Math. Program. 85: 15–33 (1999) / DOI 10.1007/s10107980006a

Hugues Marchand · Laurence A. Wolsey

The 0-1 Knapsack problem with a single continuous variable

Received May 22, 1997 / Revised version received December 22, 1997
Published online November 24, 1998

Abstract. Constraints arising in practice often contain many 0-1 variables and one or a small number of continuous variables. Existing knapsack separation routines cannot be used on such constraints. Here we study such constraint sets, and derive valid inequalities that can be used as cuts for such sets, as well for more general mixed 0-1 constraints.

Specifically we investigate the polyhedral structure of the knapsack problem with a single continuous variable, called the *mixed 0-1 knapsack* problem. First different classes of facet-defining inequalities are derived based on restriction and lifting. The order of lifting, particularly of the continuous variable, plays an important role. Secondly we show that the flow cover inequalities derived for the single node flow set, consisting of arc flows into and out of a single node with binary variable lower and upper bounds on each arc, can be obtained from valid inequalities for the mixed 0-1 knapsack problem. Thus the separation heuristic we derive for mixed knapsack sets can also be used to derive cuts for more general mixed 0-1 constraints. Initial computational results on a variety of problems are presented.

Key words. mixed 0-1 Knapsacks – valid inequalities – lifting – restriction

1. Introduction

Here we study the polyhedral structure of one of the simplest mixed 0-1 sets, a binary knapsack set with a single unbounded continuous variable (called the *mixed 0-1 knapsack* set):

$$Y = \left\{ (y, s) \in B^n \times R^1_+ : \sum_{j \in N} a_j y_j \leq b + s \right\}$$

where $a_j > 0$ for $j \in N = \{1, \dots, n\}$, and $b \geq 0$.

When $s = 0$, Y reduces to a binary knapsack set for which a large number of facet-defining inequalities have been derived [1] [11] [20] [21]. For such sets, the concepts of restriction, lifting and cover have played an important role. Using separation heuristics, the resulting inequalities have been successfully used as cutting planes in various branch-and-bound and branch-and-cut systems [6] for problems containing constraints involving only binary variables.

In mixed 0-1 programming, so-called single node flow sets, of the form

H. Marchand: CORE and FSA, Université Catholique de Louvain. Present address: LSE, Houghton St., London WC2A 2AE, England, e-mail: H.Marchand@lse.ac.uk
Part of this work was carried out during a one year stay at Columbia Business School, financed by a CIM doctoral fellowship.

L.A. Wolsey: CORE and FSA, Université Catholique de Louvain, Voie du Roman Pays 34, 1348 Louvain-la-Neuve, Belgium. e-mail: Wolsey@core.ucl.ac.be

$$Z = \left\{ (x, y) \in R_+^n \times B^n : \sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq b, l_j y_j \leq x_j \leq u_j y_j \ j \in N^+ \cup N^- \right\},$$

where $n = | N^+ \cup N^- |$, have been used to derive flow cover inequalities [15], [18] which have also been integrated in branch-and-bound [19] and branch-and-cut [16], [5] systems.

The set Y studied here is in many ways simpler than the mixed integer set Z. It also turns out to have a richer structure than the binary knapsack set, and to be a useful relaxation of both Z and mixed 0-1 sets $K = \{(x, y) \in R_+^p \times B^n : \sum_{=1}^p g_j x_j + \sum_{=1}^n a_j y_j \leq b, x_j \leq u_j \ j = 1, \ldots, p\}$. An outline of the paper now follows. In Section 2 we introduce an example used as motivation in the later sections, and make some basic observations about the set Y. In Section 3 we derive two families of facet-defining inequalities from the underlying knapsack problem and the complemented knapsack problem respectively. Here the continuous variable is introduced (lifted) last. In Section 4 we develop three more families of valid inequalities. We essentially fix all but one of the binary variables, leaving just one binary and one continuous variable. The resulting set has just one nontrivial facet which is then lifted by reintroducing the other variables. It turns out that if all variables fixed to one are lifted before all variables fixed to zero, or vice versa, the lifting coefficients, and hence the resulting inequalities, called *continuous cover* and *continuous reverse cover* inequalities, are unique. With other lifting sequences, the facets obtained are sequence dependent. For the small example, the five classes suffice to completely define the convex hull.

In Section 5 we show that the set Y can be viewed as a relaxation of the mixed integer set Z, and that certain continuous cover inequalities for Y produce flow cover inequalities for Z. By varying the relaxation and using lifting, new variants of the flow cover inequalities are obtained. In Section 6 we present computational results showing the behaviour of a separation heuristic for Y producing inequalities of the type derived in Sections 3 and 4. The instances tackled include mixed knapsack sets, single node flow sets and mixed 0-1 problems from the MIPLIB library [3]. The results are comparable with those obtained using flow-cover separation routines. Finally we terminate with a discussion of possible extensions.

2. The mixed 0-1 Knapsack problem

Consider the set:

$$Y = \left\{ (y, s) \in B^n \times R^+ : \sum_{j \in N} a_j y_j \leq b + s \right\}$$

where $a_j > 0$ for $j \in N = \{1, \ldots, n\}$, $b \geq 0$, and we assume throughout that $\sum_{j \in N} a_j > b$.

First we observe that conv(Y) is full-dimensional. The following characterisation of the trivial facets is straightforward.

Proposition 1.

i) The inequality $\sum_{j\in N} a_j y_j \le b+s$ *defines a facet of conv(Y) if* $a_j \le \sum_{i\in N} a_i - b$ *for all* $j \in N$.
ii) The inequality $s \ge 0$ *defines a facet of conv(Y) if* $a_j \le b$ *for all* $j \in N$.
iii) The inequality $y_j \ge 0$ *defines a facet of conv(Y).*
iv) The inequality $y_j \le 1$ *defines a facet of conv(Y).*

We now introduce an example to which we will refer in the following sections.

Example 1. Let

$$Y = \left\{(y,s) \in B^5 \times R^1_+ : 7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 \le 11 + s\right\}.$$

A complete description of the convex hull is given by the trivial facets $y_j \ge 0$ and $y_j \le 1$ for all $j = \{1 \dots 5\}$, $s \ge 0$, the defining inequality $7y_1+6y_2+5y_3+3y_4+2y_5 \le 11+s$, and the following system of inequalities:

$$\begin{array}{lr}
2y_1 + 2y_2 \le 2 + s & (1)\\
y_1 + y_3 \le 1 + s & (2)\\
y_1 + y_4 + y_5 \le 2 + s & (3)\\
3y_1 + 3y_2 + 3y_3 + 3y_4 \le 6 + s & (4)\\
2y_1 + 2y_2 + 2y_3 + 2y_5 \le 4 + s & (5)\\
3y_1 + 2y_2 + 2y_3 + y_4 + y_5 \le 4 + s & (6)\\
6y_1 + 5y_2 + 5y_3 + 3y_4 + 2y_5 \le 10 + s & (7)\\
5y_1 + 4y_2 + 3y_3 + y_4 + 2y_5 \le 7 + s & (8)\\
6y_1 + 5y_2 + 4y_3 + 3y_4 + y_5 \le 9 + s & (9)\\
4y_1 + 4y_2 + 2y_3 + 2y_5 \le 6 + s & (10)\\
5y_1 + 5y_2 + 3y_3 + 3y_4 \le 8 + s & (11)\\
3y_1 + 2y_2 + y_3 \le 3 + s & (12)\\
4y_1 + 3y_2 + 4y_3 + 3y_4 \le 7 + s & (13)\\
3y_1 + 2y_2 + 3y_3 + 2y_5 \le 5 + s & (14)\\
4y_1 + 3y_2 + 2y_3 + y_4 + y_5 \le 5 + s & (15)\\
5y_1 + 4y_2 + 4y_3 + 3y_4 + y_5 \le 8 + s & (16)\\
4y_1 + 3y_2 + 3y_3 + y_4 + 2y_5 \le 6 + s & (17)
\end{array}$$

This description was obtained using the code of [7].

□

3. Knapsack and complemented Knapsack facets

One way to generate facets of conv(Y) is to adapt facets of the underlying knapsack polytope, and reintroduce the continuous variable s by a lifting procedure. Let $\tilde{N} = \{j \in N : a_j \le b\}$ and $Y_0 = \{(y,s) \in Y : s = 0,\ y_j = 0,\ j \in N \setminus \tilde{N}\}$.

Proposition 2. *If* $\sum_{j\in\tilde{N}} \pi_j y_j \leq \pi_0$ *with* $\pi \geq 0$, $\sum_{j\in\tilde{N}} \pi_j > \pi_0 > 0$ *defines a facet of conv(Y_0), then*

$$\sum_{j\in N} \pi_j y_j \leq \pi_0 + \frac{s}{\beta} \tag{18}$$

defines a facet of conv(Y), where $\beta = \min_{s>0} \frac{s}{\eta(s)-\pi_0}$, $\eta(s) = max \left\{ \sum_{j\in\tilde{N}} \pi_j y_j : \sum_{j\in\tilde{N}} a_j y_j \leq b + s, y \in B^{|\tilde{N}|} \right\}$ *and* $\pi_j = \pi_0 + \frac{a_j - b}{\beta}$ *for* $j \in N \setminus \tilde{N}$.

Proof. By [4], we observe that $\sum_{j\in\tilde{N}} \pi_j y_j \leq \pi_0 + \frac{s}{\beta}$ is a facet of the convex hull of the set Y where $y_j = 0$ for $j \in N \setminus \tilde{N}$. Moreover $\beta\pi_0 \leq b$ and $\beta\pi_j \leq a_j$ for all $j \in \tilde{N}$.

Now suppose that there is a point $(y, s) \in Y$ with $y_k = 1$ for some $k \in N \setminus \tilde{N}$ and $\sum_{j\in N} \beta\pi_j y_j > \beta\pi_0 + s$. Note that we also have $\beta\pi_j = \beta\pi_0 + (a_j - b) \leq a_j$ for $j \in N \setminus \tilde{N}$ as $\beta\pi_0 \leq b$. Therefore $s < \sum_{j\in N} \beta\pi_j y_j - \beta\pi_0 \leq \sum_{j\in N\setminus\{k\}} a_j y_j + (\beta\pi_0 + (a_k - b))y_k - \beta\pi_0 = \sum_{j\in N} a_j y_j - b$ and then $(y, s) \notin Y$, a contradiction.

Hence (18) is a valid inequality for Y. Moreover as the points $y_k = 1, s = a_k - b$ for $k \in N \setminus \tilde{N}$ are tight, it defines a facet of $conv(Y)$.

□

Thus each nontrivial facet of the knapsack set gives rise to a distinct facet of conv(Y).

Example (continued). The underlying knapsack set is

$$Y_0 = \left\{ y \in B^5 : 7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 \leq 11 \right\}$$

A complete description of the convex hull is given by the trivial facets $y_j \geq 0$ for all $j = \{1 \ldots 5\}$, $y_4 \leq 1$, $y_5 \leq 1$ and the following inequalities.

$$\begin{aligned}
y_1 + y_2 &\leq 1 && (19)\\
y_1 + y_3 &\leq 1 && (20)\\
y_1 + y_4 + y_5 &\leq 2 && (21)\\
y_1 + y_2 + y_3 + y_4 &\leq 2 && (22)\\
y_1 + y_2 + y_3 + y_5 &\leq 2 && (23)\\
3y_1 + 2y_2 + 2y_3 + y_4 + y_5 &\leq 4 && (24)
\end{aligned}$$

Note that whereas conv(Y) has 17 nontrivial facets, conv(Y_0) has only 6.

For the knapsack facet (19), $\beta = 2$ and so by Proposition 2 the corresponding facet-defining inequality for the mixed knapsack set Y is $2y_1 + 2y_2 \leq 2 + s$. In this way the six inequalities (19) − (24) of the knapsack set Y_0 lead to the six facet-defining inequalities (1) − (6) for conv(Y).

□

To derive another class of facets, we introduce the slack variable $\bar{s} = b + s - \sum_{j\in N} a_j y_j$. Complementing the variables using $\bar{y}_j = 1 - y_j$ leads to the equation

$\sum_{j\in N} a_j\bar{y}_j + s = \left(\sum_{j\in N} a_j - b\right) + \bar{s}$, and after dropping the nonnegative variable s, we obtain the set

$$\bar{Y} = \left\{(\bar{y}, \bar{s}) \in B^n \times R^1_+ : \sum_{j\in N} a_j\bar{y}_j \le \left(\sum_{j\in N} a_j - b\right) + \bar{s}\right\}.$$

Now we can apply the same lifting approach to $\bar{Y}$. Again let $\tilde{N} = \{j \in N : a_j \le \sum_{k\in N} a_k - b\}$ and $\bar{Y}_0 = \{(\bar{y}, \bar{s}) \in \bar{Y} : \bar{s} = 0, \bar{y}_j = 0, j \in N \setminus \tilde{N}\}$. If $\sum_{j\in\tilde{N}} \mu_j\bar{y}_j \le \mu_O$ is a facet-defining inequality for $\mathrm{conv}(\bar{Y}_0)$, we obtain a facet-defining inequality $\sum_{j\in N} \mu_j\bar{y}_j \le \mu_0 + \bar{s}/\bar{\beta}$ for $\mathrm{conv}(\bar{Y})$. Recomplementing variables leads to a facet-defining inequality

$$\sum_{j\in N}(a_j - \bar{\beta}\mu_j)y_j \le b + \bar{\beta}\left(\mu_0 - \sum_{j\in N}\mu_j\right) + s$$

for $\mathrm{conv}(Y)$.

Example (continued). In the complemented space $(\overline{y}, \overline{s})$, the underlying mixed knapsack set is

$$\bar{Y} = \left\{(\overline{y}, \overline{s}) \in B^5 \times R^1_+ : 7\overline{y}_1 + 6\overline{y}_2 + 5\overline{y}_3 + 3\overline{y}_4 + 2\overline{y}_5 \le 12 + \overline{s}\right\}$$

A complete description of the convex hull of the associated knapsack set $\bar{Y}_0$ is given by the trivial facets $\overline{y}_j \ge 0$ for all $j = \{1 \ldots 5\}$, $\overline{y}_j \le 1$ for all $j = \{3 \ldots 5\}$ and the following system of inequalities,

$$\overline{y}_1 + \overline{y}_2 \le 1 \tag{25}$$
$$\overline{y}_1 + \overline{y}_2 + \overline{y}_3 + \overline{y}_4 \le 2 \tag{26}$$
$$\overline{y}_1 + \overline{y}_2 + \overline{y}_3 + \overline{y}_5 \le 2 \tag{27}$$

For the facet (25), $\mu = (1, 1, 0, 0, 0)$, $\mu_0 = 1$ and $\bar{\beta} = 1$. So in the original space of variables we obtain

$$(7-1)y_1 + (6-1)y_2 + (5-0)y_3 + (3-0)y_4 + (2-0)y_5 \le 11 + 1(1-2) + s$$

or $6y_1 + 5y_2 + 5y_3 + 3y_4 + 2y_5 \le 10 + s$, facet-defining inequality (7) of $\mathrm{conv}(Y)$.

In the same way, using Proposition 2, the three inequalities $(25) - (27)$ lead to the three facet-defining inequalities $(7) - (9)$ of $\mathrm{conv}(Y)$.

□

4. Continuous cover inequalities

Here we examine facets of conv(Y) obtained by first fixing all but one of the binary variables, then deriving a facet-defining inequality in the two-dimensional space consisting of one binary variable and the continuous variable s, and finally lifting back the binary variables.

We start by reviewing some results on sequential lifting. For a more thorough introduction and basic references on 0-1 lifting and superadditivity, see [13]. Let

$$W(d) = \left\{ (z, s) \in B^{|N^+|+|N^-|} \times R^1_+ : \sum_{j\in N^+} a_j z_j + \sum_{j\in N^-} a_j z_j \leq d + s \right\}$$

with $a_j > 0$ for $j \in N^+$, $a_j < 0$ for $j \in N^-$ and $N = N^+ \cup N^-$. Now suppose that $M^+ \subseteq N^+$, $M^- \subseteq N^-$, $M = M^+ \cup M^-$ and

$$\sum_{j\in M} \pi_j z_j \leq \pi_0 + s \tag{28}$$

is a tight valid inequality for the set $W_M(d) = W(d) \cap \{z : z_j = 0 \; j \notin M\}$.

The lifting question is then to determine values $\{\alpha_j\}_{j\in N\setminus M}$ so that

$$\sum_{j\in M} \pi_j z_j + \sum_{j\in N\setminus M} \alpha_j z_j \leq \pi_0 + s$$

is valid for $W(d)$.

One way to lift variables is to introduce them one at a time (sequential lifting). Suppose that we start with some variable z_k with $k \in N \setminus M$.

Let $G_M(u) = \max\left\{\sum_{j\in M} \pi_j z_j - s : (z, s) \in W_M(u)\right\}$, and $\phi_M(u) = G_M(d) - G_M(d-u)$ for $u \in R^1$. As inequality (28) is tight, $\pi_0 = G_M(d)$.

Proposition 3. *The inequality*

$$\alpha_k z_k + \sum_{j\in M} \pi_j z_j \leq \pi_0 + s \tag{29}$$

is valid for $W_{M\cup\{k\}}(d)$ if and only if $\alpha_k \leq \phi_M(a_k)$. If (28) *is facet-defining for $W_M(d)$, then* (29) *is facet-defining for $W_{M\cup\{k\}}(d)$ if $\alpha_k = \phi_M(a_k)$.*

Now to sequentially lift in a second variable, it is necessary to update the functions G_M and ϕ_M, so as to calculate the next lifting coefficient. Specifically if $k \in N \setminus M$ was introduced first with lifting coefficient $\alpha_k = \phi_M(a_k)$, then

$$G_{M\cup\{k\}}(u) = \max\left[G_M(u), G_M(u - a_k) + \phi_M(a_k)\right],$$

so in general the function G_M can increase as more variables are lifted in. However $G_M(d)$ does not change, and so ϕ_M can decrease. In the next two Propositions we explore two cases in which these functions remain unchanged over an important part of their domains.

Definition 1. *A function $F : D \subseteq R^m \to R^1$ is superadditive on D if $F(d_1) + F(d_2) \leq F(d_1 + d_2)$ for all $d_1, d_2, d_1 + d_2 \in D$.*

We distinguish whether the variable lifted has a positive coefficient $k \in N^+ \setminus M^+$ or negative coefficient $k \in N^- \setminus M^-$ in the description of $W(d)$.

Proposition 4. *[22] i) $\phi_M(u) = \min_{v \leq d}[G_M(v) - G_M(v-u)]$ for $u \in R^1_+$*
if and only if
ii) ϕ_M is superadditive on R^1_+.
If i) and ii) hold, then
iii) for any $k \in N^+ \setminus M^+$, $\phi_M(u) = \phi_{M\cup\{k\}}(u)$ for $u \in R^1_+$, and
iv) the inequality $\sum_{j\in M} \pi_j z_j + \sum_{j\in N^+\setminus M^+} \phi_M(a_j) z_j \leq \pi_0 + s$ is valid for $W_{M\cup(N^+\setminus M^+)}(d)$, and it is facet-defining for $W_{M\cup(N^+\setminus M^+)}(d)$ if (28) is facet-defining for $W_M(d)$.

Note that a slightly stronger statement can be made: if iii) holds for all possible values of $a_k \in R^1_+$, or if iv) holds for all values of $a_j \in R^1_+$ with $j \in N^+ \setminus M^+$ and $| N^+ \setminus M^+ | \geq 2$, then i) and ii) hold.

In a similar fashion, we obtain

Proposition 5. *The following are equivalent*
i) $\phi_M(u) = \min_{v \geq d}[G_M(v) - G_M(v-u)]$ for $u \in R^1_-$
ii) ϕ_M is superadditive on R^1_-
If i) and ii) hold and $k \in N^- \setminus M^-$, then
iii) $\phi_M(u) = \phi_{M\cup\{k\}}(u)$ for $u \in R^1_-$
iv) the inequality $\sum_{j\in M} \pi_j z_j + \sum_{j\in N^-\setminus M^-} \phi_M(a_j) z_j \leq \pi_0 + s$ is valid for $W_{M\cup(N^-\setminus M^-)}(d)$, and it is facet-defining for $W_{M\cup(N^-\setminus M^-)}(d)$ if (28) is facet-defining for $W_M(d)$.

We now apply the above results to derive facet-defining inequalities for Y. One pair of definitions is needed.

Definition 2. *A pair (k, C) is a k-cover for Y if $k \in C \subseteq N$, $\sum_{j\in C} a_j = b + \lambda$ with $\lambda > 0$, and $\sum_{j\in C\setminus\{k\}} a_j < b$.*

A pair (k, T) is a k-reverse-cover for Y if $k \in T \subseteq N$, $\sum_{j\in T} a_j = \sum_{j\in N} a_j - b + \mu$ with $\mu > 0$, and $\sum_{j\in T\setminus\{k\}} a_j < \sum_{j\in N} a_j - b$.

Note that if (k, C) is a k-cover for Y, then (k, T) is a k-reverse-cover for Y with $T = (N \setminus C) \cup \{k\}$, and vice versa, and $a_k = \lambda + \mu$. The resulting pair of k-covers, denoted (k, C, T) is called a *k-cover-pair*.

4.1. Fixing variables

Here we see that, given a k-cover-pair (k, C, T), there is a natural way to fix all the binary variables other than k. The set Y can be rewritten as

$$Y = \left\{ (y, s) \in B^n \times R^1_+ : -a_k \bar{y}_k - \sum_{j\in C\setminus\{k\}} a_j \bar{y}_j + \sum_{j\in T\setminus\{k\}} a_j y_j \leq -\lambda + s \right\},$$

or alternatively as

$$Y = \left\{ (y, s) \in B^n \times R^1_+ : a_k y_k - \sum_{j \in C \setminus \{k\}} a_j \overline{y}_j + \sum_{j \in T \setminus \{k\}} a_j y_j \leq \mu + s \right\}.$$

Now we restrict the space of feasible solutions to that of the variables (y_k, s) by setting $y_j = 1$ for $j \in C \setminus \{k\}$ and $y_j = 0$ for $j \in T \setminus \{k\}$. The resulting set is

$$Q = \left\{ (y_k, s) \in B^1 \times R^1_+ : -a_k \overline{y}_k \leq -\lambda + s \right\} = \left\{ (y_k, s) \in B^1 \times R^1_+ : a_k y_k \leq \mu + s \right\}.$$

It is easily checked that the unique nontrivial facet-defing inequality for Q is the Gomory mixed integer [8] or MIR [14] inequality
$-\lambda \overline{y}_k \leq -\lambda + s$, or $(a_k - \mu) y_k \leq s$.

Now we lift back the variables for $j \in N \setminus \{k\}$. Different orderings will lead to different facets.

4.2. Continuous cover inequalities

Here we start by lifting in the variables that have been set to $y_j = 1$ (or $\overline{y}_j = 0$) for $j \in C \setminus \{k\}$. Let $Y_C = \{(y, s) \in Y : y_j = 0 \ j \in N \setminus C\}$ and $\tilde{C} = \{j \in C : a_j > \lambda\}$, so $k \in \tilde{C}$.

Proposition 6. *If (k, C) is a k-cover for Y, the inequality*

$$\sum_{j \in \tilde{C}} \lambda y_j + \sum_{j \in C \setminus \tilde{C}} a_j y_j \leq \left(| \tilde{C} | -1 \right) \lambda + \sum_{j \in C \setminus \tilde{C}} a_j + s \tag{30}$$

is facet-defining for conv(Y_C).

Proof. Given the facet-defining inequality

$$-\lambda \overline{y}_k \leq -\lambda + s$$

for

$$Q = \left\{ (\overline{y}_k, s) \in B^1 \times R^1_+ : -a_k \overline{y}_k \leq -\lambda + s \right\},$$

we sequentially lift the variables $\overline{y}_j \ j \in C \setminus \{k\}$. We have that $G_{\{k\}}(u) = \max\{-\lambda \overline{y}_k - s : -a_k \overline{y}_k - s \leq u, (\overline{y}_k, s) \in B^1 \times R^1_+\}$. Thus $G_{\{k\}}(u) = \min(u, 0)$ for $u \geq -\lambda$, and $\phi_{\{k\}}(u) = G_{\{k\}}(-\lambda) - G_{\{k\}}(-\lambda - u) = \max[-\lambda, u]$ for $u \in R^1_-$. As $\phi_{\{k\}}$ is superadditive on R^1_-, we obtain from Proposition 5 that the lifted inequality

$$-\sum_{j \in \tilde{C}} \lambda \overline{y}_j - \sum_{j \in C \setminus \tilde{C}} a_j \overline{y}_j \leq -\lambda + s$$

is facet-defining for Y_C. After complementation, the claim follows.

□

Next we lift the variables from $y_j = 0$ for $j \in T \setminus \{k\} = N \setminus C$. Now $G_C(u) =$

$$\max \left\{ -\sum_{j \in \tilde{C}} \lambda \bar{y}_j - \sum_{j \in C \setminus \tilde{C}} a_j \bar{y}_j - s : -\sum_{j \in C} a_j \bar{y}_j - s \le u, (\bar{y}, s) \in B^{|C|} \times R^1_+ \right\}.$$

Suppose WLOG that $\tilde{C} = \{1, \dots, r\}$ with $a_1 \ge \dots \ge a_r > \lambda$. Let $A_j = \sum_{i=1}^{j} a_i$ for $j \le r$, and $A_0 = 0$. It can be verified that

$$G_C(u) = \begin{cases} 0 \text{ if } u \ge 0 \\ -j\lambda + u + A_j \text{ if } -A_j - \lambda \le u \le -A_j \;\; j = 0, \dots, r-1 \\ -j\lambda \text{ if } -A_j < u \le -A_{j-1} - \lambda \;\; j = 1, \dots, r \\ -r\lambda + u - A_r \text{ if } u \le -A_r \end{cases}$$

and $\phi_C(u) = G_C(-\lambda) - G_C(-\lambda - u)$ is given on R^1_+ by

$$\phi_C(u) = \begin{cases} (j-1)\lambda \text{ if } A_{j-1} \le u \le A_j - \lambda \;\; j = 1, \dots, r \\ (j-1)\lambda + [u - (A_j - \lambda)] \text{ if } A_j - \lambda \le u \le A_j \;\; j = 1, \dots, r-1 \\ (r-1)\lambda + [u - (A_r - \lambda)] \text{ if } A_r - \lambda \le u. \end{cases}$$

Proposition 7. *If (k, C) is a k-cover for Y, then the inequality*

$$\sum_{j \in \tilde{C}} \lambda y_j + \sum_{j \in C \setminus \tilde{C}} a_j y_j + \sum_{j \in N \setminus C} \phi_C(a_j) y_j \le \left(\mid \tilde{C} \mid -1\right)\lambda + \sum_{j \in C \setminus \tilde{C}} a_j + s \qquad (31)$$

is facet-defining for conv(Y).

Proof. Function ϕ_C is superadditive on R^1_+ and thus by Proposition 4

$$-\sum_{j \in \tilde{C}} \lambda \bar{y}_j - \sum_{j \in C \setminus \tilde{C}} a_j \bar{y}_j + \sum_{j \in N \setminus C} \phi(a_j) y_j \le -\lambda + s$$

is facet-defining for conv(Y). Complementing variables, the claim follows.

□

Example (continued). $C = \{1, 2, 4\}$ is a 1-cover with $\lambda = 5$. By Proposition 6 we obtain the valid inequality

$$5y_1 + 5y_2 + 3y_4 \le 8 + s$$

for Y_C.

The function ϕ_C is given by

$$\phi_C(u) = \begin{cases} 0 & 0 \le u \le 2 \\ u - 2 & 2 \le u \le 7 \\ 5 & 7 \le u \le 8 \\ u - 3 & 8 \le u \end{cases}$$

Therefore $\phi_C(a_3) = \phi_C(5) = 3$, $\phi_C(a_5) = \phi_C(2) = 0$ and, by Proposition 7, the inequality

$$5y_1 + 5y_2 + 3y_3 + 3y_4 \le 8 + s$$

defines a facet of conv(Y).

Proposition 7 produces the following facets: (1) ($C = \{1, 2\}$), (2) ($C = \{1, 3\}$), (3) ($C = \{1, 4, 5\}$), (4) ($C = \{2, 3, 4\}$), (5) ($C = \{2, 3, 5\}$), (7) ($C = \{2, 3, 4, 5\}$), (10) ($C = \{1, 2, 5\}$), and (11) ($C = \{1, 2, 4\}$).

□

We can derive facets of the form (31) for the complemented problem $\overline{Y}$ and re-express the inequalities in the initial space of variables. However we choose to derive this new class directly.

4.3. *Continuous reverse cover inequalities*

Here we start by lifting in the variables that have been set to $y_j = 0$ for $j \in T \setminus \{k\}$. Let $Y_T = \{(y, s) \in Y : y_j = 1 \ j \in N \setminus T\}$. Suppose that $T = \{1 \dots p\}$, order $\{a_j\}_{j \in T}$ s.t. $a_j \geq a_{j+1}$, $j \in \{1 \dots p-1\}$ and define r=min $\{j \in T : a_j > \mu\}$. Note that $r \geq 1$ as $a_k > \mu$.

Proposition 8. *If* (k, T) *is a k-reverse-cover for* Y*, the inequality*

$$\sum_{j \in T} (a_j - \mu)^+ y_j \leq s$$

is facet-defining for conv(Y_T).

Proof. Given the facet-defining inequality $(a_k - \mu) y_k \leq s$ for $Q = \{(y_k, s) \in B^1 \times R^1_+ : a_k y_k \leq \mu + s\}$, we sequentially lift the variables y_j $j \in T \setminus \{k\}$. We have that $G_{\{k\}}(u) = \max \left\{ (a_k - \mu) y_k - s : a_k y_k - s \leq u, (y_k, s) \in B^1 \times R^1_+ \right\}$. Thus $G_{\{k\}}(u) = \min(0, u)$ for $u \leq \mu$, and $\phi_{\{k\}}(u) = G_{\{k\}}(\mu) - G_{\{k\}}(\mu - u) = (u - \mu)^+$ for $u \geq 0$. As $\phi_{\{k\}}$ is superadditive on R^1_+, the claim follows immediately from Proposition 4.

□

Next we lift the variables $\overline{y}_j$ for $j \in C \setminus \{k\} = N \setminus T$ with coefficient $-a_j$. $G_T(u) = \max \left\{ \sum_{j \in T} (a_j - \mu)^+ y_j - s : \sum_{j \in T} a_j y_j - s \leq u, (y, s) \in B^{|T|} \times R^1_+ \right\}$ and $\phi_T(u) = G_T(\mu) - G_T(\mu - u)$. It is straightforward to verify that $\phi_T(u) = -\psi_T(-u)$ on R^1_- where ψ_T is defined on R^1_+ as follows:

$$\psi_T(u) = \begin{cases} u - i\mu & \text{if } A_i \leq u \leq A_{i+1} - \mu \quad \text{for } i = 0 \dots r-1 \\ A_i - i\mu & \text{if } A_i - \mu \leq u \leq A_i \qquad \text{for } i = 1 \dots r-1 \\ A_r - r\mu & \text{if } u \geq A_r - \mu \end{cases}$$

where $A_0 = 0$ and $A_i = \sum_{j=1}^{i} a_j$ for $i = 1 \dots r$.

Proposition 9. *If* (k, T) *is a k-reverse-cover for* Y*, then the inequality*

$$\sum_{j \in T} (a_j - \mu)^+ y_j + \sum_{j \in N \setminus T} \psi_T(a_j) y_j \leq \sum_{j \in N \setminus T} \psi_T(a_j) + s \tag{32}$$

is facet-defining for Y.

Proof. The function ϕ_T is superadditive on R^1_- and hence by Propositions 5 and 8,

$$\sum_{j\in T}(a_j-\mu)^+y_j+\sum_{j\in N\setminus T}\phi_T(-a_j)\overline{y}_j\le s$$

is facet-defining for Y. The claim follows after complementation and substitution.

□

Example (continued). $T=\{2,3,5\}$ is a 2-reverse cover with $a_2=6>\mu=1$. Therefore by Proposition 8,

$$5y_2+4y_3+1y_5\le s$$

is facet-defining for Y_T. Now

$$\psi_T(d)=\begin{cases} d & 0\le d\le 5\\ 5 & 5\le d\le 6\\ d-1 & 6\le d\le 10\\ 9 & 10\le d\le 11\\ d-2 & 11\le d\le 12\\ 10 & 12\le d\end{cases}$$

So $\psi_T(a_1)=\psi_T(7)=6$, $\psi_T(a_4)=3$ and so the inequality

$$6y_1+5y_2+4y_3+3y_4+y_5\le 9+s$$

obtained by Proposition 9 defines a facet of conv(Y).

Proposition 9 produces the following facets: (1) $T=\{1,3,4,5\}$, (2) $T=\{1,2,4,5\}$, (3) $T=\{1,2,3\}$, (9) $T=\{2,3,5\}$, (7) $T=\{1,2\}$, (8) $T=\{2,3,4\}$, (10) $T=\{1,3,4\}$, (11) $T=\{1,3,5\}$, (12) $T=\{2,3,4,5\}$, (13) $T=\{1,2,5\}$, (14) $T=\{1,2,4\}$, (10) $T=\{1,3,4\}$ and (15) $T=\{1,4,5\}$.

□

4.4. *Lifted continuous cover inequalities*

Again let (k,C,T) be a k-cover pair, and suppose now that the lifting order is arbitrary. We start from $Q=\left\{(y,s)\in B^1\times R^1_+ : a_ky_k\le\mu+s\right\}$, and lift terms a_jy_j for $j\in T\setminus\{k\}$ and $-a_j\overline{y}_j$ for $j\in C\setminus\{k\}$.

Proposition 10. *For each lifting sequence, a facet-defining inequality of the form*

$$(a_k-\mu)y_k+\sum_{j\in C\setminus\{k\}}\gamma_jy_j+\sum_{j\in T\setminus\{k\}}\gamma_jy_j\le\sum_{j\in C\setminus\{k\}}\gamma_j+s$$

is obtained, with $0\le\gamma_j\le(a_j-\mu)^+$ *for* $j\in T\setminus\{k\}$, *and* $a_j\ge\gamma_j\ge\min[\lambda,a_j]$ *for* $j\in C\setminus\{k\}$.

Proof. From Proposition 3, $\phi_{\{k\}}(u)$ provides an upper bound on the lifting coefficient. Another result from [22] states that $\gamma_{\{k\}}(u) = \min_{v \in R^1}[G_{\{k\}}(v) - G_{\{k\}}(v-u)]$ always produces a feasible value and hence lower bound for the lifting coefficent. As $G_{\{k\}}(u) = \min\{u, \min[(u-\mu)^+, \lambda]\}$, it follows that $\gamma_{\{k\}}(u) = \min(u, 0)$. Therefore for $j \in T \setminus \{k\}$, $u = a_j > 0$, so $\gamma_{\{k\}}(u) = 0 \leq \gamma_j \leq \phi_{\{k\}}(u) = (a_j - \mu)^+$, and for $j \in C \setminus \{k\}$, $u = -a_j < 0$, so $\gamma_{\{k\}}(u) = -a_j \leq -\gamma_j \leq \phi_{\{k\}}(u) = \max[-\lambda, -a_j] = -\min[\lambda, a_j]$.

□

Example (continued). $(\{5\}, \{1,4,5\}, \{2,3,5\})$ is a 5-cover-pair with $\lambda = \mu = 1$. The set Y now can be written as $(y, s) \in B^5 \times R^1_+$ satisfying

$$-7\overline{y}_1 + 6y_2 + 5y_3 - 3\overline{y}_4 + 2y_5 \leq 1 + s.$$

Setting $y_1 = y_4 = 1$, $y_2 = y_3 = 0$, we obtain $Q = \{(y_5, s) \in B^1 \times R^1_+ : 2y_5 \leq 1 + s\}$ with facet-defining inequality $y_5 \leq s$. Lifting in the variables in the order y_4, y_3, y_2, y_1, and calculating the coefficients one by one by repeated use of Proposition 3, we obtain

$$-4\overline{y}_1 + 3y_2 + 2y_3 - \overline{y}_4 + y_5 \leq s, \text{ or}$$

$$4y_1 + 3y_2 + 2y_3 + y_4 + y_5 \leq 5 + s.$$

Starting from the same k-cover-pair, and lifting in the order y_3, y_1, y_4, y_2 leads to facet-defining inequality (16).

Finally taking the k-cover-pair $(4, \{1,4,5\}, \{2,3,4\})$ and lifting in the order y_3, y_1, y_5, y_2 leads to (17).

We have explained all the facets of conv(Y) in our example.

□

5. The mixed Knapsack problem and the flow model

Here we show how valid inequalities for Y can be used to derive valid inequalities for single node flow sets, and in particular generate a large class of flow cover inequalities.

The mixed knapsack set Y can be seen as a relaxation of the flow model

$$Z = \left\{ (x, y) \in R^n_+ \times B^n : \sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq b, l_j y_j \leq x_j \leq u_j y_j \ \ j \in N^+ \cup N^- \right\},$$

with $l_j, u_j \geq 0$ for $j \in N^+ \cup N^-$. This means that valid inequalities for Y provide valid inequalities for the flow model Z. We first present a simple example.

Example 2. Consider an instance of the flow model Z consisting of the points $(x, y) \in R^5_+ \times B^5$ satisfying:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 11$$
$$x_1 \leq 7y_1, x_2 \leq 6y_2, x_3 \geq 5y_3, x_4 \geq 3y_4, x_5 \geq 2y_5$$

After substituting $x_j + t_j = u_j y_j$ with $t_j \geq 0$ for $j = 1, 2$, and $x_j \geq l_j y_j$ for $j = 3, 4, 5$, we obtain

$$7y_1 + 6y_2 + 5y_3 + 3y_4 + 2y_5 \leq 11 + (t_1 + t_2)$$

of the form Y. The valid inequality (1) for Y gives $2y_1 + 2y_2 \leq 2 + (t_1 + t_2)$ which after eliminating t_1, t_2 gives

$$x_1 + x_2 - 5y_1 - 4y_2 \leq 2$$

which is a basic flow cover inequality for Z, see [15].

□

We now describe a general procedure:
i) Let (N_l^+, N_u^+, N_b^+) be a partition of N^+ and (N_l^-, N_u^-, N_b^-) a partition of N^-.
ii) For $j \in N_l^+ \cup N_l^-$, substitute $x_j = l_j y_j + s_j$ with $s_j \geq 0$.
iii) For $j \in N_u^+ \cup N_u^-$, substitute $x_j = u_j y_j - t_j$ with $t_j \geq 0$.
The resulting relaxation is:

$$\tilde{Z} = \Bigg\{ (x, y, s, t) : \sum_{j \in N_l^+} l_j y_j + \sum_{j \in N_u^+} u_j y_j - \sum_{j \in N_l^-} l_j y_j - \sum_{j \in N_u^-} u_j y_j$$

$$+ \sum_{j \in N_b^+} x_j + \sum_{j \in N_l^+} s_j + \sum_{j \in N_u^-} t_j$$

$$\leq b + \sum_{j \in N_u^+} t_j + \sum_{j \in N_l^-} s_j + \sum_{j \in N_b^-} x_j,$$

$$y_j \in \{0, 1\}\ j \in N_l^- \cup N_l^+ \cup N_u^- \cup N_u^+, x, s, t \geq 0 \Bigg\}.$$

iv) Complement the binary variables with negative coefficients, set $b' = b + \sum_{j \in N_l^-} l_j + \sum_{j \in N_u^-} u_j$, define a nonnegative variable $s' = \sum_{j \in N_u^+} t_j + \sum_{j \in N_l^-} s_j + \sum_{j \in N_b^-} x_j$, and define $a_j = l_j$ for $j \in N_l^+ \cup N_l^-$ and $a_j = u_j$ for $j \in N_u^+ \cup N_u^-$.
The final relaxation is

$$\tilde{Y} = \{(y, s') : \sum_{j \in N_l^+ \cup N_u^+} a_j y_j + \sum_{j \in N_l^- \cup N_u^-} a_j \overline{y}_j \leq b' + s',$$

$$y_j \in \{0, 1\}\ j \in N_l^- \cup N_l^+ \cup N_u^- \cup N_u^+, s' \geq 0\}$$

Example 3. Consider a set Z consisting of points $(x, y) \in R_+^7 \times B^7$ satisfying

$$x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7 \leq 1$$

$$7y_1 \leq x_1 \leq 12y_1, x_2 \leq 6y_2, x_3 \leq 5y_3, 5y_4 \leq x_4 \leq 9y_4,$$

$$1y_5 \leq x_5 \leq 3y_5, 4y_6 \leq x_6 \leq 9y_6, x_7 = 2y_7$$

Setting $N_l^+ = \{1\}$, $N_u^+ = \{2\}$, $N_b^+ = \{3\}$, $N_l^- = \{4\}$, $N_u^- = \{5, 7\}$ and $N_b^- = \{6\}$, we obtain after substitution and complementation

$$7y_1 + 6y_2 + 5\overline{y}_4 + 3\overline{y}_5 + 2\overline{y}_7 + s_1 + x_3 + t_5 + t_7 \leq 11 + t_2 + s_4 + x_6$$

and after further relaxation

$$7y_1 + 6y_2 + 5\overline{y}_4 + 3\overline{y}_5 + 2\overline{y}_7 \leq 11 + s'.$$

The mixed knapsack inequality (11): $5y_1 + 5y_2 + 3\overline{y}_4 + 3\overline{y}_5 \leq 8 + s'$ gives after substituting back the inequality

$$5y_1 + (x_2 - y_2) \leq 2 + (x_4 - 2y_4) + 3y_5 + x_6$$

that is valid for Z.

□

Having the relaxation $\tilde{Y}$, we can in particular derive continuous cover (31) and continuous reverse cover (32) inequalities for $\tilde{Y}$, and then convert them back into valid inequalities for Z. In the next part of this section we express both inequalities in the original space Z.

A k-cover C for Z is a set of the form $C = C_l^+ \cup C_u^+ \cup C_l^- \cup C_u^- \neq \emptyset$ where $C_l^- \subseteq N_l^-$, etc.,

$$\sum_{j \in C_l^+} l_j + \sum_{j \in C_u^+} u_j - \sum_{j \in N_l^- \setminus C_l^-} l_j - \sum_{j \in N_u^- \setminus C_u^-} u_j - b = \lambda > 0$$

and either $k \in C_l^+ \cup C_l^-$ with $l_k > \lambda$ or $k \in C_u^+ \cup C_u^-$ with $u_k > \lambda$.

Now (k, C) is a k-cover for $\tilde{Y}$, and ϕ_C can be defined as before to generate the continuous cover inequality (31).

Proposition 11. *If C is a k-cover for Z, then*

$$\sum_{j \in C_u^+} [x_j - (u_j - \lambda)^+ y_j] + \sum_{j \in C_l^+} \min\{l_j, \lambda\} y_j$$

$$+ \sum_{j \in N_u^+ \setminus C_u^+} [x_j - (u_j - \phi_C(u_j)) y_j] + \sum_{j \in N_l^+ \setminus C_l^+} \phi_C(l_j) y_j$$

$$\leq \sum_{j \in C_u^+} \min\{u_j, \lambda\} + \sum_{j \in C_l^+} \min\{l_j, \lambda\} - \lambda + \sum_{j \in C_l^-} [x_j - (l_j - \lambda)^+ y_j] + \sum_{j \in C_u^-} \min\{u_j, \lambda\} y_j$$

$$+ \sum_{j \in N_l^- \setminus C_l^-} [x_j - l_j y_j - \phi_C(l_j)(1 - y_j)] - \sum_{j \in N_u^- \setminus C_u^-} \phi_C(u_j)(1 - y_j) + \sum_{j \in N_b^-} x_j \quad (33)$$

is a valid inequality for Z.

Proof. Remembering that $a_j = l_j$ for $j \in C_l^+ \cup C_l^-$ and $a_j = u_j$ for $j \in C_u^+ \cup C_u^-$, the inequality (31) gives:

$$\sum_{j \in C_u^+} [\min(\lambda, u_j) y_j - t_j] + \sum_{j \in C_l^+} [\min(\lambda, l_j) y_j] + \sum_{j \in N_u^+ \setminus C_u^+} [\phi_C(u_j) y_j - t_j]$$
$$+ \sum_{j \in N_l^+ \setminus C_l^+} \phi_C(l_j) y_j + \sum_{j \in C_u^-} \min(\lambda, u_j) \overline{y}_j + \sum_{j \in C_l^-} [\min(\lambda, l_j) \overline{y}_j - s_j]$$
$$+ \sum_{j \in N_u^- \setminus C_u^-} \phi_C(u_j) \overline{y}_j + \sum_{j \in N_l^- \setminus C_l^-} [\phi_C(l_j) \overline{y}_j - s_j]$$
$$\leq \sum_{j \in C_u^+ \cup C_u^-} \min(\lambda, u_j) + \sum_{j \in C_l^+ \cup C_l^-} \min(\lambda, l_j) - \lambda + \sum_{j \in N_b^-} x_j.$$

The result follows by substituting back for s_j, t_j and complementing the variables $\overline{y}_j$. $\square$

Similarly a k-reverse-cover T for Z is a set of the form $T = T_l^+ \cup T_u^+ \cup T_l^- \cup T_u^- \neq 0$ where $T_l^- \subseteq N_l^-$, etc.,

$$b + \sum_{j \in T_l^-} l_j + \sum_{j \in T_u^-} u_j - \sum_{j \in N_l^+ \setminus C_l^+} l_j - \sum_{j \in N_u^+ \setminus T_u^+} u_j = \mu > 0$$

and either $k \in T_l^+ \cup T_l^-$ with $l_k > \mu$ or $k \in T_u^+ \cup T_u^-$ with $u_k > \mu$.

Now (k, T) is a k-reverse-cover for $\tilde{Y}$ and ψ_T can also be defined as before.

Proposition 12. *If T is a k-reverse-cover for Z, then*

$$\sum_{j \in T_l^+} (l_j - \mu)^+ y_j + \sum_{j \in T_u^+} [x_j - (u_j - (u_j - \mu)^+) y_j]$$
$$- \sum_{j \in N_l^+ \setminus T_l^+} \psi_T(l_j)(1 - y_j) + \sum_{j \in N_u^+ \setminus T_u^+} [x_j - \psi_T(u_j)(1 - y_j) - u_j y_j]$$
$$\leq - \sum_{j \in T_u^-} (u_j - \mu)^+ (1 - y_j) + \sum_{j \in T_l^-} [x_j - (l_j y_j + (l_j - \mu)^+ (1 - y_j))]$$
$$+ \sum_{j \in N_l^- \setminus T_l^-} [x_j - (l_j - \psi_T(l_j)) y_j] + \sum_{j \in N_u^- \setminus T_u^-} \psi_T(u_j) y_j + \sum_{j \in N_b^-} x_j \tag{34}$$

is a valid inequality for Z.

In many cases, valid inequalities (33) and (34) define faces of high dimensions of conv(Z). In particular we can prove,

Proposition 13. *Suppose $N^- = \emptyset$, $u_j \leq b$ and $l_j = 0$ for all $j \in N^+$. Then inequality (34) defines a facet of conv(Z).*

More generally we observe that the inequalities (33) and (34) are complementary in exactly the same way as the inequalities of Van Roy and Wolsey [18] and Stallaert [17] respectively. Under certain conditions inequality (33) dominates the Van Roy-Wolsey inequality, but in general neither dominates. By complementarity, the same remark applies to (34) and the inequality of Stallaert.

6. Computational aspects

Separation heuristics have been developed for the knapsack and continuous cover inequalities described in Sections 3 and 4. Also by applying these routines to the complementary set $\bar{Y}$, complemented knapsack and continuous reverse cover inequalites are also obtained. The heuristics follow standard lines. The cover C is generated greedily from the linear programming relaxation of the mixed knapsack constraint, or by greedily choosing elements with the smallest coefficient a_j.

To apply the heuristics to flow models of the form Z, it is necessary to obtain a set of the form $\tilde{Y}$ as in Section 5. Therefore one needs to choose the sets $N_b^+, N_u^+, N_l^+, N_b^-, N_u^-, N_l^-$. The approach is as described in [4]. Two candidate sets $\tilde{Y}$ are generated. The idea of the first is to try to create a pure surrogate knapsack constraint. Suppose (x^*, y^*, s^*) is the linear programming solution. If $x_j^* = u_j y_j^*$, then put $j \in N_u^+$ or N_u^- as appropriate. If $x_j^* = l_j y_j^*$, then put $j \in N_l^+$ or N_l^-. Otherwise if $j \in N^+$, put $j \in N_l^+$, and if $j \in N^-$, put $j \in N_u^-$. The second is based on minimizing the difference between each continuous variable and its simple or variable bounds i.e. if $l_j y_j^* + s_j^* = x_j^* = u_j y_j^* - t_j^*$, and $\min\{s_j^*, t_j^*\} = s_j^*$ or $\min\{s_j^*, t_j^*\} = t_j^*$, then we put $j \in N_l$ or $j \in N_u$ respectively.

Initial tests have been carried out on two problem sets using the MIPO [2] code running on a Sun Sparc Ultra 1 with the separation routine added to generate cuts at the top node only.

The first set consists of randomly generated mixed integer knapsack problems involving optimization over sets of the form

$$K = \left\{ (x, y) \in R_+^p \times B^n : \sum_{j=1}^{p} g_j x_j + \sum_{j=1}^{n} a_j y_j \le b, x_j \le u_j \quad j = 1, \ldots, p \right\}.$$

Adding cuts as described above, the problems with 30 variables generated as in [23] are all solved at the top node. To generate some more difficult instances, the objective coefficients have been correlated with those of the constraint, the number of variables increased and the value of b modified. Results for six more difficult instances are presented.

In Table 1 results are given using pure branch-and-bound (BB) without cuts, and cut-and-branch (CB) with the mixed knapsack separation routines called only at the top node. By default MIPO includes a simple primal heuristic at the top node, uses reduced cost fixing in the tree, and uses a cut pool. This means that cuts added at the top node and later discarded, may be added again later at some node of the tree. A pure best bound strategy is used to choose the next node. LP, XLP and IP are the values of the linear programming relaxation, the value after adding cuts, and the optimal value respectively. The instance names begin with the numbers n and p of $0-1$ and continuous variables. Thus $25.5k2$ is instance number 2 with $n = 25$ and $p = 5$. (The data for the two largest instances are given in the appendix of [12]).

Table 1. Mixed Integer Knapsacks-Cut and Branch with MIPO

Inst	LP	XLP	IP	BBnod	BBSec	CBnod	CBSec	Cuts
25.5k1	553358.2	553283.2	553283.2	93	0.13	1	0.11	13
25.5k2	553624.1	553479.1	553479.1	215	0.22	1	0.15	14
50.5k1	1106620.2	1106524.4	1106524.4	653	0.78	1	0.19	15
50.5k2	1106691.7	1106589.4	1106575.5	1269	1.53	7	0.57	30
100.5k1	1911261.1	1911164.0	1911164.0	447	0.95	1	0.23	15
100.5k2	1108425.9	1108272.7	1108187.4	+100000	******	5323	35.63	55

The second set are standard mixed integer problems from MIPLIB [3] that have been preprocessed by MINTO [16]. The results with MIPO are again obtained running the cut routine only at the top node. Comparative results using flow cover separation routines can be found in [10].

Table 2. MIPLIB problems-Cut and Branch with MIPO

instance	lp	xlp	ip	nb cut	CB node	CB time
egout	511.62	567.29	568.1	35	5	0.35
fixnet3	5413.77	51973	51973	156	1	6.34
fixnet4	7703.48	8675.61	8936	350	67	33.04
fixnet6	3192.04	3624.35	3983	299	1661	131.28
fxch.3	152.01	187.89	197.98	408	667	22.43
mod013	256.02	270.57	280.95	52	183	2.07
modglob	20430947.6	20681347.1	20740508.1	361	1789	61.73
khb05250	95919464	106741250.4	106940226	205	31	6.81
rgn	48.80	68.00	82.20	72	2337	19.29
set1al	11651.63	15867.25	15869.75	400	35	8.66
set1ch	35118.11	40680.22	54537.75	306	+5000	****
set1cl	2183.57	6484.25	6484.25	400	1	5.54
utrans.1	158.66	177.86	183.337	93	111	2.35
utrans.2	207.74	233.68	239.217	116	155	3.93
utrans.3	388.31	416.39	432.285	135	645	13.15
vpm1	16.43	19.50	20	96	243	5.94

7. Comments

We have shown that the mixed knapsack model Y is somewhat richer in structure than the pure knapsack model, and that it also allows us to derive inequalities for the more general flow model Z studied again recently in [9]. In the example treated in detail most of the inequalities found belong to more than one family, but it is clear that in general each of the five families contributes new facet-defining inequalities.

The inequalities we have derived are very basic in that we have only examined sequential lifting in which the continuous variable is essentially introduced last or first. The great advantage of the superadditivity is that one avoids solving knapsack problems to calculate each lifting coefficient. Using other sequential lifting orders and non-sequential lifting are also theoretically possible, but are computationally more complicated.

In practice we have shown that four classes of inequalities can be separated and used efficiently, and for mixed 0-1 rows such as the flow model Z, the use of the mixed knapsack model combined with appropriate choices of substitutions gives results comparable with those obtained with flow cover inequality separation routines.

To extend the results on facet-defining inequalities from Y to the more general mixed 0-1 set K, we can first, by complementing variables if necessary, assume that K is in canonical form in which for all variables x_j with $g_j > 0$, $u_j = \infty$. Now two natural sets to consider are $Y^+ = \{(y, s, s^+) \in B^n \times R^1_+ \times R^1_+ : \sum_{j \in N} a_j y_j + s^+ \leq b + s\}$ and $Y(u) = Y \cap \{(y, s) : s \leq u\}$. It is easily seen that all tight valid inequalities for Y^+ have a zero coefficient for s^+, and so the nontrivial facet-defining inequalities of Y^+ and Y are the same. This indicates that we can essentially ignore terms $g_j x_j$ with $g_j > 0$ in the description of K. $Y(u)$ is more interesting. Letting $Y_0(u)$ denote the knapsack set $\{y \in B^n : \sum_{j \in N} a_j y_j \leq b + u\}$, it follows from the observation that $\{y \in R^n : \exists s \text{ with } (y, s) \in Y(u)\} = Y_0(u)$ that the facet-defining inequalities of conv($Y(u)$) in which s has a coefficient of zero are exactly the facet-defining inequalities of conv($Y_0(u)$). It is also immediate that the facet-defining inequalities of conv(Y) in which s has a non-zero coefficient are valid, but not necessarily facet-defining, for $Y(u)$. Now if $P = \{j \in \{1, \ldots, p\} : g_j < 0\}$ and $P_1 \subseteq P$, all facets of K of the form $\sum_{j=1}^n \pi_j y_j \leq \pi_0 + \alpha \left[\sum_{j \in P_1} (-g_j x_j)\right]$ can be obtained by examining relaxations of K of the form

$$\tilde{K} = \left\{(y, s') \in B^n \times R^1_+ : \sum_{j=1}^n a_j y_j \leq b' + s', s' \leq u'\right\}$$

where $b' = \left[b + \sum_{j \in P \setminus P_1} (-g_j) u_j\right]$, $s' = \sum_{j \in P_1} (-g_j) x_j$ and $u' = \sum_{j \in P_1} (-g_j) u_j$ which is precisely of the form $Y(u)$.

The approach of restricting and lifting can also be used to obtain strong valid inequalities for 0-1 knapsacks with GUBs, and integer knapsacks, and also allows one to generate flow cover inequalities for the corresponding single node generalizations of these models.

Acknowledgements. The authors are grateful to Sebastian Ceria and Yves Pochet for their helpful comments.

References

1. Balas, E. (1975): Facets of the Knapsack Polytope. Math. Program. **8**, 146–164
2. Balas, E., Ceria, S., Cornuéjols, G. (1996): Mixed 0–1 Programming by Lift-and-Project in a Branch-and-cut Framework. Manage. Sci. **42**, 1229–1246

3. Bixby, R.E., Boyd, E.A., Indovina, R.R. (1992): MIPLIB "A test set of mixed integer programming problems". SIAM News, March, 16
4. Ceria, S., Cordier, C., Marchand, H., Wolsey, L.A. (1998): Cutting Planes for Integer Programs with General Integer Variables. Math. Program. **81**, 201–214
5. Cordier, C., Marchand, H., Laundy, R., Wolsey, L.A. (1997): $bc-opt$: a Branch-and-Cut Code for Mixed Integer Programs. CORE DP9778, Université Catholique de Louvain, Louvain-la-Neuve
6. Crowder, H., Johnson, E.L., Padberg, M.W. (1983): Solving Large Scale Zero-One Linear Programming Problems. Oper. Res. **31**, 803–834
7. Fukuda, K., Prodon, A. (1996): Double Description Method Revisited. In: Combinatorics and Computer Science, Lecture Notes in Computer Science no 1120. Springer
8. Gomory, R.E. (1960): Solving Linear Programming Problems in Integers. In: Bellman, R.E., Hall, M., Jr., eds., Combinatorial Analysis, pp. 211–216. American Mathematical Society
9. Gu, Z., Nemhauser, G.L., Savelsbergh, M.W.P. (1995): Sequence Independent Lifting. LEC-95-08. Georgia Institute of Technology
10. Gu, Z., Nemhauser, G.L., Savelsbergh, M.W.P. (1996): Lifted Flow Cover Inequalities for Mixed 0-1 Integer Programs. LEC-96-05. Georgia Institute of Technology
11. Hammer, P.L., Johnson, E.L., Peled, U.N. (1975): Facets of Regular 0-1 Polytopes. Math. Program. **8**, 179–206
12. Marchand, H., Wolsey, L.A. (1997): The 0-1 Knapsack Problem with a Single Continuous Variable. CORE DP9720, Université Catholique de Louvain, Louvain-la-Neuve.
13. Nemhauser, G.L., Wolsey, L.A. (1988): Integer and Combinatorial Optimization. Wiley
14. Nemhauser, G.L., Wolsey, L.A. (1990): A Recursive Procedure for Generating all Cuts for 0-1 Mixed Integer Programs. Math. Program. **46**, 379–390
15. Padberg, M.W., Van Roy, T.J., Wolsey, L.A. (1985): Valid Linear Inequalities for Fixed Charge Problems. Oper. Res. **33**, 842–861
16. Savelsbergh, M.W.P., Nemhauser, G.L. (1993): Functional description of MINTO, a Mixed INTeger Optimizer, Report COC-91-03A. Georgia Institute of Technology, Atlanta, Georgia
17. Stallaert, J. (1992): On the Polyhedral Structure of Capacited Fixed Charge Network Problems. Ph.D. Thesis, Univ. of California
18. Van Roy, T.J., Wolsey, L.A. (1986): Valid Inequalities for Mixed 0-1 Programs. Discrete Appl. Math. **4**, 199–213
19. Van Roy, T.J., Wolsey, L.A. (1987): Solving Mixed 0-1 Problems by Automatic Reformulation. Oper. Res. **35**, 45–57
20. Weismantel, R. (1997): On the 0-1 Knapsack Polytope. Math. Program. **77**, 49–68
21. Wolsey, L.A. (1975): Faces for a Linear Inequality in 0-1 Variables. Math. Program. **8**, 165–178
22. Wolsey, L.A. (1977): Valid Inequalities and Superadditivity for 0/1 Integer Programs. Math. Oper. Res. **2**, 66–77
23. Yan, X-Q., Boyd, E.A. (1998): Cutting Planes for Mixed-Integer Knapsack Polyhedra. Math. Program. **81**, 257–262