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ON THE FACIAL STRUCTURE OF SET PACKING POLYHEDRA

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In this paper we address ourselves to identifying facets of the set packing polyhedron, i.e., of the convex hull of integer solutions to the set covering problem with equality constraints and/or constraints of the form "$\leqslant$". This is done by using the equivalent node-packing problem derived from the intersection graph associated with the problem under consideration. First, we show that the cliques of the intersection graph provide a first set of facets for the polyhedron in question. Second, it is shown that the cycles without chords of odd length of the intersection graph give rise to a further set of facets. A rather strong geometric property of this set of facets is exhibited.

1. Introduction

Set covering and set packing problems have found considerable attention in recent literature, see e.g. [2] for numerous references. There appear to be two reasons for this renewed interest: (i) set covering and set packing problems have a large variety of practical applications; (ii) much stronger *structural* properties are present in these problems than in the general integer linear programming problem (see [2, 3, 23]). Both reasons suggest to compare the position of the set covering problem vis-à-vis the general integer linear programming problem to that of the transportation problem vis-à-vis the general linear programming problem. Different from the transportation problem, however, the set covering polyhedron has *non-integer* extreme points, so that it becomes necessary to find the hyperplanes which in addition to the constraints of the associated linear programming problem define the convex hull of integer solutions. In this paper we identify two sets of such hyperplanes, "facets", for the set packing problem, i.e., the integer linear program of the following form:

(P) $\quad \max\ cx,$

$$\text{subject to } Ax \leqslant e, \quad x_j = 0 \text{ or } 1 \quad \text{for all } j \in N = \{1, ..., n\}$$

where A is an $m \times n$ matrix of zeros and ones, c is an arbitrary n-vector and $e^{\mathrm{T}} = (1, ..., 1)$ is an m-vector. Obviously, if the convex hull of integer solutions to (P) is found, we have also a characterization of the convex hull of integer solutions to (P_0), the equality constrained set covering problem, i.e., the problem obtained from (P) by substituting $Ax = e$ for $Ax \leqslant e$. (It has been shown in [11, 19] that, by an appropriate modification of the objective function, problem (P_0) can always be brought into the form (P).) Also we consider here one type of the covering problem, where $Ax \geqslant e$ is substituted for $Ax \leqslant e$, namely the node-covering problem. In this instance, A has exactly two +1 entries in each row, so that using the substitution $x' = e - x$ we can transform this particular covering problem into a "node-packing" problem.

In fact, using the notion of the intersection graph, it is possible to show that every problem of the form (P) can be transformed into an equivalent node-packing problem (P_G). In Section 2 we briefly review the definition of the intersection graph and those properties relevant for the asserted equivalence. Next it is shown that the *cliques* of the intersection graph give rise to facets of the convex hull of integer solutions to the node-packing problem (P_G) (and hence to the original set covering problem (P) from which we have constructed (P_G)). Using a partition of the intersection graph into cliques, we obtain a further problem (P_K) which is equivalent to (P), but which has the advantage that all constraints constitute facets of the convex hull of integer solutions to (P), i.e., are as "tight" as possible.

In Section 3 we show that a second set of facets of the convex hull of integer solutions to (P) is obtained from *odd cycles without chords* of the intersection graph. Finally, we state some interesting properties of the facets for problem (P) obtained in this paper.

2. Cliques of the intersection graph and facets

With problem (P) we can associate the graph $G = (N, E)$, where (i) we define a node $j \in N$ for every column of A, and (ii) we define an edge $(i, j) \in E$ joining node i and node j if and only if $a^i a^j \geqslant 1$, i.e., column i and column j of A have at least one +1 entry in common. G is generally referred to as the intersection graph (see [14, p. 19; 7, p. 498]; we note here that Ore's intersection graph [18] denotes something entirely different). Despite the fact that the concept of the intersection graph has been known for quite some time, its use from an integer programming point of view appears to be new; see [11, 16, 20]. Denote by A_G the edge (rows) versus nodes (columns) incidence matrix of G and by (P_G) the integer programming problem which results from (P) by replacing A with A_G and e with e_G, where $e_G^{\mathrm{T}} = (1, ..., 1)$ is dimensioned compatibly with A_G.

Remark 2.1. Every feasible solution to (P_G) is a feasible solution to (P) and vice versa. In particular, every optimal solution to (P_G) is an optimal solution to (P) and vice versa.

Problem (P_G) is generally referred to as the problem of determining a maximum weighted independent node set in G (see [18]) and is an equivalent formulation of the problem of determining a minimum weighted node covering [18] (Proof: Use the substitution $x' = e - x$, where $e^{\mathrm{T}} = (1, ..., 1)$ is the n-vector having all n entries equal to $+1$.) Denoting by P_I (resp. P_I^G) the convex hull of solutions to problem (P) (resp. problem (P_G)), we have an implication for the facets of P_I and P_I^G, respectively (we note that $\dim P_I = n$): by a facet of P_I we denote an $(n-1)$-dimensional face of P_I. As customary in literature, we use the term facet synonymously for the *inequality* $\pi x \leqslant \pi_0$ which produces the facet. By definition of an $(n-1)$-dimensional face of P_I there exist n affinely independent vertices of P_I satisfying such inequality $\pi x \leqslant \pi_0$ with equality, while all $x \in P_I$ satisfy $\pi x \leqslant \pi_0$.

Remark 2.2. $P_I = P_I^G$, i.e., every facet of P_I is a facet of P_I^G and vice versa.

We can now proceed and identify a first set of facets for problem (P) by means of the cliques in G.

Definition 2.3. A *clique* in a graph G is a maximal complete subgraph of G.

Theorem 2.4. *An inequality*

$$\sum_{j \in K} x_j \leqslant 1\,, \tag{2.1}$$

where $K \subseteq N$, is a facet of P_I if and only if K is the node set of a clique in G.

Proof. Since K is the node set of a clique, there exists an edge (i, j) for all $i, j \in K$. Consequently, the inequaltiy (2.1) holds for all $x \in P_I$. On the other hand, there exist at least n linearly independent zero–one solutions to (P_G) satisfying (2.1) with equality: Construct $|K|$ solutions by setting $x_j = 1$ for exactly one $j \in K$ at a time, $x_j = 0$ otherwise; for every $k \in N \setminus K$ there must exist at least one $j \in K$ such that $(k, j) \notin E$ since K is the node set of a *maximal* complete subgraph. Hence $x_k = 1$, $x_j = 1$, $x_i = 0$ otherwise yields a feasible solution to (P_G) which satisfies (2.1) with equality. By doing so for every $k \in N \setminus K$, we obtain $|N \setminus K|$ feasible solutions to (P_G) satisfying (2.4) with equality. The matrix made up of the corresponding n solutions is triangular (modulo a permutation of its rows and columns) with determinant equal to ± 1.

To prove the only if part of Theorem 2.4, we observe that the subgraph with nodes in K and those edges of G which join the nodes in K define a complete subgraph in G. Suppose that this subgraph is not a clique in G, i.e., not maximal. Let $j \in N$ be any node in G such that the nodes in $K \cup \{j\}$ with the respective edges of G form a (larger) complete subgraph. Then

$$\sum_{k \in K \cup \{j\}} x_k \leqslant 1$$

is satisfied with equality by all feasible solutions to (P_G) satisfying (2.1) with equality and at least one solution more, namely $x_j = 1$, $x_i = 0$ otherwise. Consequently, (2.1) cannot be a facet of P_I.

Corollary 2.5. *Every facet $\pi x \leqslant \pi_0$ of P_I with integral coefficients and such that $\pi_0 = 1$ is of the form* (2.1).

Proof. We remark that the non-negativity of A implies that $\pi_j \geqslant 0$ for all $j \in N$, for every facet of P_I. But then $\pi_j = 0$ or 1 for all $j \in N$ since every vector of the form $x_j = 1$, $x_k = 0$ for all $k \in N \setminus \{j\}$, provides a feasible solution for (P) for every $j \in N$.

Remark 2.6. As G. Nemhauser has kindly pointed out to me, Theorem 2.4 can be obtained from [9, Theorem 8]. The approach taken here is, however, quite different from the methods employed by Fulkerson [9]. In particular, the constructive proof of Theorem 2.4 can be generalized (see Theorem 3.3, below).

Define A_K to be the incidence matrix of a set of cliques in G such that every edge of G is contained in at least one of the cliques considered versus the nodes of G (columns of A_K), i.e., $A_K = (a_{ij}^K)$, where

$$a_{ij}^K = \begin{cases} 1 \text{ if node } j \text{ is contained in clique } i, \\ 0 \text{ otherwise.} \end{cases}$$

Denote by (P_K) the integer programming problem which results from substituting A_K for A in problem (P) and e_K for e, where $e_K^{\mathrm{T}} = (1, ..., 1)$ is dimensioned compatibly with A_K.

Corollary 2.7. *Problem* (P_K) *is equivalent to problem* (P), *i.e., every (optimal) solution to* (P_K) *is an (optimal) solution to* (P) *and vice versa.*

Proof. Note that, by construction, every (optimal) solution to (P_K) is an (optimal) solution to (P_G). Consequently, by Remark 2.1, (P_K) and (P) are equivalent.

The obvious implication of Corollary 2.7 is that generally many (seemingly different) problems of the form (P) lead to the same problem (P_G), which, of course, is uniquely defined. Since the constraints of the form (2.1) correspond to facets of the underlying polytope P_I, the constraint set of the problem (P_K) is as "tight" as possible. No matter whether one uses an enumerative or an algebraic approach to solving (P), it appears advantageous to check first whether or not the rows of A represent cliques in the associated graph G. In an earlier version [20] of this paper we have outlined an algorithm which reduces any matrix A to an equivalent matrix A_K whose rows correspond to cliques of the associated intersection graph G.

Remark 2.8. If problem (P) was originally in equality form (P_0), obviously all zero–one solutions to (P_0) must satisfy the inequalities provided by the cliques of the associated intersection graph. Though these "valid" inequalities need not be facets of the convex hull of solutions to (P_0), they can generally be used in order to *fix* some of the variables of (P_0) at the value zero. More precisely, let $M_i = \{j \in N: a_{ij} = 1\}$ for $i \in M' \subseteq \{1, ..., m\}$ be such that there exists a clique in the associated intersection graph with node set K such that K properly contains M_i, $i \in M'$. Then we have that

$$x_j = 0 \qquad \text{for all } j \in K \setminus M^* ,$$

where $M^* = \bigcap_{i \in M'} M_i$ and the $|M'|$ constraints

$$\sum_{j \in M_i} x_j = 1 , \qquad i \in M'$$

can be replaced by a single one, namely

$$\sum_{j \in M^*} x_j = 1 \tag{2.2}$$

Remark 2.9. A further conclusion can be drawn from relation (2.2). For suppose problem (P_0), the equality constrained set covering problem, is such that M^* defined as in Remark 2.8 is the empty set. Then it follows from relation (2.2) that (P_0) cannot have a feasible solution. This is, however, a sufficient, but not a necessary condition for (P_0) to have no feasible solution. For let C be the edge (rows) versus nodes (columns) incidence matrix of a cycle having 5 nodes and no chords. Then $Cx = e$, $x_j = 0$ or 1, $j = 1, ..., 5$, is unfeasible, though there does not exist a clique K in the associated intersection graph and sets M_i satisfying $M^* = \emptyset$. More systematic use of the "reduction" possibilities of the properties of the intersection graph as applied to the problem (P_0) will be made in a forthcoming paper. We leave it to the reader to verify that various "reduction rules" which have been developed in [1, 10, 13, 21, 22] for the equality constrained set covering problem (P_0) have a natural interpretation as operations on the intersection graph associated with the problem in question.

3. Odd cycles and facets

In this section we show that odd cycles of the intersection graph associated with (P) give rise to a further class of facets of P_I.

Definition 3.1. A *cycle without chords* in a graph G is a cycle in G satisfying the additional requirement that there do not exist edges of G

connecting two non-consecutive nodes of the cycle in question.

We shall call the incidence matrix C of a (subgraph consisting of a) cycle without chords *cyclic*. By means of a suitable permutation of its rows and columns, any cyclic matrix can be brought to the following form:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \end{bmatrix}$$

That is, C has (up to permutation) +1 entries only in the main diagonal and in the first subdiagonal and a single +1 entry in the upper right corner.

Remark 3.2. A cyclic matrix C is nonsingular if and only if it is of odd size.

Denote by (LP_G) the linear programming problem obtained from (P_G) by dropping the integrality requirement. Any *cycle* (with or without chords) in the associated graph G gives rise to a cyclic submatrix of A_G and vice versa. Let S' be the node set of a cycle in G. Obviously, the vector $\bar{x}$ given by $\bar{x}_j = \frac{1}{2}$ for all $j \in S'$, $\bar{x}_j = 0$ for $j \in N \setminus S'$, is a feasible solution to (LP_G). Furthermore, if the cycle is of *odd* length, we then have by Remark 3.2 that $\bar{x}$ is a *basic* feasible solution to (LP_G), i.e., $\bar{x}$ is a vertex of the polytope P_C defined by the constraint set of (LP_G). Clearly, all feasible solutions to (P_G) must satisfy the following inequality:

$$\sum_{j \in S'} x_j \leq \tfrac{1}{2}(|S'| - 1) . \tag{3.1}$$

Though (3.1) is a *valid* inequality for P_I, it generally does not provide a facet for P_I. In order to construct facets for P_I, we proceed as follows: Let S be the index set of any set of nodes of G which define a *cycle without chords* in the graph G. Let A_G be defined as in Section 2 and denote by a^j column j of A_G. Define T to be the following index set:

$$T = \{j \in N \setminus S \colon a^j a^k > 0 \text{ for some } k \in S\}, \tag{3.2}$$

i.e., T is the set of all nodes different from those in S which are linked to some (or more) node(s) of S by an edge of G. Define T^q to be

$$T^q = T^{q-1} \cup \{j_q\} \quad \text{for } j_q \in T \setminus T^{q-1} ,$$

and for $q = 1, ..., Q = |T|$, with the convention that $T^0 = \emptyset$. Note that we do not require any specific order for the elements of T. Then let (H_q) be the following problem:

$$(\mathrm{H}_q) \qquad \max z_q = \sum_{k \in S} x_k + \sum_{k \in T^{q-1}} \beta_k x_k ,$$

$$\text{subject to} \quad \sum_{k \in S+T^{q-1}} a^k x_k \leqslant e_G - a^{j_q} ,$$

$$x_k = 0 \text{ or } 1, \qquad k \in S \cup T^{q-1} ,$$

where the β_k are defined recursively by

$$\beta_{j_q} = s - \bar{z}_q , \tag{3.3}$$

$s = \frac{1}{2}(|S| - 1)$, and $\bar{z}_q$ is the optimal value of the objective function of problem (H_q). The problems (H_q) are generally easy to solve; see Example 3.4 for a numerical example.

Theorem 3.3. *Given a subset $S \subseteq N$ such that the nodes contained in S define an odd cycle without chords in the graph G with the edge–node incidence matrix A_G. Then the following inequality provides a facet of P_I:*

$$\sum_{j \in S} x_j + \sum_{j \in T} \beta_j x_j \leqslant s , \tag{3.4}$$

where T is defined by (3.2), $s = \frac{1}{2}(|S| - 1)$ *and the β_j are obtained by solving the problems* (H_q), $q = 1, ..., Q$.

Proof. We have to show first that every solution to (P_G) satisfies (3.4). It is clear that we can restrict ourselves to considering the submatrix A' of A_G which is made up of the columns with associated indices in $S \cup T$, i.e., to the system of inequalities

$$A' x' \leqslant e' , \tag{3.5}$$

where x' and e' have been truncated accordingly. Let $\bar{x}'$ be any zero–one solution to (3.5) which does not satisfy (3.4) and suppose that the β_j have been computed in the order $\{j_1, ..., j_Q\}$. Let $q < Q$ be the largest index q such that

$$\sum_{k \in S} x_k + \sum_{k \in T^q} \beta_k x_k \leqslant s \tag{3.6}$$

for the particular solution $\bar{x}'$. But then $\beta_{j_{q+1}} = s - \bar{z}_{q+1} \geqslant 0$ is obtained from solving (H_{q+1}) and consequently (3.6) must be fulfilled with $q + 1$ substituted for q. Hence a largest index q strictly less than Q with the above property cannot exist. It remains to be shown that there exists at least n linearly independent zero–one solutions satisfying (3.4) with equality. Since S is the node set of an odd cycle without chords, it is easy to show that there exist exactly $|S|$ linearly independent solutions to (P_G) which have $x_j = 0$ for all $j \in N \setminus S$ and satisfy (3.4) with equality. Define $N^1 = N \setminus (S \cup T)$ and $n^1 = |N^1|$. Take any particular solution which satisfies (3.4) with equality and has $x_j = 0$ for all $j \in N \setminus S$. From this particular solution we can construct n^1 linearly independent solutions to (P_G) which satisfy (3.4) with equality by, e.g., setting $x_j = 1$ for exactly one $j \in N^1$ at a time. Finally, by construction of the coefficients β_j in (3.4), we obtain $|T|$ solutions to (P_G) which have $x_{j_q} = 1$ and $x_j = 0$ for all $j \in N^1 \cup (T \setminus T^{q+1})$, where q successively assumes the values $1, ..., Q$ and satisfy (3.4) with equality. Clearly, these solutions must be linearly independent among themselves and from the ones constructed above.

Remark 3.3. Using the construction of Theorem 3.3, the facets of Section 2 with $|K| \geqslant 3$ can be generated by any one of the 3-cycles (without chords) contained in such cliques of G, i.e., if $|S| = 3$, (3.4) yields as facet a clique containing S. This means, of course, that in this case the solution of (H_q) is reduced to finding any clique of G that contains S. Also, different orderings of T are seen to be responsible for different facets, generally, since a given triangle (3-cycle) may well belong to two (or more) different cliques of G.

Example 3.4. To give an example for the determination of facets of P_I, consider the graph G with $n = 7$ nodes as shown in Fig. 1. G con-

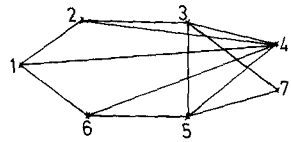

Fig. 1.

tains exactly seven odd cycles without chords with respective index sets $S_1 = \{1, 2, 4\}$, $S_2 = \{1, 4, 6\}$, $S_3 = \{2, 3, 4\}$, $S_4 = \{3, 4, 5\}$, $S_5 = \{4, 5, 6\}$, $S_6 = \{3, 5, 7\}$, and $S_7 = \{1, 2, 3, 5, 6\}$. The node sets $S_1, ..., S_6$ define cliques in G, hence all inequalities

$$\sum_{j \in S_i} x_j \leqslant 1 , \qquad i = 1, ..., 6,$$

define facets of the associated P_I. In order to remove e.g. the fractional vertices caused by S_7, we have to solve (H_q), $q = 1, 2$, with $T_7 = \{4, 7\}$. We leave it to the reader to verify that the resulting inequality of the form (3.4) is given by

$$\sum_{j \in S_7} x_j + 2x_4 + 0x_7 \leqslant 2$$

In particular, a permutation of the elements in T_7 does not produce a different inequality. (This is, however, *not* true in general).

We shall show next that the facets of P_I given by Theorems 2.4 and 3.3 possess a remarkable geometric property. Some of the following results can be found in greater detail in [19].

Lemma 3.5. *Let D be a nonsingular matrix of zeros and ones such that every row of D has exactly two* +1 *entries. Then D contains at least one nonsingular cyclic submatrix.*

Proof. By a straightforward argument we have that D contains a nonsingular submatrix E, say, whose row and column sums equal two. Since both E and its transpose E^{T} are edge–node incidence matrices of a regular graph of degree 2, the assertion follows.

Denote by P_C the solution set of (LP_G). Let vert P_I be the set of integer solutions to (LP_G) and $\operatorname{vert}(P_C|x')$ be the set of vertices of P_C which are adjacent to $x' \in P_{C'}$, i.e., connected to x' by a 1-dimensional face ("edge") of P_C.

Lemma 3.6. *Given* $x^1 \in \operatorname{vert} P_I$ *and* $x^2 \in \operatorname{vert}(P_C|x^1)$, x^2 *fractional. Let* B_i, $i = 1, 2$, *be any two adjacent bases associated with* x^i, *i.e.*, B_1 *and* B_2 *differ in exactly one column. Then* B_2 *can be written as follows (after, possibly, permuting some rows and columns):*

$$B_2 = \begin{bmatrix} D & 0 & 0 \\ 0 & G & 0 \\ F_1 & F_2 & I \end{bmatrix}, \tag{3.7}$$

where D has exactly two +1 *entries in each row and contains exactly one cyclic submatrix of odd order.*

Proof. After reordering rows and columns we can write B_2 as

$$B_2 = \begin{bmatrix} E & 0 \\ F & I \end{bmatrix},$$

where the identity matrix is the one associated with the basic slack variables. Define I_2 to be the index set of all basic variables in the solution x^2. We then can partition I_2 as

$$I_{21} = \{i \in I_2 : 0 < x_i^2 < 1\},$$

$$I_{22} = \{i \in I_2 : x_i^2 = 0\}, \qquad I_{23} = \{i \in I_2 : x_i^2 = 1\}.$$

Since x^2 is fractional, we have that $I_{21} \neq \emptyset$. The partition of I_2 gives rise to a similar partition of $E = (E_1\ E_2\ E_3)$, again after, possibly, reordering the columns of E. It is now easy to show that $(E_1)^{\mathrm{T}} E_3 = 0$, which gives rise to the following vertical partition of E, after, possibly, rearranging the rows of E:

$$E = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & E_{23} \end{bmatrix},$$

where the zero matrices in the top row come from the fact that in each row of E_{11} we have exactly two +1 entries. Let $D = E_{11}$ and $G = (E_{22}\ E_{23})$, then we have the general form (3.7). All that remains to be shown is that D contains exactly one cyclic submatrix of odd order. By Lemma 3.5, we have that D contains at least one cyclic submatrix of odd order. To show that it contains exactly one such submatrix we proceed as follows: Denote by a^k that column of B_1 which is in B_1 but not in B_2. (Suppose that the row interchanges applied to B_2 have also been applied to a^k). Suppose now $a_{ki} = 0$ for $i = 1, \ldots, h$, where h is the size of D. Then we have that

$$D\zeta^1 = D\zeta^2 ,$$

where $\zeta^i_j = x^i_j$, for all $j \in I_{21}$, $i = 1, 2$; now ζ^1 has only integer components, whereas ζ^2 has only fractional components. Hence a contradiction to $|D| \neq 0$. We consequently have that $a_{ki} \neq 0$ for at least one $i \in \{1, \ldots, h\}$. Since both A_G and D have exactly two +1 entries in each row, this implies that the variable associated with a^k must necessarily be a slack variable. Furthermore, by the same argument, the +1 entry of a^k must occur in a row of D which is part of a nonsingular cyclic submatrix of D, and since a^k contains exactly one +1 entry, D can contain at most one such submatrix.

Whereas Lemma 3.5 characterizes *fractional* vertices of P_C that are *adjacent* to a given integral vertex of P, the following remark gives a complete characterization of all vertices of P_C (for proofs see [19]).

Remark 3.7. Let $\bar{x} \in \operatorname{vert} P_C$. Define x^* to be the vector of all strictly fractional components of $\bar{x}$, i.e., $x^*_j = \bar{x}_j$ for all $j \in \{1, \ldots, n\}$ such that $0 < \bar{x}_j < 1$, $x^*_j = 0$ otherwise. Then $x^* \in \operatorname{vert} P_C$, i.e., x^* is a vertex of P_C. (This is immediate from the assumption that $\bar{x} \in \operatorname{vert} P_C$ and the non-negativity of A_G). Suppose that $x^* \neq 0$ and let B^* be the submatrix of A_G made up of those columns a^j of A_G such that the associated component x^*_j of x^* is strictly positive. Then B^* can be decomposed as follows

$$B^* = \begin{bmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & B_p & 0 \\ E_1 & & \cdots & E_p & E_{p+1} \end{bmatrix}$$

after suitable row and column permutations, where the B_i, $i \in \{1, ..., p\}$, are nonsingular cyclic submatrices of A_G, $p \geqslant 1$, and the E_i, $i \in \{1, ..., p\}$, are such that every row of E_i contains at most one positive entry. Furthermore, E_{p+1} does not contain a nonsingular cyclic submatrix. Define S_i to be index set of the columns of B_i in the decomposition of B^*, $i \in \{1, ..., p\}$, and x^i to be given by $x^i_j = x^*_j$ for all $j \in S_i$, $x^i_j = 0$ otherwise. Then $x^i \in \operatorname{vert} P_C$. In [19], x^i has been called a "minimal cyclic component" of the fractional vertex $\bar{x}$ of P_C. (We note that the minimal cyclic components of a fractional vertex $\bar{x}$ of P_C are generally *not* unique, i.e., it is in general possible to decompose B^* in several different ways, all of which lead to the same *general form* of the decomposition stated above). In graphical terms, each index set S_i represents the node set of a *cycle without chords* in the graph G for $i \in \{1, ..., p\}$.

Define V to be the set of variables with associated columns in the nonsingular cyclic submatrix contained in D of Lemma 3.6. Let $\alpha_0 = \frac{1}{2}(|V| - 1)$, where $|V|$ denotes the number of elements in the set V. We then have

$$\sum_{j \in V} x^1_j = \alpha_0 , \qquad \sum_{j \in V} x^2_j = \alpha_0 + \tfrac{1}{2} > \alpha_0 . \tag{3.8}$$

Theorem 3.8. *Given $x^1 \in \operatorname{vert} P_I$ and $x^2 \in \operatorname{vert}(P_C|x^1)$, x^2 fractional. Then for at least one inequality $\pi x \leqslant \pi_0$ of the general form* (3.4) *such that $\pi x^2 > \pi_0$ we have that $\pi x^1 = \pi_0$.*

Proof. We have to show that there exists an odd cycle without chords of odd length with index set S in the graph G with edge–node incidence matrix A_G such that

$$\sum_{j \in S} x^1_j = \tfrac{1}{2}(|S| - 1), \qquad \sum_{j \in S} x^2_j > \tfrac{1}{2}(|S| - 1) . \tag{3.9}$$

By construction of the inequalities $\pi x \leqslant \pi_0$ of the general form (3.4) we then have the desired result. From Lemma 3.6 we know a cycle of odd length with index set V such that (3.8) holds. Suppose that there are at least two nodes in V linked by an edge which is not an edge of the cycle with index set V. This edge partitions the cycle with index set V into two cycles, one of odd length and one of even length. Denote by S_1 the index set of nodes which define the cycle of odd length in question. Since x^1 satisfies (3.8), we then have necessarily that

$$\sum_{j \in S_1} x_j^1 = \tfrac{1}{2}(|S_1| - 1) .$$

For suppose that

$$\sum_{j \in S_1} x_j^1 < \tfrac{1}{2}(|S_1| - 1) .$$

Then

$$\begin{aligned} \tfrac{1}{2}(|V| - 1) = \sum_{j \in V} x_j^1 &= \sum_{j \in S_1} x_j^1 + \sum_{j \in V \setminus S_1} x_j^1 \\ &< \tfrac{1}{2}(|S_1| - 1) + \sum_{j \in V \setminus S_1} x_j^1 \\ &\leqslant \tfrac{1}{2}(|S_1| - 1) + \tfrac{1}{2}|V \setminus S_1| , \end{aligned}$$

or equivalently,

$$|V| < |S_1| + |V \setminus S_1| ,$$

which is the desired contradiction. If the resulting cycle of odd length with index set S_1 has no chords, we are done, i.e., we have found an odd cycle without chords which satisfies (3.9) with $S = S_1$. The second condition is satisfied since $x_j^2 = \frac{1}{2}$ for all $j \in S_1 \subset V$. If the cycle with index set S_1 has at least one chords, we then can reapply the above argument until we have found an odd cycle without chords which satisfies (3.9).

Despite the strong property of the facets of P_I identified so far, not *all* facets of P_I have been identified. We demonstrate this statement by means of the following example.

Example 3.9. Consider the graph G with 7 nodes and 14 edges as shown in Fig. 2. The inequality $\Sigma_{j=1}^{7} x_j \leqslant 2$ provides a facet for the associated P_I which, however, is not generated by an odd cycle without chords since G does not contain a cycle without chords of length 5.

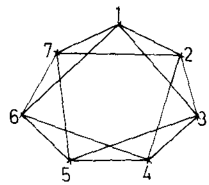

Fig. 2.

4. Conclusions

Despite the fact that we have not yet obtained a complete characterization of P_I, the form of the facets obtained thus far allow for some interesting conclusions. First of all we note that the coefficients of the facets for P_I are not all zero or one. This is a basic difference to the facets for the closely related edge-matching problem for which Edmonds [8] has derived the complete set of linear inequalities which characterize the associated convex hull of integer solutions. In fact, the β_j in (3.4) can assume all integral values between 0 and s. There is a second difference to the edge-matching case [8]. For due to different orderings of T in (3.4) different facets for P_I result: The same odd cycle in the graph G is responsible generally for several facets of P_I, whereas in case of the edge-matching problem every odd cycle produces one facet, no matter how many chords the cycle in question has. (In the edge-matching case, the chords of odd cycles play the role of the set T in (3.4)). This, of course, makes it unlikely that an equally elegant and efficient algorithm for the node-covering problem can be found as the one Edmonds has developed for the edge-matching problem and which uses *implicitly* the facets of the matching polyhedron. This observation coincides with the conclusions reached along different lines by Balinski [5] and more recently by Karp [16]. An algorithm not relying on cutting planes, but using the strong structural properties stated in [2, 23], is currently being developed [3].

A second observation pertains to the group-theoretic approach to solving integer programming problems; see [12]. Whereas this approach yields *all* facets for the edge-matching polytope [8], this is not the case for problem (P_G). For if Gomory's theory [12] is applied to determine facets of the convex hull of integer solutions of the problem (LP_G), one only obtains the first part of the inequality (3.4), i.e., $\Sigma_{j\in S} x_j \leqslant s$. This follows directly from the fact that the group associated with any fractional solution to (LP_G) is of order 2^k, k integer. In fact, if one writes down a basis associated with $\bar{x}$, where $\bar{x}_i = \frac{1}{2}$ for $i \in S$, all other x_j, $j \in N \setminus S$, being nonbasic, it is clear that the columns of A_G associated with any T as defined by (3.2) are mapped into the zero group element, thus producing zero coefficients in the desired inequality. Thus, if the group theoretic approach is used to determine facets for (P_G) and hence (P), one will generally obtain lower-dimensional faces of the convex hull of solutions to (P_G). (In fact, in the case of Example 3.9 the inequality obtained via group-theoretic arguments to cut off $x_1 = ... = x_7 = \frac{1}{2}$ is $\Sigma_{j=1}^{7} x_j \leqslant 3$ for which $P_I \cap \{x \in R^7 : \Sigma_{j=1}^{7} x_j = 3\} = \emptyset$, i.e., this inequality does not even provide a support of P_I.) The more or less obvious reason for this lies in the fact that the fractional vertex from which we generate the group problem and hence the Gomory cuts is generally degenerate in the sense that part of the tight constraints have zero basic slacks. These constraints are not used in the group-theoretic approach though the shape of the cone defined by the fractional vertex and the (feasible) edges emanating from it may depend critically (and, obviously, does so in the case of the node-covering problem) on this set of constraints. This indicates a possible direction in which to extend the group-theoretic approach. First steps in that direction – though not with the aim of identifying facets for the convex hull of integer solutions – have been undertaken by Wolsey [24].

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