Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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To cite this article:
Manfred W. Padberg, (1975) Technical Note—A Note on Zero-One Programming. Operations Research 23(4):833-837. http://dx.doi.org/10.1287/opre.23.4.833

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A Note on Zero-One Programming

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(Received original May 15, 1973; final, March 3, 1975)

We generalize a method for constructing facets for the convex hull of integer solutions to set packing problems to arbitrary zero-one problems having nonnegative constraint-matrices. A particular class of facets is obtained explicitly and illustrated by a numerical example.

WE CONSIDER the convex polytope in R^n defined by

$$P = \{x\epsilon R^n | \sum_{j=1}^{j=n} \boldsymbol{a}_j x_j \leqq \boldsymbol{a}_0,\ 0 \leqq x_j \leqq 1,\ j\epsilon N = \{1, \cdots, n\}\}, \qquad (1)$$

where $\boldsymbol{a}_j \epsilon R_+{}^m$, $j=0, 1, \cdots, n$, are nonnegative vectors with m *integral* components. Denote by P_I the convex hull of the integer vertices of P, i.e.

$$P_I = \mathrm{conv}\{x\epsilon P | x_j = 0 \quad \text{or} \quad 1, \forall j\epsilon N\}. \qquad (2)$$

In this note we describe a general method that can be used to obtain 'facets' of P_I. An inequality $\pi x \leqq \pi_0$ is called a *facet* of P_I if (i) $x\epsilon P_I$ implies $\pi x \leqq \pi_0$ and (ii) there exist exactly d affinely independent vertices x^i of P_I satisfying $\pi x^i = \pi_0$, $i=1, \cdots, d$, if dim $P_I = d$. Any inequality $\pi x \leqq \pi_0$ for which (i) holds is called a valid inequality for P_I. We note that 'cutting planes' used in integer programming generally constitute *valid* inequalities,[1] whereas *facets* of P_I constitute 'deepest' cutting planes. Facets are not only deepest in the sense that they cannot be pushed further into feasible set P without cutting off at least one feasible integer point of P, but they also belong to the class of inequalities that uniquely determines the polytope P_I. Whenever the dimension of P_I coincides with the number of variables in (1), i.e., if dim $P_I=n$, then there is a uniquely defined class of facets of P_I.

We observe first that the inequalities $x_j\geqq 0$ are facets of P_I, provided that $a_j\leqq a_0$ for all $j=1, \cdots, n$. We shall call the inequalities $x_j\geqq 0$, $j=1, \cdots, n$, 'trivial' facets of P_I. Observe that for any nontrivial facet of P_I we have $\pi_j\geqq 0$, $j=1, \cdots, n$ and $\pi_0>0$. Consequently, requirement (ii) in the definition of a facet states that there must exist d linearly independent vertices of P_I satisfying the condition in (ii). We now assume explicitly that $a_j\leqq a_0$ for all $j\epsilon N$. Hence, dim $P_I=n$.

Let T be a nonempty proper subset of N. Denote by P^T the polytope obtained from P by setting the variables x_j, $j\epsilon T$, equal to zero, i.e.,

$$P^T=P\cap\bigcap_{j\epsilon T}\{x\epsilon R^n|x_j=0\} \tag{3}$$

and define P_I^T to be the convex hull of the zero-one points of P^T.

Let $T=\{j_1, \cdots, j_t\}$, where $t=|T|$ and the elements of T are arbitrarily ordered. For $q=1, \cdots, t$ define T_q to be

$$T_q=T_{q-1}\cup\{j_q\}, \tag{4}$$

with the convention that $T_0=\emptyset$. Similar to P^T and P_I^T, we denote by P^{T-T_q} the polytope obtained from P by setting the variables x_j, $j\epsilon T-T_q$, equal to zero and define $P_I^{T-T_q}$ to be the convex hull of the zero-one points of P^{T-T_q}. Note that with the above definitions $P^{T-T_0}=P^T$ and $P^{T-T_t}=P$. Furthermore, by the above assumptions we have that dim $P^{T-T_q}=$ dim $P_I^{T-T_q}=(n-t+q)$.

Let $\pi x\leqq\pi_0$ be any valid inequality for P_I that is a (nontrivial) *facet* for the $(n-t)$-dimensional polytope P_I^T and consider the zero-one problem

$$z=\max\ \pi x,\ x\epsilon P_I^{T-T_1}\cap\{x\epsilon R^n|x_{j_1}=1\}, \tag{5}$$

where we have set the variables x_j, $j\epsilon T-T_1$, equal to zero and the variable x_{j_1} equal to $+1$. (Remember $T_1=\{j_1\}$.)

Define the vector π^1 as follows: $\pi_j^1=\pi_j$, $\forall j\epsilon N-T$, $\pi_{j_1}^1=\pi_0-\bar{z}$, $\pi_j^1=0$ otherwise, where $\bar{z}$ is the optimal objective function value of (5). It follows easily that $\pi_{j_1}^1\geqq 0$ and that the inequality $\pi^1 x\leqq\pi_0$ is a valid inequality for P_I, which is a (nontrivial) *facet* for the $(n-t+1)$-dimensional polytope $P_I^{T-T_1}$. Continuing the above process with j_2, etc., until T is exhausted, we obtain a (nontrivial) facet for the n-dimensional polytope P_I. To be more specific, suppose that π_j, $j\epsilon N-T$, and $\pi_0>0$ are given. Define a sequence of maximization problems (H_q) as

$$\begin{aligned} z_q=\max\ & \textstyle\sum_{j\epsilon N-T}\pi_j x_j+\sum_{j\epsilon T_{q-1}}\pi_j x_j \\ & \textstyle\sum_{j\epsilon N-T}a_j x_j+\sum_{j\epsilon T_{q-1}}a_j x_j\leqq a_0-a_{j_q} \\ & x_j=0 \text{ or } 1,\ \forall j\epsilon(N-T)\cup T_{q-1}, \end{aligned} \tag{H_q}$$

where the π_j, $j\epsilon T_{q-1}$, are defined recursively by

$$\pi_{j_q}=\pi_0-z_q \tag{6}$$

and z_q is the optimal value of the objective function of problem (H_q), $q=1, \cdots, t$. The following theorem generalizes Theorem 3.3 of reference 3. (See also reference 4, Theorem 3.11.)

THEOREM. *Let $T=\{j_1, \cdots, j_t\}$, where $1\leqq t=|T|\leqq n-1$, be an arbitrarily ordered subset of N and let $\pi x\leqq\pi_0$ be a nontrivial facet of P_I^T as defined in (3). Let π' be defined by $\pi_j'=\pi_j$ for all $j\epsilon N-T$, $\pi'_{j_q}=\pi_0-z_q$ for $q=1, \cdots, t$, where z_q are obtained by solving the problems (H_q) for $q=1, \cdots, t$. Then $\pi' x\leqq\pi_0$ is a (nontrivial) facet of P_I.*

The proof of the theorem closely follows the argument used in the proof of Theorem 3.3 in reference 3. It also follows from Theorem 4.2 of reference 13 since if we take $S=N$ and $\mathcal{I}=\{I|I\subseteq N$ and $\sum_{j\epsilon I} a_j\leqq a_0\}$, the pair $(S, \mathcal{I})$ is easily verified to define an *independence system*, i.e., $I_1\subseteq I_2\epsilon\mathcal{I}$ implies $I_1\epsilon\mathcal{I}$.

In the following we show how the above result can be used to construct a special class of facets of P_I. Let N' be any subset of N satisfying

$$\sum_{j\epsilon N'} \boldsymbol{a}_j\nleqq\boldsymbol{a}_0, \tag{7}$$

where the symbol '$\nleqq$' is meant to say that for *at least one component* of the vectors involved in (7) the inequality $\leqq$ is violated.

Proposition. Let N' be any subset of N satisfying (7) and suppose

$$\sum_{j\epsilon Q} \boldsymbol{a}_j\leqq\boldsymbol{a}_0 \tag{8}$$

holds for all $Q\subset N'$ such that $|Q|=\pi_0$, but for no $Q\subset N'$ such that $|Q|>\pi_0$. Then

$$\sum_{j\epsilon N'} x_j\leqq\pi_0 \tag{9}$$

is a facet of $P_I^{N^*}$, where $N^*=N-N'$.

Proof. From the assumption that (7) holds, we have that for every zero-one vector x, $\sum_{j\epsilon N'} x_j>\pi_0$ implies $x\notin P_I^{N^*}$; hence (9) is satisfied by all $x\epsilon P_I^{N^*}$. By this assumption, furthermore, $n'=|N'|\geqq\pi_0+1$. Consequently, letting $k\epsilon N'-Q$ for some $Q\subseteq N'$ with $|Q|=\pi_0$, we find π_0+1 linearly independent zero-one vectors x^i satisfying (9) with equality and with the property that $j\epsilon N-(Q\cup\{k\})$ implies $x_j{}^i=0$ for $i=1, \cdots, \pi_0+1$. The remaining $n'-(\pi_0+1)$ vectors x^i for $i=\pi_0+2, \cdots, n'$ are constructed so as to ensure a lower triangular structure in the rows and columns that correspond to the index set $\{\pi_0+2, \cdots, n'\}$. The latter construction is possible, since (8) holds for all subsets of N' with cardinality π_0. The proposition also follows from Corollary 4.4 of reference 13.

Example. Suppose that $m=1$ in (1) and consider the inequality in zero-one variables

[illegible] $x_2+66x_3+66x_4+66x_5+65x_6+38x_7\leqq 205.$ (10)

Taking N' to be the index set $N'=\{2, 3, 4, 5, 6\}$, we easily check that N' satisfies the requirements of the proposition. Consequently, the inequality

$$x_2+x_3+x_4+x_5+x_6 \leqq 3 \tag{11}$$

provides a facet for $P_I^{\{1,7\}}$. In order to 'extend' (11) to a facet of P_I, we set $T=\{7, 1\}$ and solve (H_q), $q=1, 2$. Hence

$$z_1 = \max\ x_2+x_3+x_4+x_5+x_6,$$

$$71x_2+66x_3+66x_4+66x_5+65x_6 \leqq 167 = 205-38,$$

$$x_i=0 \text{ or } 1,\ i=2, \cdots, 6,$$

yields $z_1=2$. Consequently, a new inequality is given by

$$x_2+x_3+x_4+x_5+x_6+x_7 \leqq 3$$

and this inequality is a facet for $P_I^{\{1\}}$. Solving (H_2), we obtain finally

$$2x_1+x_2+x_3+x_4+x_5+x_6+x_7 \leqq 3. \tag{12}$$

The inequality provided by (12) is a facet for the convex hull of zero-one solutions to (10). (In fact, we have chosen an example where (10) and (12) have identical zero-one solutions.) The other (and in this case the only other) facets of P_I are the trivial facets $x_j \geqq 0$ and the nontrivial facets $x_j \leqq 1$, $j=1, \cdots, 7$. In our particular example, the order of T did not affect the resulting facet. This is, however, not true in general. Finally, we would like to point out that the upper bound on the number of facets needed to describe P_I for an arbitrary underlying polytope P is given by 2^{n-1} (see reference 2).

NOTES

1. Since this paper was originally written (March 1973), a number of papers have appeared that deal with topics very closely related to the subject of this note. A (possibly incomplete) list of papers that the interested reader should consult is given in the "Additional References" below.
2. The results given here were independently and simultaneously obtained by Professor Padberg and G. L. Nemhauser and L. E. Trotter, Jr.[13] The presentation in reference 13 is somewhat more general, but also more abstract. To avoid duplication, this note is an abbreviated version of the original paper. The Editor.

ACKNOWLEDGMENT

I am indebted to Egon Balas for helpful discussions and remarks on the topic of this paper. I did the research for this paper while visiting the University of Waterloo in March-April, 1973. Financial support by NRC [illegible] gratefully acknowledged.

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