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TECHNICAL NOTE

Sequence Independent Lifting for Mixed-Integer Programming

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We show that superadditive lifting functions lead to sequence independent lifting of inequalities for general mixed-integer programming. As an application, we note that mixed-integer rounding (MIR) may be viewed as sequence independent lifting. Consequently, we obtain facet conditions for MIR inequalities for mixed-integer knapsacks.

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1. Introduction

Lifting is a procedure for deriving strong valid inequalities for a closed set from inequalities that are valid for its lower dimensional restrictions. It is arguably one of the most effective ways of strengthening linear programming relaxations of 0–1 programming problems. Wolsey (1977) and Gu et al. (2000) show that superadditive lifting functions lead to sequence independent lifting of valid inequalities for monotone 0–1 programming and for monotone mixed 0–1 programming, respectively. We show that this property holds for general mixed-integer programming (MIP) as well if lower dimensional restrictions are obtained by setting integer variables to a bound.

Lifting with general integer variables is computationally harder than lifting with 0–1 variables, because the former requires the solution of nonlinear integer problems rather than linear integer problems. Here we see that nonlinearity in lifting problems is resolved easily with superadditive lifting functions. The results presented here may pave the way for efficient applications of lifting with general integer variables.

2. Sequential Lifting

Consider a mixed-integer set $P = \{x \in \mathbb{R}^C \times \mathbb{Z}^I\colon Ax \leqslant b\}$, where A is a rational matrix with m rows, and b is a rational m column vector. Let $F \subseteq I$, $R = C \cup I \backslash F$, and $P' = \{x_R \in \mathbb{R}^C \times \mathbb{Z}^{I\backslash F}\colon A_R x_R \leqslant d\}$ ($\neq \varnothing$) be the restriction of P, obtained by setting $x_F = y_F$, where $y_i \in \mathbb{Z}$, $l_i \leqslant y_i \leqslant u_i$, $i \in F$, and l_i and u_i are the minimum and maximum values of x_i in P; thus $d = b - A_F y_F$. Let $\pi_R x_R \leqslant \pi_o$ be a valid inequality for the restriction P'. Then, Theorem 1 can be used to lift $\pi_R x_R \leqslant \pi_o$ with the fixed variables $\{x_i\}_{i\in F}$ one at a time, in some sequence, to derive a valid inequality for P. For brevity, we let $Ri = R \cup \{i\}$, $\min_{x\in T} f(x) = +\infty$, and $\max_{x\in T} f(x) = -\infty$ if $T = \varnothing$. Also, for $R \subseteq S \subseteq C \cup I$ and $h \in \mathbb{R}^m$, let $P_S(h) = \{x_S \in \mathbb{R}^C \times \mathbb{Z}^{S\backslash C}\colon A_S x_S \leqslant h\}$; so $P' = P_R(d)$.

Theorem 1 (Wolsey 1976). *$\pi_R x_R + \pi_i(x_i - y_i) \leqslant \pi_o$ is a valid inequality for $P_{Ri}(d + A_i y_i)$ if and only if $\underline{\pi}_i \leqslant \pi_i \leqslant \bar{\pi}_i$, where*

$$\bar{\pi}_i = \min\left\{\frac{\pi_0 - \pi_R x_R}{x_i - y_i}\colon x_i > y_i,\, x_{Ri} \in P_{Ri}(d + A_i y_i)\right\},$$

$$\underline{\pi}_i = \max\left\{\frac{\pi_0 - \pi_R x_R}{x_i - y_i}\colon x_i < y_i,\, x_{Ri} \in P_{Ri}(d + A_i y_i)\right\}.$$

Moreover, if $\pi_i = \underline{\pi}_i > -\infty$ or $\pi_i = \bar{\pi}_i < +\infty$ and $\pi_R x_R \leqslant \pi_o$ defines a k-dimensional face of $\operatorname{conv}(P_R(d))$, *then $\pi_R x_R + \pi_i(x_i - y_i) \leqslant \pi_o$ defines an at least $k+1$-dimensional face of* $\operatorname{conv}(P_{Ri}(d + A_i y_i))$.

Remark 1. In order to use Theorem 1 for a given sequence of fixed variables, one needs to solve *two nonlinear* (fractional) mixed-integer lifting problems for each variable in the sequence. However, lifting an inequality with a 0–1 variable requires the solution of a *single linear* mixed-integer program, because $x_i - y_i \in \{-1, 1\}$ and one of the above problems is infeasible.

Remark 2. For a particular $i \in F$, the later x_i is introduced to the inequality in a lifting sequence, the smaller $\bar{\pi}_i$ is and the larger $\underline{\pi}_i$ is. Therefore, different sequences may lead to different lifted inequalities for P and not all sequences may lead to a valid lifted inequality as it might be the case that $\underline{\pi}_i > \bar{\pi}_i$ for some $i \in F$ in some sequence.

REMARK 3. For many structured MIP problems, lifting problems for the *first* variable in the lifting sequence can be solved efficiently by exploiting the special structure of the function $\pi_R x_R$ in the objective. However, once new lifting coefficients are introduced to the inequality, the structure in the objective function may be lost.

3. Bounds on the Lifting Coefficients

To simplify the notation, after changing variables as $z_F = x_F - y_F$ and $z_R = x_R$, we rewrite P as $Q = \{z \in \mathbb{R}^C \times \mathbb{Z}^I\colon Az \leqslant d\}$ and define $Q_S(h) = \{z_S \in \mathbb{R}^C \times \mathbb{Z}^{S\setminus C}\colon A_S z_S \leqslant h\}$ for $R \subseteq S \subseteq C \cup I$ and $h \in \mathbb{R}^m$. Hence, $Q_R(d)$ is obtained by setting $z_F = 0$ and $\pi_R z_R + \pi_F z_F \leqslant \pi_o$ is valid for Q if and only if $\pi_R x_R + \pi_F(x_F - y_F) \leqslant \pi_o$ is valid for P.

Consider the value function $v(h) = \max\{\pi_R z_R\colon z_R \in Q_R(h)\}$ and define two functions $\Phi\colon \mathbb{R}^m \mapsto \mathbb{R} \cup \{+\infty\}$ and $\Psi\colon \mathbb{R}^m \mapsto \mathbb{R} \cup \{+\infty\}$ for $\pi_R x_R \leqslant \pi_o$ as

$$\Phi(a) = \pi_o - v(d-a),$$

$$\Psi(a) = \min\left\{\frac{\Phi(u+ka) - \Phi(u)}{k}\colon u \in D, k \in \mathbb{Z}_{++}\right\},$$

where $D = \{a \in \mathbb{R}^m\colon Q_R(d-a) \neq \varnothing\}$ and $\mathbb{Z}_{++}$ is the set of positive integer numbers. $\Phi < +\infty$ on D, and because $\pi_R z_R \leqslant \pi_o$ is valid for $Q_R(d)$, $\Phi > -\infty$ on $\mathbb{R}^m$. Hence, $\Psi > -\infty$ on $\mathbb{R}^m$ and $\Psi < +\infty$ on D, by taking $k = 1$ and $u = 0$ as $0 \in D$.

Suppose $\pi_R z_R \leqslant \pi_o$ is lifted with variables indexed with $E \subset F$ in some sequence. Now, consider the lifting problems associated with z_i, $i \in F\setminus E$:

$$\bar{\pi}_i = \min\left\{\frac{\pi_0 - \pi_R z_R - \pi_E z_E}{z_i}\colon z_i > 0, z_{REi} \in Q_{REi}(d)\right\},$$

$$\underline{\pi}_i = \max\left\{\frac{\pi_0 - \pi_R z_R - \pi_E z_E}{z_i}\colon z_i < 0, z_{REi} \in Q_{REi}(d)\right\},$$

which are bounded because $\pi_R z_R + \pi_E z_E \leqslant \pi_o$ is valid for $Q_{RE}(d)$.

LEMMA 2. *For any $i \in F$, $\Psi(A_i) \leqslant \bar{\pi}_i \leqslant \Phi(A_i)$ and $-\Phi(-A_i) \leqslant \underline{\pi}_i \leqslant -\Psi(-A_i)$ hold independent of E.*

PROOF. If $A_i \in D$, the minimization lifting problem has a feasible solution such that $z_i = 1$, $z_E = 0$ and has objective value $\Phi(A_i)$; else $\Phi(A_i) = +\infty \geqslant \bar{\pi}_i$. If $-A_i \in D$, the maximization lifting problem has a feasible solution such that $z_i = -1$, $z_E = 0$ and has objective value $-\Phi(-A_i)$; else $-\Phi(-A_i) = -\infty \leqslant \underline{\pi}_i$.

If the minimization lifting problem is infeasible, $\bar{\pi}_i = +\infty \geqslant \Psi$. Otherwise, let $\bar{z}_{Ri\cup E}$ be an optimal solution. Then,

$$\begin{aligned}
\bar{\pi}_i &= \min\left\{\frac{\pi_0 - \pi_R z_R - \pi_E \bar{z}_E}{z_i}\colon z_i > 0, z_{Ri} \in Q_{Ri}(d - A_E\bar{z}_E)\right\}\\
&\geqslant \min\left\{\frac{\Phi(A_E\bar{z}_E + kA_i) - \pi_E\bar{z}_E}{k}\colon k \in \mathbb{Z}_{++}\right\}\\
&\geqslant \min\left\{\frac{\Phi(A_E\bar{z}_E + kA_i) - \Phi(A_E\bar{z}_E)}{k}\colon k \in \mathbb{Z}_{++}\right\}\\
&\geqslant \Psi(A_i).
\end{aligned}$$

The second inequality holds because $\pi_R z_R + \pi_E z_E \leqslant \pi_o$ is valid for Q_{RE} if and only if $\pi_E z_E \leqslant \pi_o - v(d - A_E z_E) = \Phi(A_E z_E)$.

If the maximization lifting problem is infeasible, $\underline{\pi}_i = -\infty \leqslant -\Psi$. Otherwise, let $\bar{z}_{Ri\cup E}$ be an optimal solution. Then,

$$\begin{aligned}
\underline{\pi}_i &= \max\left\{\frac{\pi_0 - \pi_R z_R - \pi_E \bar{z}_E}{z_i}\colon z_i < 0, z_{Ri} \in Q_{Ri}(d - A_E\bar{z}_E)\right\}\\
&\leqslant \max\left\{\frac{\pi_E\bar{z}_E - \Phi(A_E\bar{z}_E - kA_i)}{k}\colon k \in \mathbb{Z}_{++}\right\}\\
&\leqslant \max\left\{\frac{\Phi(A_E\bar{z}_E) - \Phi(A_E\bar{z}_E - kA_i)}{k}\colon k \in \mathbb{Z}_{++}\right\}\\
&\leqslant -\Psi(-A_i). \quad \square
\end{aligned}$$

REMARK 4. Because the bounds on $\underline{\pi}_i$ and $\bar{\pi}_i$ are independent of E, they hold for any $i \in F$, *independent of the lifting sequence.*

DEFINITION 1. A function $f\colon \mathbb{R}^m \mapsto \mathbb{R}$ is *superadditive* if $f(a) + f(b) \leqslant f(a+b)$ for all $a, b \in \mathbb{R}^m$.

LEMMA 3. *$\Psi = \Phi$ if and only if Φ is superadditive.*

PROOF. In general, for any $a \in \mathbb{R}^m$, $\Psi(a) \leqslant \Phi(a) - \Phi(0) \leqslant \Phi(a)$, which follows from the definition of Ψ by taking $k = 1$ and $u = 0$ and from $\Phi(0) \geqslant 0$ as $\pi_R z_R \leqslant \pi_o$ is valid for $Q_R(d)$.

For $a \in \mathbb{R}^m$, let $\Psi(a) = (\Phi(u^* + k^* a) - \Phi(u^*))/k^*$. Then, by superadditivity of Φ,

$$\Psi(a) \geqslant \frac{\Phi(u^*) + k^*\Phi(a) - \Phi(u^*)}{k^*} = \Phi(a).$$

For the other direction, if $\Phi(a) \leqslant \Psi(a)$ for all $a \in \mathbb{R}^m$, then $\Phi(a) \leqslant \Phi(u+a) - \Phi(u)$ for all $a, u \in \mathbb{R}^m$. $\square$

COROLLARY 4. *Let $\bar{F} = \{i \in F\colon\ l_i < y_i < u_i\}$ and suppose Φ is superadditive. Then, there exists a valid inequality lifted from $\pi_R x_R \leqslant \pi_o$ if and only if $\Phi(-A_i) = -\Phi(A_i)$ for all $i \in \bar{F}$.*

PROOF. From Lemmas 2 and 3, if Φ is superadditive then $\bar{\pi}_i = \Phi(A_i)$ and $\underline{\pi}_i = -\Phi(-A_i)$. However, because Φ is superadditive, $\Phi(-A_i) + \Phi(A_i) \leqslant \Phi(0) = 0$, and hence $\underline{\pi}_i \geqslant \bar{\pi}_i$. The result follows because $\underline{\pi}_i \leqslant \bar{\pi}_i$ for validity. $\square$

4. Sequence Independent Lifting

Corollary 4 suggests that if Φ is superadditive, lifting of inequality is possible only under strong conditions if some integer variable x_i, $i \in F$, is fixed to a value y_i that is not equal to either its lower bound or upper bound in the restriction. Therefore, we now consider a nonempty restriction $P_R(d)$ of P obtained by setting *integer* variables x_L to their lower bounds and x_U to their upper bounds; thus,

$F = L \cup U \subseteq I$ and $d = b - A_L l_L - A_U u_U$, where l_L and u_U are finite lower bound and upper bound vectors for x_L and x_U, respectively.

We generalize a result by Gu et al. (2000) for monotone mixed 0–1 programming, which states that it suffices to use a superadditive lower bound on Φ for deriving valid lifting coefficients for all $i \in F$ from $P_R(d)$. Below we give a simple proof of this result for general MIP.

THEOREM 5. *Let $\pi_R x_R \leqslant \pi_0$ be a valid inequality for $P_R(d)$ and Φ be its lifting function as defined before. Let ϕ: $\mathbb{R}^m \mapsto \mathbb{R}$ be a superadditive function such that $\phi \leqslant \Phi$. Then, the lifted inequality $\sum_{i\in L} \phi(A_i)(x_i - l_i) + \sum_{i\in U} \phi(-A_i)(u_i - x_i) + \pi_R x_R \leqslant \pi_o$ is valid for P. Moreover, if $\phi(A_i) = \Phi(A_i)$ for all $i \in L$, $\phi(-A_i) = \Phi(-A_i)$ for all $i \in U$, and $\pi_R x_R \leqslant \pi_o$ defines a k-dimensional face of $\operatorname{conv}(P_R(d))$, then the lifted inequality defines a face of $\operatorname{conv}(P)$ of dimension at least $k + |L| + |U|$.*

PROOF. Let $z_L = x_L - l_L$, $z_U = x_U - u_U$, and $z_F = x_F$. For any $z \in Q$, $v(d - A_L z_L - A_U z_U) \geqslant \pi_R z_R$. Therefore,

$$\begin{aligned}\pi_o - \pi_R z_R &\geqslant \pi_o - v(d - A_L z_L - A_U z_U)\\ &= \Phi(A_L z_L + A_U z_U)\\ &\geqslant \phi(A_L z_L + A_U z_U)\\ &\geqslant \sum_{i\in L} \phi(A_i z_i) + \sum_{i\in U} \phi(A_i z_i)\\ &\geqslant \sum_{i\in L} \phi(A_i) z_i - \sum_{i\in U} \phi(-A_i) z_i.\end{aligned}$$

The last inequality follows from superadditivity of ϕ, nonnegativity and integrality of z_i, $i \in L$, and nonpositivity and integrality of z_i, $i \in U$.

For the second part of the theorem, observe that there exists an optimal solution to the lifting problem for $i \in L$ with $z_i = 1$ and $z_j = 0, j \in F\backslash\{i\}$ and objective value $\Phi(A_i)$, and for $i \in U$ with $z_i = -1$ and $z_j = 0, j \in F\backslash\{i\}$ and objective value $-\Phi(-A_i)$ for $i \in U$. These points are affinely independent with the points of the face of $\operatorname{conv}(P_R(d))$ induced by $\pi_R x_R \leqslant \pi_o$, because $z_i = 0$ for all $i \in F$ on this face. □

REMARK 5. Under the conditions of the second part of Theorem 5, by Lemma 2, the lifted inequality $\sum_{i\in L} \Phi(A_i) \cdot (x_i - l_i) + \sum_{i\in U} \Phi(-A_i)(u_i - x_i) + \pi_R x_R \leqslant \pi_o$ is the *unique* facet-defining inequality that can be obtained by sequential lifting of $\pi_R x_R \leqslant \pi_o$ in *any* sequence.

REMARK 6. If Φ is superadditive on D, then the lifting problem for z_i, $i \in F$, reduces to computing $\Phi(A_i)$ (or $\Phi(-A_i)$), which is a *linear* mixed-integer problem rather than a nonlinear one. Also, any special structure in $\pi_R x_R$ can be exploited for *all* fixed variables, not just for the first one in the lifting sequence. This may help to compute all lifting coefficients efficiently.

REMARK 7. For validity of the lifted inequality in Theorem 5, $\pi_R x_R \leqslant \pi_o$ need not be tight for $P_R(d)$. Validity of $\pi_R x_R \leqslant \pi_o$ for $P_R(d)$ implies that $\Phi(0) \geqslant 0$. On the other hand, superadditivity of ϕ implies that $\phi(0) \leqslant 0$. Therefore, if $\Phi(0) = \phi(0)$, then $v(d) = \pi_o$, i.e., $\pi_R x_R \leqslant \pi_o$ is tight for $P_R(d)$.

5. Application: Mixed-Integer Rounding

Mixed-integer rounding (MIR) is a general procedure for deriving valid inequalities for MIP problems. For a constraint of a MIP problem with nonnegative variables,

$$\sum_{i\in I} a_i x_i + \sum_{i\in C} g_i y_i \leqslant b, \tag{1}$$

where I is the index set of integer variables and C is the index set of continuous variables, the MIR inequality (Gomory 1960, Nemhauser and Wolsey 1990) is stated as

$$\sum_{i\in I}\left(\lfloor a_i \rfloor + \frac{(f_i - f)^+}{1-f}\right) x_i + \sum_{i\in C:\, g_i<0} \frac{g_i}{1-f} y_i \leqslant \lfloor b \rfloor, \tag{2}$$

where $f = b - \lfloor b \rfloor$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i \in I$. MIR cuts may also be viewed as disjunctive cuts (Balas 1979) or split cuts (Cook et al. 1990); also see Cornuéjols and Li (2001), Marchand and Wolsey (2001), and Nemhauser and Wolsey (1990). The MIR inequality obtained after multiplying (1) with $\lambda > 0$ is called the λ-MIR inequality (Cornuéjols et al. 2000).

It is well known that the coefficient of variable x_i in the MIR inequality is a superadditive function of a_i; see Nemhauser and Wolsey (1988, 1990). Here, to illustrate Theorem 5, we observe that the MIR inequality (2) can be derived by sequence independent lifting, which allows us to make a statement about its strength.

Let $z_1 = \sum_{i\in C:\, g_i>0} g_i y_i$, $z_2 = -\sum_{i\in C:\, g_i<0} g_i y_i$, and consider the mixed-integer set

$$S_c = \{w \in \mathbb{Z}, z \in \mathbb{R}^2_+ \colon cw + z_1 - z_2 \leqslant b, l \leqslant w \leqslant u\}.$$

Observe that the LP relaxation of S_c has a fractional vertex $(b/c, 0, 0)$ if and only if $b/c \notin \mathbb{Z}$ and $l < b/c < u$. Suppose $c > 0$ (the argument is symmetric otherwise) and let $\eta = \lceil b/c \rceil$ and $r = b - \lfloor b/c \rfloor c$. Then, the $1/c$-MIR inequality for $cw + z_1 - z_2 \leqslant b$,

$$(c - r)w - z_2 \leqslant b - \eta r, \tag{3}$$

cuts off the fractional vertex $(b/c, 0, 0)$ and is sufficient to describe $\operatorname{conv}(S_c)$ together with the original inequalities of S_c. The lifting function for inequality (3),

$$\Phi_c(a) := b - \eta r - \max\{(c - r)w - z_2 \colon cw + z_1 - z_2 \leqslant b - a,\\ l \leqslant w \leqslant u, w \in \mathbb{Z}, z \in \mathbb{R}^2_+\},$$

can be expressed explicitly as

$$\Phi_c(a) = \begin{cases} (\eta - u - 1)(c - r) & \text{if } a < b - uc, \\ k(c - r) & \text{if } kc \leqslant a < kc + r, \\ a - (k+1)r & \text{if } kc + r \leqslant a < (k+1)c, \\ a - (\eta - l)r & \text{if } a \geqslant b - lc, \end{cases}$$

where $k \in \{\eta - u - 1, \eta - u, \ldots, \eta - l\}$ (Atamtürk 2003). Φ_c is not superadditive on $\mathbb{R}$; however, by Theorem 5, one can use a superadditive lower bound on Φ_c to lift inequality (3). For example, relaxing the bound constraints as $-\infty \leqslant w \leqslant +\infty$, the following superadditive lower bound is obtained:

$$\phi_c(a) = \begin{cases} k(c - r) & \text{if } kc \leqslant a < kc + r, \\ a - (k+1)r & \text{if } kc + r \leqslant a < (k+1)c. \end{cases}$$

For any $c > 0$, if necessary by introducing a fictitious variable w with coefficient c and projecting it back to 0, we may use ϕ_c to lift inequality (3) to inequality

$$\sum_{i \in I} \phi_c(a_i) x_i + \sum_{i \in C: g_i < 0} g_i y_i \leqslant b - \eta r. \tag{4}$$

Observe that for $c = 1$, we have $\eta = \lceil b \rceil$, $r = f$, and in this case ϕ_c reduces to

$$\phi_1(a) = \begin{cases} \lfloor a \rfloor (1 - f) & \text{if } \lfloor a \rfloor \leqslant a < \lfloor a \rfloor + f, \\ a - \lceil a \rceil f & \text{if } \lfloor a \rfloor + f \leqslant a < \lceil a \rceil \end{cases}$$
$$= \lfloor a_i \rfloor (1 - f) + (f_i - f)^+$$

and the right-hand side of (4) becomes $\lfloor b \rfloor (1 - f)$. Thus, ϕ_c is the same superadditive function used to define the $1/c$-MIR inequalities; see §II.1.7 in Nemhauser and Wolsey (1988).

Deriving MIR inequalities as lifted inequalities using an approximate lifting function allows us to make a statement about their strength by giving conditions under which the approximate lifting coefficients equal the exact lifting coefficients. The following proposition is an immediate consequence of Theorem 5, because $\phi_{a_k}(a_i) = \Phi_{a_k}(a_i)$ under the stated conditions, which is seen by taking $l = 0$, $u = +\infty$, and $c = a_k$.

Proposition 6. *The $1/|a_k|$-MIR inequality ($k \in I$) is facet defining for* $\operatorname{conv}\{(x, y) \in \mathbb{Z}_+^I \times \mathbb{R}_+^C$: (x, y) *satisfies* (1)} *if $a_k, b > 0$ and $a_i \leqslant \lceil b/a_k \rceil a_k$ for all $i \in I$, or if $a_k, b < 0$ and $a_i \geqslant \lceil b/a_k \rceil a_k$ for all $i \in I$.*

See Atamtürk (2003) for superadditive lower-bounding lifting functions that exploit the bounds of integer variables for deriving strong inequalities for mixed-integer knapsack sets.

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