A NOTE ON THE KNAPSACK PROBLEM WITH SPECIAL ORDERED SETS

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The knapsack problem with special ordered sets and arbitrarily signed coefficients is shown to be equivalent to a standard problem of the same type but having all coefficients positive. Two propositions are proven which define an algorithm for the linear programming relaxation of the standard problem that is a natural generalization of the Dantzig solution to the problem without special ordered sets. Several properties of the convex hull of the associated zero-one polytope are derived.

Knapsack problem, special ordered sets, GUB, algorithms, facets

Introduction

Consider the following problem (KPSOS):

$$\begin{aligned} \max \quad & \sum_{i\in I}\sum_{j\in K_i} c_j x_j, \\ \text{subject to} \quad & \sum_{i\in I}\sum_{j\in K_i} a_j x_j \le a_0, \\ & \sum_{j\in K_i} x_j \le 1 \quad \text{for all } i \in I, \\ & x_j \ge 0 \quad \text{for all } j\in K_i,\ i\in I, \end{aligned}$$

where the K_i satisfy $|K_i| \ge 1$ for all $i \in I$, $K_i \cap K_j = \emptyset$ holds for all $i \ne j$ and the data c_j and aj are arbitrarily signed real numbers. This problem has received much attention in recent literature, see e.g. Zemel [6] for a survey of the literature. In this note we first show that (KPSOS) can always be brought into a 'standard' form with positive data and prove two propositions which define an algorithm for (KPSOS) that is a natural extension of Dantzig's solution [2] to (KPSOS) when $|K_i| = 1$ for all $i \in I$. Note that arbitrary upper bounds u_i on the special ordered sets can be dealt with by scaling. We then consider the zero-one-variables version of (KPSOS) and derive several properties of the facets of the associated convex hull of zero-one solutions to (KPSOS).

The algorithm

Note that any special ordered set constraint of the form $\sum_{j\in K_i} x_j = 1$ can be brought to the inequality form by eliminating one of the variables in such set. Next we show that (KPSOS) can always be brought to a standard form with all data positive. Define

$$a_{j_i} = \min\{a_j \mid j \in K_i\} \quad \text{for } i \in I \tag{1.1}$$

and let $H = \{i \in I \mid a_{j_i} < 0\}$. For all $i \in H$ we set

$$a'_j = \begin{cases} a_j - a_{j_i} & \text{for all } j \in K_i - \{j_i\}, \\ -a_{j_i} & \text{for } j = j_i, \end{cases}$$

$$c'_j = \begin{cases} c_j - c_{j_i} & \text{for all } j \in K_i - \{j_i\}, \\ -c_{j_i} & \text{for } j = j_i, \end{cases} \tag{1.2}$$

$$a'_0 = a_0 - \sum_{i\in H} a_{j_i}.$$

Let $K = \bigcup_{i\in H} K_i$ and define $a'_j = a_j$, $c'_j = c_j$ for all $j \notin K$. Consider the problem (KPSOS*):

$$\begin{aligned} \max \quad & \sum_{i\in I}\sum_{j\in K_i} c'_j z_j, \\ \text{subject to} \quad & \sum_{i\in I}\sum_{j\in K_i} a'_j z_j \le a'_0, \end{aligned}$$

$$\sum_{j\in K_i} z_j \le 1 \quad \text{for all } i \in I,$$
$$z_j \ge 0 \quad \text{for all } j \in K_i,\, i \in I.$$

Problem (KPSOS*) satisfies $a'_j \ge 0$ for all j and the variable substitution

$$\begin{aligned} &x_j = z_j && \text{for } j \notin K \\ &\left.\begin{aligned} &x_j = z_j && \text{for } j \in K_i - \{j_i\} \\ &x_j = 1 - \sum_{j\in K_i} z_j && \text{for } j = j_i \end{aligned}\right\} && \text{for all } i \in H \end{aligned} \tag{1.3}$$

provides a one-to-one mapping between feasible solutions to (KPSOS) and (KPSOS*), respectively, which preserves—up to the constant term $\Sigma_{i\in H} c_{j_i}$—the value of the objective function.

If $c'_j \le 0$ and $a'_j \ge 0$ in (KPSOS*) we set $z_j = 0$; if $c'_j > 0$ and $a'_j = 0$ for some $j \in K_i$ and some $i \in I$, it follows that every optimum solution to (KPSOS*) satisfies $\Sigma_{j\in K_i} z_j = 1$ and letting

$$c'_{j_*} = \max\{c'_j \mid a'_j = 0, j \in K_i\} \tag{1.4}$$

we can eliminate variable z_{j_*} from the problem. The eliminated variable is recorded and its value determined at the end of the calculation. After this reduction (possibly used repeatedly) we thus obtain the linear knapsack problem with special ordered sets in the following standard form (KPSOSX):

$$\begin{aligned} \max \quad & \sum_{i\in L}\sum_{j\in S_i} d_j z_j, \\ \text{subject to} \quad & \sum_{i\in L}\sum_{j\in S_i} g_j z_j \le g_0, \\ & \sum_{j\in S_i} z_j \le 1 \quad \text{for all } i \in L, \\ & z_j \ge 0 \quad \text{for all } j \in S_i,\, i \in L, \end{aligned}$$

where $d_j > 0$, $g_j > 0$, $g_0 > 0$ for all j and $|S_i| \ge 1$ for all $i \in L$. Let for $i \in L$

$$\frac{d_{j_i}}{g_{j_i}} = \max\left\{\frac{d_j}{g_j} \middle| j \in S_i\right\} \tag{1.5}$$

and suppose that the special ordered sets have been ordered such that

$$\frac{d_{j_1}}{g_{j_1}} \ge \frac{d_{j_2}}{g_{j_2}} \ge \cdots \ge \frac{d_{j_h}}{g_{j_h}} \tag{1.6}$$

where $h = |L|$.

Proposition 1.1. *There exists an optimal solution z^* to* (KPSOSX) *such that $\Sigma_{k\in S_1} z^*_k = 1$ or an optimal solution is given by $z^*_{j_1} = g_0/g_{j_1}$, $z^*_j = 0$ for all $j \ne j_1$.*

Proof. Suppose that $\Sigma_{k\in S_1} z_k > 1$ holds in every optimal solution to (KPSOSX). Let z be any optimal solution and $z_j > 0$ for some $j \ne j_1$. Define z^* by $z^*_{j_1} = z_{j_1} + \epsilon$, $z^*_j = z_j - \epsilon g_{j_1}/g_j$, $z^*_k = z_k$ for all $k \ne j_1, j$ where

$$\epsilon = \min\left\{1 - \sum_{k\in S_1} z_k, z_j g_j/g_{j_1}\right\}. \tag{1.7}$$

Clearly $\epsilon > 0$ and z^* is a feasible solution to (KPSOSX). We then get

$$\begin{aligned} \sum_{\text{all } K} d_k z^*_k - \sum_{\text{all } k} d_k z_k &= d_{j_1}\epsilon - d_j \epsilon g_{j_1}/g_j \\ &= \left(\frac{d_{j_1}}{g_{j_1}} - \frac{d_j}{g_j}\right) g_{j_1}\epsilon \ge 0 \end{aligned} \tag{1.8}$$

and thus z^* is optimal as well. If $\epsilon = 1 - \Sigma_{k\in S_1} z_k$ holds, then $\Sigma_{k\in S_1} z^*_k = 1$. Else we can iterate unless $z^*_{j_1}$ is the only positive variable and $z^*_{j_1} = g_0/g_{j_1}$ holds since $\Sigma_{k\in S_1} z_k < 1$ holds in every optimal solution. The proposition follows.

Proposition 1.2. *If $g_{j_1} \ge g_0$, then $z_{j_1} = g_0/g_{j_1}$, $z_k = 0$ for all $k \ne j_1$ is an optimal solution to* (KPSOSX).

Proof. If $g_{j_1} \ge g_0$ holds, the solution of Proposition 1.2 is basic and a feasible basis is obtained by making all slack variables of the special ordered sets basic. Denoting s_0 the slack of the knapsack constraint we compute

$$\sum_{i\in L}\sum_{j\in S_i} d_j z_j = d_{j_1}\frac{g_0}{g_{j_1}} + \sum_{j\ne j_1}\left(d_j - g_j\frac{d_{j_1}}{g_{j_1}}\right) z_j - \frac{d_{j_1}}{g_{j_1}} s_0$$

and thus the reduced cost of all nonbasic variables are nonpositive, establishing optimality of the proposed solution. The proposition follows.

Propositions 1.1 and 1.2 define an algorithm for the linear knapsack problem with special ordered sets which is a natural generalization of Dantzig's method for the linear knapsack problem with variables having upper bounds. Let $j_1, j_2, \ldots, j_h$ be defined as in (1.6). If $g_{j_1} \ge g_0$ holds, we are done by Proposition 1.2. Else $g_{j_1} < g_0$ holds, thus we know by proposition 1.1 that

$$\sum_{j\in S_1} z_j = 1 \tag{1.9}$$

holds at the optimum. If $|S_1| = 1$ holds, we fix the corresponding variable at 1, record the change of g_0, and iterate. Else we eliminate from S_1 the variable j_* defined by

$$d_{j_*} = \max\{d_j \mid g_j = g^*\} \quad \text{where } g^* = \min\{g_j \mid j \in S_1\}. \tag{1.10}$$

We replace S_1 by the set

$$S_1^* = \left\{ j \in S_1 \mid d_j > d_{j_*} \right\}, \tag{1.11}$$

replace d_j by $d_j - d_{j_*}$, g_j by $g_j - g_{j_*}$ for all $j \in S_1^*$ and replace g_0 by $g_0 - g_{j_*}$. If $S_1^* = \emptyset$ holds, we fix z_{j_*} at value 1 and iterate. If $S_1 \neq \emptyset$ holds, we record the eliminated variable and compute

$$\frac{d_{j_+}}{g_{j_+}} = \max\left\{ \frac{d_j}{g_j} \middle| j \in S_1^* \right\}. \tag{1.12}$$

We merge the special ordered set S_1^* into its proper place according to the value of this ratio among the remaining (decreasingly ordered) ratios to be worked on and iterate.

During the subsequent iterations two cases may occur. In the first case, the (reduced) special ordered set S_1^* is *not* used again for the elimination of a variable. In this case, the recorded variable is set equal to one. The second case is that the set S_1^* is used again for the elimination of a variable. Here we have again two possibilities: either the algorithm stops when this occurs or we continue to iterate. If the algorithm stops, then, by the stopping rule, the current $g_{j_1} \geq g_0$. In this case the variable with the current index j_1 is set equal to g_0/g_{j_1}, and the previously recorded variable is assigned the value $1 - g_0/g_{j_1}$. If the algorithm continues to iterate, then the variable j_* which is currently eliminated becomes the recorded variable for its special ordered set while the previously recorded variable is set equal to zero and is dropped from further considerations.

Proposition 1.1 implies (1.9). Any variable in S_1^* could be eliminated, but the variable j_* is chosen so that the coefficients remain positive after elimination of j_*. Furthermore, if the reduced set S_1^* is used again for the elimination of a variable, then Proposition 1.1 implies that either (1.9) holds with S_1 replaced by S_1^*, or the algorithm terminates. If (1.9) holds with S_1 replaced by S_1^*, then all of the variables in $S_1 - S_1^*$ must be equal to zero and can be dropped from the problem. Consequently, for each special ordered set, only the last eliminated variable is recorded. That is, each special ordered set has at most one recorded variable. On the other hand, if the algorithm terminates, then Proposition 1.2 implies that the current $z_{j_1} = g_0/g_{j_1}$, and if there is a recorded variable for that special ordered set, then it must have value $1 - z_{j_1}$. It follows that the algorithm stops with at most two variables at fractional values, both of which are in the same special ordered set. When all the special ordered sets are singletons, the above algorithm is precisely Dantzig's solution method [2].

2. Properties of facets for the zero-one polytope

We consider again the general knapsack problem with special ordered sets and require that all variables be zero-one variables. As we are interested in the convex hull of zero-one solutions we dispense with the objective function and consider the constraint set

$$\text{(c1)} \quad \begin{aligned} &\sum_{i \in I} \sum_{j \in K_i} a_j x_j \leq a_0, \\ &\sum_{j \in K_i} x_j \leq 1 \quad \text{for all } i \in I, \\ &x_j = 0 \text{ or } 1 \quad \text{for all } j \in K_i,\, i \in I. \end{aligned}$$

Using the variable substitution (1.3) we bring (c1) into the equivalent form

$$\text{(c2)} \quad \begin{aligned} &\sum_{i \in I} \sum_{j \in K_i} \alpha_j z_j \leq \alpha_0, \\ &\sum_{j \in K_i} z_j \leq 1 \quad \text{for all } i \in I, \\ &z_j = 0 \text{ or } 1 \quad \text{for all } j \in K_i,\, i \in I, \end{aligned}$$

where the α_j are nonnegative and correspond to the a_j' of (1.2).

Proposition 2.1. *Let $\pi z \leq \pi_0$ be a facet (a valid inequality) of the convex hull of* (c2), *then*

$$\sum_{i \in H} (-\pi_{j_i}) x_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\pi_j - \pi_{j_i}) x_j + \sum_{j \notin K} \pi_j x_j \leq \pi_0 - \sum_{i \in H} \pi_{j_i}$$

is a facet (a valid inequality) of the convex hull of (c1). *Conversely, let $\mu x \leq \mu_0$ be a facet (a valid inequality) of the convex hull of* (c1), *then*

$$\sum_{i \in H} (-\mu_{j_i}) z_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\mu_j - \mu_{j_i}) z_j \sum_{j \notin K} \mu_j z_j \leq \mu_0 - \sum_{i \in H} \mu_{j_i}$$

is a facet (a valid inequality) of the convex hull of (c2), *where the notation of Section* 1 *is used.*

Proof. The variable substitution (1.3) can be written as $x = f + Az$, where f is a zero-one vector and A a nonsingular matrix. On the other hand one verifies that $z = f + Ax$ holds as well. Validity of the respective inequalities follows from the fact that the variable substitution provides a one-to-one mapping of feasible solutions. The facet defining property of the respective inequalities follows from the non-singularity of A and thus the proposition follows.

We assume now without restriction of generality that $\alpha_j \leq \alpha_0$ holds for all j. It follows that the inequalities

$$\sum_{j\in K_i} z_j \leq 1 \quad \text{for all } i \in I,$$
$$-z_j \leq 0 \quad \text{for all } j \in K_i, i \in I$$

define facets of the convex hull of (c2). We call those facets the *trivial* facets. Due to the non-negativity of the constraint system (c2) we know, e.g. from [3], that every facet $\pi z \leq \pi_0$ of (c2) which is not a non-negativity condition satisfies $\pi \geq 0$ and $\pi_0 < 0$.

Proposition 2.2 *If $\alpha_j = 0$ for some $j \in K_i$, $i \in I$, then $\pi_j = 0$ in every nontrivial facet $\pi z \leq \pi_0$ of the convex hull of* (c2).

Proof. Suppose the proposition is false and let $\pi z \leq \pi_0$ be a facet which has $\pi_j > 0$ for some $j \in K_i$, $i \in I$, with $\alpha_j = 0$. Since $\pi z \leq \pi_0$ is nontrivial there exists a feasible solution z^* to (c2) such that $\pi z^* = \pi_0$ and $\sum_{j\in K_i} z_j^* = 0$, since otherwise we have $\pi z = \sum_{j\in K_i} z_j$ and $\pi_0 = 1$. But then z^{**} defined by $z_k^{**} = z_k^*$ for all $k \neq j$ and $z_j^{**} = 1$ is a feasible solution to (c2) satisfying $\pi z^{**} = \pi_0 + \pi_j > \pi_0$, a contradiction.

Consequently, we can purge all zero α_j in (c2) and assume that $\alpha_0 \geq \alpha_j > 0$ holds for all $j \in K_i$ and all $i \in I$.

Assume next that the K_i are indexed such that

$$k_i = \{j_i, j_i+1, \ldots, j_i+t_i\},$$
$$\alpha_{j_i} \leq \alpha_{j_i+1} \leq \cdots \leq \alpha_{j_i+t_i},$$

where $t_i \geq 0$ holds.

Proposition 2.3. *If $\pi z \leq \pi_0$ is a nontrivial facet of the convex hull of* (c2) *and for some $i \in I$ and $j \in K_i$ we have $\pi_j > 0$, then*

$$\pi_k \geq \pi_j \quad \text{for all } k \in K_i, k > j \tag{2.5}$$

holds.

Proof. Suppose the proposition is false. Since $\pi z \leq \pi_0$ is nontrivial, there exists a feasible solution z^* to (c2) with $\pi z^* = \pi_0$ and $z_k^* = 1$ since otherwise $\pi z = -z_k$ and $\pi_0 = 0$ holds. Since $\alpha_j \leq \alpha_k$ it follows that z^{**} defined by $z_l^{**} = z_l^*$ for all $l \neq k, j$ and $z_k^{**} = 0$, $z_j^{**} = 1$ is feasible for (c2) and that $\pi z^{**} = \pi_0 + \pi_j - \pi_k > \pi_0$ holds, a contradiction.

To state the next proposition we need the following auxiliary problem (AUXKT): For $k \in I$ and $l \in K_k$ let

$$\zeta_l = \max \sum_{\substack{i\in I\\ i\neq k}} \sum_{j\in K_i} \alpha_j z_j,$$
$$\text{subject to} \quad \sum_{\substack{i\in I\\ i\neq k}} \sum_{j\in K_i} \alpha_j z_j \leq \alpha_0 - \alpha_l,$$
$$\sum_{j\in K_i} z_j \leq 1 \quad \text{for all } i \in I, i \neq k,$$
$$z_j = 0 \text{ or } 1 \quad \text{for all } j.$$

It should be clear from the definition of ζ_l that if $\zeta_j < \zeta_l$ for some $j, l \in K_k$, then it must follow that $\alpha_j \leq \alpha_l$. Hence, by Proposition 2.3, it follows that $\pi_j \leq \pi_l$ for every nontrivial facet of the convex hull of (c2). The next proposition gives a sharper result for the equality case: $\zeta_j = \zeta_l$.

Proposition 2.4. *If $\zeta_j = \zeta_l$ for $l \neq j$, $l \in K_k$, $j \in K_k$ and some $k \in I$, then $\pi_j = \pi_l$ for every nontrivial facet $\pi z \leq \pi_0$ of the convex hull of* (c2).

Proof. Suppose the proposition is false and let $\pi z \leq \pi_0$ be a facet with $\pi_j \neq \pi_l$. Without restriction of generality let $\pi_j < \pi_l$. Since $\pi z \leq \pi_0$ is a nontrivial facet of the convex hull of (c2), there exists a feasible zero-one vector z with $z_j = 1$ and $\pi z = z_0$. Define z^* by $z^*_l = 1$, $z_j^* = 0$, $z_h^* = z_h$ for all $h \neq l, j$. Then

$$\sum_{i\in I} \sum_{h\in K_i} \alpha_h z_h^* \leq \alpha_l + \zeta_j = \alpha_l + \zeta_l \leq \alpha_0$$

holds and thus z^* is feasible. But $\pi z^* = \pi_0 + \pi_l - \pi_j > \pi_0$ holds, a contradiction. Thus the proposition follows.

The assumption of Proposition 2.4 is satisfied, e.g. if $\alpha_j = \alpha_l$ holds. It follows that we can purge all but one of the variables in any special ordered set which have identical coefficients when we are interested in finding a minimal linear constraint set for the convex hull of (c2). We Initially conjectured a stronger property, namely that all facets of the convex hull of (c2) could be obtained by 'lifting' facets from associated knapsack inequalities with all special ordered sets consisting of singletons. While the conjecture is true for facets derived from minimal covers, see [3,4], and from $(1,k)$ configurations, see [5], the following example shows that the conjecture is false in general.

Example 2.5. Consider the inequalities

$$5x_1 + 3x_2 + 3x_3 + x_4 + 2x_5 \leq 7,$$
$$x_1 + x_2 \leq 1,$$
$$0 \leq x_i \leq 1 \text{ and integer}, \quad i = 1, \ldots, 5.$$

The inequality $2x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ defines a facet of the corresponding convex hull none of whose projections into a lower-dimensional space, however, defines a facet for the associated lower-dimensional zero-one problem.

The algorithm for the linear knapsack problem with special ordered sets as well as the properties of facets of the convex hull of (c2) are currently used in a cutting-plane based approach to large-scale zero-one programming problems [1].

References

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