INFORMS Journal on Computing

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Lifted Cover Inequalities for 0-1 Integer Programs: Complexity

Zonghao Gu, George L. Nemhauser, Martin W. P. Savelsbergh,

INFORMS Journal on Computing
Vol. 11, No. 1, Winter 1999

0899-1499/99/1101-0117 $05.00

Lifted Cover Inequalities for 0-1 Integer Programs: Complexity

ZONGHAO GU, GEORGE L. NEMHAUSER, AND MARTIN W.P. SAVELSBERGH / *School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0205, Email: {martin.savelsbergh,george.nemhauser}@isye.gatech.edu*

(Received: October 1994; revised: December 1995, September 1998; accepted: November 1998)

We investigate several complexity issues related to branch-and-cut algorithms for 0-1 integer programming based on lifted-cover inequalities (LCIs). We show that given a fractional point, determining a violated LCI over all minimal covers is NP-hard. The main result is that there exists a class of 0-1 knapsack instances for which any branch-and-cut algorithm based on LCIs has to evaluate an exponential number of nodes to prove optimality.

Consider the set P of feasible solutions to a 0-1 knapsack problem with integer coefficients, i.e.,

$$P = \left\{ x \in B^n : \sum_{j \in N} a_j x_j \leqslant b \right\}$$

where, without loss of generality, we assume $a_j > 0$ for $j \in N$ (because 0-1 variables can be complemented) and $a_j \leqslant b$ for $j \in N$ (because $a_j > b$ implies $x_j = 0$). A set $C \subseteq N$ is called a *cover* if $\Sigma_{j \in C} a_j > b$. A cover C is minimal if it is minimal with respect to this property. For any minimal cover C, the inequality

$$\sum_{j \in C} x_j \leqslant |C| - 1$$

is called a *cover inequality* and is valid for the convex hull of P, which we call the *0-1 knapsack polytope.*

Assuming, without loss of generality, that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$, Laurent and Sassano[13] show that if the sequence $a_n, a_{n-1}, \ldots, a_1$ is weakly superincreasing, i.e., satisfies $a_n + \ldots + a_q \leqslant a_{q-1}$ for $q = n, \ldots, 2$, then $conv(P)$ is fully described by the cover inequalities.

Cover inequalities can be strengthened considerably by a process called lifting. A sequential *lifted-cover inequality* (LCI) is of the form

$$\sum_{j \in C_1} x_j + \sum_{j \in N \setminus C} \alpha_j x_j + \sum_{j \in C_2} \gamma_j x_j \leqslant |C_1| - 1 + \sum_{j \in C_2} \gamma_j,$$

where (C_1, C_2) is a partition of a cover C with $|C_1| \geqslant 2$ and α_j for $j \in N \setminus C$ and γ_j for $j \in C_2$ are nonnegative integers. An LCI is obtained by starting from the cover inequality $\Sigma_{j \in C_1} x_j \leqslant |C_1| - 1$ and maximally lifting up all variables in $N \setminus C$, i.e., making the coefficients α_j as large as possible, and maximally lifting down all variables in C_2, i.e., making the coefficients γ_j as small as possible, in some specified order. An LCI defines a facet of the 0-1 knapsack polytope. See Gu, Nemhauser, and Savelsbergh[7] for a more detailed discussion of LCIs.

An important special case arises when we take $C_2 = \emptyset$ so that $C_1 = C$. In this case, the LCI has the form

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leqslant |C| - 1 \tag{1}$$

and is called a *simple LCI.*

Padberg[18] generalized simple LCIs to a class of facets called $(1, k)$-configuration inequalities. He also gave a condition on the knapsack coefficients for $conv(P)$ to be described by these inequalities. It can easily be shown that $(1, k)$-configuration inequalities are in fact LCIs, although not simple LCIs.

LCIs have been used successfully in branch-and-cut algorithms for the solution of 0-1 integer programs (Crowder, Johnson, and Padberg,[5] Hoffman and Padberg,[10] and Gu, Nemhauser, and Savelsbergh[7]). However, there are still many interesting open questions associated with LCIs.

It is well-known that in general LCIs do not completely describe $conv(P)$. Weismantel[19] studies the 0-1 knapsack polytopes P^i for $i \in \{1, \ldots, \lfloor \frac{b}{2} \rfloor\}$ associated with the 0-1 knapsack constraint

$$\sum_{k \in N_1} x_k + \sum_{\lfloor \frac{b}{i+1} \rfloor < j \leqslant \lfloor \frac{b}{i} \rfloor} \sum_{k \in N_j} j x_k \leqslant b,$$

where N_j denotes the set of variables with coefficient j. He identifies a class of facet-inducing inequalities that cannot be obtained as LCIs, but are necessary in a complete description of the polytopes P^1 and P^2. Balas and Zemel[2] show that all the facets of a 0-1 knapsack polytope can be obtained by *simultaneous* lifting (Padberg,[17] Zemel[21]) and complementing (Wolsey[20]) of cover inequalities.

In this article, we focus on three complexity issues associated with simple LCIs.

For a given minimal cover, different lifting sequences may result in different simple LCIs. Therefore, it is natural to ask whether it is hard, given a fractional point and a minimal cover, to determine a simple LCI that is violated by this fractional point. A closely related question is whether it is hard, given a fractional point, to determine a violated simple

Subject classifications: Integer programming.
Other key words: Cutting plane, complexity of separation, separation algorithms.

LCI over all minimal covers. In Section 1, we show that both problems are NP-hard.

In Section 2, we show that not all simple LCIs can be obtained directly from the original knapsack constraint, i.e., there exist higher-order simple LCIs that can only be obtained from previously generated simple LCIs.

Although the 0-1 knapsack problem is NP-complete (Garey and Johnson[6]), it is well-known that most instances can be successfully solved by dynamic programming or branch-and-bound, see for instance Martello and Toth.[14] However, there exist classes of difficult 0-1 knapsack problems, see for instance Jereslow,[11] Chvátal,[4] and Chung et al.[3] Chvátal[4] presents a class of 0-1 knapsack problems for which he shows that linear programming based branch-and-bound algorithms have to evaluate an exponential number of nodes to prove optimality. In Section 3, we strengthen Chvátal's result by presenting a class of 0-1 knapsack problems for which branch-and-cut algorithms based on simple LCIs have to evaluate an exponential number of nodes to prove optimality.

1. Complexity of Identifying Violated Simple LCIs

In this section, we are concerned with the complexity of identifying violated simple LCIs.

An important result in this area is the fact that for a given minimal cover C a sequential simple LCI can be computed in $O(n^2)$ by dynamic programming, see Nemhauser and Wolsey[16] and Zemel.[21] However, in a companion paper (Gu, Nemhauser, and Savelsbergh[8]), we show that this dynamic programming algorithm may take exponential time to compute a sequential LCI that is not simple. It is still an open question whether an arbitrary LCI can be computed in polynomial time for a given minimal cover C. Some related results can be found in Hartvigsen and Zemel.[9] They discuss, among other things, the complexity of recognizing simple LCIs, i.e., the complexity of deciding whether a given inequality is a simple LCI, and show that this can also be done in $O(n^2)$ time.

Let $C = \{j_1, j_2, \ldots, j_r\}$ be a minimal cover with $a_{j_1} \geqslant a_{j_2} \geqslant \ldots \geqslant a_{j_r}$, $\mu_0 = 0$, $\mu_h = \Sigma_{i_1}^{h} a_{j_i}$ for $h = 1, \ldots, r$, and $\lambda = \mu_r - b$.

Theorem 1. (Balas[1]) Every facet-defining simple LCI satisfies the following conditions:

1. If $\mu_h \leqslant a_j \leqslant \mu_{h+1} - \lambda$, then $\alpha_j = h$.
2. If $\mu_{h+1} - \lambda + 1 \leqslant a_j \leqslant \mu_{h+1} - 1$, then $\alpha_j \in \{h, h+1\}$.

Although Theorem 1 nearly determines all the lifting coefficients, it does not give sufficient conditions on the α_j to obtain facets because it does not specify whether certain α_js can be equal to $h + 1$ or must be equal to h to preserve validity. The following theorem extends Theorem 1 by giving necessary and sufficient conditions on which subsets of α take the larger values in facet-defining simple LCIs. We need the following: for $k \in N\backslash C$, let $\beta_k = h$ if $\mu_h \leqslant a_k \leqslant \mu_{h+1} - 1$ and for $Q \subseteq N\backslash C$ define $\beta(Q) = \Sigma_{i \in Q}(\beta_i + 1)$. A set $S \subseteq N\backslash C$ is called *independent* if for all nonempty $Q \subseteq S$,

$$\sum_{i \in Q} a_i > \mu_{\beta(Q)} - \lambda. \tag{2}$$

Theorem 2. (Nemhauser and Vance[15]) A simple LCI is facet-defining for P if and only if $\alpha_j = \beta_j + 1$ for all j in a maximal independent set $S \subseteq N\backslash C$ and $\alpha_j = \beta_j$ for all $j \in N\backslash(C \cup S)$.

As a consequence of Theorem 2, the problem of determining whether there exists a violated simple LCI for a given fractional point and a given minimal cover is equivalent to determining whether there exists an independent set S such that $\Sigma_{j \in S} x_j^* > |C| - 1 - \Sigma_{j \in C} x_j^* + \Sigma_{j \in N\backslash C} \beta_j x_j^*$, because $\Sigma_{j \in C} x_j^* + \Sigma_{j \in N\backslash C} \alpha_j x_j^*$ is equal to $\Sigma_{j \in C} x_j^* + \Sigma_{j \in N\backslash(C \cup S)} \beta_j x_j^* + \Sigma_{j \in S}(\beta_j + 1) x_j^*$.

Theorem 3. Given a 0-1 knapsack constraint, a feasible fractional point x^*, and a minimal cover C, deciding whether there exists a violated simple LCI is NP-complete.

Proof. Transformation from KNAPSACK.

KNAPSACK:

Instance. A set $N = \{1, \ldots, n\}$, for each $j \in N$ a weight $w_j \in Z^+$ and a profit $p_j \in Z^+$, positive integers P and W, and $P \geqslant p_j > 0$ and $W \geqslant w_j > 0$ for each $j \in N$.

Question: Does there exist a set $S \subseteq N$ such that $\Sigma_{j \in S} w_j < W$ and $\Sigma_{j \in S} p_j \geqslant P$?

Without loss of generality, we assume $\Sigma_{1 \leqslant j \leqslant n} w_j \geqslant W$ and $\Sigma_{1 \leqslant j \leqslant n} p_j \geqslant P$. Consider the 0-1 knapsack constraint

$$\sum_{j=1}^{2n} a_j x_j \leqslant b,$$

where

$$a_j = \begin{cases} 2W - w_j, & j = 1, \ldots, n, \\ 2W, & j = n+1, \ldots, 2n, \end{cases}$$

$$b = (2n - 1)W,$$

the fractional point

$$x_j^* = \begin{cases} \dfrac{p_j}{2\sum_{j=1}^{n} p_j}, & j = 1, \ldots, n, \\ \dfrac{1}{n}\left(n - 1 - \dfrac{P - 1}{2\sum_{j=1}^{n} p_j}\right), & j = n+1, \ldots, 2n, \end{cases}$$

and the minimal cover

$$C = \{n + 1, \ldots, 2n\}.$$

Note that x^* is a feasible fractional solution to the LP relaxation of the knapsack constraint because

$$\begin{aligned} \sum_{j=1}^{2n} a_j x_j^* &= \sum_{j=1}^{n} (2W - w_j) \frac{p_j}{2\sum_{j=1}^{n} p_j} \\ &\quad + \sum_{j=n+1}^{2n} 2W \frac{1}{n}\left(n - 1 - \frac{P - 1}{2\sum_{j=1}^{n} p_j}\right) \\ &\leqslant W + 2W(n - 1) \\ &= b. \end{aligned}$$

Furthermore, $\mu_h = 2hW$ for $h = 0, \ldots, n$, $\lambda = W$ and $\mu_0 \leqslant a_j \leqslant \mu_1 - 1$ for $j = 1, \ldots, n$, so $\beta_j = 0$ for $j = 1, \ldots, n$ and $\beta(Q) = |Q|$. Inequality (2) becomes

$$\sum_{i \in Q} a_i = \sum_{i \in Q} (2W - w_i) > 2|Q|W - W = \mu_{|Q|} - \lambda,$$

which is equivalent to

$$\sum_{i \in Q} w_i < W.$$

Because $\Sigma_{i \in S} w_i < W$ implies $\Sigma_{i \in Q} w_i < W$ for all nonempty $Q \subseteq S$, S is independent if and only if $\Sigma_{i \in S} w_i < W$.

Now observe that

$$\sum_{j \in S} x_j^* > |C| - 1 - \sum_{j \in C} x_j^* + \sum_{j \in N \setminus C} \beta_j x_j^* \ \text{N}$$

$$\sum_{j \in S} \frac{p_j}{2\sum_{j=1}^n p_j} > n - 1 - \sum_{n+1 \leqslant j \leqslant 2n} \left(\frac{1}{n} \left(n - 1 - \frac{P-1}{2\sum_{j=1}^n p_j} \right) \right) \ \text{N}$$

$$\sum_{j \in S} \frac{p_j}{2\sum_{j=1}^n p_j} > n - 1 - n + 1 + \frac{P-1}{2\sum_{j=1}^n p_j} \ \text{N}$$

$$\sum_{j \in S} \frac{p_j}{2\sum_{j=1}^n p_j} > \frac{P-1}{2\sum_{j=1}^n p_j} \ \text{N}$$

$$\sum_{j \in S} p_j \geqslant P.$$

Consequently, there exists a violated simple LCI if and only if there is a set $S \subseteq N$ such that $\Sigma_{j \in S} w_j < W$ and $\Sigma_{j \in S} p_j \geqslant P$, i.e., KNAPSACK has a solution. ■

Note that we have not shown that x^* is a realizable fractional point, i.e., a fractional point that occurs as the optimal solution to the current linear programming relaxation. Although it may not be if we are just solving a 0-1 knapsack problem, in the context of solving general 0-1 integer programs the assumption is reasonable. Recently Klabjan et al.[12] have shown that the separation problem for cover inequalities is NP-hard. Their proof explicitly deals with fractional x^* that are feasible to the LP relaxation.

Now we show that being able to choose the cover does not make the problem easier.

Theorem 4. Given a 0-1 knapsack constraint and a feasible fractional point x^*, deciding whether there exists a violated simple LCI is NP-complete.

Proof. We use the instance defined in the proof of Theorem 3. Let $C = \{n + 1, \ldots, 2n\}$ be the given minimal cover, and C' be an arbitrary minimal cover. We prove the theorem by showing

1. If $|C'| = n$, then the set of simple LCIs for C' is contained in the set of simple LCIs for C.
2. If $|C'| > n$, then any simple LCI for C' is not violated by the given fractional solution x^*.

Because

$$\sum_{j=1}^n a_j = \sum_{j=1}^n (2W - w_j) = 2nW - \sum_{j=1}^n w_j < 2nW - W = b,$$

C' contains at least one element of $\{n + 1, \ldots, 2n\}$, so $\mu_1 = 2W$. We use α'_j for the coefficients of a simple LCI with minimal cover C' and α_j for those of a simple LCI with minimal cover C. By Theorem 1, $\alpha'_j \leqslant 1$ for $j = 1, \ldots, n$ and $\alpha'_j = 1$ for $j = n + 1, \ldots, 2n$.

Case 1. For an arbitrary simple LCI with C', let $S_0 = \{j : \alpha'_j = 0, 1 \leqslant j \leqslant n\}$ and $S_1 = \{1, \ldots, n\} \setminus S_0$. Then $\alpha'_j = 1$ for $j \in S_1$, because $\alpha'_j \leqslant 1$. We show that this simple LCI is equal to the simple LCI obtained from C when the variables in S_1 are lifted first and the variables in S_0 are lifted last; the order within S_1 and S_0 is arbitrary. Suppose that the lifting order for S_1 is $\{j_1, j_2, \ldots, j_k\}$. We first show that $\alpha_j = 1$ for $j \in S_1$. Suppose not, then let j_l be the first element in $\{j_1, j_2, \ldots, j_k\}$ such that $\alpha_{j_l} = 0$. From the simple LCI with C', we know that

$$\sum_{j=n+1}^{2n} x_j + \sum_{i=1}^{l} x_{j_i} \leqslant n - 1$$

is valid, so $\alpha_{j_l} \geqslant 1$, which contradicts $\alpha_{j_l} = 0$. Next, we show that $\alpha_j = 0$ for $j \in S_0$. Suppose not, then the simple LCI with C is stronger than the simple LCI with C', which contradicts the fact that any simple LCI defines a facet for the knapsack polytope.

Case 2. The violation of the simple LCI for C' is less than or equal to

$$\sum_{j=1}^{2n} a_j x_j^* - n = \sum_{j=1}^{n} \frac{p_j}{2\sum_{j=1}^n p_j} + \sum_{j=n+1}^{2n} \frac{1}{n} \left(n - 1 - \frac{P-1}{2\sum_{j=1}^n p_j} \right) - n$$

$$= -\frac{1}{2} - \frac{P-1}{2\sum_{j=1}^n p_j} < 0. \quad ■$$

2. Higher Order Simple LCIs

In this section, we make some observations related to the question of how well the polyhedron defined by simple LCIs approximates $conv(P)$. We show that there exist facet-inducing inequalities of $conv(P)$ that are not derivable as a simple LCI from the original knapsack constraint, but that are derivable from other simple LCIs. This leads to the notion of a hierarchy of simple LCIs.

Example 1

Consider the set P of feasible solutions to the 0-1 knapsack constraint

$$\sum_{j=1}^{3} 13x_j + \sum_{j=4}^{7} 9x_j + \sum_{j=8}^{14} 5x_j + 2x_{15} + \sum_{j=16}^{42} x_j \leqslant 28. \quad (3)$$

Then $C = \{15, 16, \ldots, 42\}$ is a minimal cover with $r = |C| = 28$. This gives $\mu_0 = 0$, $\mu_h = h + 1$, for $h = 1, 2, \ldots, 28$, and

$\lambda = \mu_{28} - b = 28 + 1 - 28 = 1$. By Theorem 1, if $h + 1 \leqslant a_j \leqslant (h + 1) + 1 - 1$, i.e., $a_j = h + 1$, then $\alpha_j = h = a_j - 1$. So we get a simple LCI given by

$$\sum_{j=1}^{3} 12x_j + \sum_{j=4}^{7} 8x_j + \sum_{j=8}^{14} 4x_j + \sum_{j=15}^{42} x_j \leqslant 27. \tag{4}$$

For this inequality, $C = \{8, 9, \ldots, 14\}$ is a minimal cover with $r = |C| = 7$. This gives $\mu_h = 4h$ for $h = 0, 1, \ldots, 7$, and $\lambda = \mu_7 - b = 28 - 27 = 1$. By Theorem 1, we have $\alpha_j = h$ if $4h \leqslant a_j \leqslant 4(h + 1) - 1 = 4h + 3$. So

$$\alpha_j = \begin{cases} 3 & \text{for} \quad j = 1, 2, 3 \\ 2 & \text{for} \quad j = 4, \ldots, 7 \\ 0 & \text{for} \quad j = 15, \ldots, 42. \end{cases}$$

Thus we have derived a simple LCI from (4) given by

$$\sum_{j=1}^{3} 3x_j + \sum_{j=4}^{7} 2x_j + \sum_{j=8}^{14} x_j \leqslant 6. \tag{5}$$

Note that this simple LCI cannot be derived directly from the original knapsack constraint (3) because $\{8, 9, \ldots, 14\}$ is not a minimal cover. However, it does give a facet-defining inequality for P. Consider the matrix:

$$A = \begin{pmatrix} B & & & \\ & C & & \\ D & E & F & \\ G & & & H \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$D = (1_{7\times1}, 0_{7\times2})$, $E = (1_{7\times1}, 0_{7\times3})$, $G = (1_{28\times2}, 0_{28\times1})$, $F = I_7$, and $H = I_{28}$, where $a_{n\times m}$ represents an $n \times m$ matrix with elements equal to a and I_n represents an $n \times n$ identity matrix. It is easy to see that each row of A defines a feasible solution to (3) that satisfies (5) at equality, and that $det(A) \neq 0$ (because $det(B) \neq 0$ and $det(C) \neq 0$). Hence (5) defines a facet of P.

This example also shows that we can obtain a facet-inducing inequality of P by lifting a non-minimal cover inequality and that this facet-inducing inequality cannot be obtained by lifting a minimal cover inequality. If we take $C = \{8, 9, \ldots, 14\}$ as a (non-minimal) cover and lifting sequence $\{1, 2, \ldots 7, 15, 16, \ldots, 42\}$, we obtain (5).

We call facet-inducing inequalities order 1 if they are derivable as a simple LCI directly from the original knapsack constraint, and we call facet-inducing inequalities order $k + 1$ if they are derivable as a simple LCI from a simple LCI of order k, but not derivable as a simple LCI from the original knapsack constraint or from a simple LCI of order $k - 1$ or lower. We have shown that simple LCIs of order 2 exist, but we conjecture that even higher order simple LCIs exist. Note that the set of facet-inducing inequalities derivable as simple LCIs (of arbitrary order) does not define $conv(P)$, because the maximal coefficient of a simple LCI is less than n.

It is not true that a second order simple LCI can always be obtained by lifting a non-minimal cover inequality, as shown by the following example.

Example 2
Consider the set P of feasible solutions to the 0-1 knapsack constraint

$$\sum_{j=1}^{3} 3kx_j + \sum_{j=4}^{7} 2kx_j + \sum_{j=8}^{14} (k + 1)x_j + 2x_{15} + \sum_{j=16}^{16+6k} x_j \leqslant 6k + 2, \tag{6}$$

where $k \geqslant 5$. Similar to the above example, we can show that (5) is a second order simple LCI of (6), but it cannot be obtained by lifting any cover inequalities of (6).

3. Branch-and-Cut Based on Simple LCIs

Chvátal[4] identified a class of instances of the 0-1 knapsack problem for which linear programming based branch-and-bound algorithms have to evaluate an exponential number of nodes to prove optimality. In this section, we strengthen Chvátal's result by showing that there exists a class of instances of the 0-1 knapsack problem for which branch-and-cut algorithms based on simple LCIs have to evaluate an exponential number of nodes to prove optimality. More precisely, we show that there exists a class of instances with $20n$ variables for which all nodes in the search tree of depth less than or equal to n cannot be fathomed. Therefore, the number of nodes that has to be evaluated to prove optimality is greater than or equal to 2^n.

Let P_1^n be the set of feasible solutions to the family of 0-1 knapsack constraints

$$\sum_{j=1}^{20n} a_j x_j = \sum_{j=1}^{12n} 2x_j + \sum_{j=12n+1}^{20n} 3x_j \leqslant 6n, \tag{7}$$

for $n \geqslant 10$.

First, we present two propositions concerning P_1^n. Let $N = \{1, 2, \ldots, 20n\}$, $N_1 = \{1, 2, \ldots, 12n\}$, $N_2 = \{12n + 1, \ldots, 20n\}$, $I \subset N$ and $0 \leqslant |I| \leqslant n$.

Proposition 5. If we fix a set of variables x_j, $j \in I$, either at their lower bounds or at their upper bounds, then (7) becomes

$$\sum_{j\in N\setminus I} a_j x_j \leqslant 6n - \sum_{j\in I} a_j \bar{x}_j, \tag{8}$$

where $\bar{x}_j$ for $j \in I$ denotes the value of the fixed variable x_j, and (8) induces a facet of the 0-1 knapsack polytope defined by the 0-1 knapsack constraint (8) for $n \geqslant 10$.

Proof. Let $\bar{b} = 6n - \sum_{j\in I} a_j \bar{x}_j$, let P_I denote the set of 0-1 solutions to (8), let $F = \{x \in P_I: \sum_{j\in N\setminus I} a_j x_j = \bar{b}\}$, and let $F' = \{x \in P_I: \sum_{j\in N\setminus I} \alpha_j x_j = \bar{b}\}$ be an arbitrary facet of $conv(P_I)$ containing F, i.e., $F' \supseteq F$.

For convenience, we assume $\bar{b}$ is even. At the end of the proof it will be clear that the same proof technique can be applied in case $\bar{b}$ is odd.

Because $\bar{b} \leqslant 6n$, we need to set at most $3n$ variables in N_1 to 1 to construct a feasible point $x \in F$. Furthermore, because $|I| \leqslant n$, we have $|N_1 \backslash I| \geqslant 11n$. Define

$$S_1 = \{j \in N_1 \backslash I : a_j = \alpha_j\},$$

$$S_2 = \{j \in N_1 \backslash I : a_j < \alpha_j\},$$

and

$$S_3 = \{j \in N_1 \backslash I : a_j > \alpha_j\}.$$

Suppose $|S_1| < 3n$, then either $|S_2| \geqslant 3n$ or $|S_3| \geqslant 3n$. Suppose $|S_2| \geqslant 3n$ (the case $|S_3| \geqslant 3n$ is handled analogously). Construct a feasible point $x \in F$ using only variables with an index in S_2. Obviously, $x \notin F'$, which contradicts $F' \supseteq F$.

Now suppose $3n \leqslant |S_1| < |N_1 \backslash I|$. Consider any variable x_k with k in $S_2 \cup S_3$, i.e., any variable with $\alpha_j \neq a_j$. Construct a feasible point $x \in F$ using x_k and variables with an index in S_1. Obviously, $x \notin F'$, which contradicts $F' \supseteq F$.

Consequently, $|S_1| = |N_1 \backslash I|$ and $\alpha_j = a_j$ for all $j \in N_1 \backslash I$.

Analogously, we can show that $\alpha_j = a_j$ for all $j \in N_2 \backslash I$. Therefore, $\alpha_j = a_j$ for all $j \in N$ and $F = F'$, i.e., F is a facet of *conv*(P). ■

Proposition 6. The facet-defining inequality (8) cannot be obtained by lifting a cover inequality plus fixing at most n variables. (The cover inequality can be for the original knapsack constraint as well as for a simple LCI.)

Proof. Compare the simple LCI (1) with (8). If we fix some variables of (1) and move the corresponding terms to the right-hand side, then the value of the right-hand side cannot increase. Suppose that we obtain (8) by fixing variables of (1). Because the right-hand side of (8) is greater than or equal to $3n$, $|C| > 3n$ in (1). Because (8) has no variables with coefficients equal to one, we need to fix at least $3n$ variables to eliminate all the variables with coefficients equal to one in (1). However, we are only allowed to fix at most n variables. ■

Next, we present two propositions concerning slightly perturbed versions of the above class of 0-1 knapsack constraints, namely

$$\sum_{j=1}^{12n} (2 \cdot 2^n - \delta_j)\, x_j + \sum_{j=12n+1}^{20n} (3 \cdot 2^n - \delta_j)\, x_j \leqslant 6n \cdot 2^n, \quad (9)$$

for $n \geqslant 10$, $\delta_j \in \{1, \ldots, \lfloor 2^{n-1}/3n \rfloor\}$ for $1 \leqslant j \leqslant 20n$. Let $\delta = (\delta_1, \ldots, \delta_{20n})$ and $P_2^n(\delta)$ be the set of solutions to (9).

Proposition 7. $P_1^n = P_2^n(\delta)$ for all δ.

Proof. If $x \in P_1^n$, then $x \in P_2^n(\delta)$ because $\delta_j \geqslant 0$, $j \in N$. If $x \in P_2^n(\delta)$ for all δ_j, then

$$\sum_{j=1}^{20n} \left[2 \cdot 2^n - \frac{2^{n-1}}{3n} \right] x_j \leqslant 6n \cdot 2^n$$

$$\text{f} \sum_{j=1}^{20n} x_j \leqslant 3n + \sum_{j=1}^{20n} \frac{1}{12n} x_j \leqslant 3n + \frac{20n}{12n} = 3n + \frac{5}{3}$$

$$\text{f} \sum_{j=1}^{20n} x_j \leqslant 3n + 1.$$

By Eq. 9, we have

$$\sum_{j=1}^{12n} 2x_j + \sum_{j=12n+1}^{20n} 3x_j \leqslant 6n + \frac{1}{2^n} \sum_{j=1}^{20n} \delta_j x_j$$

$$\leqslant 6n + \frac{1}{2^n} \cdot \frac{2^{n-1}}{3n} \sum_{j=1}^{20n} x_j$$

$$\leqslant 6n + \frac{1}{6n} \cdot (3n + 1)$$

$$< 6n + 1.$$

Hence $x \in P_1^n$. ■

Note that if we fix some variables to the same values in P_1^n and $P_2^n(\delta)$, they are still equal. Therefore, we have the next proposition.

Proposition 8. Inequality (8) defines a facet of the polytope for $P_2^n(\delta)$ for $n \geqslant 10$ where all variables x_j, $j \in I$ are fixed.

Now, we are ready to introduce the class of instances for which we will show that branch-and-cut algorithms based on simple LCIs have to evaluate an exponential number of nodes to prove optimality. Consider the following class of instances of 0-1 knapsack problems $KP(\delta, \xi)$:

$$\max \sum_{j=1}^{12n} (2\theta - \xi_j)\, x_j + \sum_{j=12n+1}^{20n} (3\theta - \xi_j)\, x_j$$

$$\sum_{j=1}^{12n} (2 \cdot 2^n - \delta_j)\, x_j + \sum_{j=12n+1}^{20n} (3 \cdot 2^n - \delta_j)\, x_j \leqslant 6n \cdot 2^n$$

$$x \in B^{20n},\ n \geqslant 10,\ \delta_j \in \{1, \ldots, \lfloor 2^{n-1}/3n \rfloor\} \text{ for } 1 \leqslant j \leqslant 20n$$

$$\theta = (60n \cdot 2^n)^{20n+1},\ \xi_j \in \{1, \ldots, 2^n\} \text{ for } 1 \leqslant j \leqslant 20n$$

Proposition 9. Let $z_{KP(\delta,\,\xi)}$ be the optimal value of $KP(\delta\, \xi)$, then $z_{KP(\delta,\,\xi)} \leqslant 6n\theta$.

Proof. Because, by Proposition 8, $\Sigma_{j=1}^{12n} 2x_j + \Sigma_{j=12n+1}^{20n} 3x_j \leqslant 6n$ holds for the solution vector x, $z_{KP(\delta,\,\xi)} \leqslant 6n\theta$. ■

Consider the following branch-and-cut algorithm for solving 0-1 knapsack problems of the form

$$\max \sum_{j \in N} c_j x_j,\ \sum_{j \in N} a_j x_j \leqslant b,\ x \in B^n,\ n = |N|.$$

Notation:

AN: set of active nodes of the search tree (0 is the root node and other nodes are numbered 1, 2, ...).

TN: set of nodes of the search tree.

F^i: set of constraints for node i, i.e., $\Sigma_{j \in N} a_j x_j \leqslant b$ plus LCIs (of any order) that have been added to the linear programming relaxation.

I_0^i: set of variables which are fixed to 0 at node i.
I_1^i: set of variables which are fixed to 1 at node i.
I^i: $I_0^i \cup I_1^i$.
z_{ip}: the best objective value of all feasible solutions found so far.

Algorithm:

1. Initialization.
$k := 0$, $AN := \{0\}$, $TN := \{0\}$, $z_{ip} := -\infty$, $I_0^0 := \phi$, $I_1^0 := \phi$ and $F^0 := \{\sum_{j\in N} a_j x_j \leq b\}$.
2. While $AN \neq \phi$ pick a node $i \in AN$ and do the following steps:
2.1 Solve the LP for node i,

$$z_{lp}^i = \max \sum_{j\in N} c_j x_j$$

$$\begin{aligned} s.t. \quad & \text{all constraints} \in F^i \\ & 0 \leq x_j \leq 1, \text{ for } j \in N \\ & x_j = 1, \text{ if } j \in I_1^i \\ & x_j = 0, \text{ if } j \in I_0^i \end{aligned}$$

to get an optimal solution x^* with optimal value z_{lp}^i.

2.2 If x^* is integral and $\lfloor z_{lp}^i \rfloor > z_{ip}$ then
$z_{ip} := z_{lp}^i$ and $AN := \{j : z_{lp}^j > z_{ip}, j \in AN\}$.
2.3 If $\lfloor z_{lp}^i \rfloor \leq z_{ip}$ then fathom node i, i.e. $AN := AN \backslash \{i\}$.
2.4 If x^* is not integral and $\lfloor z_{lp}^i \rfloor > z_{ip}$ then
2.4.1 Generate violated simple LCIs from all constraints $\in F^i$ and add them to F^i, go to step 2.1.
2.4.2 If we cannot generate violated simple LCIs, then pick a fractional variable x_j, $j \neq I^i$ and branch as follows:
$AN := AN \cup \{k+1, k+2\} \backslash \{i\}$
$TN : TN \cup \{k+1, k+2\}$
$I_0^{k+1} := I_0^i \cup \{j\}$, $I_1^{k+1} := I_1^i$
$I_0^{k+2} := I_0^i$, $I_1^{k+2} := I_1^i \cup \{j\}$
$F^{k+1} := F^{k+2} := F^i$
$k := k + 2$.

This description of a branch-and-cut algorithm based on simple LCIs for 0-1 knapsack problems allows us to state the main result of this section.

Theorem 10. A branch-and-cut algorithm based on simple LCIs for $KP(\delta, \xi)$ has to evaluate at least 2^n nodes for any (δ, ξ).

Proof. By Propositions 6, 7, and 8, (8) is a facet for the polytope at node i, if $I = I^i$ and $0 \leq |I^i| \leq n$, and it cannot be obtained by lifting a cover inequality and fixing variables in I^i, i.e. Eq. 8 $\notin F^i$. So there is a feasible fractional solution x^* to the LP at node i and a positive ϵ such that

$$\sum_{j\in N\backslash I^i} a_j x_j^* = 6n - \sum_{j\in I^i} a_j x_j^* + \epsilon$$

$$\text{f} \sum_{j=1}^{12n} 2x_j^* + \sum_{j=12n+1}^{20n} 3x_j^* = 6n + \epsilon$$

Obviously all simple LCIs in F^i have coefficients $\leq 20n$, so the largest value of all coefficients of all constraints in F^i is $3 \cdot 2^n$. By Proposition 3.1 in Chapter I.5 of Nemhauser and Wolsey,[16]

$$\epsilon > \frac{1}{(20n \cdot 3 \cdot 2^n)^{20n}} = \frac{1}{(60n \cdot 2^n)^{20n}}.$$

Then the optimal objective value of the LP at node i:

$$\begin{aligned} z_{lp}^i &\geq \sum_{j=1}^{12n} (2\theta - \xi_j) x_j^* + \sum_{j=12n+1}^{20n} (3\theta - \xi_j) x_j^* \\ &= 6n\theta + \theta\epsilon - \sum_{j=1}^{20n} \xi_j x_j^* \\ &\geq 6n\theta + \frac{(60n \cdot 2^n)^{20n+1}}{(60n \cdot 2^n)^{20n}} - 20n \cdot 2^n \\ &= 6n\theta + 40n \cdot 2^n. \end{aligned}$$

Hence by Proposition 9, $\lfloor z_{lp}^i \rfloor \geq 6n\theta + 40n \cdot 2^n > z_{KP(\delta,\xi)}$, i.e., node i cannot be fathomed, so all nodes of the branching tree at depth $\leq n$ cannot be fathomed. Hence the number of the nodes of the branching tree is at least 2^n. ■

It is easy to see that the input length of $KP(\delta, \xi)$ is $t = O(n^3)$, so

$$|TN| = \Omega(2^n) = \Omega(2^{\sqrt[3]{t}}).$$

Note that even if we add functions commonly found in integer programming solvers, such as preprocessing and primal heuristics, to the above described branch-and-cut algorithm Theorem 10 remains valid.

Acknowledgements

This research was supported by U.S. Army Research Office DAAH04-94-G-0017 and NSF Grant No. DDM-9115768.

References

1. E. BALAS, 1975. Facets of the Knapsack Polytope, *Mathematical Programming 8*, 146–164.
2. E. BALAS and E. ZEMEL, 1984. Lifting and Complementing Yields All the Facets of Positive Zero-One Programming Polytopes, in *Mathematical Programming*, R.W. Cottle, H.L. Kennington, and B. Korte (eds.), North-Holland, Amsterdam, 13–24.
3. C.S. CHUNG, M.S. HUNG, and W.O. ROM, 1988. A Hard Knapsack Problem, *Naval Research Logistics Quarterly 35*, 85–98.
4. V. CHVATAL, 1980. Hard Knapsack Problems, *Operations Research 28*, 1402–1411.
5. H. CROWDER, E. JOHNSON, and M. PADBERG, 1983. Solving Large-Scale Zero-One Linear Programming Problems, *Operations Research 31*, 803–834.
6. M.R. GAREY and D.S. JOHNSON, 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness,* W.H. Freeman and Company, San Francisco.
7. Z. GU, G.L. NEMHAUSER, and M.W.P. SAVELSBERGH, 1998. Lifted Cover Inequalities for 0-1 Integer Programs: Computation, *INFORMS Journal on Computing 10*, 427–437.

8. Z. GU, G.L. NEMHAUSER, and M.W.P. SAVELSBERGH, 1999. Lifted Cover Inequalities for 0-1 Integer Programs: Algorithms, Technical Report, Georgia Institute of Technology, in preparation.
9. D. HARTVIGSEN and E. ZEMEL, 1992. The Complexity of Lifted Inequalities for the Knapsack Problem, *Discrete Applied Mathematics 39*, 113–123.
10. K. HOFFMAN and M. PADBERG, 1991. Improving LP-Representations of Zero-One Linear Programs for Branch-and-Cut, *ORSA Journal of Computing 3*, 121–134.
11. R.G. JERESLOW, 1974. Trivial Integer Programs Unsolved by Branch and Bound, *Mathematical Programming 6*, 105–109.
12. D. KLABJAN, G.L. NEMHAUSER, and C. TOVEY, 1998. The Complexity of Cover Inequality Separation, *Operations Research Letters 23*, 35–40.
13. M. LAURENT and A. SASSANO, 1992. A Characterization of Knapsacks with the Max-Flow-Min-Cut Property, *Operations Research Letters 11*, 105–110.
14. S. MARTELLO and P. TOTH, 1990. *Knapsack Problems: Algorithms and Computer Implementations*, John Wiley and Sons, New York.
15. G.L. NEMHAUSER and P.H. VANCE, 1994. Lifted Cover Facets of the 0-1 Knapsack Polytope with GUB Constraints, *Operations Research Letters 16*, 255–263.
16. G.L. NEMHAUSER and L.A. WOLSEY, 1988. *Integer and Combinatorial Optimization*, John Wiley and Sons, New York.
17. M.W. PADBERG, 1975. A Note on Zero-One Programming, *Operations Research 23*, 833–837.
18. M.W. PADBERG, 1980. $(1, k)$-Configurations and Facets for Packing Problems, *Mathematical Programming 18*, 94–99.
19. R. WEISMANTEL, 1994. On the 0/1 Knapsack Polytope, Technical Report SC 91-1, Konrad-Zuse-Zentrum fur Informationstechnik Berlin, Berlin, Germany.
20. L.A. WOLSEY, 1975. Faces for a Linear Inequality in 0-1 Variables, *Mathematical Programming 8*, 165–178.
21. E. ZEMEL, 1989. Easily Computable Facets of the Knapsack Polytope, *Mathematics of Operations Research 14*, 760–765.