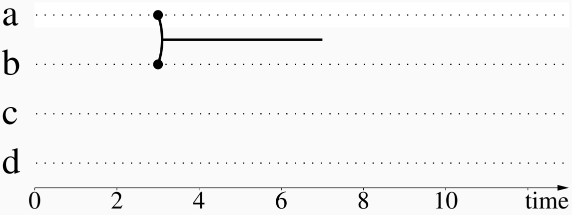
1 - Maximal clique enumeration in link streams



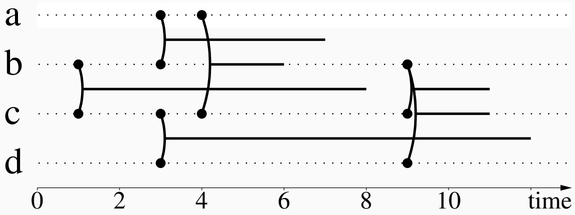
Link stream

- Vertices : a, b, c et d
- Time period : [0,12]
- Interaction : temporal links
  - a, b linked over $[3, 7]$

Advantages

- deals directly with the stream of interactions
- no arbitrary choice of time scale
- time is continuous

1 - Maximal clique enumeration in link streams
> Link stream: model temporal interactions



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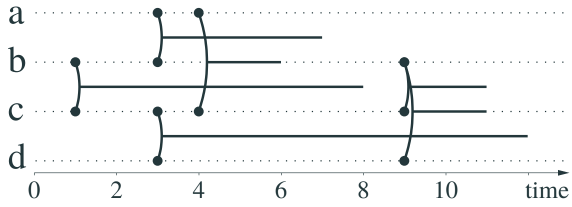

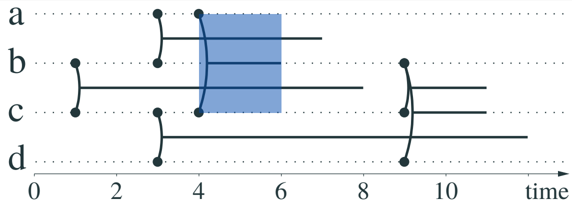

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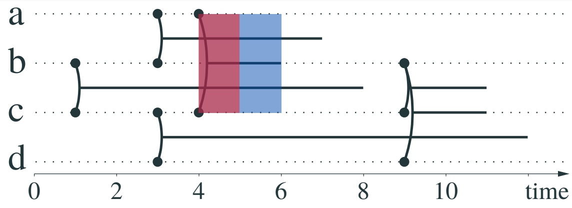

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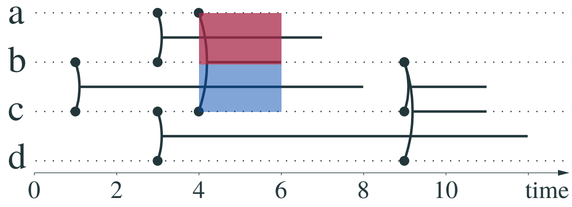

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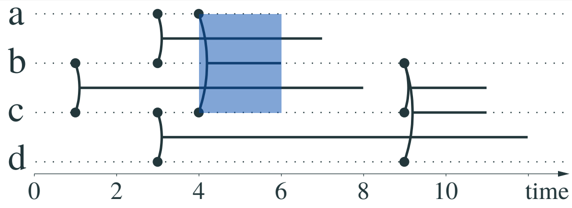

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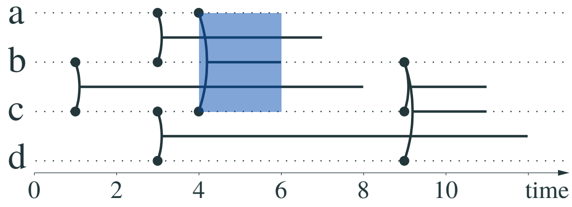

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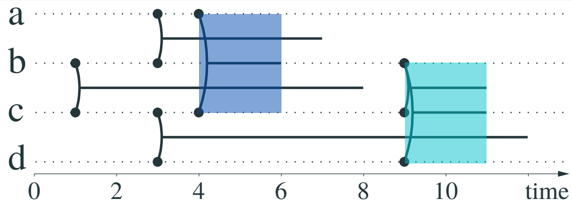

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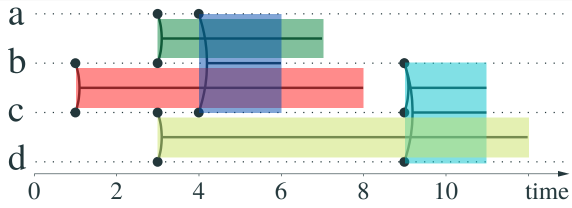

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How to efficiently enumerate maximal cliques in massive real-world link streams?

Input: A link stream.

Output: List of all its maximal cliques.

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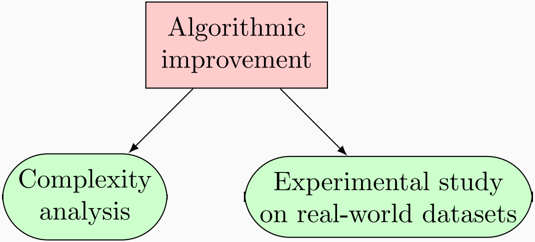
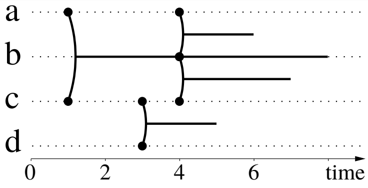


2 - New algorithm

List of temporal links sorted chronologically



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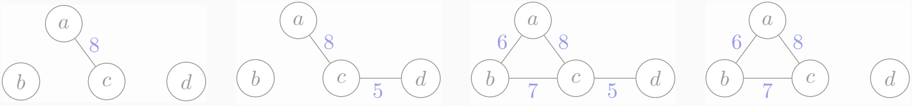
Evolving instantaneous graph G_t, with end dates



$G_1 : t = 1$ $G_3 : t = 3$ $G_4 : t = 4$ $G_5 : t = 5$

2 - New algorithm
> Input datastructure

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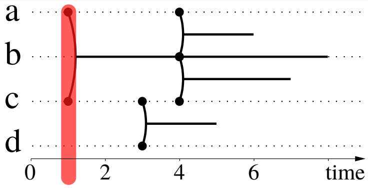

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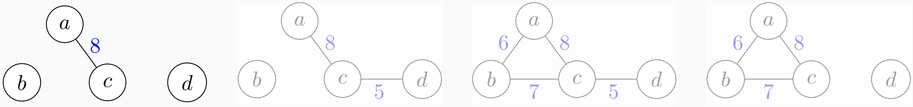

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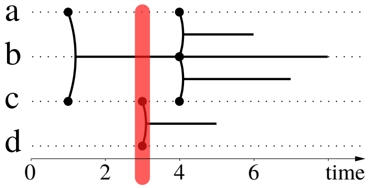

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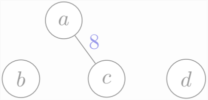

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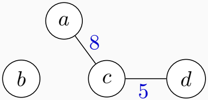
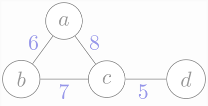
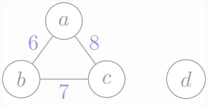


$G_3 : t = 3$



$G_4 : t = 4$



$G_5 : t = 5$

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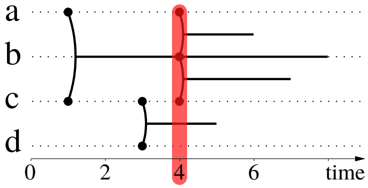
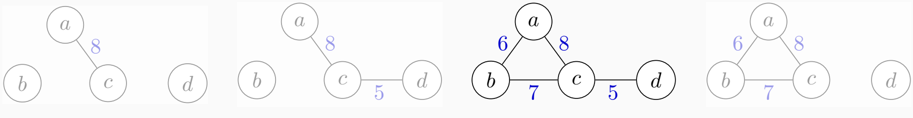


Evolving instantaneous graph G_t, with end dates



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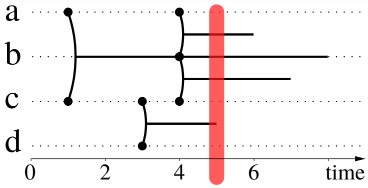

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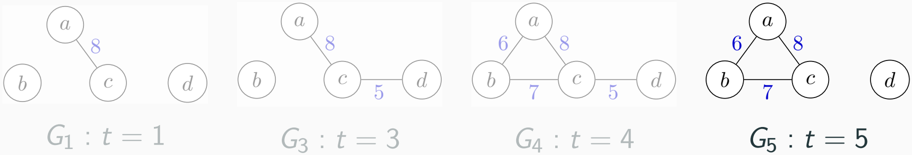
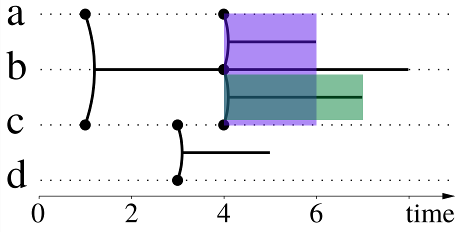


2 - New algorithm
> Maximal cliques that begin at each time t



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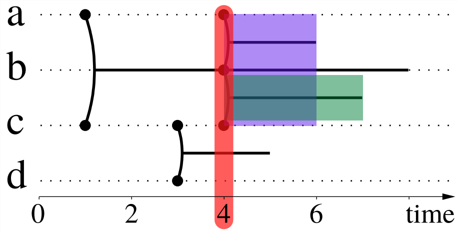
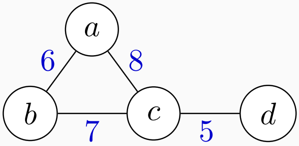


Instantaneous graph G_4



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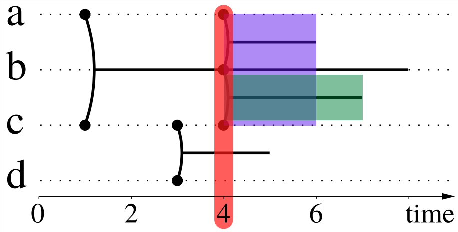

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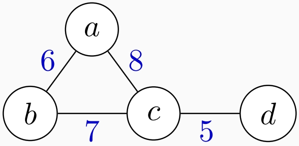


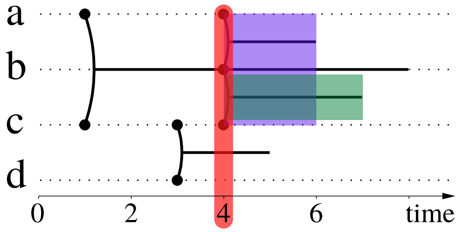
Cliques of G_4 that contain a new edge $\Longleftrightarrow$ Time-maximal cliques that begin at $t = 4$

Graph algorithm

Instantaneous graph G_4

Filter

Maximal cliques that begin at $t = 4$



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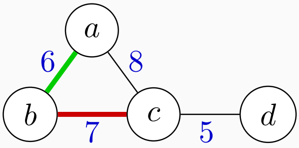
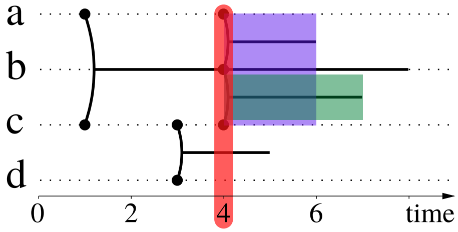


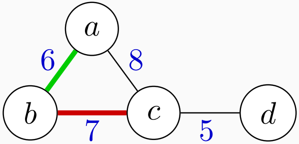
$(\{b, c\},\quad), (\{a, b, c\},\quad), (\{a, b\},\quad)$

$\rightarrow$ cliques of G_4 containing a new edge

2 - New algorithm

$>$ Maximal cliques that begin at each time t





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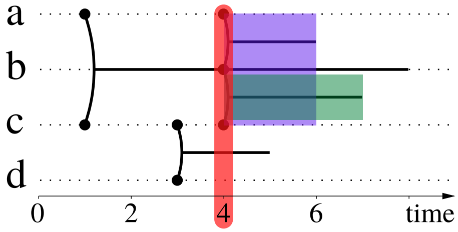
$\rightarrow$ starting time: $t = 4$



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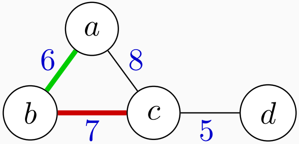

({b, c}, [4, 7]), ({a, b, c}, [4, 6]), ({a, b}, [4, 6])

→ cliques of G_4 containing a new edge

→ starting time: $t = 4$

→ end time: min of edge end times

2 - New algorithm

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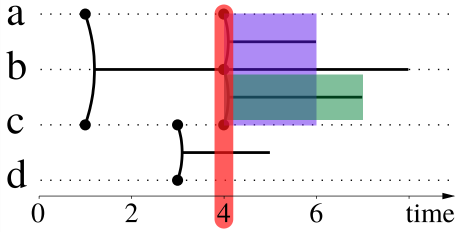

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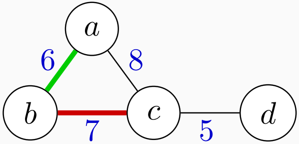

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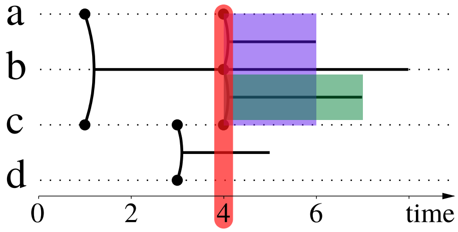
- $\rightarrow$ end time: min of edge end times
- $\rightarrow$ filter vertex-maximal cliques



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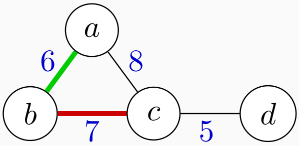

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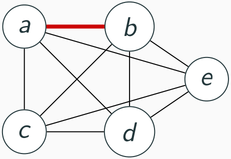
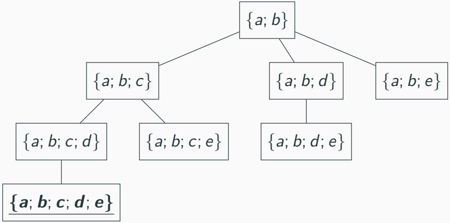
Back to a graph problem: clique enumeration in G_t.

> Pruning the enumeration in G_t

Clique enumeration

in some G_t





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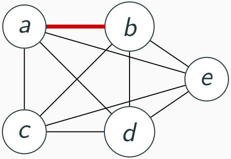

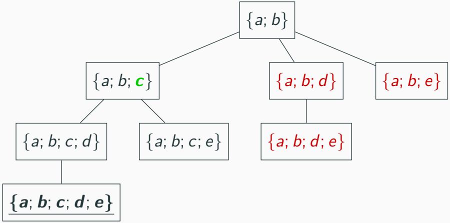

c chosen as pivot at first step

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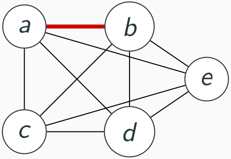

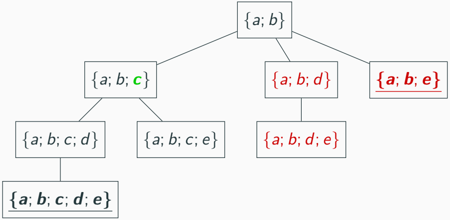

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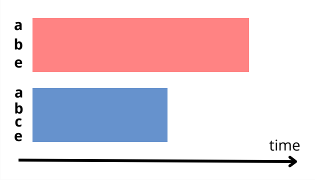
Example 1: do not prune $\{a; b; e\}$



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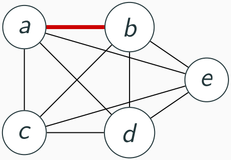

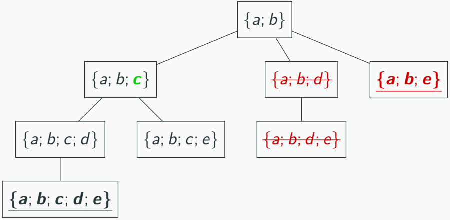

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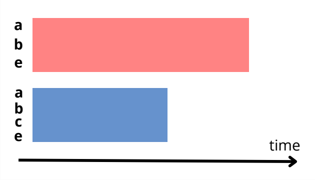

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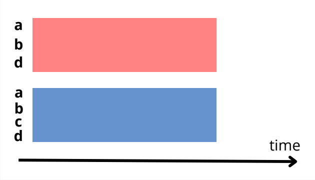
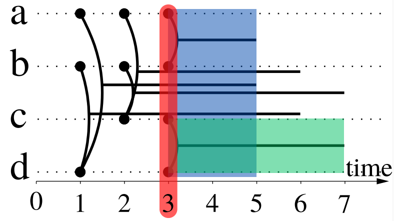
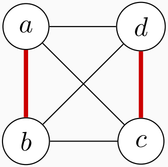
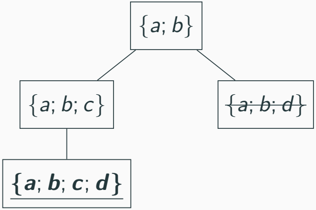


2 - New algorithm
> Pruning branches with already processed edges



Instantaneous graph G_3





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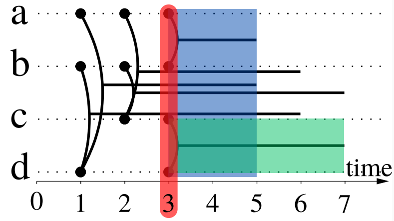

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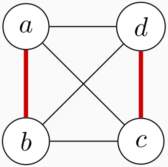

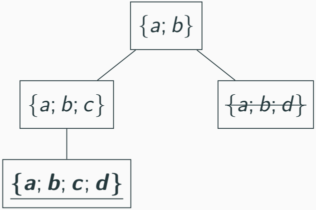

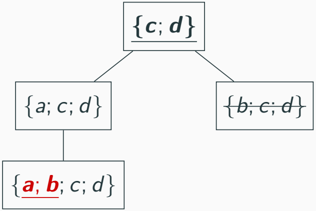

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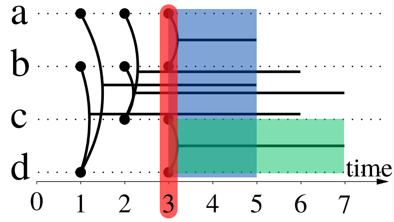
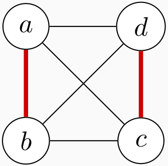
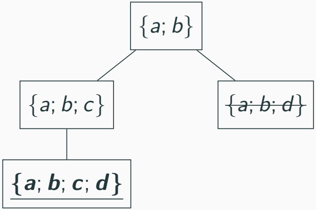
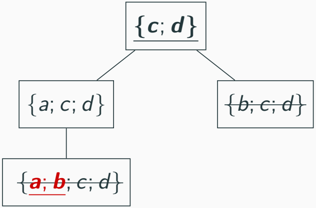


Instantaneous graph G_3







Summary of the new algorithm

$\rightarrow$ cliques not stored in memory ✓

$\rightarrow$ interactions reduced at each time step ✓

Summary of the new algorithm

→ cliques not stored in memory ✓

→ interactions reduced at each time step ✓

State of the art : four main works

- Viard *et al.* 2016
- Viard *et al.* 2018

Store all cliques
⇒ too costly in memory

- Himmel *et al.* 2017
- Bentert *et al.* 2019

Need all past and future interactions when processing a vertex.

3 - Complexity analysis

3 - Complexity analysis
> From input characteristics

Input characteristics

$\boldsymbol{d}$: maximal instantaneous degree

$\boldsymbol{m}$: number of links

3 - Complexity analysis
> From input characteristics

Input characteristics

***d*:** maximal instantaneous degree

***m*:** number of links

Algorithm: for each link $(u \xrightarrow{[t_0,t_1]} v)$:

$\rightarrow$ List and process cliques in G_{t_0} that contain $\{u, v\}$

- Number of those cliques: $\mathcal{O}\left(2^d\right)$
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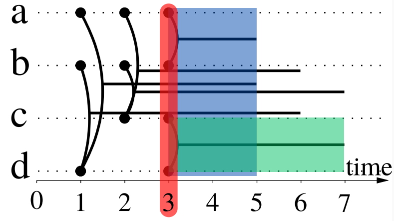
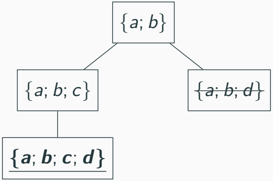
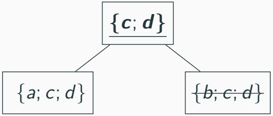
3 - Complexity analysis
> From output characteristics







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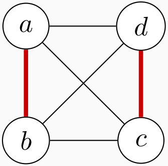

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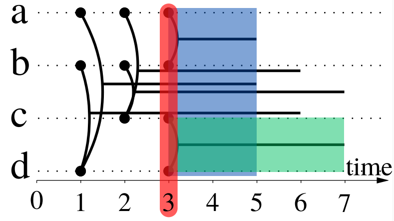

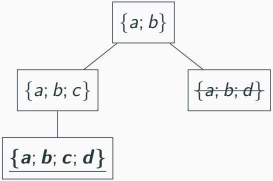

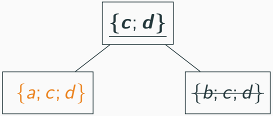

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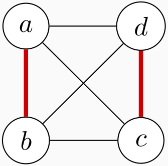

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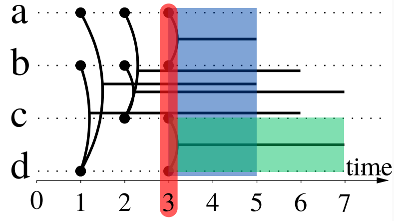

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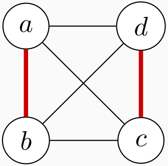
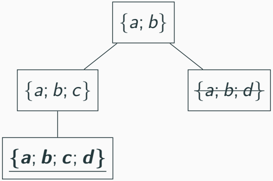
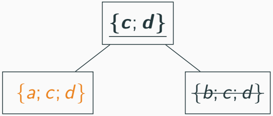




Ratio "good" leaves

$$r = \frac{\text{nb maximal clique leaves}}{\text{nb leaves}}$$



Here: $r = \frac{1}{4}$; optimal: $r = 1$.

Output characteristics

α**:** number of maximal cliques

$\boldsymbol{q}$**:** maximal number of vertices in a clique

3 - Complexity analysis
$>$ From output characteristics

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$\rightarrow$ Nb nodes: $\mathcal{O}\left((\text{max depth}) \cdot (\text{nb leaves})\right)$

- max depth $= q$
- nb leaves $= \frac{1}{r} \cdot (\text{nb maximal clique leaves}) = \mathcal{O}\left(\frac{1}{r} \cdot \alpha\right)$

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$\rightarrow$ Cost of a node: $\mathcal{O}\left(d^2\right)$

$\Rightarrow$ Complexity of the algorithm: $\boxed{\mathcal{O}\left(\frac{1}{r} \cdot \alpha \cdot q \cdot d^2\right)}$

From input characteristics

$$\boxed{\mathcal{O}\left(m \cdot d^2 \cdot 2^d\right)}$$

From output characteristics

$$\boxed{\mathcal{O}\left(\frac{1}{r} \cdot d^2 \cdot q \cdot \alpha\right)}$$

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($c \leq d$ degeneracy, $|T|$ number of time steps)

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$$\boxed{\mathcal{O}\left(\frac{1}{r} \cdot d^2 \cdot q \cdot \alpha\right)}$$

- output size $= \mathcal{O}\left(q \cdot \alpha\right) \Rightarrow$ factor $\frac{1}{r} \cdot d^2$ from output size

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From output characteristics

$$\boxed{\mathcal{O}\left(\frac{1}{r} \cdot d^2 \cdot q \cdot \alpha\right)}$$

- output size $= \mathcal{O}\left(q \cdot \alpha\right) \Rightarrow$ factor $\frac{1}{r} \cdot d^2$ from output size
- $1 \leq \frac{1}{r} \leq 2^q$ but $\frac{1}{r} \approx 1.1$ in practice (experiments)

4 - Experimental study: performance gains

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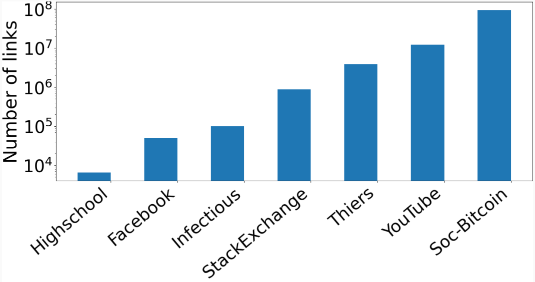
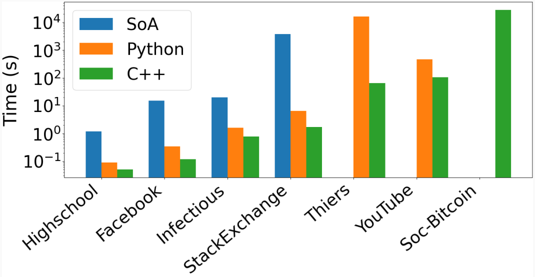
Datasets: state of the art + massive link streams

- communication networks
- human interactions



4 - Experimental study: performance gains



Experimental protocol

- Code in `Python` and `C++`
- Maximum 24h and 390Gb RAM.

5 - Conclusion and perspectives

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> Contributions

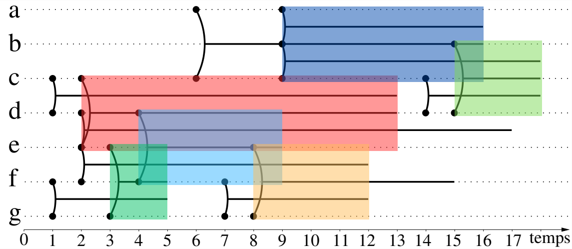
New algorithm
→ cliques not stored
→ interactions at each time step

Complexity
→ input characteristics
→ output characteristics

Experiments
→ real-world datasets
→ gain orders of magnitude:
– time computation
– size datasets

Communities in link streams by clique percolation

[Baudin et al. 2023]



→ temporal data analysis

→ anomaly detection

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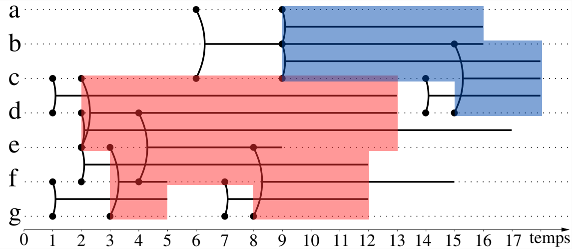

→ temporal data analysis

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Improve enumeration by ordering nodes

Ordering the nodes of each instantaneous graph G_t.

[Eppstein *et al.* 2010]: core ordering of vertices.

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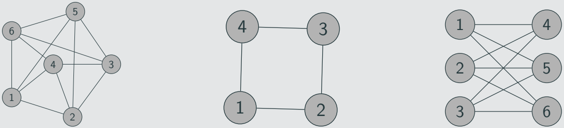
Other temporal enumeration using a same framework

Enumerate other motifs in each instantaneous graph G_t.



Thank you for your attention!

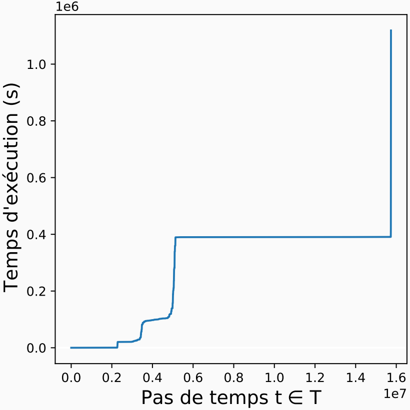
Code in open-access:

`https://gitlab.lip6.fr/baudin/maxcliques-linkstream`

Contact: `alexis.baudin@lip6.fr`

Appendix

Study the limits of computation



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