MATHEMATICS OF OPERATIONS RESEARCH
Vol. 2, No. 1, February, 1977
Printed in U.S.A.

VALID INEQUALITIES AND SUPERADDITIVITY FOR 0-1 INTEGER PROGRAMS*

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CORE

It is shown that valid inequalities for 0-1 problems can be essentially characterized by two underlying functions, one of which is superadditive. These functions are essential to the characterization of maximal inequalities, the projection of valid inequalities and the definition of a master polytope. Similar properties are shown to hold for 0-1 group problems.

1. Introduction. The first major characterization of valid inequalities for an integer programming problem was the work of Gomory [7], in which he showed subadditivity to be the crucial property for the asymptotic group problem. More recently Gomory and Johnson [8], [9], Johnson [13]–[15], Durand-Araoz [1] and others have extended this approach to the mixed group, mixed integer, and certain unbounded all-integer problems, convexity being the additional vital property when continuous variables are involved.

Here we attempt to obtain a characterization of valid inequalities for 0-1 problems. In particular, we show how dynamic programming recursions can play a most important role in this characterization, and help to unify two recent approaches to the 0-1 problem, in particular the work of Hammer et al. [11] and Johnson [16] on Master Polytopes, and certain papers on projection, in particular that of Balas and Zemel [4].

We show that superadditivity is still the crucial property in characterizing valid inequalities, though this is less evident than in the non-0-1 case. For most of the paper we shall restrict ourselves to a special subset of 0-1 problems, monotone problems with nonnegative integer coefficients, for ease of exposition. However, the extension of the results and their proofs is in general evident.

In §2 we associate two functions to a given inequality, one of which is superadditive, and develop the properties of these functions, along with their relation to maximal inequalities. In §3 we restate one of the basic results on the projection of valid inequalities within the dynamic programming framework, and then show how the two functions defined above are crucial in characterizing projected inequalities, and their possible uniqueness. In §4 we apply the earlier results to a Master Polytope, and examine the problem of obtaining some or all facets of this polytope. In §5 we look at the 0-1 group problem, and more general 0-1 problems to indicate that super/subadditivity is also the basic property underlying their valid inequalities. Finally, we terminate with some suggestions for further research.

2. Valid inequalities and superadditivity. Throughout the next three sections we shall consider the set P:

$$\sum_{j=1}^{n} a_j x_j \leqslant b, \qquad x_j \in \{0, 1\}, \tag{P}$$

where $\{a_j\}_{j=1}^{n}$, $b \in Z_m^+$ (the set of nonnegative integer m-vectors).

* Received November 14, 1975; revised September 14, 1976.
AMS 1970 subject classification. Primary 90C10.
IAOR 1973 subject classification. Main: Nonlinear programming.
Key words. Integer programs, groups, superadditivity, inequalities, dynamic programming.

DEFINITION 2.1. An inequality

$$\sum_{j=1}^{n} \pi_j x_j \leqslant \pi_0 \tag{1}$$

is a *valid inequality* for P (denoted $(\pi;\pi_0)$) if every feasible point in P satisfies the inequality. A valid inequality is a *facet* of P if $\exists n$ affinely independent points of P satisfying it with equality. Note that P has a facet if and only if $\operatorname{conv}(P)$ is of full dimension, i.e., $a_j \leqslant b\ \ \forall j = 1, 2, \ldots, n$.

The basic purpose of this paper is to examine the properties of the valid inequalities and facets of P. Also by considering $\sum_{j=1}^{n} \pi_j x_j$ as an objective function various elements of sensitivity analysis can be obtained.

NOTATION 2.2.

$$Z_m^+(b) = \{\lambda \mid \lambda \in Z_m^+, \lambda \leqslant b\},$$

$$P_r(\lambda) = \left\{x \mid \sum_{j=1}^{r} a_j x_j \leqslant \lambda,\ x_j \in \{0, 1\}\right\},$$

$$G_r(\lambda) = \max\left\{\sum_{j=1}^{r} \pi_j x_j \mid x \in P_r(\lambda)\right\}.$$

Note that $P \equiv P_n(b)$, and that $(\pi;\pi_0)$ is a valid inequality if and only if $\pi_0 \geqslant G_n(b)$. Below we shall always assume that (1) is tight for P, i.e., $\pi_0 = G_n(b)$. The reader will recognize $G_r(\lambda)$ from dynamic programming, and remember that:

$$G_r(\lambda) = \max\{G_{r-1}(\lambda), G_{r-1}(\lambda - a_r) + \pi_r\},$$

with appropriate initial conditions.

We also note that as the inequalities $(\pi;\pi_0)$ and $(\lambda\pi;\lambda\pi_0)$ define the same half space in R^n $\forall\lambda > 0$, the set of valid inequalities forms a cone. Throughout the paper we shall treat $(\pi;\pi_0)$ and $(\lambda\pi;\lambda\pi_0)$ with $\lambda > 0$ as one unique inequality.

DEFINITION 2.3. The valid inequality $(\pi;\pi_0)$ *dominates* the valid inequality $(\pi';\pi_0')$ if $\pi_j \geqslant \pi_j'$, $j = 1, 2, \ldots, n$, and $\pi_0 \leqslant \pi_0'$. The valid inequality $(\pi;\pi_0)$ is *maximal* if it is not dominated by any other valid inequality.

DEFINITION 2.4. The valid inequality $(\pi;\pi_0)$ is an *extreme* valid inequality if it cannot be expressed as the midpoint of two other distinct valid inequalities.

It appears natural to try and restrict attention to the maximal and extreme valid inequalities, especially as it is well known that: (a) the valid inequalities form a convex set, (b) the extreme valid inequalities are the facets of P, and (c) the extreme valid inequalities are maximal [14].

Given the set P, it is clear that $\pi_0 \geqslant 0$. It is also well known that aside from the trivial inequalities $x_j \geqslant 0$, we can take $\pi_0 = G_n(b) > 0$, and also for any maximal inequality $\{\pi_j\}_{j=1}^{n}$ are nonnegative. Often it is convenient to normalize with $\pi_0 = 1$, so that the set of nontrivial valid inequalities $(\pi;\pi_0)$ with $\pi_0 = 1$, and $\pi_j \geqslant 0$, $j = 1, \ldots, n$, is a bounded convex set when P is of full dimension.

DEFINITION 2.5. A function $\delta : Z_m^+ \to R$ is *superadditive* on $Z_m^+(b)$ if $\delta(\lambda) + \delta(\mu) \leqslant \delta(\lambda + \mu)$ whenever $\lambda, \mu, \lambda + \mu \in Z_m^+(b)$.

Now as stated in the introduction we shall associate two families of functions from $Z_m^+(b)$ to R to the inequality (1), which are crucial to the underlying structure.

DEFINITION 2.6.

$$\alpha_r(\lambda) = \min_{\mu}\{G_r(\lambda + \mu) - G_r(\mu) \mid \mu, \lambda + \mu \in Z_m^+(b)\},$$

$$\beta_r(\lambda) = G_r(b) - G_r(b - \lambda).$$

THEOREM 2.7. (i) *$\alpha_n \leqslant \beta_n$ on $Z_m^+(b)$, $\alpha_n \leqslant G_n$ on $Z_m^+(b)$;*
(ii) *α_n is superadditive on $Z_m^+(b)$;*
(iii) *$\alpha_n = \beta_n$ on $Z_m^+(b)$ if and only if β_n is superadditive on $Z_m^+(b)$.*

PROOF. (i) Follows immediately from the definitions, as $G_n(0) = 0$.
(ii) To show that $\alpha_n(\lambda) + \alpha_n(\lambda') \leqslant \alpha_n(\lambda + \lambda')$ on $Z_m^+(b)$ let

$$\alpha_n(\lambda + \lambda') = \min_{\mu} \{ G_n(\lambda + \lambda' + \mu) - G_n(\mu) \} = G_n(\lambda + \lambda' + \mu'') - G_n(\mu'');$$

then

$$\begin{aligned} \alpha_n(\lambda) + \alpha_n(\lambda') &= \min_{\mu} \{ G_n(\lambda + \mu) - G_n(\mu) \} + \min_{\mu'} \{ G_n(\lambda' + \mu') - G_n(\mu') \} \\ &\leqslant \{ G_n(\lambda + \mu'') - G_n(\mu'') \} + \{ G_n(\lambda' + \lambda + \mu'') - G_n(\lambda + \mu'') \} \\ &= G_n(\lambda + \lambda' + \mu'') - G_n(\mu'') = \alpha_n(\lambda + \lambda'). \end{aligned}$$

(iii) $\beta_n(\lambda)$ is superadditive
if and only if $\beta_n(\lambda) + \beta_n(\mu) \leqslant \beta_n(\lambda + \mu)$, for all $\lambda, \mu, \lambda + \mu \in Z_m^+(b)$,
if and only if $\beta_n(\lambda) + G_n(b) - G_n(b - \mu) \leqslant G_n(b) - G_n(b - \lambda - \mu)$,
if and only if $\beta_n(\lambda) \leqslant G_n(b - \mu) - G_n(b - \mu - \lambda)$,
if and only if $\beta_n(\lambda) = \min_v \{ G_n(v) - G_n(v - \lambda) \}$,
if and only if $\beta_n(\lambda) = \alpha_n(\lambda)$.

DEFINITION 2.8. The valid inequality (1) $(\pi; \pi_0)$ is said to be *superadditive* if β_n is superadditive on $Z_m^+(b)$.

Let $q(\lambda)$ = the number of times λ occurs in the set $\{a_j\}_{j=1}^n$. If $q(\lambda) > 0$, denote the corresponding coefficients by $\pi_\lambda^1 \geqslant \pi_\lambda^2 \geqslant \cdots \geqslant \pi_\lambda^{q(\lambda)}$.

Let $k(\lambda)$ = the largest integer such that $k(\lambda) \cdot \lambda \leqslant b$.

LEMMA 2.9. (i) *If inequality* (1) *is maximal, then $\pi_j \geqslant \beta_n(a_j)$ $\forall j = 1, 2, \ldots, n$.*
(ii) *If inequality* (1) *is maximal, and for some $\lambda = a_j$, $\pi_\lambda^{q(\lambda)} = \alpha_n(\lambda)$, then $\alpha_n(\lambda) = \beta_n(\lambda)$.*

PROOF. (i) Consider an inequality (1) with $\pi_j < \beta_n(a_j)$. Then we claim that the coefficient π_j can be increased to $\beta_n(a_j)$. Let $G_n^*(\lambda)$ be the corresponding values with $\beta_n(a_j)$ replacing π_j. Let $G_{n+1}(\lambda)$ be the values with $(\pi_{n+1}, a_{n+1}) = (\beta_n(a_j), a_j)$. Then $G_{n+1} \geqslant G_n^* \geqslant G_n$, and by definition of $\beta_n(a_j)$ and the recurrence formula for G_{n+1}, $G_{n+1}(b) = G_n(b)$. Hence $G_n^*(b) = G_n(b)$, and the inequality is still valid after the substitution. Hence (1) is not maximal.

(ii) From (i) we have that $\alpha_n(\lambda) = \pi_\lambda^{q(\lambda)} \geqslant \beta_n(\lambda)$ $\forall \lambda$ with $q(\lambda) > 0$. But $\alpha_n \leqslant \beta_n$, and the result follows.

A natural question to ask is whether the converse of (ii) holds. The following result is weak in that it fails to characterize a large number of maximal inequalities.

LEMMA 2.10. *If for all λ with $q(\lambda) > 0$, $\alpha_n(\lambda) = \beta_n(\lambda) \leqslant \pi_\lambda^{q(\lambda)}$, and for each such λ either*
(i) *$q(\lambda) \geqslant k(\lambda)$, or*
(ii) *$k(\lambda) > 1$, $\pi_\lambda^{q(\lambda)} = \alpha_n(\lambda)$ and $2\alpha_n(\lambda) < \beta_n(2\lambda)$,*
then inequality (1) *is maximal.*

PROOF. (i) If (1) is not maximal, $\pi_\lambda^{q(\lambda)}$ can be increased to $\pi_\lambda'^{q(\lambda)}$. However, as $x_\lambda^j = 1$, $j = 1, 2, \ldots, k(\lambda)$, is not a feasible solution to $P_n(b - \lambda)$, and $\pi_\lambda^1 \geqslant \pi_\lambda^2 \geqslant \cdots \geqslant \pi_\lambda^{q(\lambda)}$, $\exists$ an optimal solution with value $G_n(b - \lambda)$ with $x_\lambda^{q(\lambda)} = 0$. But then $\pi_\lambda'^{q(\lambda)} \leqslant G_n(b) - G_n(b - \lambda) = \alpha_n(\lambda)$, a contradiction.

(ii) Suppose that $\pi_\lambda^{q(\lambda)}$ can be increased to $\pi_\lambda'^{q(\lambda)}$, and the variables are ordered so that $x_\lambda^{q(\lambda)} = x_n$. If $G_{n-1}(b - \lambda) = G_n(b - \lambda)$, the same argument as in (i) can be used, namely $\pi_\lambda'^{q(\lambda)} \leqslant G_n(b) - G_{n-1}(b - \lambda) = \beta_n(\lambda) = \alpha_n(\lambda)$. Therefore we need only consider the case where $G_{n-1}(b-\lambda) < G_n(b-\lambda)$, which implies from the DP recursion that $G_n(b-\lambda) = G_{n-1}(b-2\lambda) + \alpha_n(\lambda)$. By definition $\alpha_n(\lambda) \leqslant G_n(b-\lambda) - G_n(b-2\lambda)$ and hence $G_{n-1}(b-2\lambda) = G_n(b-\lambda) - \alpha_n(\lambda) \geqslant G_n(b-2\lambda)$. But $G_n \geqslant G_{n-1}$, and therefore $G_{n-1}(b-2\lambda) = G_n(b-2\lambda)$. Finally $\alpha_n(\lambda) = G_n(b-\lambda) - G_n(b-2\lambda) = \beta_n(2\lambda) - \alpha_n(\lambda)$, contradicting assumption (ii).

We conclude this section with a further property of the superadditive functions α_r which will be useful later.

LEMMA 2.11. *$\alpha_{r+1} \geqslant \alpha_r$ on $Z_m^+(b)$.*

PROOF.

$$\alpha_{r+1}(\lambda) = \min_{\mu} \eta(\mu) \quad \text{where} \quad \eta(\mu) = G_{r+1}(\lambda+\mu) - G_{r+1}(\mu)$$
$$= \max\left[G_r(\lambda+\mu), G_r(\lambda+\mu-a_{r+1}) + \pi_{r+1}\right] - \max\left[G_r(\mu), G_r(\mu-a_{r+1}) + \pi_{r+1}\right].$$

Either $G_r(\mu) \geqslant G_r(\mu - a_{r+1}) + \pi_{r+1}$, implying that $\eta(\mu) \geqslant G_r(\lambda+\mu) - G_r(\mu) \geqslant \alpha_r(\lambda)$, or $G_r(\mu) < G_r(\mu - a_{r+1}) + \pi_{r+1}$, implying that

$$\eta(\mu) \geqslant \{G_r(\lambda+\mu-a_{r+1}) + \pi_{r+1}\} - \{G_r(\mu-a_{r+1}) + \pi_{r+1}\} \geqslant \alpha_r(\lambda).$$

Hence in either case $\eta(\mu) \geqslant \alpha_r(\lambda)\ \forall \mu$, and hence $\alpha_{r+1}(\lambda) \geqslant \alpha_r(\lambda)$.

EXAMPLE (1a). $6x_1 + 4x_2 + 3x_3 + 3x_4 + 1x_5 + 1x_6 + 1x_7 \leqslant 8$ with $x_j \in \{0, 1\}$ with valid inequality $10x_1 + 5x_2 + 6x_3 + 4x_4 + 3x_5 + 1x_6 + 1x_7 \leqslant 14$.

λ	1	2	3	4	5	6	7	8
$G_7(\lambda)$	3	4	6	9	10	11	13	14
$\beta_7(\lambda)$	1	3	4	5	8	10	11	14
$\alpha_7(\lambda)$	1	2	4	5	8	10	11	14

The conditions of Lemma 2.10 are satisfied, and hence the inequality is maximal.

EXAMPLE (1b). $\binom{1}{0}x_1 + \binom{0}{1}x_2 + \binom{0}{1}x_3 + \binom{1}{1}x_4 + \binom{0}{2}x_5 \leqslant \binom{1}{2}$, $x_j \in \{0, 1\}$, with valid inequality: $2x_1 + 4x_2 + 1x_3 + 3x_4 + 4x_5 \leqslant 7$,

λ	$\binom{1}{0}$	$\binom{0}{1}$	$\binom{1}{1}$	$\binom{0}{2}$	$\binom{1}{2}$
$G_5(\lambda)$	2	4	6	5	7
$\beta_5(\lambda)$	2	1	3	5	7

with β_5 superadditive on $Z_2^+\binom{1}{2}$. However, $\pi_5 < \beta_5\binom{0}{2}$, and by Lemma 2.9(i) the inequality is not maximal.

3. Projecting valid inequalities. Methods of obtaining valid inequalities by projection have recently received a large amount of attention [2], [7], [10], [16], [17], [18]–[22], [24]. Below we see that the functions α_n and β_n play an important role in characterizing such inequalities.

Suppose the set: $Z = \{(x, y) \mid \sum_{j=1}^n a_j x_j + \sum_{k=1}^p d_k y_k \leqslant b,\ x_j \in \{0, 1\},\ y_k \in \{0, 1\}\}$ with $a_j \in Z_m^+(b)$, $d_k \in Z_m^+(b)$ is given.

The basic *projection* problem is to characterize the coefficients $\{\gamma_k\}_{k=1}^p$ such that the inequality

$$\sum_{j=1}^n \pi_j x_j + \sum_{k=1}^p \gamma_k y_k \leqslant \pi_0 \tag{2}$$

is valid for Z, when inequality (1) is valid for P with $\pi_0 = G_n(b)$.

A complete characterization, or equivalent definition, easily expressed with the function G_n, has been given by Zemel:

THEOREM 3.1 [24]. *Inequality* (2) *is valid for Z, or a projection of* (1), *if and only if* $\gamma\cdot y \leqslant G_n(b) - G_n(b - Dy)$ $\forall y \in Y$, *where* $Y = \{y \mid (x, y) \in Z$ *for some* $x\}$, $\gamma = (\gamma_1, \dots, \gamma_p)$ *the vector of coefficients appearing in inequality* (2), *and D is the m by p matrix with columns* $\{d_k\}_{k=1}^{p}$ *appearing in the definition of the set Z.*

If we now consider the question of how to find the coefficients $\gamma_k = \gamma(\lambda)$ in (2) for arbitrary columns $\lambda = d_k \in Z_m^+(b)$, we see that α_n and β_n play a natural role.

THEOREM 3.2. *If* (1) *is a tight valid inequality for* (P), *and* (2) *is a valid inequality for Z,*

(i) $\alpha_{n+r} \leqslant \alpha_{n+r+1} \leqslant \beta_{n+r+1} \leqslant \beta_{n+r}$ *on* $Z_m^+(b)$.

(ii) $\alpha_n(\lambda) \leqslant \gamma(\lambda) \leqslant \beta_n(\lambda)$ *in the sense that if* $\gamma(\lambda) < \alpha_n(\lambda)$, *then* $\gamma(\lambda)$ *can be immediately raised to* $\alpha_n(\lambda)$ *without loss of validity independently of all other entries, and if* $\gamma(\lambda) > \beta_n(\lambda)$, *inequality* (2) *cannot be valid.*

(iii) $\exists$ *a unique valid inequality of the given form if and only if* $\alpha_n(d_k) = \beta_n(d_k)$ $\forall k = 1, 2, \dots, p$. *In that case* $G_n = G_{n+1} = \cdots = G_{n+p}$ *on* $Z_m^+(b)$.

(iv) $\exists$ *a unique valid inequality of the given form independent of* $\{d_k\}_{k=1}^{p}$ *if and only if inequality* (1) *is superadditive.*

REMARK. Uniqueness in (iii) and (iv) is under the assumption that $\gamma(\lambda) \geqslant \alpha_n(\lambda)$ which is possible by (ii).

PROOF. (i) $\alpha_{n+r} \leqslant \alpha_{n+r+1}$ is shown in Lemma 2.11. As $G_{n+r+1} \geqslant G_{n+r}$ and $G_{n+r+1}(b) = G_{n+r}(b)$, $\beta_{n+r+1} \leqslant \beta_{n+r}$.

(ii) Taking $d_1 = \lambda$, $\gamma_1 = \gamma(\lambda)$, we show that if $\gamma_1 < \alpha_n(\lambda)$, there can be no tight feasible solution with $y_1 = 1$. As $\gamma_1 < \alpha_n(\lambda)$, and $\alpha_n(\lambda) \leqslant G_n(b) - G_n(b - d_1) \leqslant G_{n+1}(b) - G_n(b - d_1)$, the required conclusion follows. The case for $\gamma_k = \gamma(\lambda)$ with $k > 1$ holds similarly as $\alpha_n \leqslant \alpha_{n+1} \leqslant \cdots \leqslant \alpha_{n+k}$.

(iii) From (ii) $\gamma(d_k) = \alpha_n(d_k)$. To show that $G_n = G_{n+1} = \cdots = G_{n+p}$, it suffices to show that $G_n(\lambda) = G_{n+1}(\lambda)$ $\forall \lambda \in Z_m^+(b)$. Now

$$G_n(\lambda - d_1) + \gamma(d_1) = G_n(\lambda - d_1) + \alpha_n(d_1) = G_n(\lambda - d_1) + \min_{\mu}\{G_n(\mu) - G_n(\mu - d_1)\}$$

$$\leqslant G_n(\lambda - d_1) + \{G_n(\lambda) - G_n(\lambda - d_1)\} = G_n(\lambda),$$

and as $G_{n+1}(\lambda) = \max\{G_n(\lambda), G_n(\lambda - d_1) + \gamma(d_1)\}$ the result follows.

(iv) follows from (iii) and the definition of a superadditive inequality.

REMARK 3.3. Upper and lower bounds on $\gamma(\lambda)$ have been calculated for special facets of the knapsack polytope by Balas and Zemel [4], and Peled [20]. Naturally the bounds they derived correspond to $\beta_n(\lambda)$ and $\alpha_n(\lambda)$, respectively.

REMARK 3.4. The functions α_n and β_n can be seen to have immediate applications to sensitivity analysis. $\beta_n(\lambda)$ is the marginal price level required to maintain profit that one wishes to use once some new activity requires λ units of the resources. $\alpha_n(\lambda)$ is a lower bound on the marginal price level if there is an unlimited supply of this new activity.

ASSUMPTION. *From now on we shall always assume that* $\pi_j \geqslant \beta_n(a_j)$ *and* $\gamma(\lambda) \geqslant \alpha_n(\lambda)$ *in any inequality* (1) *or* (2).

EXAMPLE 2 (Balas and Zemel [4]). (a) Starting from:

$$4x_1 + 4x_2 + 4x_3 + 4x_4 \leqslant 13, \qquad x_j \in \{0, 1\}, \tag{P}$$

with the valid inequality

$$x_1 + x_2 + x_3 + x_4 \leqslant 3, \tag{1}$$

we have:

λ	1	2	3	4	5	6	7	8	9	10	11	12	13
$G_4(\lambda)$	0	0	0	1	1	1	1	2	2	2	2	3	3
$\beta_4(\lambda)$	0	1	1	1	1	2	2	2	2	3	3	3	3
$\alpha_4(\lambda)$	0	0	0	1	1	1	1	2	2	2	2	3	3

(b) As $\alpha_4(\lambda) = \beta_4(\lambda)$ for $\lambda = 5, 8, 9$, it follows from Theorem 3.2 (iii) that $x_1 + x_2 + x_3 + x_4 + y_1 + 2y_2 + 2y_3 \leqslant 3$ is the unique projection of (1) that is a valid inequality for:

$$4x_1 + 4x_2 + 4x_3 + 4x_4 + 5y_1 + 8y_2 + 9y_3 \leqslant 13, \qquad x_j, y_k \in \{0, 1\},$$

and $G_7 = G_4$ on $Z_1^+(13)$.

(c) If we choose $\lambda = d_4 = 2$, with coefficient $\gamma_4 = \beta_7(2) = 1$, we obtain

λ	1	2	3	4	5	6	7	8	9	10	11	12	13
$G_8(\lambda)$	0	1	1	1	1	2	2	2	2	3	3	3	3
$\beta_8(\lambda)$	0	0	0	1	1	1	1	2	2	2	2	3	3

and β_8 is superadditive.

Therefore again by Theorem 3.2,

$$x_1 + x_2 + x_3 + x_4 + y_1 + 2y_2 + 2y_3 + y_4 + y_5 + y_6 + 0y_7 + 0y_8 \leqslant 3$$

is the unique projection of (1) with $\gamma(2) = \gamma_4 = 1$ that is a valid inequality for:

$$Z: 4x_1 + 4x_2 + 4x_3 + 4x_4 + 5y_1 + 8y_2 + 9y_3 + 2y_4 + 7y_5 + 6y_6 + 3y_7 + 1y_8 \leqslant 13,$$
$$x_j, y_k \in \{0, 1\}.$$

4. The master polytope. Following Hammer et al. [11] for the knapsack polytope, we define the Master Polytope on $Z_m^+(b)$ as follows:

$$\sum_{\lambda \in Z_m^+(b)} \sum_{j=1}^{k(\lambda)} \lambda x_\lambda^j \leqslant b, \qquad x_\lambda^j \in \{0, 1\}. \tag{MP}$$

Valid inequalities for MP are written as:

$$\sum_{\lambda \in Z_m^+(b)} \sum_{j=1}^{k(\lambda)} \pi_\lambda^j x_\lambda^j \leqslant \pi_0 \tag{3}$$

where $\pi_\lambda^1 \geqslant \pi_\lambda^2 \geqslant \cdots \geqslant \pi_\lambda^{k(\lambda)}$ and we let $N = \sum_\lambda k(\lambda)$, $\pi_0 = G_N(b)$.

The idea behind the Master Polytope is that (3) gives a valid inequality for any problem P, namely the inequality $\sum_{\lambda \in Z_m^+(b)} \sum_{j=1}^{q(\lambda)} \pi_\lambda^j x_\lambda^j \leqslant \pi_0$, where $\pi_\lambda^j = \pi_\lambda^{k(\lambda)}$ if $j > k(\lambda)$. This is justified by Theorem 4.1 below coupled with the unique projection of superadditive inequalities (Theorem 3.3(iv)).

THEOREM 4.1. *Inequality* (3) *is maximal if and only if it is superadditive, and* $\pi_\lambda^{k(\lambda)} \geqslant a_N(\lambda)$ $\forall \lambda \in Z_m^+(b)$. *In addition* $\pi_\lambda^{k(\lambda)} = \alpha_N(\lambda)$ $\forall \lambda \in Z_m^+(b)$.

PROOF. By maximality, and Lemma 2.9, $\pi_\lambda^{k(\lambda)} \geqslant \beta_N(\lambda)$. Now suppose $\alpha_N(\lambda) = G_N(\lambda + \mu^*) - G_N(\mu^*)$. By the construction of MP, and $k(\lambda)$, there exists an optimal feasible solution giving value $G_N(\mu^*)$ with $x_\lambda^{k(\lambda)} = 0$. Hence $G_N(\lambda + \mu^*) \geqslant G_N(\mu^*) + \pi_\lambda^{k(\lambda)}$, and therefore $\alpha_N(\lambda) \geqslant \pi_\lambda^{k(\lambda)}$. As $\alpha_N \leqslant \beta_N$, the result follows.

The converse is a restatement of Lemma 2.10(i) with $q(\lambda) = k(\lambda)$.

We now turn to the problem of obtaining maximal valid inequalities or facets for MP. In particular we can restrict our attention to superadditive functions on $Z_m^+(b)$. First we note certain properties of the maximal inequalities.

LEMMA 4.2. *If the inequality* (3) *is maximal for MP, then*
(i) $\alpha_N(\lambda) \leqslant \pi_\lambda^1 \leqslant \min_\mu \{G_N(\lambda+\mu) - \alpha_N(\mu)\} \leqslant G_N(\lambda)$.
(ii) *If* $\alpha_N(\lambda) + \alpha_N(b-\lambda) = \alpha_N(b)$, *or equivalently* $\alpha_N(\lambda) = G_N(\lambda)$, $\pi_\lambda{}^j = \alpha_N(\lambda)$ $\forall j = 1, 2, \ldots, k(\lambda)$.
(iii) *If* $k(\lambda) = 1$, $\pi_\lambda^1 = \alpha_N(\lambda)$.
(iv) *If* $\lambda = e_j$ *and* $b_j > 2$, $\pi_\lambda^1 = G_N(\lambda)$.

PROOF. (i) $\pi_\lambda^1 + \alpha_N(\mu) \leqslant G_N(\lambda+\mu)$ via the solution $(x_\lambda^1 = 1, x_\mu^{k(\mu)} = 1)$.
(ii) The equivalence is immediate using $\alpha_N = \beta_N$, and $G_N(0) = 0$. Then as $\alpha_n(\lambda) \leqslant \pi_\lambda{}^j \leqslant \pi_\lambda^1$ the result follows from (i).
(iii) See Theorem 4.1.
(iv) From maximality, there exists a tight feasible solution with $x_\mu^1 = 1$ for $\mu = b - e_j$. If $b_j > 2$, $k(\mu) = 1$. By (iii) $\pi_{b-e_j}^1 = \alpha_N(b - e_j)$. Hence $\pi_{e_j}^1 = G_N(b) - \alpha_N(b - e_j) = G_N(e_j)$.

Any inequality (3) can also be viewed as a projection of the valid inequality:

$$\sum_\lambda \alpha^*(\lambda) x_\lambda^{k(\lambda)} \leqslant \alpha^*(b), \tag{4}$$

for $\{x \mid \sum_\lambda \lambda x^{k(\lambda)} \leqslant b,\ x_\lambda^{k(\lambda)} \in \{0,1\}\}$ where $\alpha^*(\lambda) = \pi_\lambda^{k(\lambda)}$ $\forall \lambda \in Z_m^+(b)$.

Let $\theta = \prod_{i=1}^m (b_i + 1) - 1$ be the number of variables in (4), or the number of nonzero DP states.

LEMMA 4.3. *Suppose inequality* (3) *is obtained by projection from an inequality* (4) *where* $\alpha^*(\lambda) = \pi_\lambda^{k(\lambda)}$, $\lambda \in Z_m^+(b)$, *is superadditive. Inequality* (3) *is maximal if and only if*
(i) $G_\theta = \alpha_\theta = \alpha_N = \alpha^*$,
(ii) $G_N = \beta_\theta$.

PROOF. If (3) is maximal, then by Theorem 4.1, $\alpha^* = \alpha_N = \beta_N$, and by superadditivity $G_\theta = \alpha_\theta = \alpha^*$. Finally using the definition of β_r, and the fact that $G_\theta(b) = G_N(b)$, we obtain $G_N(\lambda) = G_N(b) - \beta_N(b - \lambda) = G_\theta(b) - G_\theta(b-\lambda) = \beta_\theta(\lambda)$.

Conversely if $\alpha^* = \alpha_N$, then by Theorem 4.1, inequality (3) is maximal.

In an effort to obtain the facets of MP one might hope (in analogy with the case for asymptotic groups [7]) that the extreme rays of the polyhedral cone of superadditive functions on $Z_m^+(b)$, namely the extreme rays of

$$\Delta_m^+(b) = \{\delta \mid \delta(\lambda) + \delta(\mu) \leqslant \delta(\lambda+\mu),\ \delta \geqslant 0 \quad \text{with } \lambda, \mu, \lambda+\mu \in Z_m^+(b)\}$$

coupled with projection, would suffice to give all the nontrivial facets of MP. Unfortunately this is not the case, and some other superadditive functions also give facets.

However, as we see below it is often possible to obtain at least one different facet from each extreme superadditive function.

Let Π be the polyhedral cone of nontrivial valid inequalities for MP

$$\Pi = \left\{(\pi_\lambda; \pi_0) \mid \sum_\lambda \sum_j \xi_\lambda{}^j \pi_\lambda{}^j \leqslant \pi_0 \ \ \forall \xi \in P_N(b),\ \pi_\lambda{}^j \geqslant 0\right\}.$$

In particular we know from polarity that the maximal extreme rays of Π are the nontrivial facets of MP, see for example [5].

Let $\Pi^{\mathrm{MP}} = \{(\pi_\lambda{}^j; \pi_0) \mid (\pi_\lambda{}^j; \pi_0) \in \Pi,\ \pi_\lambda^1 \geqslant \pi_\lambda^2 \geqslant \cdots \geqslant \pi_\lambda^{k(\lambda)}\}$. This polyhedral cone is easier to work with, but has the disadvantage that its maximal extreme rays may include some valid equalities which are not facets of MP.

Let $\Sigma^{\mathrm{MP}} = \{(\pi_\lambda{}^j; \pi_0) \mid (\pi_\lambda{}^j; \pi_0) \in \Pi^{\mathrm{MP}}, \pi_\lambda^{k(\lambda)} + \pi_\mu^{k(\mu)} \leqslant \pi_{\lambda+\mu}^{k(\lambda+\mu)}\}$.

LEMMA 4.4. *If $(\pi_\lambda{}^j; \pi_0)$ is a maximal valid inequality, it is an extreme ray of Π^{MP} if and only if it is an extreme ray of Σ^{MP}.*

PROOF. As $\Sigma^{\mathrm{MP}} \subseteq \Pi^{\mathrm{MP}}$, and maximal inequalities lie in Σ^{MP}, extreme in Π^{MP} immediately implies extreme in Σ^{MP}. Conversely, suppose the inequality is extreme in Σ^{MP}, but not in Π^{MP}. Then $(\pi_\lambda{}^j; \pi_0) = \frac{1}{2}(\rho_\lambda{}^j; \rho_0) + \frac{1}{2}(\sigma_\lambda{}^j; \sigma_0)$ where $(\rho_\lambda{}^j; \rho_0), (\sigma_\lambda{}^j; \sigma_0) \in \Pi^{\mathrm{MP}}$. Now every maximal valid inequality which can be written as the midpoint of two valid inequalities is necessarily the midpoint of two maximal valid inequalities. Hence $(\rho_\lambda{}^j; \rho_0), (\sigma_\lambda{}^j; \pi_0)$ are maximal, and therefore both lie in Σ^{MP}, contradicting the assumption.

Let $\Sigma^{\mathrm{MP}}(\alpha^*) = \{(\pi_\lambda{}^j; \pi_0) \mid (\pi_\lambda{}^j; \pi_0) \in \Sigma^{\mathrm{MP}}, \pi_\lambda^{k(\lambda)} = \alpha^*(\lambda)\ \forall \lambda \in Z_m^+(b)\}$ where α^* is superadditive on $Z_m^+(b)$.

LEMMA 4.5. *If α^* is an extreme ray of $\Delta_m^+(b)$, and $(\bar{\pi}_\lambda{}^j; \bar{\pi}_0)$ is an extreme ray of $\Sigma^{\mathrm{MP}}(\alpha^*)$, then if $(\bar{\pi}_\lambda{}^j; \bar{\pi}_0)$ is maximal it is an extreme ray of Π^{MP}.*

PROOF. Suppose that $(\bar{\pi}_\lambda^j; \bar{\pi}_0) = \frac{1}{2}(\rho_\lambda{}^j; \bar{\rho}_0) + \frac{1}{2}(\sigma_\lambda{}^j; \bar{\sigma}_0)$ with $(\rho_\lambda{}^j; \bar{\rho}_0), (\sigma_\lambda{}^j; \bar{\sigma}_0) \in \Pi^{\mathrm{MP}}$. Then as above it follows that $(\rho_\lambda{}^j; \bar{\rho}_0), (\sigma_\lambda{}^j; \bar{\sigma}_0) \in \Sigma^{\mathrm{MP}}$. But as α^* is extreme, $\alpha^*(\lambda) = \rho_\lambda^{k(\lambda)} = \sigma_\lambda^{k(\lambda)}$. But then the extremality of $(\bar{\pi}_\lambda{}^j; \bar{\pi}_0)$ in $\Sigma^{\mathrm{MP}}(\alpha^*)$ implies that the rays $(\bar{\pi}_\lambda{}^j; \bar{\pi}_0)$, $(\rho_\lambda{}^j; \bar{\rho}_0)$, $(\sigma_\lambda{}^j; \bar{\sigma}_0)$ are identical, and $(\bar{\pi}_\lambda{}^j; \bar{\pi}_0)$ is extreme in Π^{MP}.

This suggests that one way to obtain facets of MP is to start from an extreme ray α^* of $\Delta_m^+(b)$. From superadditivity it provides a valid inequality (4) for $\{x \mid \sum_\lambda \lambda x_\lambda^{k(\lambda)} \leqslant b, x_\lambda^{k(\lambda)} \in \{0, 1\}\}$. Projection can be used to generate maximal extreme rays of $\Sigma^{\mathrm{MP}}(\alpha^*)$. (For example, see [24], by projecting one variable at a time, and giving each new variable y_k the maximum possible coefficient $\gamma_k = \beta_{n+k-1}(d_k)$.) Then if the resulting inequalities are maximal (or $\alpha_N = \alpha^*$), they are extreme in Π^{MP} by Lemma 4.5, and provided they are independent of the order inequalities, namely extreme in Π, they are facets of MP.

EXAMPLE 3 (Hammer and Peled [12], Johnson [16]). Take $m = 1$, $b = 5$. Then $\Delta_1^+(5)$:

$$\begin{aligned} \alpha(1) + \alpha(1) &\leqslant \alpha(2), & \alpha(2) + \alpha(2) &\leqslant \alpha(4),\\ \alpha(1) + \alpha(2) &\leqslant \alpha(3), & \alpha(1) + \alpha(4) &\leqslant \alpha(5),\\ \alpha(1) + \alpha(3) &\leqslant \alpha(4), & \alpha(2) + \alpha(3) &\leqslant \alpha(5), \end{aligned}$$

$$\alpha \geqslant 0,$$

has extreme rays:

λ	1	2	3	4	5	No.
$\alpha(\lambda)$:	1	2	3	4	5	1
	0	0	0	0	1	2
	0	0	0	1	1	3
	0	0	1	1	1	4
	0	1	1	2	2	5
	0	1	2	2	3	6

Taking extreme ray No. 6, $\alpha^* = (0\ 1\ 2\ 2\ 3)$ we have from Lemma 4.3 that a resulting maximal inequality for MP must satisfy

$$\begin{array}{rl} \lambda: & 1\ 2\ 3\ 4\ 5 \\ \alpha_N = G_5 = \alpha^* = \alpha_5(\lambda): & 0\ 1\ 2\ 2\ 3 \\ G_N(\lambda): & 1\ 1\ 2\ 3\ 3 \end{array}$$

and from Lemma 4.2

(iv) $\pi_1^1 = 1$,
(ii) $\pi_2^j = 1 \ \forall j, \ \pi_3^1 = 2, \ \pi_5^1 = 3$,
(iii) $\pi_4^1 = 2$.

Hence all the coefficients except $\pi_1^j, j \geq 2$, are already fixed. Introducing x_1^1 with coefficient $\pi_1^1 = 1 = \beta_5(1)$, we obtain

$$\begin{array}{rccccc} \lambda: & 1 & 2 & 3 & 4 & 5 \\ G_6(\lambda): & 1 & 1 & 2 & 3 & 3 \\ \beta_6(\lambda): & 0 & 1 & 2 & 2 & 3. \end{array}$$

Now we can set $\pi_1^j = 0, j \geq 2$. Noting that $\alpha_6 = \beta_6 = \alpha^*$, we now know that the valid inequality (3) $1x_1^1 + 0x_1^2 + 0x_1^3 + 0x_1^4 + 0x_1^5 + 1x_2^1 + 1x_2^2 + 2x_3^1 + 2x_4^1 + 3x_5^1 \leq 3$ is maximal, and from the projection procedure extreme in $\Sigma^{\mathrm{MP}}(\alpha^*)$. Hence by Lemma 4.5 it is extreme in Π^{MP}.

For this example, each of the six extreme rays of $\Delta_1^+(5)$ gives rise to a unique facet of MP, but MP has six other facets.

5. 0-1 Polytopes and superadditivity. In the previous sections we have restricted our attention to the polytope P with coefficients limited to $Z_m^+(b)$. However, an examination of the definitions and proofs shows that the only basic condition is the dynamic programming recursion. Here we indicate that the results extend in large part to the more general 0-1 problem, and in at least one case, that of a 0-1 group problem, a restricted Master Problem can also be defined.

Consider the set:

$$\sum_{j=1}^{n} a_j x_j \in S, \qquad x_j \in \{0, 1\}, \tag{Q}$$

again with valid inequality

$$\sum_{j=1}^{n} \pi_j x_j \leq \pi_0. \tag{1}$$

DEFINITION 5.1.

$$Q_n(\lambda) = \left\{ x \mid \sum_{j=1}^{n} a_j x_j = \lambda, \qquad x_j \in \{0, 1\} \right\},$$

$$H_n(\lambda) = \max \left\{ \sum_{j=1}^{n} \pi_j x_j \mid x \in Q_n(\lambda) \right\},$$

$$\bar{\beta}_n(\lambda) = \sup_{\xi \in S} H_n(\xi) - \sup_{\xi \in S} H_n(\xi - \lambda),$$

$$\bar{\alpha}_n(\lambda) = \inf_{\mu} \left[\sup_{\xi \in S} H_n(\xi - \mu) - \sup_{\xi \in S} H_n(\xi - \lambda - \mu) \right].$$

Clearly (1) is a valid inequality for $Q = \bigcup_{\xi \in S} Q_n(\xi)$ if and only if $\pi_0 \geq \sup_{\xi \in S} H_n(\xi)$.

Given suitable domains of definition of S and the $\{a_j\}_{j=1}^n$ all the results of §§2 and 3 carry over to this general case, except Lemma 2.10(i) as $k(\lambda)$ and the Master Polytope cannot be obviously generalized.

We now examine various special cases of S, where the structure allows some further observations to be made.

Case 1. $S = \{\lambda \mid \lambda \in Z_m^+(b)\}$ with $a_j \in Z_m^+(b) \ \forall j$. Here

$$\sup_{\xi \in S} H_n(\xi - \mu) = \max \left\{ \sum_{j=1}^{n} \pi_j x_j \mid \sum_{j=1}^{n} a_j x_j \leq b - \mu, \ x_j \in \{0, 1\} \right\} = G_n(b - \mu)$$

as defined in §2. Therefore this corresponds to the case considered earlier, with $\alpha_n = \bar{\alpha}_n$, $\beta_n = \bar{\beta}_n$ and $P = Q$.

Case 2. $S = \{\lambda \in R_m \mid \lambda \leqslant b\}$ with $a_j \in R_m$ $\forall j$. Again $\sup_{\xi \in S} H_n(\xi - \mu) = G_n(b - \mu)$.

LEMMA 5.2. *$\bar{\alpha}_n(\lambda) \geqslant 0$ when $\lambda \geqslant 0$, and $\bar{\alpha}_n(\lambda) \leqslant 0$ when $\lambda < 0$, and $\bar{\beta}_n$ has the same property.*

PROOF. $G_n(\lambda)$ is monotonic nondecreasing.

Case 3. $S = \{b\}$ with $b \in R_m$, $a_j \in R_m$ $\forall j$. $\sup_{\xi \in S} H_n(\xi - \mu) = H_n(b - \mu)$.

Case 4. $S = \{g_0\}$ with $a_j, g_0 \in H$ a finite abelian group. $\max_{\xi \in S} H_n(\xi - \mu) = H_n(g_0 - \mu)$.

LEMMA 5.3. *$\bar{\alpha}_n \leqslant 0$ on H.*

PROOF. Let $H_n(g^*) = \max_{h \in H} H_n(h)$. Then $\bar{\alpha}_n(g) \leqslant H_n(g^* + g) - H_n(g^*) \leqslant 0$.

We now show that for this particular case it is again possible to define a Master Problem.

DEFINITION 5.4. The Master Group Problem is defined as

$$\sum_{h \in H} \sum_{j=1}^{k(h)} h x_h^j = g_0, \qquad x_h^j \in \{0, 1\}, \qquad \mathrm{MP}(H)$$

where $k(h)$ is the order of h in H, and $K = \sum_h k(h)$ is the number of variables, with valid inequality

$$\sum_{h \in H} \sum_{j=1}^{k(h)} \pi_h^j x_h^j \leqslant \pi_0, \quad \text{where } \pi_h^1 \geqslant \pi_h^2 \geqslant \cdots \geqslant \pi_h^{k(h)}. \tag{5}$$

THEOREM 5.5. (i) *If the inequality* (5) *for* MP(H) *is maximal, and $\sum_{j=1}^{k(h)} \pi_h^j \leqslant 0$ $\forall h$, then $\bar{\alpha}_K(h) = \bar{\beta}_K(h) = \pi_h^{k(h)}$ $\forall h \in H$.*

(ii) *If $\bar{\alpha}_K = \bar{\beta}_K$ on H, and $\pi_h^{k(h)} \geqslant \bar{\alpha}_K(h)$ $\forall h \in H$, then inequality* (5) *is maximal.*

Note. If $H_K(0) = 0$, then necessarily $\sum_{j=1}^{k(h)} \pi_h^j \leqslant 0$.

PROOF. Suppose the inequality is maximal. Then $\pi_g^{k(g)} \geqslant \bar{\beta}_K(g)$. Now let $\bar{\alpha}_K(g) = H_K(g + h^*) - H_K(h^*)$. As $\sum_{j=1}^{k(g)} g = 0$, $\sum_{j=1}^{k(g)} \pi_g^j \leqslant 0$, and $\pi_g^1 \geqslant \cdots \geqslant \pi_g^{k(g)}$, there exists an optimal solution for $H_K(h^*)$ with $x_g^{k(g)} = 0$. Therefore $H_K(g + h^*) \geqslant H_K(h^*) + \pi_g^{k(g)}$, and $\pi_g^{k(g)} \leqslant \bar{\alpha}_K(g)$. Hence, as $\bar{\alpha}_K \leqslant \bar{\beta}_K$, $\bar{\alpha}_K(g) = \bar{\beta}_K(g) = \pi_g^{k(g)}$.

Conversely, suppose $\bar{\alpha}_K = \bar{\beta}_K$, but that (5) is not maximal. Then $\pi_g^{k(g)} \geqslant \bar{\alpha}_K(g)$ can be increased to $\pi_g'^{k(g)}$ for some $g \in H$. However $H_K(g_0 - g)$ has an optimal solution with $x_g^{k(g)} = 0$, using the same argument as above with h^* replaced by $g_0 - g$, and hence $\pi_g'^{k(g)} \leqslant H_K(g_0) - H_K(g_0 - g) = \bar{\alpha}_K(g)$, a contradiction.

(N.B. Inequalities of the form $\sum_{j=1}^{n} \pi_j x_j \geqslant \pi_0$ can be obtained either by the substitution of $\bar{x}_j = 1 - x_j$, or replacing max/min, superadditivity/subadditivity in the definitions of H_n, $\bar{\alpha}_n$, etc., above.)

REMARK 5.6. Various extreme subadditive functions have been calculated in [7], [8] for unrestricted group problems. As shown in the above references valid inequalities for different g_0 can be obtained from automorphisms of H. Also, subadditive functions for infinite groups on $\{0, 1\}^m$ can be used to calculate valid 0-1 inequalities as in [8], [9].

We choose to look at inequalities of the form $\sum_{j=1}^{n} \pi_j x_j \geqslant \pi_0$ as they generally have this form in previous work on the group problem.

EXAMPLE 4. $1x_1 + 2x_2 + 3x_3 = 3 \pmod 4$, $x_j \in \{0, 1\}$;

λ		1	2	3	4
$\alpha^*(\lambda)$	:	1	2	1	0

is subadditive on Z_4. Hence $1x_1 + 2x_2 + 1x_3 \geqslant 1$ is a valid inequality. Computing

$$\begin{array}{rrrrrl} \lambda: & 1 & 2 & 3 & 4 & \\ \alpha^*(\lambda) = \alpha_3(\lambda) = H_3(\lambda): & 1 & 2 & 1 & 0 & \text{(by subadditivity)} \\ \beta_3(\lambda): & -1 & 0 & 1 & 0, & \end{array}$$

we see by Theorem 3.2(ii) that $1x_1 + 2x_2 + 1x_3 - 1y_1 \geqslant 1$ is a valid inequality for: $1x_1 + 2x_2 + 3x_3 + 1y_1 = 3 \pmod 4$. Recomputing

$$\begin{array}{rrrrr} \lambda: & 1 & 2 & 3 & 4 \\ \mathrm{H}_4(\lambda): & -1 & 0 & 1 & 0 \\ \beta_4(\lambda): & 1 & 2 & 1 & 0 \\ \alpha_4(\lambda): & 1 & 2 & 1 & 0 \end{array}$$

we see by Theorem 3.2(iv) that the last inequality has a unique projection.

6. Conclusions. In [23] we have shown how dynamic programming gives one way of representing the valid inequalities for a 0-1 integer program. Here we have shown how each inequality is essentially characterized by the two functions $\bar{\alpha}_n$ and $\bar{\beta}_n$. The superadditivity of $\bar{\alpha}_n$ can be seen as the essential tie with problems with unrestricted variables, and when the inequality, or $\bar{\beta}_n$ becomes superadditive, the introduction of further 0-1 variables is not different from the introduction of new nonnegative integer variables. In other words, the asymptotic stage in the dynamic program for computing $\bar{\beta}_n$ has been reached.

We terminate by raising a few of the questions only touched upon earlier. From §3 it appears that DP is one natural way to present the projection of valid inequalities, and we note that Theorem 3.1 holds under much more general conditions, i.e., if the y variables are unrestricted integer, or continuous. Also examining the more general type of projection not just onto the zero vector (see [24]) may show that α_n and β_n have other interpretations for sensitivity analysis. Though it is shown in §4 how some facets for the Master Polytope can be calculated, and how only superadditive inequalities need be considered, a good way to calculate all the facets is still needed.

Finally, it would be of interest to see whether functions resembling $\bar{\alpha}_n$ exist for other problems amenable to solution by dynamic programming, and to relate the DP view of the general problem Q with the other subadditive [13], [14], and disjunctive set [3] approaches.

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