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FACES FOR A LINEAR INEQUALITY IN 0–1 VARIABLES

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Given a linear inequality in 0–1 variables we attempt to obtain the faces of the integer hull of 0–1 feasible solutions. For the given inequality we specify how faces of a variety of lower-dimensional inequalities can be raised to give full-dimensional faces. In terms of a set, called a "strong cover", we obtain necessary and sufficient conditions for any inequality with 0–1 coefficients to be a face, and characterize different forms that the integer hull must take.

In general the suggested procedures fail to produce the complete integer hull. Special subclasses of inequalities for which all faces can be generated are demonstrated. These include the "matroidal" and "graphic" inequalities, where a count on the number of such inequalities is obtained, and inequalities where all faces can be derived from lower dimensional faces.

0. Introduction

In this paper we attempt to generate the faces of the integer hull of solutions to a linear inequality in binary variables. After introducing the notation and certain definitions we show in Section 2 how to augment certain lower-dimensional faces to give one or more faces of full dimension. Following this it is shown how every "strong cover" gives rise to at least one different face, as well as giving a simple necessary and sufficient condition for an inequality with 0–1 coefficients to be a face. We then consider the complete integer hull and in Sections 5–7 special subclasses of inequalities for which the above methods of generating faces give the complete integer hull are examined, as well as cases where the problem can be reduced to one of smaller dimension. Examples and certain special cases are also presented.

1. Definitions and notation

The basic inequality to be considered is:

$$\sum_{j=1}^{n} a_j x_j \leqslant a_0, \quad x_j \in \{0, 1\}, \quad j = 1, 2, ..., n. \tag{1}$$

Assumption 1.1. By substitution of $x'_j = 1 - x_j$, $a_j \geqslant 0, j = 0, 1, 2, ..., n$. We also assume throughout that $a_{j-1} \geqslant a_j$, $j = 1, 2, ..., n$.

Definition 1.2. The inequality

$$\sum_{j=1}^{n} \alpha_j x_j \leqslant \alpha_0 \tag{2}$$

is called a $(n - 1)$-dimensional face of (1) or a face if all 0–1 feasible solutions to (1) satisfy (2), and n affinely independent 0–1 feasible solutions satisfy (2) with equality.

It is easily shown that all faces, apart from the trivial faces $x_i \geqslant 0$, $i = 1, 2, ..., n$, have the above form with $\alpha_j \geqslant 0, j = 1, 2, ..., n$, and $\alpha_0 > 0$. Also multiplying through by a positive scalar gives an identical inequality.

Definition 1.3. Let $N = \{1, 2, ..., n\}$. Let S be a subset of N, and $S^c = N \sim S$, its complement. Denote

$$S = \{j_1, j_2, ..., j_r\}, \qquad S^c = \{i_1, i_2, ..., i_{n-r}\},$$

where in general $j_1 \leqslant j_2 \leqslant ... \leqslant j_r$, and $i_1 \leqslant i_2 \leqslant ... \leqslant i_{n-r}$.

S is called a *strong cover of* (1) if

(i) $\Sigma_{j \in S}\, a_j > a_0$,

(ii) $\Sigma_{j \in S}\, a_j - a_{j^*} \leqslant a_0$ for all $j^* \in S$,

(iii) if $\kappa \in S^c$ is the smallest integer greater than j_1,

$$\sum_{j \in S} a_j - a_{j_1} + a_\kappa \leqslant a_0 .$$

N.B. Throughout the paper, when considering subsets of N it will be assumed that the elements are arranged in increasing order, unless otherwise stated. Also $r = |S|$ denotes the cardinality of the set S, and by Assumption 1.1 that $a_0 \geqslant a_1$, faces of full dimension exist, and $r \geqslant 2$.

Definition 1.4. A strong cover S of (1) is called a *strong q-cover of* (1) if $j_1 = q$. $\mathcal{S}$ is the set of all *strong covers* of (1).

Definition 1.5. Given a strong q-cover S, the set $E(S) = S \cup S^1$ is called the *extension* of S, where $S^1 = \{1, 2, ..., q-1\}$.

2. Augmenting faces

In this section we show how, given certain faces of dimension k, it is always possible to construct from them faces of dimension $k+1$.

Theorem 2.1. *Given a face* $\Sigma_{j=1}^{k} \alpha_j x_j \leqslant \alpha_0$ *for the inequality*

$$\sum_{j=1}^{k} a_j x_j \leqslant a_0 - a_{k+1}, \qquad x_j \in \{0, 1\},$$

it can be raised to a face

$$\sum_{j=1}^{k+1} \alpha_j x_j \leqslant \alpha_0 + \alpha_{k+1} \quad \textit{of} \quad \sum_{j=1}^{k+1} a_j x_j \leqslant a_0, \quad x_j \in \{0, 1\},$$

if and only if $\alpha_{k+1} = Z - \alpha_0$, *where*

$$\text{(P)} \qquad \begin{aligned} Z = \max \sum_{j=1}^{k} \alpha_j x_j, \\ \text{s.t.} \sum_{j=1}^{k} a_j x_j \leqslant a_0, \qquad x_j \in \{0, 1\}. \end{aligned}$$

N.B. α_{k+1} is not required to satisfy the order or the non-negativity conditions of Assumption 1.1.

Proof. Let $X^1, X^2, ..., X^k$ be k linearly independent feasible points on the face of dimension $k-1$. Let X^* be an optimal solution of (P). Then the $k + 1$ vectors $\{(X^i, 1)\}_{i=1}^k$, $(X^*, 0)$ of dimension $(k + 1)$ are linearly independent, and feasible, and satisfy $\Sigma_{j=1}^{k+1} \alpha_j x_j = \alpha_0 + \alpha_{k+1}$. As $\Sigma_{j=1}^k \alpha_j x_j \leqslant \alpha_0$ is a face, all feasible vectors $(X, 1)$ satisfy $\Sigma_{j=1}^{k+1} \alpha_j x_j \leqslant \alpha_0 + \alpha_{k+1}$. Also from (P) all feasible vectors $(X, 0)$ satisfy it.

Hence the new inequality is a face. It also follows from the construction that if $\alpha_{k+1} \neq Z - \alpha_0$, the inequality cannot be a face.

Corollary 2.2. *Taking the complement* $\bar{x}_{k+1} = 1 - x_{k+1}$ *we obtain the projection result of Padberg* [13, 14] *and Trotter* [15] *for non-negative coefficients, giving faces of* $\Sigma_{j=1}^{k+1} a_j x_j \leqslant a_0$, $x_j \in \{0, 1\}$ *from faces of* $\Sigma_{j=1}^k a_j x_j \leqslant a_0$, $x_j \in \{0, 1\}$.

It is also possible that if the above extension is applied more than once to a face the order of introducing the variables may allow one to obtain different faces.

The reverse operation of dropping dimensionality only holds in special cases.

Lemma 2.3. *If* (2) *is a face of* (1) *with* $\alpha_t = 0$ *for some* $t \in N$, *then* $\Sigma_{j \in N - \{t\}} \alpha_j x_j \leqslant \alpha_0$ *is a* $(n - 2)$*-dimensional face of* $\Sigma_{j \in N - \{t\}} a_j x_j \leqslant a_0$.

Proof. Let $\{X^i\}_{j=1}^n$ be n linearly independent feasible points defining the face (2). Let X^{it} be the vector obtained from X^i by dropping its t^{th} entry. Now the $(n - 1) \times n$ matrix $M^t = (X^{1t}, X^{2t}, ..., X^{nt})$ has rank $(n - 1)$, and hence $\{X^{it}\}_{i=1}^n$ contains $(n - 1)$ linearly independent vectors. Each of these satisfies $\Sigma_{j \in N - \{t\}} \alpha_j x_j = \alpha_0$; all feasible points satisfy $\Sigma_{j \in N - \{t\}} a_j x_j \leqslant \alpha_0$, and hence it is a face.

3. Canonical faces

In the following two sections we examine certain connections between faces, strong covers and the integer hull, which can be proved using the results of the previous section.

Definition 3.1. Faces of the form $x_i \geqslant 0$ or $x_i \leqslant 1$ are called *elementary faces*. By Assumption 1.1, $x_i \geqslant 0$ is a face $i = 1, 2, ..., n$. Evidently $x_i \leqslant 1$ is a face if and only if $a_1 + a_i \leqslant a_0$ for $i = 2, 3, ..., n$, and $x_1 \leqslant 1$ is a face if and only if $a_1 + a_2 \leqslant a_0$.

The following theorem gives a simple classification of a subset of the non-elementary faces of (1) with 0–1 coefficients.

Theorem 3.2. [18]. *$\Sigma_{j\in T} x_j \leqslant |T| - 1$ is a non-elementary face of (1) if and only if T is a strong 1-cover for (1).*

This result in conjunction with those of the previous section leads naturally to a complete classification:

Theorem 3.3 [19]. *$\Sigma_{j\in T} x_j \leqslant |T| - k$ is a non-elementary face of (1) if and only if*

(i) *$T = E(S)$ is the extension of some set $S \subseteq N$.*
(ii) *S is a strong k-cover for (1).*
(iii) *$|T| - k = |S| - 1$.*
(iv) *$a_1 + \Sigma_{j\in S} a_j - a_{j_1} - a_{j_2} \leqslant a_0$, where we again use $S = \{j_1, j_2, ..., j_r\}$, and have by (ii), $j_1 = k$.*

Proof. See [2] and [11] for two alternative proofs.

Lemma 3.4. *Every strong cover gives rise to at least one different face.*

Proof. Every strong k-cover is a strong 1-cover for the inequality $\Sigma_{j=k}^{n} a_j x_j \leqslant a_0$, $x \in \{0, 1\}$. Applying Theorem 3.2 and Corollary 2.2, the result follows.

Example 3.5. $3x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leqslant 5$.

From the strong covers (124), (125), (134), (135), (2345) one has by Theorem 3.3 the faces

$$\begin{aligned} x_1 + x_2 \qquad + x_4 \qquad &\leqslant 2, \\ x_1 + x_2 \qquad\qquad + x_5 &\leqslant 2, \\ x_1 \qquad + x_3 + x_4 \qquad &\leqslant 2, \\ x_1 + x_2 + x_3 + x_4 + x_5 &\leqslant 3. \end{aligned}$$

By combining Lemma 3.4 with Theorem 2.1 it is often possible to obtain additional faces.

Setting $x_3 = 1$, one obtains a strong cover (245) for the lower-dimensional inequality $3x_1 + 2x_2 + x_4 + x_5 \leqslant 3, x \in \{0, 1\}$ and applying Lemma 3.4 this has a face $2x_1 + x_2 + x_4 + x_5 \leqslant 2$. Applying Theorem 2.1 we finally obtain the full-dimensional face

$$2x_1 + x_2 + 2x_3 + x_4 + x_5 \leqslant 4.$$

Similarly with $x_2 = 1$, one obtains the face

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 \leqslant 4 .$$

Setting $x_1 = 1$, one has two strong covers (34) and (35) in lower dimension giving faces

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 \quad &\leqslant 3, \\ 2x_1 + x_2 + x_3 \quad + x_5 &\leqslant 3 . \end{aligned}$$

The original inequality which happens to be the only other non-elementary face cannot be obtained in this manner.

4. The integer hull

Above we have seen how to generate a large number of faces. However as shown in the previous example it is not always possible to obtain all faces in this way. Here we consider the different forms that the integer hull can take, and then examine certain cases where it can be obtained explicitly.

First we emphasize the importance of the set of strong covers $\mathcal{S}$, as indicated by the following recent result of Glover.

Theorem 4.1 [9]. *Let*

$$\chi = \left\{ X \colon \sum_{j \in E(S)} x_j \leqslant |S| - 1 \text{ for all } S \in \mathcal{S},\ 0 \leqslant x_i \leqslant 1 \right\} .$$

A 0–1 *point X is a solution of* (1) *if and only if* $X \in \chi$. χ *is the minimum set of inequalities in* 0–1 *coefficients with this property.*

From this and Lemma 3.4 the following cases can now be distinguished:

Type a. χ is the integer hull of (1).

Type b. Each strong cover of (1) gives one or more faces of (1), and there are no other non-elementary faces.

Type c. Every non-elementary face of (1) can be obtained from a strong cover of (1), or by projecting up faces obtained from strong covers or elementary faces in lower dimension.

Type d. There exist faces which cannot be projected up from lower dimensions.

As a corollary of Theorems 3.2 and 4.1 we have:

Theorem 4.2. *If the integer hull of* (1) *only has inequalities with* 0–1 *coefficients,* χ *is the integer hull.*

Evidently the above procedures allow us to generate all faces of Type a, b or c. One would like to classify inequalities into each of these four types. Below we mainly consider Type a.

5. Matroidal inequalities

Theorem 5.1. χ *is the integer hull of* (1) *if the family* $\{E(S)\colon S \in \mathcal{S}\}$ *of sets can be partitioned into two families* V_p, $p = 1, 2$ *such that given any two members of* V_p *one is a subset of the other.*

Proof. The condition implies that the constraints of χ are totally unimodular; see [8] on the intersection of two matroids.

Corollary 5.2. χ *is the integer hull of* (1) *when the inequality* (1) *has only one or two strong covers.*

The following theorem suggested and simultaneously proved by Edmonds is covered by the above theorem, but is itself of independent interest. It is also placed in the more general context of regular 0–1 polytopes to which the earlier results of this paper extend.

Definition 5.3 [11]. Given a polytope

$$Y = \left\{x\colon \sum_{j=1}^{n} a_j x_j \leqslant a_0,\ x \in \{0, 1\}\right\},$$

where $\{a_i\}$ and a_0 are m-dimensional column vectors, Y is called *regular* if $\Sigma_{j\in S}\, a_j \leqslant a_0$ implies

(i) $\Sigma_{j\in T}\, a_j \leqslant a_0$ for all $T \subset S$,
(ii) $\Sigma_{j\in S}\, a_j - a_{j*} + a_\kappa \leqslant a_0$ for all $j^* \in S$, $\kappa \in S^c$ with $j^* < \kappa$.

We remark immediately that the polytope defined by inequality (1) is regular.

Definition 5.4 [3]. Given a regular polytope Y, and a set $S \subseteq N$ such that

(i) $\Sigma_{i\in S}\, a_i \leqslant a_0$,

(ii) if $t \in S^c$, $\Sigma_{i\in S}\, a_i + a_t \nleqslant a_0$,

(iii) if $t \in S^c$ and $t+1 \in S$, $\Sigma_{i\in S}\, a_i + a_t - a_{t+1} \nleqslant a_0$, then S is called a *ceiling* of Y.

Theorem 5.5. *The feasible* 0–1 *solutions of a regular polytope Y are the independent sets of a matroid if and only if Y has only one ceiling.* (*If Y has this property, it will be called matroidal.*)

Proof. Let $C = (c_1, c_2, ..., c_r)$ be the ceiling. Let $A = (d_1, d_2, ..., d_q)$ be a subset of N. Construct the set $\Gamma = (\gamma_1, \gamma_2, ..., \gamma_{r(A)})$ as follows:

$$\gamma_k = \min_{d_j \geqslant c_k} d_j \qquad \text{for } k = 1, 2, ... ,$$

until A is exhausted.

Consider $F = (f_1, f_2, ..., f_s) \subseteq A$. Define the 0–1 vector X^F by $x_i^F = 1$ if and only if $i \in F$. X^F satisfies (1) if $\gamma_i \leqslant f_i$, $i = 1, 2, ..., s$, and $s \leqslant r(A)$. To show the converse, suppose X^F lies in Y but violates this condition. Then $c_i \leqslant f_i$ for all $i = 1, 2, ..., s$ is impossible, and the uniqueness of the ceiling C is contradicted. Hence Γ is the unique ceiling of $\Sigma_{j\in A}\, a_j x_j \leqslant a_0$. But it is evident that for an inequality with one ceiling the maximal feasible solutions all have the magnitude of the ceiling. Hence the feasible solutions in Y form a matroid.

Conversely, suppose $C = (c_1, c_2, ..., c_r)$ and $D = (d_1, d_2, ..., d_s)$ are two ceilings of Y. Necessarily by the matroid property, $r = s$. Suppose that $c_i = d_i$, $i = r, r-1, ..., k+1$, but $c_k < d_k$. Let $A = C \cup D - \{d_k\}$. C and $D - \{k\}$ are maximal feasible sets in A of different cardinality, contradicting the matroid property.

Lemma 5.6. *The number of distinct regular* 0–1 *polytopes in n variables that are matroidal is 2^{n-2}.*

Proof. Let $C = (c_1, c_2, ..., c_r)$ be the ceiling. From the full-dimensionality of Y it follows that $c_1 = 1$, and $c_r \leqslant n-1$. The number of ceilings of cardinality k is then ${}^{n-2}C_{k-1}$. Hence summing over k, the number of ceilings is 2^{n-2}.

Specialising this result to the single linear inequality (1) we have:

Theorem 5.7. *An inequality* (1) *is matroidal if and only if it has exactly one ceiling* $(c_1, c_2, ..., c_r)$ *of the form*

(a) $C = (1, 2, ..., r-1, r), \quad r = 1, 2, ..., n-1$;

(b) $C = (1, 2, ..., r-1, c_r)$, *where* $r < c_r \leqslant n-1, r = 2, ..., n-2$;

(c) $C = (1, 2, ..., p, n-r+p, n-r+p+1, ..., n-1)$, *where* $1 \leqslant p \leqslant n-4, r \geqslant 3$.

Proof. Case (a) correponds to the inequality $\Sigma_{j=1}^n x_j \leqslant r$. Suppose now that C consists of two distinct sequences

$$C = (c_1, c_2, ..., c_p, c_{p+1}, ..., c_r),$$

where $c_{i+1} = c_i + 1$ for all $i \neq p$, but $c_{p+1} > c_p + 1$. Let $\Sigma_{j=1}^n b_j x_j \leqslant b_0$ be the corresponding linear inequality. As C is the unique ceiling we can take $b_j = b_1$ for $j < c_{p+1}$, and $b_j = b_n$ for $j \geqslant c_{p+1}$.

Consider now the sets

$$C_1 = (c_1, c_2, ..., c_p, c_p + 1, c_{p+1}, ..., c_{r-2}) \quad \text{if } c_{p+1} < c_r ;$$

$$C_2 = (c_1, c_2, ..., c_{p-1}, c_{p+1}, ..., c_r, c_r + 1, c_r + 2) \quad \text{if } c_r < n-1 .$$

If C_1 and C_2 exist, we have that $\Sigma_{j \in C_1} b_j > b_0$ and $\Sigma_{j \in C_2} b_j > b_0$ as C is the unique ceiling. Hence

$$b_0 < \tfrac{1}{2}\Big(\sum_{j \in C_1} b_j + \sum_{j \in C_2} b_j\Big) = \sum_{j \in C} b_j ,$$

contradicting the fact that C is a ceiling.

A similar argument shows up a contradiction whenever C splits up into three or more distinct sequences. Therefore we are left with:

Case (b). If $c_{p+1} = c_r$ and C has the form $(1, 2, ..., r-1, c_r)$, it is easily verified to be the unique ceiling of

$$\sum_{j=1}^{c_r - 1} k x_j + \sum_{j=c_r}^{n} (k-1) x_j \leqslant rk - 1 \quad \text{for } k \text{ sufficiently large} .$$

Case (c). If $c_r = n-1$ and C has the form

$$(1, 2, ..., p, n-r+p, n-r+p+1, ..., n-1),$$

it is the unique ceiling of

$$\sum_{j=1}^{n-r+p-1} k x_j + \sum_{j=n-r+p}^{n} x_j \leqslant pk + r - p \quad \text{for } k \text{ sufficiently large.}$$

Lemma 5.8. *The number of distinct matroidal inequalities* (1) *in n variables is* $n^2 - 5n + 8$.

Proof. Case (a). We obtain $(n-1)$ inequalities.

Case (b). For fixed r, c_r can take $n-r-1$ values. Summing for $r = 2, ..., n-2$, we obtain $\frac{1}{2}(n-3)(n-2)$ different inequalities.

Case (c). For fixed p we obtain $n-p-3$ inequalities. Summing for $p = 1, ..., n-4$, we obtain $\frac{1}{2}(n-4)(n-3)$ different inequalities.

Corollary 5.9. *If inequality* (1) *has only one ceiling,* χ *is the integer hull.*

Example 5.10. For $n = 5$ there are 75 different full-dimensional knapsack inequalities listed in [12]. Of these, 8 are matroidal, and 20 are the intersection of two matroids.

The non-trivial matroids are:

$$\begin{array}{lll} \text{Case (b) } C = & (13) & 3x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 \leqslant 5, \\ C = & (14) & 2x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leqslant 3, \\ C = & (124) & 2x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leqslant 5, \\ \text{Case (c) } C = & (134) & 3x_1 + 3x_2 + x_3 + x_4 + x_5 \leqslant 5\,. \end{array}$$

6. Graphic inequalities

Another class of inequalities (1) for which a complete classification can be found are the graphic inequalities (corresponding to 1-threshold graphs in [5]).

Definition 6.1. The inequality (1) is *graphic* if and only if for some t

(i) $a_{t-1} + a_t > a_0$,

(ii) $\Sigma_{j=t}^n a_j \leqslant a_0$.

Theorem 6.2. χ *is the integer hull of* (1) *if the inequality* (1) *is graphic.*

Proof. From (i) and (ii) it follows that the inequality (1) has only strong covers of cardinality 2, and these are necessarily the sets (k_0, t) $(k_1, t+1)$ $(k_2, t+2)$ ... (k_{n-t}, n), where $t - 1 = k_0 \geqslant k_1 \, ... \geqslant k_{n-t} \geqslant 1$. The resulting set of 0–1 inequalities χ is of the form:

$$\begin{array}{llllll}
x_1 + x_2 + \ldots + x_{k_0} & + x_t & & & & \leqslant 1, \\
x_1 + x_2 + \ldots + x_{k_1} & & + x_{t+1} & & & \leqslant 1, \\
x_1 + x_2 + \ldots + x_{k_2} & & & + x_{t+2} & & \leqslant 1, \\
\vdots & & & & \ddots & \\
x_1 \quad + \ldots + x_{k_{n-t}} & & & & + x_n & \leqslant 1 .
\end{array} \qquad (3)$$

or $(A, I)x \leqslant e$. Subtracting row $p + 1$ of A from row p, $p = 1, 2, \ldots, n - 1$ it is evident that A, and hence (A, I) is totally unimodular, and the result follows.

Lemma 6.3. *The number of different graphic inequalities in n variables is* 2^{n-2}.

Proof. It follows from [5], or the fact that every set of inequalities of the form (3) is equivalent to a single linear inequality (1), that the number of graphic inequalities equals the number of different possible forms of (3).

For fixed t, the number of ways to choose $(n - t)$ integers $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_{n-t}$, where $k_1 \leqslant t - 1$ and $k_{n-t} \geqslant 1$ is ${}^{(n-t)+(t-1)-1}C_{n-t} = {}^{n-2}C_{n-t}$. Summing over $t = 2, 3, \ldots, n$, we obtain 2^{n-2} inequalities.

Example 6.4. The inequality $5x_1 + 4x_2 + 2x_3 + 2x_4 + x_5 \leqslant 5$ is graphic with integer hull:

$$\begin{array}{llllll}
x_1 + x_2 + x_3 & & & \leqslant 1, \\
x_1 + x_2 & + x_4 & & \leqslant 1, \\
x_1 & & + x_5 & \leqslant 1; \\
x \geqslant 0 . & & &
\end{array}$$

For arbitrary n, a graphic inequality (1) is matroidal if and only if $t = n$ in the definition of graphic, while it is the intersection of two matroids if and only if $t = n - 1$. Hence we see that the overlap between the graphic, and the one and two matroid problems is small; one and $(n - 2)$ inequalities respectively.

7. Lower-dimensional inequalities

The above theorems characterize certain inequalities of type (a). A further question that it is natural to ask is when can all faces be obtained from faces of lower dimension, perhaps allowing a classification into types (a), (b) or (c).

Theorem 7.1. *If $\{j, n\}$ is a strong cover for some j, all the non-trivial faces are of the form $\alpha_0 x_1 + \Sigma_{j=2}^n \alpha_j x_j \leqslant \alpha_0$, where $\Sigma_{j=2}^n \alpha_j x_j \leqslant \alpha_0$ is a face of $\Sigma_{j=2}^n a_j x_j \leqslant a_0$, $x_j \in \{0, 1\}$.*

Proof. Suppose $\Sigma_{j=1}^n \alpha_j x_j \leqslant \alpha_0$ is a face. $X^1 = (1, 0, ..., 0)$ is the only feasible point with $x_1 = 1$. It must lie on the face (as it is of full dimension), and hence $\alpha_1 = \alpha_0$. Let $\{X^i\}_{i=2}^n$ be $(n-1)$ other linearly independent points defining the face, each with $X_1^i = 0$. Evidently they define the face $\Sigma_{j=2}^n \alpha_j x_j \leqslant \alpha_0$.

Conversely we have:

Theorem 7.2. *If all the strong covers are 1-covers, all non-elementary faces can be obtained by raising the faces, of*

$$\sum_{j \in R} a_j x_j \leqslant a_0 - a_1 , \quad \text{where } R \subseteq N - \{1\} .$$

Proof. Let $\Sigma_{j=1}^n \alpha_j x_j \leqslant \alpha_0$ be a non-elementary face of (1). Suppose $\alpha_j \neq 0$ for all $j \in T$, $T \subseteq N$. By Lemma 2.3, $\Sigma_{j \in T} \alpha_j x_j \leqslant \alpha_0$ is a face of $\Sigma_{j \in T} a_j x_j \leqslant a_0$. As all strong covers are 1-covers, $X = (0, 1, ..., 1)$ is feasible, and therefore $1 \in T$ and X is the only point with $x_1 = 0$ lying on the face. But this implies that there exist $|T| - 1$ linearly independent solutions of $\Sigma_{j \in T - \{1\}} a_j x_j \leqslant a_0 - a_1$ satisfying $\Sigma_{j \in T - \{1\}} \alpha_j x_j = \alpha_0 - \alpha_1$.

Example 7.3. $4x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leqslant 4$.

By Theorem 7.1 all faces can be obtained from the faces of

$$2x_2 + 2x_3 + x_4 + x_5 \leqslant 4 .$$

This inequality has two strong covers (234) and (235), and hence by Theorem 5.1 its integer hull is

$$x_2 + x_3 + x_4 \leqslant 2, \qquad x_2 + x_3 + x_5 \leqslant 2 ,$$

$$x_i \leqslant 1, \quad i = 2, 3, 4, 5; \quad x_i \geqslant 0 \quad \text{for all } i.$$

Now applying Theorem 7.1 we obtain the integer hull

$$
\begin{aligned}
2x_1 + x_2 + x_3 + x_4 \quad\quad &\leqslant 2,\\
2x_1 + x_2 + x_3 \quad\quad + x_5 &\leqslant 2,\\
x_1 + x_2 \quad\quad\quad\quad &\leqslant 1,\\
x_1 \quad + x_3 \quad\quad\quad &\leqslant 1,\\
x_1 \quad\quad + x_4 \quad\quad &\leqslant 1,\\
x_1 \quad\quad\quad\quad + x_5 &\leqslant 1;\\
x_i \geqslant 0, \qquad i = 1, 2, \ldots, 5 &.
\end{aligned}
$$

Hence the inequality is in fact of type (b).

8. Conclusions

The effort in this paper to characterize the integer hull of 0–1 solutions to a linear inequality is evidently motivated by the work of Edmonds on the matching problem, and matroid problems more generally [6, 7, 8] and also recent work on the vertex packing and covering problems [13, 14, 16]. The extension of such results to other, and more general independence systems is of considerable interest.

Methods to obtain other faces, particularly for inequalities of type (d) are evidently needed. For self-dual inequalities [3, 18] any minimal (extremal) representation is known to be a face, but otherwise little is known. It does seem likely that problems with a small number of strong covers are easier to solve, but from [3] it appears that in general the number grows exponentially with n.

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