Mathematical Programming 15 (1978) 268–277.
North-Holland Publishing Company

LIFTING THE FACETS OF ZERO-ONE POLYTOPES

Eitan ZEMEL

Northwestern University, Evanston, IL, U.S.A.

Received 28 February 1975
Revised manuscript received 12 May 1978

We discuss a procedure to obtain facets and valid inequalities for the convex hull of the set of solutions to a general zero-one programming problem. Basically, facets and valid inequalities defined on lower dimensional subpolytopes are lifted into the space of the original problem. The procedure generalizes the previously known techniques for lifting facets in two respects. First, the general zero-one programming problem is considered rather than various special cases. Second, the procedure is exhaustive in the sense that it accounts for all the facets (valid inequalities) which are liftings of a given lower dimensional facet (valid inequality).

Key words: Integer Programming, Zero–One Programming, Facets, Valid Inequalities, Lifting.

1. Introduction

We discuss a procedure to generate facets and valid inequalities for the convex hull of the set of solutions of a general 0–1 integer program, by lifting facets and valid inequalities of lower dimensional polytopes. To be more specific, let G_I be the solution set of any 0–1 program, i.e., G_I is an arbitrary subset of $\{0, 1\}^N$, where $N = \{1, \dots, n\}$ is the index set for the problem variables. Let also $G = \operatorname{conv}(G_I)$, where for any set H we denote by $\operatorname{conv}(H)$ the convex hull of H. For every subset N' of $N, \emptyset \subseteq N' \subseteq N$, define:

$$G_I(N') = \{x \in G_I \mid x_i = 0, i \in N\backslash N'\},$$
$$G(N') = \operatorname{conv}(G_I(N')).$$

We will alternately treat $G_I(N')$ as a subset of $\{0, 1\}^N$ and of $\{0, 1\}^{N'}$.

Suppose we have a facet (valid inequality) of $G(N')$:

$$\sum_{j\in N'} b_j x_j \le b_0 \tag{1}$$

and we are interested in obtaining facets (valid inequalities) of G of the form:

$$\sum_{j\in N'} b_j x_j + \sum_{j\in N\backslash N'} b_j x_j \le b_0. \tag{2}$$

We call the inequality (2) a *lifting* of (1).

The existence of facets which are liftings of lower dimensional facets was observed by Pollatschek [12] with regard to independence systems. Padberg [9, 10, 11] proposed a procedure for calculating the lifting coefficients. His results were first proved for the Set Packing Polytope and then extended to any 0–1 linear program whose coefficient matrix is nonnegative. The same procedure, for different special cases, is also treated in Nemhauser and Trotter [8], Trotter [14], Balas [2], Hammer, Johnson and Peled [7] and Wolsey [15]. In addition, Hammer, Johnson and Peled [7] treat the lifting of valid inequalities. A generalization of the procedure by Wolsey [16] is valid for any linear integer program (not necessarily 0–1), and allows for changes in the right-hand side of (1) too.

All the above mentioned authors use the common technique of introducting the new variables (i.e., those of $N \backslash N'$) one at a time. It is known that the facets (valid inequalities) thus obtained depend on the sequence in which these variables are introduced. Generally, even if all the possible sequences are followed, we do not exhaust the family of facets (valid inequalities) of the form (2).

The purpose of this paper is to suggest a generalized procedure capable of an exhaustive characterization of the facets (valid inequalities) which are liftings of (1), i.e., are of the form (2). Rather than introducing the variable of $N \backslash N'$ sequentially the generalized procedure is a "parallel" one, introducing sets of variables simultaneously. In the context of 0–1 programming it subsumes the above mentioned results and is able to generate additional facets that are unobtainable by sequential lifting. The procedure, which we denote here by *simultaneous* lifting, uses polarity and in that sense it is related to [1], [3], [6].

The added flexibility of the simultaneous lifting procedure is not free of a computational toll. If one is to get *all* the facets which can be lifted from a given lower dimensional facet (1), the number of integer programs one has to solve can be prohibitively large. Even so, each of these integer programs involves the variables of N' only, and they all have the same structure. The procedure becomes more effective, however, when it is applied to problems whose special structure helps to cut down the number of integer programs one has to solve, or simplifies their solution. Indeed, the results of this paper, coupled with those of [2] and some direct observations, yield some very interesting (and computationally appealing) properties of the facets of the knapsack problems. This is the subject of another paper [5] and is not discussed here.

In the sequel we present the simultaneous lifting procedure for valid inequalities and facets (Theorems 1 and 2) and then consider its relation to the sequential procedure (Corollaries 1, 2, 3). Finally we consider two examples.

2. The simultaneous lifting procedure

Let us consider a subset $N' \subseteq N$ which is arbitrary but fixed.
For a subset $M \subseteq N \backslash N'$ let:

$$G_I(N', M) = \{x \in G_I \mid x_j = 1, j \in M, x_j = 0, j \in N \backslash (M \cup N')\},$$

i.e., $G_I(N', M)$ is the (possibly empty) set of vertices of G_I for which the components indexed by M are set to 1 and those indexed by $N \backslash (N' \cup M)$ are set to 0. The components indexed by N' are left free. Let us also denote by $\mathcal{M}$ the family of those subsets of $N \backslash N'$, M, for which $G_I(N', M)$ is not empty, i.e.,

$$\mathcal{M} = \{M \subseteq N \backslash N' \mid G_I(N', M) \neq \emptyset\}.$$

The members of $\mathcal{M}$ induce a partition of the vertices of G_I such that every vertex of G_I belongs to exactly one subset $G_I(N', M)$.

With every set $M \subseteq N \backslash N'$, we associate an integer program IP_M, whose variables are those of N' and whose feasible set is $G_I(N', M)$:

$$(\mathrm{IP}_M) \qquad Z'_M = \max \sum_{j \in N'} b_j x_j, \qquad \text{s.t. } x \in G_I(N', M)$$

with the convention: $Z'_M = -\infty$ if $G_I(N', M) = \emptyset$ (i.e., if $M \notin \mathcal{M}$). Let

$$\pi'_M = b_0 - Z'_M$$

and let PL be the polyhedral set:

$$\mathrm{PL} = \left\{ b \in R^{N \backslash N'} \;\middle|\; \sum_{j \in M} b_j \leq \pi'_M, \text{ for every } M \in \mathcal{M} \right\}.$$

Theorem 1. *Let $0 \subset N' \subset N$, and let* (1) *be a valid inequality for $G(N')$. Then* (2) *is a valid inequality for G iff $b \in \mathrm{PL}$.*

Proof. Suppose $b \in \mathrm{PL}$. Let $x \in G_I$. Then $x \in G_I(N', M)$ for a unique $M \in \mathcal{M}$. We then have

$$\begin{aligned} \sum_{j \in N} b_j x_j &= \sum_{j \in N'} b_j x_j + \sum_{j \in N \backslash N'} b_j x_j \\ &= \sum_{j \in N'} b_j x_j + \sum_{j \in M} b_j \leq Z'_M + \sum_{j \in M} b_j \\ &= b_0 - \pi'_M + \sum_{j \in M} b_j \leq b_0 \end{aligned}$$

where the last inequality follows our assumption that $b \in \mathrm{PL}$, i.e., that $\sum_{j \in M} b_j \leq \pi'_M$. Since x is arbitrary we get that (2) is valid for G_I and hence for G.

Conversely, suppose $b \notin \mathrm{PL}$. Then, for some $M \in \mathcal{M}$ we have that

$$\sum_{j \in M} b_j > \pi'_M..$$

Let $x \in G_I(N', M)$ be any point for which the maximum of IP_M is attained, i.e., for which

$$\sum_{j \in N'} b_j x_j = Z'_M.$$

Then

$$\sum_{j\in N} b_j x_j = \sum_{j\in N'} b_j x_j + \sum_{j\in N\setminus N'} b_j x_j = Z'_M + \sum_{j\in M} b_j > Z'_M + \pi'_M = b_0.$$

Hence x violates (2).

We now give a full characterization of lifted facets:

Theorem 2. *Let $\emptyset \subset N' \subset N$ be such that $G(N')$ is full ($|N'|$) dimensional and let* (1) *be a facet of $G(N')$. Denote the linear dimension of G by k, $|N'| \le k \le N$. Then* (2) *is a* (k*-dimensional*) *facet of G iff*:

(a) $b \in \mathrm{PL}$,

(b) *there exist $k - |N'|$ linearly independent subsets of $\mathcal{M}$, M_i, $i = 1, \dots, k - |N'|$, such that $\sum_{j\in M_i} b_j = \pi'_{M_i}$.*

Proof. By Theorem 1 we have that (a) is a necessary and sufficient condition for (2) to be valid on G. Suppose, in addition to (a), that (b) holds too. To show that (2) is a facet of G we have to present k affinely independent vertices of G_I for which (2) holds with equality. Let A be the ($|N'| \times |N'|$) matrix whose rows are the incidence vectors of the $|N'|$ affinely independent vertices of $G_I(N')$ for which (1) holds with equality. Let M be the 0–1 matrix whose rows are the incidence vectors of the $k - |N'|$ linearly independent sets M_i assumed in (b). For every set M_i, $i = 1, \dots, k - |N'|$ let x^i be any vertex of $G_I(N', M_i)$ which attains the maximum of IP_{M_i}, i.e., for which

$$\sum_{j\in N'} b_j x_j^i = Z'_{M_i}.$$

Let w^i be the vertex obtained from x^i by considering the components of N' only and let W be the matrix whose rows are the incidence vectors of w^i, $i = 1, \dots, k - |N'|$. Consider the matrix

$$Y = \begin{array}{cc} N' & N\setminus N' \\ \hline \multicolumn{1}{|c|}{A} & \multicolumn{1}{c|}{0} \\ \hline \multicolumn{1}{|c|}{W} & \multicolumn{1}{c|}{M} \\ \hline \end{array}$$

Let y^i, $i = 1, \dots, k$ be the vertices of G_I which correspond to the rows of Y. By our construction y^i, $i = 1, \dots, k$, satisfies (2) with equality. Also since the rows of A are affinely, and the rows of M are linearly, independent we get that the rows of Y are affinely independent.

Conversely, let (2) be a (k-dimensional) facet of G. Let us rearrange the indices of N so that those of N' appear first. Let Y be the matrix whose rows are the incidence vectors of all the vertices of G_I for which (2) holds with equality. In particular, since (1) is a facet of $G(N')$, Y must contain a set of $|N'|$ affinely independent vertices of $G_I(N')$ for which (1) (and hence (2)) hold with equality. Let us assume that those vertices are the first $|N'|$ rows and let their incidence matrix be

$$\begin{array}{cc} N' & N\backslash N' \\ \end{array}$$
$$\begin{array}{|c|c|} \hline A & 0 \\ \hline \end{array}$$

Then Y decomposes:

$$\begin{array}{ccc} & N' & N\backslash N' \\ & & \end{array}$$
$$Y = \begin{array}{|c|c|} \hline A & 0 \\ \hline W & M \\ \hline \end{array}$$

Let us first see that M contains $k-|N'|$ linearly independent rows:

Case (a). $b_0 \neq 0$. In this case the row degree of Y is k and the row degree of A is $|N'|$. Hence M must contain $k-|N'|$ linearly independent rows.

Case (b). $b_0 = 0$. Let $j \in N'$ such that $b_j \neq 0$. By complementing x_j (i.e., replacing x_j by $\bar{x}_j = 1 - x_j$) the homogeneous facet (2) is transformed into a non-homogeneous facet in the new variables $(x_i,\ i \neq j,\ \bar{x}_j)$ and the arguments of case *a* follow.

We now show that the sets M_i, $i = 1, \ldots, k-|N'|$ defined by the rows of M satisfy

$$\sum_{j \in M_i} b_j = \pi'_{M_i}.$$

We already have by Theorem 1 that

$$\sum_{j \in M_i} b_j \leq \pi'_{M_i}.$$

Suppose that for some $i \in 1, \ldots, k-|N'|$ we have

$$\sum_{j \in M_i} b_j < \pi'_{M_i}.$$

Let y^i be the vertex of G_I whose incidence vector is the row of Y which corresponds to M_i

$$b_0 = \sum_{j \in N} b_j y^i_j = \sum_{j \in N'} b_j y^i_j + \sum_{j \in N \backslash N'} b_j y^i_j$$

$$< \sum_{j \in N'} b_j y^i_j + \pi'_{M_i} \leq Z'_{M_i} + \pi'_{M_i} = b_0$$

a contradiction.

Remark. (1) Theorem 2 is stated in its simplest form. Closely related variants of this theorem can be stated for different special cases which do not satisfy the requirements of Theorem 2. In particular, one may allow the lower dimensional inequality (1) to be a face rather than a facet of $G(N')$. Alternatively, (1) can be a facet (face) of $G(N')$ which may be less than full ($|N'|$) dimensional. In both cases the lifted inequality (2) is a face of G, and a lower bound on the number of affinely independent vertices of G_I which satisfy (2) with equality can be trivially obtained. Finally, the set N' may be empty, and one may lift the "empty inequality"

$$\sum_{j\in\emptyset} b_j x_j \leq b_0 \quad (\text{for } b_0 > 0)$$

by using a slightly modified version of Theorem 2.

(2) Following [3] we observe the following geometric interpretation of conditions (a) and (b). Let L denote the Lineality space of PL ([10]) and let $L^\perp$ be its orthogonal complement. Then (a) and (b) imply that the projection of b into $L^\perp$ is an extreme point of $\mathrm{PL} \cap L^\perp$. If G is full ($|N|$) dimensional, then $L^\perp = R^{N\setminus N'}$ and we get that b is an extreme point of PL.

3. The sequential lifting procedure

The sequential lifting procedure can be now obtained as a simple consequence of Theorems (1) and (2). To see this let $N = N' \cup \{i\}$. We observe that

$$G_I(N', \{i\}) = \{x \in G_I \mid x_i = 1\}$$

and $\mathrm{IP}_{\{i\}}$ reduces to IP_i:

$$(\mathrm{IP}_i) \qquad \begin{aligned} & Z'_i = \max \sum_{j\in N'} b_j x_j, \\ & \text{s.t. } x \in G_I, \quad x_i = 1 \\ & (Z'_i = -\infty \text{ if } \{x \in G_I \mid x_i = 1\} = \emptyset) \end{aligned}$$

We then observe (proofs are omitted):

Corollary 1. *Let $N = N' \cup \{i\}$ and let* (1) *be valid in $G(N')$. Then*

$$\sum_{j\in N'} b_j x_j + b_i x_i \leq b_0 \tag{3}$$

is valid for G iff $b_i \leq \pi'_i$.

Corollary 2. *Let $N = N' \cup \{i\}$, and such that $G(N')$ is full dimensional. Let* (1) *be a facet of $G(N')$. Then* (3) *is a facet of G iff*

$$b_i = \begin{cases} \pi'_i & \text{if } \pi'_i < \infty, \\ \text{arbitrary} & \text{if } \pi'_i = \infty. \end{cases}$$

In the second case, i.e., when $\pi'_i = \infty$, G is less than full dimensional.

Consider now the case where $|N| - |N'| > 1$. One can lift a facet of $G(N')$ by sequentially introducing the variables of $N \backslash N'$ and using Corollary 2. Note that at each iteration the set N' is enlarged to include the variables whose coefficients were already calculated. (In the simultaneous procedure of Theorem 2 the set N' is the same for all the Problems IP_{M}.) The sequential procedure may fail to yield a facet of G if at some iteration we get $\pi'_i = \infty$, since this event indicates a less than full dimensional subpolytope. In this case Corollary 2 can fail to produce facets of G even if G itself is full dimensional and even if it does have facets which are liftings of (1). This situation is demonstrated in Example 2.

Finally we observe that if $|N - N'| \leq 2$ the only lifted facets of a given lower dimensional facet are sequentially lifted facets. This fact was stated in [7], for monotone polytopes and for nonhomogeneous facets. However the statement is valid in general:

Corollary 3. *Let $N' \subseteq N$ and such that $|N| - |N'| \leq 2$ and $G(N')$ is full dimensional, and let* (1) *be a facet of $G(N')$. Let G itself be full dimensional and let* (2) *be a facet of G. Then* (2) *can be obtained from* (1) *by sequential lifting.*

Proof. If $|N| - |N'| = 1$, then Corollary 3 reduces to Corollary 2. Otherwise let $N \backslash N'$ be $\{i, k\}$. By Theorem 2 we must have 2 linearly independent subsets of $\{i, k\}$ for which $\sum_{j \in M} b_j = \pi'_M$. By their linear independence one of those sets must be a singleton, let us say $M = \{i\}$. We then have

$$b_i = \pi'_i$$

We therefore have, by Corollary 2, that (3) is a facet of $G(N' \cup \{i\})$. Applying Corollary 2 again we obtain the facet 2.

4. Examples

Example 1. Let

$$G_I = \{x \in \mathbf{R}^8 \mid 7x_1 + 6x_2 + 4x_3 + 4x_4 + 4x_5 + 4x_6 + 3x_7 + 2x_8 \leq 13,$$
$$x_j = 0 \text{ or } 1,, \ j = 1, \dots, 8\}.$$

Note that

$$x_3 + x_4 + x_5 + x_6 \leq 3 \tag{1'}$$

is a facet of $G(N')$ for $N' = \{3, 4, 5, 6\}$. By choosing different sequences for introducing the variables of $N \backslash N'$ we obtain by sequential lifting (the sequence for each facet is indicated)

$$2x_1+x_2+x_3+x_4+x_5+x_6+x_7 \leq 3 \quad (1,2,7,8), \tag{2'a}$$

$$x_1+2x_2+x_3+x_4+x_5+x_6 \leq 3 \quad (2,1,7,8), \tag{2'b}$$

$$x_1+x_2+x_3+x_4+x_5+x_6 + x_8 \leq 3 \quad (8,7,2,1). \tag{2'c}$$

The other sequences of $\{1,2,7,8\}$ do not yield new facets. 2'a, b, c are thus all the facets of G which can be obtained from (1') by sequential lifting. For the simultaneous procedure we note that

$$\mathcal{M}=\Big\{M \mid \subseteq \{1,2,7,8\}, \sum_{j\in M} a_i \leq a_0\Big\}$$

i.e., $\mathcal{M}$ with the corresponding values of π'_M are given by

$$\mathcal{M}=\{\{1\},\{2\},\{7\},\{8\},\{1,2\},\{1,7\},\{1,8\},\{2,7\}\{2,8\},\{7,8\}\{1,7,8\},\{2,7,8\}\}.$$

$$\pi'_M=\{2,\quad 2,\quad 1,\quad 1,\quad 3,\quad 3,\quad 2,\quad 2,\quad 2,\quad 1,\quad 3,\quad 3\}.$$

Thus, the inequalities which define PL are:

$$\begin{array}{llllll}
b_1 & & & & \leq 2, \\
 & b_2 & & & \leq 2, \\
 & & b_7 & & \leq 1, \\
 & & & b_8 & \leq 1, \\
b_1 + & b_2 & & & \leq 3, \\
b_1 + & & b_7 & & \leq 3^*, \\
b_1 + & & & b_8 & \leq 2, \\
 & b_2 + & b_7 & & \leq 2, \\
 & b_2 + & & b_8 & \leq 2, \\
 & & b_7 + & b_8 & \leq 1, \\
b_1 + & & b_7 + & b_8 & \leq 3^*, \\
 & b_2 + & b_7 + & b_8 & \leq 3^*,
\end{array}$$

where the inequalities marked by a star are obviously redundant. The extreme points of PL are $(2,1,1,0)$, $(1,2,0,0)$, $(1,1,0,1)$ and $(1.5,1.5,0.5,0.5)$. The first three extreme points correspond, of course, to the three sequential facets we have already obtained. The fourth facet, namely

$$1.5x_1+1.5x_2+x_3+x_4+x_5+x_6+0.5x_7+0.5x_8 \leq 3 \tag{2'd}$$

can be obtained by the simultaneous lifting procedure exclusively.

Example 2.

$$G_I = \left\{x \in \{0,1\}^5 \,\middle|\, \begin{array}{r} x_1+x_2+3x_3-2x_4-2x_5 \le 1 \\ x_1+x_2-2x_3+3x_4-2x_5 \le 1 \\ x_1+x_2-2x_3-2x_4+3x_5 \le 1 \\ x_3+\;\; x_4+\;\; x_5 \le 2 \end{array}\right\}$$

Note that

$$x_1+x_2 \le 1 \tag{1''}$$

is a facet of $G(N')$ for $N'=\{1,2\}$. Since $G(N',\{i\})=\emptyset$, $i=3$, 4 or 5, we have $\pi'_i=\infty$, $i=3,4,5$. Thus

$$x_1+x_2+b_1 \le 1, \quad i=3,4 \text{ or } 5$$

is a (less than full dimensional) facet of $G(N'\cup\{i\})$ for any choice of b_i. No sequence of calculation, however, yields a facet of G which is a sequential lifting of (1″).

For the simultaneous procedure we have

$$\mathcal{M}=\{\{3,4\},\{3,5\},\{4,5\}\}$$

with $\pi'_M=1$ for all $M\in\mathcal{M}$. Hence PL is defined by the following inequalities:

$$\begin{aligned} b_3+b_4 \qquad &\le 1, \\ b_3+\qquad b_5 &\le 1, \\ b_4+b_5 &\le 1 \end{aligned}$$

and its unique extreme point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. This extreme point corresponds to

$$x_1+x_2+\tfrac{1}{2}x_3+\tfrac{1}{2}x_4+\tfrac{1}{2}x_5 \le 1. \tag{2''}$$

which is the unique facet of G which is a lifting of 1″.

Acknowledgment

The ideas presented in this paper originated from my joint work with Professor Egon Balas who has introduced me to the subject. In addition, the useful comments of Professors V. Joseph Bowman and Robert G. Jeroslow as well as the remarks and suggestions of both the referees on an earlier version of this paper, are gratefully acknowledged.

References

[1] J.A. Araoz Durand, "Polyhedral neopolarities", Ph.D. Thesis, University of Waterloo, Faculty of Mathematics, Department of Applied Analysis and Computer Science (1974).

[2] E. Balas, "Facets of the knapsack polytope", *Mathematical Programming* 8 (1975) 146–164.

[3] E. Balas, "Disjunctive programming: properties of the convex hull of feasible points", MSRR No. 348 GSIA, Carnegie-Mellon University (July 1974).

[4] E. Balas and R.G. Jeroslow, "Canonical cuts on the unit hypercube", *SIAM Journal on Applied Mathematics* 23 (1972) 61–69.

[5] E. Balas and E. Zemel, "Facets of the knapsack polytope from minimal covers", *SIAM Journal on Applied Mathematics* 34 (1978) 119–148.

[6] D.R. Fulkerson, "Blocking and antiblocking pairs of polyhedra", *Mathematical Programming* 1 (1971) 168–194.

[7] P.L. Hammer, E.L. Johnson and U.M. Peled, "Facets of regular 0–1 polytopes", *Mathematical Programming* 8 (1975) 129–206.

[8] G.L. Nemhauser and L.E. Trotter, Jr., "Properties of vertex packing and independence systems polyhedra", *Mathematical Programming* 6 (1974) 48–61.

[9] M. Padberg, "'Simple' zero-one problems: set covering, matching and coverings in graphs", MSRR No. 235, Carnegie-Mellon University (January 1971).

[10] M. Padberg, "On the facial structure of set packing polyhedra", *Mathematical Programming* 5 (1973) 199–216.

[11] M. Padberg, "A note on 0–1 programming", *Operations Research* 23 (1975) 833–837.

[12] M.A. Pollatschek, "Algorithms on finite weighed graphs", Ph.D. Thesis, Technion-Israel Institute of Technology, Faculty of Industrial and Management Engineering (1970). [In Hebrew, with English Synopsis.]

[13] J. Stoer and C. Witzgall, *Convexity and optimization in finite dimension I* (springer, New York, 1970).

[14] L.E. Trotter, Jr., "A class of facet producing graphs for vertex packing polyhedra", *Discrete Mathematics*, to appear.

[15] L.E. Wolsey, "Faces of linear inequalities in 0–1 variables", *Mathematical Programming* 8 (1975) 165–178.

[16] L.E. Wolsey, "Facets and strong valid inequalities for integer programs", *Operations Research* 24 (1976) 367–372.