Mathematical Programming 77 (1997) 49–68

# On the 0/1 knapsack polytope

Robert Weismantel
*Konrad-Zuse-Zentrum für Informationstechnik, Heilbronner Straße 10,10711 Berlin, Germany*

Received 1 February 1994; revised manuscript received 30 May 1996

**Abstract**

This paper deals with the 0/1 knapsack polytope. In particular, we introduce the class of *weight inequalities*. This class of inequalities is needed to describe the knapsack polyhedron when the weights of the items lie in certain intervals. A generalization of weight inequalities yields the so-called "weight-reduction principle" and the class of extended weight inequalities. The latter class of inequalities includes minimal cover and $(1,k)$-configuration inequalities. The properties of lifted minimal cover inequalities are extended to this general class of inequalities. © 1997 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

*Keywords:* Complete description; Facets; Knapsack polytope; Knapsack problem; Pseudo polynomial time; Separation

## 1. Introduction and notation

Since the early seventies, many researchers have investigated the polyhedral structure of the 0/1 knapsack problem. In particular, two reasons have nourished this development: one is the increasing number of interesting applications that—at least as a subproblem—involve the single knapsack problem; the other is the discovery of beautiful concepts and results associated with minimal covers, $(1,k)$-configurations or the lifting and complementing of variables.

Most of the polyhedral studies presented so far involve two basic and general objects: minimal covers [1,10,18] and $(1,k)$-configurations [14]. Let $N$ be a subset of items, let $a_0$ denote the knapsack capacity and suppose, every item $i \in N$ has a weight $a_i > 0$. A set $S \subseteq N$ is a *cover* if $\sum_{i\in S} a_i > a_0$ holds. A cover is *minimal* if $\sum_{i\in S\setminus\{s\}} a_i \leqslant a_0$ for all $s \in S$. Let $N' \subseteq N$ be some nonempty subset of items and let $z \in N \setminus N'$. The set $N' \cup \{z\}$ is called a $(1,k)$*-configuration* if


*PII* S0025-5610(96)00064-0

$$\sum_{i \in N'} a_i \leqslant a_0;$$

$K \cup \{z\}$ is a minimal cover for all $K \subset N'$ with $|K| = k$.

These concepts together with the the lifting and complementing of variables [13,18] are a theoretical machinery to attack knapsack problems from a polyhedral point of view.

In fact, since the work of Crowder, Johnson, Padberg [5] several papers based on this polyhedral theory for the 0/1 knapsack problem have been written and are meant to turn the theory into an algorithmic tool for the solution of practical problems (for instance [15,7]). Moreover, the last decade has brought a wide range of interesting applications such as production planing problems [15], airline scheduling problems [11], vehicle routing problems, certain clustering and graph partitioning problems [9] or subproblems that arise within the design of electronic circuits or the design of mainframe computers [16,8,6], in which the 0/1 knapsack problem arises as a crucial subproblem. Indeed, the polyhedral structure of a single knapsack problem is either inherited by the polytope associated with a more complex problem or there is a (non-trivial) way to transform knapsack inequalities associated with a subproblem into inequalities for the original problem.

However, computational experiments by several researchers have revealed that often inequalities based on minimal covers and $(1,k)$-configurations in combination with sequential lifting and complementing are not sufficient for obtaining satisfactory bounds for the optimum value of a knapsack problem. One possible explanation is that minimal cover and $(1,k)$-configuration inequalities do not reflect the individual weights of the items properly, because the coefficients of all but at most one item in the support of the minimal cover or $(1,k)$-configuration inequality are fixed to 1. We show that, starting with a minimal cover, one has a variety of choices for assigning coefficients to the items in the minimal cover. Such coefficients can reflect the original weights of the items with arbitrary precision.

The paper is organized as follows. In Section 2 we present the class of *weight inequalities* for knapsack polytopes. This class of inequalities is needed to describe the polyhedron when the weight of any item is either one or bigger than half of the knapsack capacity. A generalization of weight inequalities yields the so-called "weight-reduction principle", which is discussed in Section 3. It is also shown how to separate inequalities that can be explained via this principle in pseudo-polynomial time. In Section 4 we introduce the class of extended weight inequalities which includes minimal cover and $(1,k)$-configuration inequalities as special cases. Many properties of lifted minimal cover inequalities are shown to extend to this very general class of inequalities. Finally, we prove in Section 5 the complete description of the knapsack polyhedron when the weight of any item is either one or within one third and half of the knapsack capacity. For this, the weight-reduction principle introduced in Section 3 plays a crucial role. Quite often we will give small examples of knapsack polyhedra for which we have computed the convex hull. These computations were performed with the computer program *porta* [4].

### 1.1. Notation

Given a set $N$ of items and a capacity $a_0 \in \mathbb{N}$, let $N_j$ be the subset of items in $N$ with weight $j$ ($j = 1, \ldots, a_0$). The 0/1 knapsack polytope—that we denote by $P$—is the convex hull of all 0/1 vectors that satisfy the *knapsack inequality* $\sum_{j=1}^{a_0} \sum_{i \in N_j} j x_i \leqslant a_0$. For every $i \in N$ we also use the symbol $a_i$ to denote the weight of $i$, i.e., $a_i = j$ if $i \in N_j$. We say a face $F$ of the polytope $P$ is induced by the inequality $cx \leqslant \gamma$, if $F = \{x \in P \mid cx = \gamma\}$. Every $x \in F$ is called a *root* of $cx \leqslant \gamma$. The inequalities $x_i \leqslant 1$, $i \in N$ and $x_i \geqslant 0$, $i \in N$ are called *trivial*. With a subset $S \subseteq N$ we associate the knapsack polytope $P_S := \operatorname{conv}\{x \in P : x_i = 0 \text{ for all } i \in N \setminus S\}$. For any integer $1 \leqslant s \leqslant \lfloor a_0/2 \rfloor$, we define $B^s$ as the interval $[\lfloor a_0/(s+1) \rfloor + 1, \lfloor a_0/s \rfloor]$. $\mathcal{K}^s$ denotes the knapsack problem with $N_j = \emptyset$ for all $j \notin \{1\} \cup B^s$ and $P^s$ is the convex hull of all feasible solutions to the (special) knapsack problem $\mathcal{K}^s$. For real numbers $\tau_j$, $j = 1, \ldots, n$ we define $\sum_{j=v}^{w} \tau_j := 0$ if $v > w$. Finally, for $I \subseteq \{1, \ldots, n\}$ we use the notation $\tau(I) := \sum_{i \in I} \tau_i$ with $\tau(\emptyset) = 0$. The $i$th unit vector in $\mathbb{R}^n$ is always denoted $e_i$.

## 2. The basic form of weight inequalities

In this section we introduce a class of inequalities that is valid for any knapsack problem, the *weight inequalities*. We also show that the weight inequalities, the trivial inequalities, the knapsack constraint and *one* cover inequality describe $P^1$. We start with an example motivating the definition of weight inequalities.

**Example 2.1.** Consider the knapsack polytope defined as the convex hull of all 0/1 vectors satisfying

$$x_1 + x_2 + x_3 + x_4 + 3x_5 + 4x_6 \leqslant 4.$$

A complete inequality description is given by the trivial inequalities $x_i \geqslant 0$, $i = 1, \ldots, 6$ and the following system of inequalities:

$$x_5 + x_6 \leqslant 1, \tag{1}$$

$$x_4 + x_6 \leqslant 1, \tag{2}$$

$$x_3 + x_6 \leqslant 1, \tag{3}$$

$$x_2 + x_6 \leqslant 1, \tag{4}$$

$$x_1 + x_6 \leqslant 1, \tag{5}$$

$$x_3 + x_4 + x_5 + 2x_6 \leqslant 2, \tag{6}$$

$$x_2 + x_4 + x_5 + 2x_6 \leqslant 2, \tag{7}$$

$$x_2 + x_3 + x_5 + 2x_6 \leqslant 2, \tag{8}$$

$$x_1 \qquad\qquad +x_4 +x_5 \ +2x_6 \leqslant 2, \tag{9}$$

$$x_1 \qquad +x_3 \qquad +x_5 \ +2x_6 \leqslant 2, \tag{10}$$

$$x_1 +x_2 \qquad\qquad +x_5 \ +2x_6 \leqslant 2, \tag{11}$$

$$x_1 +x_2 +x_3 \qquad +2x_5 +3x_6 \leqslant 3, \tag{12}$$

$$x_1 \qquad +x_3 +x_4 +2x_5 +3x_6 \leqslant 3, \tag{13}$$

$$x_1 +x_2 \qquad +x_4 +2x_5 +3x_6 \leqslant 3, \tag{14}$$

$$x_2 +x_3 +x_4 +2x_5 +3x_6 \leqslant 3, \tag{15}$$

$$x_1 +x_2 +x_3 +x_4 +3x_5 +4x_6 \leqslant 4. \tag{16}$$

**Definition 2.2** (*weight inequalities*). Let $T \subseteq N$ with $a(T) < a_0$ be given and let $r := a_0 - a(T)$. Assuming $\bigcup_{j \geqslant r+1} N_j \setminus T \neq \emptyset$, we define the *weight inequality with respect to T* as

$$\sum_{i \in T} a_i x_i + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} (j - r) x_i \leqslant a(T).$$

The set $T$ is called the *starting set.*

The name *weight inequality* reflects that the coefficients of the items in $T$ equal their weights and the symbol $r := a_0 - a(T)$ corresponds to the remaining capacity of the knapsack when $x_i = 1$ for all $i \in T$. Weight inequalities are always valid for the given knapsack polytope as we now show.

**Proposition 2.3.** *For $T \subseteq N$ with $a(T) < a_0$, $r := a_0 - a(T)$ and $\bigcup_{j \geqslant r+1} N_j \setminus T \neq \emptyset$, the weight inequality with respect to T is valid for the given knapsack polyhedron.*

**Proof.** Let $x \in P$ and consider the disjunction $\sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} x_i \leqslant 0$ or $\sum_{j \geqslant r+1} \cdot \sum_{i \in N_j \setminus T} x_i \geqslant 1$. In the first case,

$$\sum_{i \in T} a_i x_i + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} (j - r) x_i \leqslant \sum_{i \in T} a_i x_i = a(T).$$

In the second case,

$$\sum_{i \in T} a_i x_i + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} (j - r) x_i \leqslant \sum_{i \in N} a_i x_i - r \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} x_i \leqslant a_0 - r = a(T).$$

This proves the statement. □

It seems hard to derive necessary and sufficient conditions for weight inequalities to define facets of the underlying polyhedron when the numbers $a_i$ are allowed to take up arbitrary values. However, there is an important subclass of weight inequalities for which this question can be answered. Moreover, this subclass of inequalities plus the trivial inequalities and one additional cover inequality suffices to describe the polyhedron $P^1$. We now introduce this subclass of weight inequalities, the so-called 1-weight inequalities. The 1-weight inequality is a weight inequality with respect to some subset $T \subseteq N_1$, i.e., $T$ consists of items of weight 1 only.

**Definition 2.4** (*1-weight inequalities*). Let $T \subseteq N_1$ with $|T| < a_0$. Let $r := a_0 - |T|$ and $\bigcup_{j \geqslant r+1} N_j \setminus T \neq \emptyset$. The 1-weight inequality is the weight inequality with respect to $T$ and is of the form:

$$\sum_{i \in T} x_i + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} (j - r) x_i \leqslant |T|.$$

These 1-weight inequalities have been introduced in [12] in a slightly different context. They can be viewed as a special case of lifted minimal cover inequalities if $N_{r+1} \neq \emptyset$, because for $i \in N_{r+1}$, $T \cup \{i\}$ is a minimal cover (see also [1,10,18]). If $N_{r+1} = \emptyset$, but $N_j \neq \emptyset$ for some $j > r+1$, these inequalities are lifted $(1,k)$-configuration inequalities, as $T \cup \{i\}$ is a $(1,k)$-configuration with $i$ as the "special element" and $k = a_i - r - 1$ (see also [14]). For minimal cover and $(1,k)$-configuration inequalities the lifting coefficients of the items in $N \setminus T$ depend on the sequence in which the items are lifted, see [13]. However, weight inequalities have the property, that once we have fixed the starting set $T$, the coefficients of the remaining items are uniquely determined. Furthermore, it is not too difficult to verify that 1-weight inequalities define facets of $P$ if and only if $|T| = 1$ or $|T| > 1$ and there exists an item in $N \setminus T$ with weight between $r+1$ and $a_0 - 1$ (note that by assumption $N_j \neq \emptyset$ for some $j > r+1$).

One reason why 1-weight inequalities are interesting is the fact that they are necessary to describe the knapsack polyhedron $P^1$.

**Theorem 2.5.** *The system of inequalities*

$$-x_i \leqslant 0, \quad \textit{for all } i \in N,$$

$$\sum_{i \in T} x_i + \sum_{j > a_0 - |T|} \sum_{i \in N_j \setminus T} (j - a_0 + |T|) x_i \leqslant |T|, \quad \textit{for all } T \subseteq N_1,\ |T| < a_0,$$

$$\sum_{j \geqslant \lfloor a_0/2 \rfloor + 1} \sum_{i \in N_j} x_i \leqslant 1,$$

*together with the knapsack inequality* $\sum_{i \in N_1} x_i + \sum_{j \in B^1} \sum_{i \in N_j} j x_i \leqslant a_0$ *completely describe the polyhedron* $P^1$.

**Proof.** Let the inequality $cx \leqslant \gamma$ induce a non-trivial facet $F$ of $P^1$ and assume that $cx \leqslant \gamma$ is not a multiple of the knapsack inequality. This yields, in particular, that $c_i \geqslant 0$ for all $i \in N$. We define $W := \{i \in N_1 \mid c_i > 0\}$ and w.l.o.g. we assume that $W = \{1, \dots, k\}$ and $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k$. We distinguish the following cases:

*Case 1*: $\gamma < \sum_{v=1}^{k} c_v$. Then, $k > a_0$, because $k \leqslant a_0$ would imply that $\sum_{v=1}^{k} e_v$ is a feasible vector implying $\gamma \geqslant \sum_{v=1}^{k} c_v$, a contradiction. Thus, $k > a_0$ and every $x \in F$

satisfies $\sum_{i\in N_1} x_i + \sum_{j\in B^1} \sum_{i\in N_j} jx_i = a_0$, because the number of items of weight 1 with positive coefficients exceeds the knapsack capacity. Therefore the tight points for this inequality are precisely those that satisfy the knapsack inequality at equality, a contradiction.

*Case 2*: $\gamma = \sum_{v=1}^{k} c_v$. Since $F \not\subseteq \{x \in P^1 \mid \sum_{i\in N_1} x_i + \sum_{j\in B^1} \sum_{i\in N_j} jx_i = a_0\}$, we conclude that $k < a_0$ and we set $r := a_0 - k$. Moreover, we define $x^0 := \sum_{v=1}^{k} e_v$. Since $\gamma = \sum_{v=1}^{k} c_v = cx^0$, we conclude that $c_i = 0$ for all $i \in N_j$ with $2 \leqslant j \leqslant r$.

Now let $i \in N_j$, $j \geqslant r+1$ be given. Since $x := \sum_{v=1}^{k-(j-r)} e_v + e_i$ is feasible, we obtain

$$c_i \leqslant c_k + c_{k-1} + \cdots + c_{k-(j-r)+1}.$$

On the other hand, there exists a root $x'$ with $x'_i = 1$. Since $j \geqslant r+1$, there exists $S \subsetneq W$, $|S| = j - r$ with $x'_t = 0$ for all $t \in S$. As $x' - e_i + \sum_{t\in S} e_t$ is feasible, we obtain

$$c_i \geqslant c(S) \geqslant c_k + c_{k-1} + \cdots + c_{k-(j-r)+1}$$

and hence $c_i = c_k + c_{k-1} + \cdots + c_{k-(j-r)+1}$. Now it follows quite easily that every point in $F$ also satisfies the equation $\sum_{i\in W} x_i + \sum_{j\geqslant r} \sum_{i\in N_j} (j-r)x_i = k$. Since $F$ is a facet, the inequality $cx \leqslant \gamma$ (after appropriate scaling) must coincide with the 1-weight inequality with respect to $W$.

*Case 3*: $\gamma > \sum_{v=1}^{k} c_v$. By our initial assumption, $F$ is not the face induced by the knapsack inequality. Consequently, there exists a root $x^0$ with $\sum_{i\in N_1} x_i^0 + \sum_{j\in B^1} \sum_{i\in N_j} jx_i^0 < a_0$. This root satisfies the condition $x_t^0 = 1$ for all $t \in W$. Moreover, $\gamma > c(W)$ implies that there exists $i_0 \in N_{j_0}$ such that $x_{i_0}^0 = 1$, i.e., $\gamma = c(W) + c_{i_0}$. This further implies that every root $x \in F$ satisfies the equation $\sum_{j=\lfloor a_0/2\rfloor+1}^{a_0} \sum_{i\in N_j} x_i = 1$, and hence does not define a facet with $W \neq \emptyset$. This proves the statement. $\square$

When $a_0 = 3$, $B^1 = [2,3]$ and so Theorem 2.5 gives a description of the associated knapsack polytope in this case.

In the subsequent sections we generalize the weight inequality in two directions: one will be to show that cover and $(1,k)$-configuration inequalities are special cases of "extended weight inequalities"; the other direction is to extend weight inequalities with the goal to describe not only $P^1$, but also the polytope $P^2$.

## 3. The weight-reduction principle

One possibility to generalize weight inequalities is to apply a certain reduction principle that we present in this section. The basic idea can be described as follows: we choose a weight inequality with respect to some subset $T$ and an addititonal integer parameter $\psi \geqslant 0$ (meeting certain conditions). For an item $k \in T$ such that $a_k \geqslant a_i$ for all $i \in T$, we define its weight minus $\psi$ as its coefficient. For all items in $T \setminus \{k\}$ we choose the weight of the item as the coefficient. The corresponding right hand side is set to the value $a(T) - \psi$. Note that if $\psi = 0$ we end up with a weight inequality. The purpose of this section is to prove that for every such $\psi$ we obtain a valid inequality for $P$. Moreover, the coefficients for the items in $N \setminus T$ are uniquely determined by the parameter $\psi$ and the set $T$. These inequalities play a crucial role in the description of $P^2$ as we will see in Section 5.

**Definition 3.1** (*reduction principle for weight inequalities*). Let $T \subseteq N$ satisfy $a(T) < a_0$ and $\bigcup_{j \geqslant r+1} N_j \setminus T \neq \emptyset$ where $r := a_0 - a(T)$. For an item $k \in T$ such that $a_k \geqslant a_i$ for all $i \in T$ and $\psi \in [0, r]$ with $a_k - \psi > 0$ the *weight-reduction inequality with respect to T (and k) and* $\psi$ is defined as

$$\sum_{i \in T \setminus \{k\}} a_i x_i + (a_k - \psi) x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} c_j x_i \leqslant a(T) - \psi,$$

where

$$c_j = \begin{cases} (j - r), & \text{if } r < j \leqslant a_k + r - \psi, \\ (j - \psi), & \text{if } a_k + r - \psi < j \leqslant a_k + r, \\ (j - r - \psi), & \text{if } j > a_k + r. \end{cases}$$

The set $T$ is called the *starting set* and the parameter $\psi$ the *reduction parameter.*

We next illustrate these inequalities on an example. Thereafter it will be shown that weight-reduction inequalities are valid for $P$ and under certain conditions facet-defining. The section will terminate with an algorithm to solve the separation problem for this class of inequalities whose running time is pseudo-polynomial in the encoding length of the input.

**Example 3.2.** Consider the knapsack inequality in 0/1 variables

$$x_1 + x_2 + x_3 + x_4 + x_5 + 3x_6 + 4x_7 + 6x_8 + 7x_9 + 9x_{10} + 10x_{11} \leqslant 13.$$

(i) The inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_{10} + 2x_{11} \leqslant 5$ is the weight inequality with respect to $T = \{1, 2, 3, 4, 5\}$. It defines a facet of the corresponding knapsack polytope. Similarly, the facet-defining inequality $x_1 + x_2 + x_3 + x_4 + x_{11} \leqslant 4$ belongs to the same class with $T = \{1, 2, 3, 4\}$ as the starting set.

(ii) By setting $T = \{1, 2, 3, 4, 7\}$ we obtain $r = 13 - 4 - 4 = 5$. If we choose $\psi = 3$, we obtain the inequality

$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_9 + x_{10} + 2x_{11} \leqslant 5.$$

Setting $\psi = 2$ yields the inequality

$$x_1 + x_2 + x_3 + x_4 + 2x_7 + x_8 + 2x_9 + 2x_{10} + 3x_{11} \leqslant 6.$$

Both inequalities are weight-reduction inequalities and define facets of the corresponding knapsack polytope.

(iii) Define $T := \{1,2,3,6,7\}$ and set $\psi := 1$. Thus, $r = 13 - 3 - 3 - 4 = 3$. The inequality

$$x_1 + x_2 + x_3 + 3x_6 + 3x_7 + 3x_8 + 3x_9 + 5x_{10} + 6x_{11} \leqslant 9$$

is a weight-reduction inequality that defines a facet of the corresponding knapsack polytope. If we set $\psi = 0$, we obtain the facet-defining weight inequality

$$x_1 + x_2 + x_3 + 3x_6 + 4x_7 + 3x_8 + 4x_9 + 6x_{10} + 7x_{11} \leqslant 10.$$

For the knapsack polytope considered in Example 3.2 there are many more facet defining weight inequalities and weight-reduction inequalities. In principle, for every subset $T \subseteq N$ such that $a(T) < a_0$ we can generate a series of inequalities in this class depending on the choice of the parameter $\psi$. However, not all of these inequalities will define facets. This issue is discussed now. We start by presenting a proof that weight-reduction inequalities are valid for $P$.

**Proposition 3.3.** *Let $T \subseteq N$ satisfy $a(T) < a_0$ and $\bigcup_{j \geqslant r+1} N_j \setminus T \neq \emptyset$ where $r := a_0 - a(T)$. If $k \in T$ satisfies $a_k \geqslant a_i$ for all $i \in T$, then for every $\psi \in [0, r]$ with $a_k - \psi > 0$ the weight-reduction inequality with respect to $T$ and $\psi$ is valid for the knapsack polyhedron $P$.*

**Proof.** Let $x \in P$ be given. To simplify notation we denote the coefficient of any item $i \in N$ in the weight-reduction inequality by $c_i$. Note that $c_i \leqslant a_i - r$ for all $i \in \bigcup_{j \geqslant r+1} N_j \setminus T$ and $c_i \leqslant a_k - \psi$ for all $i \in \bigcup_{r<j<a_k+r} N_j \setminus T$. Consider the following cases:

(i) If $x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} x_i \geqslant 2$, then

$$\begin{aligned} &\sum_{i \in T \setminus k} a_i x_i + (a_k - \psi) x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} c_i x_i \\ &\quad \leqslant \sum_{i \in N} a_i x_i - r\Big(x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} x_i\Big) + (r - \psi) x_k \\ &\quad \leqslant a_0 - 2r + r - \psi = a(T) - \psi. \end{aligned}$$

(ii) If $\sum_{j \geqslant a_k + r} \sum_{i \in N_j \setminus T} x_i \geqslant 1$, then

$$\begin{aligned} \sum_{i \in T \setminus k} a_i x_i + (a_k - \psi) x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} c_i x_i &\leqslant \sum_{i \in N} a_i x_i - (r + \psi) \sum_{j \geqslant a_k + r} \sum_{i \in N_j \setminus T} x_i \\ &\leqslant a_0 - r - \psi = a(T) - \psi. \end{aligned}$$

(iii) If $\sum_{j \geqslant a_k + r} \sum_{i \in N_j \setminus T} x_i \leqslant 0$ and $x_k + \sum_{j \geqslant r+1} \sum_{i \in N_j \setminus T} x_i \leqslant 1$, then

$$\sum_{i\in T\setminus k} a_i x_i + (a_k-\psi)x_k + \sum_{j\geqslant r+1}\sum_{i\in N_j\setminus T} c_i x_i$$
$$\leqslant \sum_{i\in T\setminus k} a_i x_i + (a_k-\psi)\Big(x_k + \sum_{r<j<a_k+r}\sum_{i\in N_j\setminus T} x_i\Big)$$
$$\leqslant a(T\setminus k) + (a_k-\psi) = a(T) - \psi. \qquad \square$$

Under certain conditions one can also show that weight-reduction inequalities define facets of $P$.

**Proposition 3.4.** *Let $T \subseteq N$, $a(T) < a_0$ and $r := a_0 - a(T)$. If $L := T\cap N_1$, $l = |L|$ and $k\in T$ satisfies $a_k \geqslant a_i$ for all $i\in T$, then for every $\psi\in[0,r]$ such that $a_i \leqslant a_k - \psi < l$ for all $i \in T\setminus k$, the weight-reduction inequality with respect to $T$ and $\psi$ defines a facet of $P$ provided that $N_{a_i+r}\setminus T \neq \emptyset$ for all $i\in T\setminus k$ and $N_{a_k+r-\psi}\setminus\{k\} \neq \emptyset$.*

**Proof.** We have already shown in Proposition 3.3 that weight-reduction inequalities are valid for $P$. We now verify that they are facet-defining under the above requirements. We can assume that $L$ is a strict subset of $T$, because otherwise the inequality is a 1-weight inequality (see the previous section). We denote by $cx \leqslant \gamma$ the weight-reduction inequality and by $x^0$ the vector $\sum_{v\in T} e_v$ ($cx^0 = \gamma$). Let us further assume that $dx \leqslant \delta$ is a facet defining inequality of $P$ such that $F := \{x\in P \mid cx = \gamma\} \subseteq \{x \in P \mid dx = \delta\}$. We show that both inequalities are equal up to multiplication by a scalar by showing that

(i) $d_i = 0$ for all $i\in N_j\setminus T$ with $j\leqslant r$;
(ii) $d_u = d_v$ for all $u, v\in L$;
(iii) $d_u = \lambda(a_u - r)$ for all $u\in N\setminus T$ such that $r+1\leqslant a_u \leqslant a_k + r - \psi$;
(iv) $d_k = \lambda(a_k-\psi)$;
(v) $d_u = d_k$ for all $u\in N\setminus T$ such that $a_k + r - \psi < a_u \leqslant a_k + r$;
(vi) $d_i = \lambda a_i$ for all $i\in T\setminus\{k\}$;
(vii) $d_u = \lambda(a_u - r - \psi)$ for all $u\in N$ such that $a_k + r + 1 \leqslant a_u$.

(i) Follows from the fact that for every $i\in N_j\setminus T$ with $j\leqslant r$ both $x^0$ and $x^0 + e_i$ are feasible points in $F$.

(ii) Let $I$ be a subset of $L$ of cardinality $a_k - \psi$. As $a_k - \psi < l$, $I$ is a strict subset of $L$. Moreover, let $i$ be an item in $N_{a_k+r-\psi}\setminus\{k\}$. Then, $x^0 + e_i - \sum_{w\in I} e_w$ is a feasible point that satisfies the weight-reduction inequality at equality. Hence, $dx = \delta$ follows. In fact, for every such $I\subseteq L$, these steps apply. Choosing two sets $I_1$ and $I_2$ of cardinality $a_k - \psi$ that differ just by two elements $u$ and $v$ in $L$, we obtain that $d_u$ and $d_v$ are equal. In the following we assume that $\lambda := d_u$ for all $u\in L$.

(iii) We choose the following two feasible points: $x^0$ and $x := x^0 + e_u - \sum_{w\in I} e_w$ where $I\subseteq L$, $|I| = a_u - r$ (note that $|I| \leqslant a_k - \psi < |L|$). As $dx = dx^0$ holds and $d_u = \lambda$ for all $u\in L$, the statement follows.

(iv) Let $i\in N_{a_k+r-\psi}\setminus\{k\}$ and choose $I\subseteq L$, $|I| = a_k - \psi$. Then $x^0 + e_i - e_k$ and $x^0 + e_i - \sum_{w\in I} e_w$ are feasible and tight points for the weight-reduction inequality.

Hence, they are both tight for $dx \leqslant \delta$. This yields the statement.

(v) Follows immediately from the fact that $x^0$ and $x := x^0 - e_k + e_u$ are feasible and tight points for the weight-reduction inequality. Therefore $dx = dx^0$ which yields $d_u = d_k$.

(vi) The condition "$N_{a_i+r} \setminus T \neq \emptyset$ for all $i \in T \setminus \{k\}$" guarantees that for every $i \in T \setminus \{k\}$ there exists $u \in N_{a_i+r} \setminus T$. Moreover, $\psi \leqslant a_k - a_j$ for all $j \in T \setminus k$ and hence, $a_i + r = a_k + r - (a_k - a_i) \leqslant a_k + r - \psi$. Therefore, $d_u = (a_u - r)\lambda$ as was shown above. Now the claim follows, since $x^0$ and $x^0 - e_i + e_u$ are feasible and tight for the weight-reduction inequality.

(vii) Choose any $I \subseteq T \setminus \{k\}$ with $a(I) = a_u - r - a_k$. The set $I$ exists since $a_u - r - a_k \leqslant a_0 - r - a_k = \sum_{i \in T\setminus\{k\}} a_i$ and the number $|L|$ of items of weight 1 in $T$ is greater or equal than $a_i$ for all $i \in T \setminus \{k\}$. The two vectors $x^0$ and $x := x^0 - e_k - \sum_{w \in I} e_w + e_u$ are feasible, because $a(T) - a_k - a(I) + a_u = a_0 - r - a_k - (a_u - r - a_k) + a_u = a_0$. Moreover, $cx = cx^0$ implying $dx = dx^0$. This shows the formula and completes the proof. □

Next we deal with the separation problem for the weight-reduction inequalities, i.e., given a fractional point $y$: we want to decide whether there does exist a set $T \subseteq N$, $a(T) < a_0$ and a parameter $\psi \in [0, r]$ $(r := a_0 - a(T))$ such that the weight-reduction inequality with respect to $T$ and $\psi$ is violated and if so, return such an inequality. Here, we investigate a variant of this problem that is: find the weight-reduction inequality with respect to some $T$ and $\psi$ such that the left hand side of the inequality evaluated at the point $y$ minus the right hand side is maximal. If this value is negative, then no weight-reduction inequality is violated. Otherwise, the corresponding inequality defines a hyperplane separating $y$ from the knapsack polyhedron.

We first investigate the problem where $T$ is fixed. As before, the item $k \in T$ satisfies $a_k \geqslant a_i$ for all $i \in T$. For every $\psi \in \{0, \ldots, r\}$ such that $a_k - \psi > 0$ let $c(\psi)x \leqslant \gamma(\psi)$ denote the weight-reduction inequality with respect to $T$ and $\psi$. Our task is to determine a value $\psi^*$ such that $c(\psi^*)y - \gamma(\psi^*) = \max\{c(\psi)y - \gamma(\psi) : \psi \in [0, r], a_k - \psi > 0\}$.

For fixed $T$, $c(\psi)y - \gamma(\psi) > c(\psi+1)y - \gamma(\psi+1)$ iff $y_k + \sum_{j \geqslant a_k+r-\psi} \sum_{i \in N_j \setminus \{k\}} y_i > 1$. Set $\Theta := y_k + \sum_{j \geqslant a_k+r} \sum_{i \in N_j} y_i$. There is a simple algorithm to compute $\psi^*$ that one of the referees has pointed out: if $\Theta > 1$, then $\psi^* = r$. If $\Theta + \sum_{j=a_k}^{a_k+r} \sum_{i \in N_j \setminus \{k\}} y_i \leqslant 1$, then $\psi^* = 0$. Otherwise, $\psi^*$ is the maximal integer in $[1, r-1]$ such that $\Theta + \sum_{j=a_k+r-\psi}^{a_k+r} \sum_{i \in N_j \setminus \{k\}} y_i > 1$. As the expression $\Theta + \sum_{j=a_k+r-\psi}^{a_k+r} \sum_{i \in N_j \setminus \{k\}} y_i$ changes if we go from $\psi$ to $\psi + 1$ iff $N_{a_k+r-\psi} \neq \emptyset$, only those values of $\psi$ must be considered for which $N_{a_k+r-\psi} \neq \emptyset$. Therefore, for a given set $T$, the value $\psi^*$ can be computed in polynomial time.

In fact, $\psi^*$ is independent of the items in $T \setminus \{k\}$, but depends only on the item $k$ and the residuum $r$ (note that $a(T \setminus \{k\}) + a_k + r = a_0$). This indicates that the separation problem for the weight-reduction inequalities can be solved in pseudo-polynomial time and space complexity via a dynamic programming approach (see [2]). Indeed, this is true as we now show. We start discussing the idea of the algorithm.

We choose some item $k$. Then, the weight of every item that will belong to the subset $T \setminus \{k\}$ is less or equal than $a_k$ and the total weight of $T \setminus \{k\}$ will lie in the interval $[1, a_0 - a_k - 1]$. Henceforth, for every value of $r \in [1, a_0 - a_k - 1]$ we need to determine the "best" $\psi$ and the "best" set $T \setminus \{k\}$ with sum of the weights equal to $a_0 - a_k - r$ (in this context "best" means to find objects so that the left hand side minus the right hand side of the inequality to be determined is maximized). However, computing the best $\psi$ and the best set $T \setminus \{k\}$ can be performed independently, since the items in $T \setminus \{k\}$ have a weight which is not bigger than $a_k$ and changing $\psi$ has an impact on the value of the coefficients of items only if the corresponding weight exceeds $a_k$. Second, all items with weight $j \leqslant a_k$ have a coefficient of $\max\{0, j - r\}$, except for the elements in $T \setminus \{k\}$. The coefficient for an item $i \in T \setminus \{k\}$ is its weight $a_i$. Hence, by solving the knapsack problem

$$\begin{aligned} \max \quad & \sum_{i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}} \min\{r, a_i\} y_i z_i, \\ \text{s.t.} \quad & \sum_{i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}} a_i z_i = a_0 - r - a_k, \\ & z_i \in \{0,1\}, \quad \text{for all } i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}, \end{aligned}$$

and setting $T := \{i \in N \setminus \{k\} : z_i = 1\} \cup \{k\}$, we find the best set $T$ such that $a(T) = a_0 - r$.

This yields Algorithm 1.

**Algorithm 1.**
For all $k \in N_{a_k}$, $2 \leqslant a_k \leqslant a_0 - 1$ perform the following steps:
  For all $r = 1$, $r \leqslant a_0 - a_k - 1$ perform the following steps:
    Determine $\psi^* \in [0, r]$ as outlined before.
    Determine the optimum solution to the following knapsack problem:

$$\begin{aligned} \max \quad & \sum_{i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}} \min\{r, a_i\} y_i z_i, \\ \text{s.t.} \quad & \sum_{i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}} a_i z_i = a_0 - r - a_k, \\ & z_i \in \{0,1\}, \quad \text{for all } i \in \bigcup_{j=1}^{a_k} N_j \setminus \{k\}. \end{aligned}$$

    Set $T := \{i \in N \setminus \{k\} : z_i = 1\} \cup \{k\}$.
    Let $cx \leqslant \gamma$ be the weight-reduction inequality with respect to $T$ and $\psi^*$ and set $g(r) = cy - \gamma$.

Determine $g(r^*) = \max\{g(r) \mid r \in \{1, \ldots, a_0 - a_k - 1\}\}$. The weight-reduction inequality with respect to the subset $T$ and the parameter $\psi^*$ that attains the value $g(r^*)$, i.e., $a(T) = a_0 - r$, is the one with maximal slack subject to the condition that $k$ is an element of $T$.

Since the knapsack problem can be solved in pseudo-polynomial time and since $r$ is bounded by the knapsack capacity $a_0$ we obtain a pseudo-polynomial running time and space complexity of this algorithm. For large problem instances (in particular, instances with big knapsack capacity) this exact method does not appear to be practical. Nevertheless, the algorithm gives rise to several heuristic procedures: instead of solving for every $r$ the knapsack problem exactly, one can restrict the number of $r$-values to be considered and apply some primal heuristic for the knapsack problem.

## 4. Extended weight inequalities

The main concern of this section is to extend weight inequalities so as to generalize cover and $(1,k)$-configuration inequalities. In particular we derive one property of extended weight inequalities, namely that lifting coefficients can be computed in polynomial time. Moreover, the exact lifting coefficient of an item coincides either with a certain lower bound or its value equals this lower bound plus 1. This result generalizes [3] in which this property was verified for the class of minimal cover inequalities.

Weight inequalities and their extensions that we discuss in this section are defined via the same basic scheme with just one difference: rather than using the weights of the items as coefficients for an inequality, we now deal with "relative weights".

We first define the extended weight inequality for a subset $S$ of $N$ and prove that it is valid for the knapsack polytope $P_S$. By sequentialy computing lifting coefficients for the items in $N \setminus S$ we extend this inequality afterwards to the original knapsack problem defined on the entire set $N$.

**Definition 4.1** (*extended weight inequalities*). Let mutually disjoint subsets $T$, $I$ and $\{z\}$ of $N$ be given that satisfy the following properties:

$$a(T \cup I) \leqslant a_0, \qquad a(T \cup I) + a_z > a_0;$$

$$a_t \leqslant a_i, \quad \text{for all } t \in T \text{ and } i \in I;$$

$$a(T) \geqslant a_i, \quad \text{for all } i \in I.$$

Setting $r := a_0 - a(T) - a(I)$, the extended weight inequality defined for $T \cup I \cup \{z\}$ is of the form

$$\sum_{i \in T} x_i + \sum_{i \in I} c_i x_i + c_z x_z \leqslant |T| + \sum_{i \in I} c_i,$$

with $c_i := \min\{|S| : S \subseteq T,\ a(S) \geqslant a_i\}$ if $i \in I$ and $c_z := \min\{|S| + \sum_{j \in J} c_j : S \subseteq T,\ J \subseteq I,\ a(S \cup J) \geqslant a_z - r\}$.

Assuming that $T = \{i_1, \ldots, i_t\}$ and $I = \{i_{t+1}, \ldots, i_l\}$ with $a_{i_1} \geqslant \cdots \geqslant a_{i_t}$ and $a_{i_{t+1}} \geqslant \cdots \geqslant a_{i_l}$, the formulas defining $c_i$, $i \in I$ and $c_z$ can be rewritten as follows: $c_i := \min\{w : \sum_{u=1}^{w} a_{i_u} \geqslant a_i\}$ if $i \in I$ and $c_z$ is the optimum solution of the knapsack problem in 0/1 variables $y_w$, $w = 1, \ldots, l$:

$$\min\Big\{\sum_{w=1}^{t} y_w + \sum_{w=t+1}^{l} c_{i_w} y_w : \sum_{w=1}^{t} a_{i_w} y_w + \sum_{w=t+1}^{l} a_{i_w} y_w \geqslant a_z - r\Big\}.$$

Assigning a coefficient of 1 to the items in $T$, $c_i$ is the minimum number of items in $T$ one needs in order to cover the weight $a_i$. In some sense, $c_i$ is the relative weight of item $i$ subject to normalizing the weight of an item in $T$ to 1. The value $c_z$ can be interpreted accordingly. The extended weight inequality defined for $T \cup I \cup \{z\}$ is valid for the knapsack polytope $P_{T\cup I\cup\{z\}}$. This statement is verified next.

**Proposition 4.2.** *The extended weight inequality defined for $T \cup I \cup \{z\}$ introduced in Definition 4.1 is valid for $P_{T\cup I\cup\{z\}}$.*

**Proof.** Let $x \in P_{T\cup I\cup\{z\}}$ be given. If $x_z = 0$, then the inequality is certainly satisfied. Otherwise, $x_z = 1$. As $\sum_{i\in T\cup I\cup\{z\}} a_i x_i \leqslant a_0$ holds, there exists a subset $S \subseteq T \cup I$ with $x_i = 0$ for all $i \in S$ and $a(S) \geqslant a_z - r$. By definition of $c_z$, we have that $|S \cap T| + \sum_{i\in S\cap I} c_i x_i \geqslant c_z$ holds. This proves the claim. □

Extended weight inequalities defined for some subsets $T \cup I \cup \{z\}$ with the above properties include minimal cover and $(1,k)$-configuration inequalities. These are precisely those extended weight inequalities that satisfy:

$I = \emptyset$;

for every subset $S \subseteq T$ of cardinality $|T| - c_z$, the condition $a(S) + a_z \leqslant a_0$ holds.

**Example 4.3.** Consider the knapsack polytope defined as convex hull of all 0/1 vectors that satisfy the constraint

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 + 3x_7 + 4x_8 + 5x_9 + 7x_{10} + 10x_{11} \leqslant 15.$$

For instance, the two inequalities

$$x_5 + x_6 + x_7 + 2x_8 + x_9 + x_{10} + 3x_{11} \leqslant 5, \tag{1}$$

$$x_1 + x_2 + x_4 + x_5 + 2x_8 + x_{10} + 3x_{11} \leqslant 6, \tag{2}$$

define facets of the corresponding polyhedron. Setting $T = \{5,6,7\}$ and $I = \{8\}$, we obtain $c_8 = 2$ and $r = 4$. Then, $c_z = 1$ if we choose $z = 9$. By computing lifting coefficients for the items 10 and 11, we end up with inequality (1). Accordingly, setting $T = \{1,2,4,5\}$ and $I = \{8\}$ yields $c_8 = 2$ and $r = 5$. Then $c_z = 1$ for $z = 10$. After computing lifting coefficients for the remaining items we obtain inequality (2).

The example indicates that many inequalities that define facets of the knapsack polyhedron $P$ can be viewed as lifted extended weight inequalities. In the following we derive properties of the lifting coefficients. For a natural number $1 \leqslant \alpha \leqslant |T| + c(I)$, we define

$$\phi(\alpha) := \min\Big\{|S| + \sum_{j\in J} c_j : S \subseteq T,\ J \subseteq I,\ a(S\cup J) \geqslant \alpha\Big\}.$$

To avoid trivial special cases, we define $\phi$ also for non-positive numbers. We set $\phi(\alpha) = 0$, if $\alpha \leqslant 0$. In the following we show that the lifting coefficient $c_i$ of an item $i$ not contained in $T \cup I$ satisfies the inequalities

$$\phi(a_i - r) - 1 \leqslant c_i \leqslant \phi(a_i - r).$$

Let $T, I$ and $\{z\}$ be given as in Definition 4.1 and let $J$ be a sequence of items in $N \setminus (T \cup I \cup \{z\})$. By $J_{<i}$ we denote the subsequence of $J$ that consists of all items before $i$ in $J$. The lifted extended weight inequality with respect to $T, I$, $\{z\}$ and $J$ is of the form

$$\sum_{i\in T\cup I} c_i x_i + \sum_{i\in\{z\}\cup J} c_i x_i \leqslant |T| + c(I),$$

where $c_i = 1$ for all $i \in T$, $c_i := \min\{|S| : S \subseteq T,\ a(S) \geqslant a_i\}$ if $i \in I$ and $c_z := \min\{|S| + \sum_{j\in J} c_j : S \subseteq T,\ J \subseteq I,\ a(S\cup J) \geqslant a_z - r\}$. For all $i \in N \setminus (T \cup I \cup \{z\})$ the lifting coefficient $c_i$ is defined as

$$c_i := |T| + c(I) - \max\Big\{\sum_{s\in T\cup I\cup\{z\}\cup J_{<i}} c_s x_s : \sum_{s\in T\cup I\cup\{z\}\cup J_{<i}} a_s x_s \leqslant a_0 - a_i\Big\}.$$

**Proposition 4.4.** *The formula $\phi(a_i - r) - 1 \leqslant c_i \leqslant \phi(a_i - r)$ holds for all items $i \in N \setminus (T \cup I)$.*

**Proof.** Let $T, I$ and $\{z\}$ be given as in Definition 4.1 and let $J$ be a sequence of items in $N \setminus (T \cup I \cup \{z\})$. We use induction on the cardinality of $J_{<i}$ to verify the formula. If $|J_{<i}| = 0$, then the formula holds, because $c_z = \phi(a_z - r)$ and the extended weight inequality defined on $T \cup I \cup \{z\}$ is valid (see Proposition 4.2). Now suppose, the formula holds for all items in $J_{<i}$. We verify it for item $i$.

Suppose that $c_i$ is not bounded from above by $\phi(a_i - r)$, i.e., $\phi(a_i - r) < c_i$. Let $S \subseteq T \cup I$ be a subset with $c(S) = \phi(a_i - r)$. Then, $a(S) \geqslant a_i - r$ and the vector $x := \sum_{s\in T\cup I\setminus S} e_s + e_i$ is feasible as $\sum_{s\in T\cup I\setminus S} a_s + a_i \leqslant \sum_{s\in T\cup I\setminus S} a_s + a(S) + r \leqslant a_0$. On the other hand, $cx = \sum_{s\in T\cup I\setminus S} c_s + c_i > \sum_{s\in T\cup I\setminus S} c_s + \phi(a_i - r) = \sum_{s\in T\cup I\setminus S} c_s + c(S) = |T| + c(I)$, a contradiction since the inequality is valid. This yields

$$c_i \leqslant \phi(a_i - r).$$

Next we derive the lower bound. In fact, we prove that the lifted extended weight inequality is valid if we set $c_i := \phi(a_i - r) - 1$. By assumption, the lifted extended weight inequality $\sum_{s\in T\cup I} c_s x_s + \sum_{s\in\{z\}\cup J_{<i}} c_s x_s \leqslant |T| + c(I)$ is valid for the knapsack polytope $P_{T\cup I\cup\{z\}\cup J_{<i}}$. Therefore, every subset $V$ such that $V \subseteq \{z\} \cup J_{<i}$ satisfies $\phi(a(V) - r) \geqslant \sum_{v\in V} c_v$ (note that otherwise, for $S \subseteq T \cup I$, $c(S) = \phi(a(V) - r)$, the vector $x := \sum_{s\in T\cup I\setminus S} e_s + \sum_{s\in V} e_s$ would violate the inequality).

Second, if $x$ is a feasible point in $P$ such that $x_i = 1$ and $x_v = 1$ for $V \subseteq \{z\} \cup J_{<i}$, then there exists a subset $S \subseteq T \cup I$ with $x_s = 0$ for all $s \in S$ and $a(S) \geqslant a(V) + a_i - r$ (because $a(S) \leqslant a_0 = a(T \cup I) + r$). Moreover, by definition of $\phi$, the inequality $\sum_{s\in S} c_s = c(S) \geqslant \phi(a(V) + a_i - r)$ is satisfied. Let $x \in P_{T\cup I\cup\{z\}\cup J_{<i}\cup\{i\}}$ be a feasible point. If $x_i = 0$, then the inequality is valid, by assumption of the induction. Suppose, that $x_i = 1$. Moreover, let $V \subseteq \{z\} \cup J_{<i}$ be the subset with $x_v = 1$. As we outlined above, there exists a subset $S \subseteq T \cup I$ with $x_s = 0$ for all $s \in S$ and $a(S) \geqslant a(V) + a_i - r$. Then,

$$\sum_{s\in S} c_s \geqslant \phi(a(V) + a_i - r) \geqslant \phi(a(V) + a_i - 2r)$$

$$\geqslant \phi(a(V) - r) + \phi(a_i - r) - 1 \geqslant \sum_{v\in V} c_v + c_i,$$

where the third "greater or equal implication" follows from Lemma 4.5. We obtain, $cx \leqslant \sum_{v\in V} c_v + c_i + \sum_{s\in T\cup I\setminus S} c_s \leqslant \sum_{s\in S} c_s + \sum_{s\in T\cup I\setminus S} c_s = |T| + c(I)$. This shows that setting $c_i := \phi(a_i - r) - 1$ yields a valid inequality which implies the lower bound. □

**Lemma 4.5.** *Let subsets $T$ and $I$ be as in Definition 4.1. For all natural numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha + \beta \leqslant a(T \cup I)$, the relation $\phi(\alpha) + \phi(\beta) \leqslant \phi(\alpha + \beta) + 1$ holds.*

**Proof.** Let $S \subseteq T$, $J \subseteq I$ be such that $|S| + c(J) = \phi(\alpha + \beta)$ and $a(S \cup J) \geqslant \alpha + \beta$. Let $J_\alpha$ be a maximal subset of $J$ such that $a(J_\alpha) < \alpha$, and let $J_\beta$ be a maximal subset of $J \setminus J_\alpha$ such that $a(J_\beta) < \beta$. We distinguish five cases.

(i) If $J = J_\alpha \cup J_\beta$, then let $S_\alpha$ be a minimal subset of $S$ such that $a(J_\alpha \cup S_\alpha) \geqslant \alpha$, and let $S_\beta = (S \setminus S_\alpha) \cup \{i\}$ for some $i \in S_\alpha$. Then from the fact that $a(J_\alpha \cup S_\alpha \setminus \{i\}) < \alpha$ and $a(J \cup S) \geqslant \alpha + \beta$ we see that $a(J_\beta \cup S_\beta) > \beta$ which implies that

$$\phi(\alpha) + \phi(\beta) \leqslant |S_\alpha| + c(J_\alpha) + |S_\beta| + c(J_\beta)$$
$$= |S| + 1 + c(J) = \phi(\alpha + \beta) + 1.$$

Otherwise, there exists $j \in J \setminus (J_\alpha \cup J_\beta)$.

(ii) If there exists $k \in J \setminus (J_\alpha \cup J_\beta)$ with $k \neq j$, then both $a(J_\alpha) + a_k \geqslant \alpha$ and $a(J_\beta) + a_j \geqslant \beta$ which implies that

$$\phi(\alpha) + \phi(\beta) \leqslant c(J_\alpha) + c_k + c_j + c(J_\beta) \leqslant c(J) \leqslant \phi(\alpha + \beta).$$

In the following we can assume that $J = J_\alpha \cup J_\beta \cup \{j\}$.

(iii) If $a(J_\alpha \cup S) \geqslant \alpha$, then since $a(J_\beta) + a_j \geqslant \beta$ we have that

$$\phi(\alpha) + \phi(\beta) \leqslant |S| + c(J_\alpha) + c_j + c(J_\beta) = |S| + c(J) = \phi(\alpha+\beta).$$

Finally, $a(J_\alpha \cup S) < \alpha$ holds. Let $S' \subseteq T$ be such that $a(S') \geqslant a_j$ and $|S'| = c_j$. We set $V := S \cap S'$, $Z := S \setminus S'$ and $W := S' \setminus S$ and obtain the following relation:

$$2a(V) + a(Z) + a(W) = a(S) + a(S') \geqslant a(S) + a_j.$$

(iv) If $a(J_\alpha \cup V) \geqslant \alpha$, then define $S_\alpha$ to be a minimal subset of $V$ such that $a(J_\alpha \cup S_\alpha) \geqslant \alpha$ and set $S_\beta = V \cup Z \cup W$. This yields $a(J_\beta \cup S_\beta) \geqslant a(J_\beta) + a_j \geqslant \beta$. Therefore,

$$\begin{aligned}\phi(\alpha) + \phi(\beta) &\leqslant c(J_\beta) + |S_\beta| + c(J_\alpha) + |S_\alpha| \\ &\leqslant c(J_\beta \cup J_\alpha) + |S| + |S'| = \phi(\alpha+\beta).\end{aligned}$$

(v) If $a(J_\alpha \cup V) < \alpha$, then define $S_\alpha$ to be a minimal subset of $S' = V \cup W$ such that $a(J_\alpha \cup S_\alpha) \geqslant \alpha$ and $V \subseteq S_\alpha$. Set $S_\beta = V \cup Z \cup (W \setminus S_\alpha) \cup \{i\}$ where $i \in S_\alpha \cap W$. Then from $a(J_\alpha \cup S_\alpha \setminus i) < \alpha$ and $a(J_\alpha \cup J_\beta \cup S) + a(S') \geqslant a(J_\alpha \cup J_\beta \cup S) + a_j = a(S \cup J) \geqslant \alpha + \beta$ we see that

$$a(J_\beta \cup S_\beta) = a(J_\beta) + a(S_\beta) = a(J_\beta) + a(S) + a(S' \setminus S_\alpha) + a_i > \beta,$$

which implies that

$$\begin{aligned}\phi(\alpha) + \phi(\beta) &\leqslant |S_\alpha| + c(J_\alpha) + |S_\beta| + c(J_\beta) \\ &= |S_\alpha| + |S| + (|S'| - |S_\alpha|) + 1 + c(J) - c_j \\ &= |S| + 1 + c(J) = \phi(\alpha+\beta) + 1. \qquad \square\end{aligned}$$

This section shows that there is a "natural" and systematic way to generalize minimal cover and $(1,k)$-configuration inequalities. In fact, these inequalities are special cases of extended weight inequalities. Extended weight inequalities have one general property: for all items there exist lower and upper bounds for the exact value of the corresponding lifting coefficient that differ by at most 1. In order to compute the exact value one can use Zemel's procedure [19] whose running time is bounded by $O(|N|\pi)$ where $\pi$ is the right hand side of the inequality. For extended weight inequalities, $\pi = |T| + c(I) \leqslant |T| + |T|^2 \leqslant |N|^2$. Therefore, each lifting coefficient can be computed in polynomial time.

## 5. The convex hull of all solutions to $\mathcal{K}_2$

In order to derive the description of the knapsack polyhedron $P^2$, we first need to extend the weight-reduction principle (cf. Section 3). Many of the arguments that we use throughout this section only apply to knapsack problems of the form $\mathcal{K}^2$, because often we will need the property that there are at most two items of weight greater than one that fit together into the knapsack!

We start generalizing weight-reduction inequalities.

As outlined in Section 3, weight-reduction inequalities are essentially weight inequalities with the difference that to the item with biggest weight in the starting set $T$ we assign a coefficient that is its weight decreased by the reduction parameter $\psi$. This change of the coefficient results in a change of the corresponding right hand side as well as a change of coefficients of certain items not contained in $T$. In Section 2, we restricted the set of possible reduction parameters to values in the interval $[0, r]$ where $r = a_0 - a(T)$ is the free capacity of the knapsack after assigning all items in $T$. This restriction is crucial, since the proof of validity of weight-reduction inequalities is strongly based on this assumption. Nevertheless, there exist knapsack problems and "weight-reduction inequalities" with a reduction parameter of value bigger than $r$.

**Example 5.1.** Consider the knapsack inequality in 0/1 variables:

$$\sum_{i=1}^{100} x_i + \sum_{i=101}^{120} ix_i + 120x_{121} + \sum_{i=122}^{151} (i-1)x_i \leqslant 300.$$

The inequality

$$\sum_{i=51}^{100} x_i + \sum_{i=101}^{110} (i-100)x_i + \sum_{i=111}^{115} 10x_i + \sum_{i=116}^{120} (i-105)x_i$$
$$+ \sum_{i=121}^{131} 15x_i + \sum_{i=132}^{136} (i-1-115)x_i + \sum_{i=137}^{141} 20x_i + \sum_{i=142}^{151} (i-1-120)x_i \leqslant 80$$

defines a facet of the corresponding polytope (checked by hand). Moreover, the vector $x$ defined via $x_{151} = 1$ and $x_i = 1$ for all $i = 51, \ldots, 100$ satisfies this inequality as an equation. Hence $r = 100$, but the coefficient of item 151 is 30 which is obviously smaller than $150 - r$. On the other hand, there exist two pairs of items $(s_1, t_1), (s_2, t_2)$, $s_1 = 110, t_1 = 136, s_2 = 120, t_2 = 121$ such that the vectors $x^1 = \sum_{i=51}^{100} e_i + e_{s_1} + e_{t_1}$ and $x^2 = \sum_{i=51}^{100} e_i + e_{s_2} + e_{t_2}$ satisfy the above inequality as an equation. Moreover, summing the weight of the items $s_1$ and $t_1$ gives a weight of 245 which is smaller than the weight of item 151 plus $r$ (= 250). The same is true for the sum of weights of the items $s_2$ and $t_2$ (summing the weights of items $s_2$ and $t_2$ yields a weight of 240 which is by 10 smaller than 250). Therefore, items with weight in the range $[121, 130]$ have a coefficient whose value is equal to $c_{s_2} = c_{t_2}$ and items with weight in the range $[111, 115]$ or $[136, 140]$ have a coefficient with value $c_{s_1}$ or $c_{t_1}$, respectively. In fact, there is a principle used to generate this inequality that we want to explain now.

**Definition 5.2.** Let $T \subseteq N_1$, let $k \in N \setminus N_1$ such that $|T| + a_k < a_0$ and set $r := a_0 - |T| - a_k$. Let $(s_1, t_1), \ldots, (s_m, t_m)$ be pairs of items in $N \setminus (N_1 \cup \{k\})$ such that

$$r < a_{s_1} < a_{s_2} < \cdots < a_{s_m} \leqslant a_{t_m} < a_{t_m - 1} < \cdots < a_{t_1} < a_k$$

and $l_i := a_k + r - a_{s_i} - a_{t_i} > 0$ for $i = 1, \ldots, m$. If these pairs satisfy the additional conditions

$$a_{s_i} + l_i < a_{s_{i+1}}, \quad \text{for } i = 1, \ldots, m-1,$$
$$a_{t_i} + l_i < a_{t_{i-1}}, \quad \text{for } i = m, \ldots, 2,$$
$$a_{t_1} + l_1 < j_0,$$

we define, for every $\mu \in [0, \min\{l_m, a_{t_m} - a_{s_m}\}]$, a *recursive weight-reduction inequality* of the form $cx \leqslant \gamma := |T| + c_k$ based on $(s_1, t_1), \ldots, (s_m, t_m)$, $\mu$, $T$ and $k$ as follows:

We set $l_0 := 0$.
For $i \in N_1$ we set $c_i := 1$ if $i \in T$.
We set $c_i := 0$ if $a_i \leqslant r$ and if $i \notin T$. For $i \in N_j$ with $r + 1 \leqslant j \leqslant a_{s_1}$ we set $c_i := j - r$.
Recursively, we determine for $i \in N_j$, $u = 1, \ldots, m-1$ the coefficient of the corresponding item.
We set $c_i := c_{s_u}$ if $a_{s_u} < j \leqslant a_{s_u} + l_u$ and $c_i := c_{s_u} + (j - a_{s_u} - l_u)$ if $a_{s_u} + l_u < j \leqslant a_{s_{u+1}}$.
For $i \in N_j \setminus \{t_m\}$ we define $c_i := c_{s_m}$ if $a_{s_m} < j \leqslant a_{s_m} + l_m$ and we set $c_i := c_{s_m} + (j - a_{s_m} - l_m)$, if $a_{s_m} + l_m < j \leqslant a_{t_m} + l_m - \mu$.
We define $c_{t_m} := c_{s_m} + (a_{t_m} - a_{s_m} - \mu)$ and we set $c_i := c_{t_m}$ if $i \in N_j$ with $a_{t_m} + l_m - \mu + 1 \leqslant j \leqslant a_{t_m} + l_m$.
For $i \in N_j$ the coefficient is $c_i := c_{t_u} + (j - a_{t_u} - l_u)$ if $a_{t_u} + l_u < j \leqslant a_{t_{u-1}}$ and $c_i := c_{t_{u-1}}$ if $a_{t_{u-1}} < j \leqslant a_{t_{u-1}} + l_{u-1}$ for $u = m, m-1, \ldots, 2$.
$c_i := c_{t_1} + (j - a_{t_1} - l_1)$ if $a_{t_1} + l_1 < j \leqslant a_k$.
We set $c_i := c_k$ if $a_k < j \leqslant a_k + r$ and define $c_i := c_k + (j - a_k - r)$ if $a_k + r < j$.

The proof of the following proposition and the theorem are lengthy and technical. We do not present them in this paper, but refer to the Habilitations-Schrift of the author [17].

**Proposition 5.3.** *The recursive weight-reduction inequality based on $(s_1, t_1), \ldots, (s_m, t_m)$, $\mu$, $T$ and $k$ is valid for $P^2$.*

**Theorem 5.4.** *The set of all 1-weight inequalities with respect to some starting set $T \subseteq N_1$, $|T| < a_0$, the set of all weight-reduction inequalities with respect to some starting set $T \subseteq \bigcup_{j \in B^2} N_j \cup N_1$, $a(T) < a_0$, $|\bigcup_{j \in B^2} N_j \cap T| = 1$ and a reduction parameter $\psi \in [0, r]$, $r = a_0 - a(T)$, the set of all recursive weight-reduction inequalities based on parameters $(s_1, t_1), \ldots, (s_m, t_m)$, $\mu$, $T$ and $k$ as stated in Definition 5.2, the knapsack inequality and the system of inequalities*

$$-x_i \leqslant 0, \quad i \in N, \qquad x_i \leqslant 1, \quad i \in N \setminus N_1,$$
$$\sum_{j \in B^2} \sum_{i \in N_j} x_i \leqslant 2,$$

*describe the polyhedron $P^2$.*

## 6. Conclusions

Our complete description of the 0/1 knapsack polytope when the weight of the items is equal to one or lies in the range of $[\lfloor a_0/3 \rfloor + 1, \lfloor a_0/2 \rfloor]$ seems to be a further step to understand the richness of possibilities according to which knapsack inequalities can be derived. It would be interesting to find extensions to more general cases. For instance, is it possible to generalize the weight-reduction principle so as to describe knapsack polyhedra $P^s$ for values $s \geqslant 3$? To prove a result like this it seems necessary to generalize the principle of reducing by a parameter to the reduction by a vector of parameters.

We have also shown that minimal cover and $(1, k)$-configuration inequalities are a special case of extended weight inequalities. Can one introduce a general reduction principle for extended weight inequalities instead of for weight inequalities? Certainly, one can, but is it true that every facet of the knapsack polyhedron is some extended weight inequality with a possible reduction? It certainly would be interesting to answer this question.

A third direction that requires further investigations is to use weight and extended weight inequalities computationally. This might appear promising since cover inequalities often approximate the weights of the items very roughly as we have seen. Here the main focus is to develop good and fast separation routines for extended weight inequalities. This topic certainly requires further efforts and a lot of testing, but at least the idea of using such inequalities computationally seems to be worth trying.

## Acknowledgements

I sincerely thank Laurence Wolsey for several valuable discussions on the subject of this paper. I am also very indebted to particularly one of the referees for the good comments, the careful reading and the suggestions that helped me to improve the paper. This research was supported in part by, Science Program SC1-CT91-620 of the EEC and contract ERB CHBGCT 920167 of the EEC Human Capital and Mobility Program.

## References

[1] E. Balas, Facets of the knapsack polytope, *Mathematical Programming* 8 (1975) 146–164.

[2] R. Bellman, Some applications of the theory of dynamic programming, *Operations Research* 2 (1954) 275–288.

[3] E. Balas and E. Zemel, Facets of the knapsack polytope from minimal covers, *SIAM Journal on Applied Mathematics* 34 (1978) 119–148.

[4] T. Christof, Ein Verfahren zur Transformation zwischen Polyederdarstellungen, Master Thesis, Universität Augsburg (1991).

[5] H. Crowder, E. L. Johnson and M. W. Padberg, Solving large-scale zero-one linear programming problems, *Operations Research* 31 (5) (1983) 803–834.

[6] C.E. Ferreira, On combinatorial optimization problems arising in computer system design, Ph.D. Dissertation, Technische Universität Berlin (1993).

[7] C.E. Ferreira, A. Martin and R. Weismantel, Solving multiple knapsack problems by cutting planes , *SIAM Journal on Optimization* 6 (3) (1996) 858–877.

[8] C.E. Ferreira, M. Grötschel, S. Kiefl, C. Krispenz, A. Martin and R. Weismantel, Some integer programs arising in the design of mainframe computers, *Zeitschrift für Operations Research* 38 (1993) 77–100.

[9] C.E. Ferreira, A. Martin, C.C. de Souza, R. Weismantel and L.A. Wolsey, Formulations and valid inequalities for the node capacitated graph partitioning problem, *Mathematical Programming* 74 (3) (1996) 247–266.

[10] P.L. Hammer, E.L. Johnson and U.N. Peled, Facets of regular 0–1 polytopes, *Mathematical Programming* 8 (1975) 179–206.

[11] K.L. Hoffman and M. Padberg, Solving airline crew-scheduling problems by branch-and-cut, Working Paper (1992).

[12] E.K. Lee, Facets of special knapsack equality polytopes, Technical Report, Rice University, Houston, TX (1993).

[13] M.W. Padberg, A note on 0–1 programming, *Operations Research* 23 (1975) 833–837.

[14] M.W. Padberg, $(1,k)$-Configurations and facets for packing problems, *Mathematical Programming* 18 (1980) 94–99.

[15] T.J. van Roy and L.A. Wolsey, Solving mixed integer programming problems using automatic reformulation, *Operations Research* 1 (1987) 45–57 .

[16] R. Weismantel, Plazieren von Zellen: Analyse und Lösung eines quadratischen 0/1-Optimierungsproblems, Ph.D. Dissertation, Technische Universität Berlin (1992).

[17] R. Weismantel, Knapsack problems, test sets and polyedra, Habilitations-Schrift, Technische Universität Berlin (1995).

[18] L.A. Wolsey, Facets of linear inequalities in 0–1 variables, *Mathematical Programming* 8 (1975) 165–178.

[19] E. Zemel, Easily computable facets of the knapsack polytope, *Mathematics of Operations Research* 14 (1989) 760–764.