SIAM J. DISC. MATH.
Vol. 8, No. 1, pp. 133–153, February 1995

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SEQUENTIAL AND SIMULTANEOUS LIFTINGS OF MINIMAL COVER INEQUALITIES FOR GENERALIZED UPPER BOUND CONSTRAINED KNAPSACK POLYTOPES *

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Abstract. A family of facets for the GUB (Generalized Upper Bound) constrained knapsack polytope that are obtainable through a lifting procedure is characterized. The sequential lifting procedure developed herein computes lifted coefficients of the variables in each GUB set simultaneously, in contrast with the usual sequential lifting procedure that lifts only one variable at a time. Moreover, this sequential lifting procedure can be implemented in polynomial time of complexity $O(nm)$, where n is the number of variables and $m(\leq n)$ is the number of GUB sets. In addition, a characterization of the facets obtainable through a simultaneous lifting procedure is derived. This characterization enables us to deduce lower and upper bounds on the lifted coefficients. In particular, for the case of the ordinary knapsack polytope, a known lower bound on the coefficients of lifted facets derived from minimal covers was further tightened.

Key words. GUB knapsack polytope, minimal GUB covers, reformulation–linearization technique, lifting, facets

AMS subject classifications. 90C10, 90C27

1. Introduction.

Consider the GUB (Generalized Upper Bound)-constrained, or multiple-choice, knapsack problem defined as follows:

$$\text{(GKP)} \qquad \text{minimize} \left\{ \sum_{j\in N} c_j x_j : \sum_{i\in M}\sum_{j\in N_i} a_j x_j \geq b,\ \sum_{j\in N_i} x_j \leq 1\ \forall\, i \in M,\right.$$
$$\left. x_j \in (0,1)\ \ \forall\, j \in N \right\},$$

where the data are all integers, $N = \{1, ..., n\}$, $M = \{1, ..., m\}$, and where $\cup_{i\in M} N_i \equiv N$, with $N_i \cap N_j = \phi$ for $i,\ j \in M,\ i \neq j$. Johnson and Padberg [7] show that any GUB knapsack problem with arbitrarily signed coefficients b and $a_j,\ j \in N$, can be equivalently transformed into a form with $b > 0$ and with $0 < a_j \leq b\,\forall\, j \in N$, and they relate facets of the transformed problem with those of the original problem. Hence, without loss of generality, we will also assume that $b > 0$ and that $0 < a_j \leq b\,\forall\, j \in N$. Note that if $|N_i| = 1\,\forall\, i \in M$, then problem GKP is the *ordinary* 0-1 *knapsack problem.*

There are many useful applications of model GKP. As suggested in Sinha and Zoltners [13], this model is appropriate for capital budgeting problems having a single resource and where the investment opportunities are divided into disjoint subsets. Balintify et al. [3] identify another application in menu planning for determining what food items should be selected from various daily menu courses to maximize an individual's food preference, subject to a calorie constraint. More importantly, model GKP frequently arises as a subset of large-scale real-world 0-1 integer programming

* Received by the editors May 6, 1992; accepted for publication (in revised form) January 13, 1994. This material is based upon work supported by National Science Foundation grant DDM-9121419 and Air Force Office of Scientific Research 2304/B1.

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problems. As demonstrated in the results of Crowder, Johnson, and Padberg [4] and Hoffman and Padberg [6], even a partial knowledge of the polyhedral structure of ordinary and GUB-constrained knapsack polytopes can significantly enhance the overall performance of branch-and-cut algorithms. Moreover, Martin and Schrage [8] and Hoffman and Padberg [6] present logical implications that can be derived from GUB-constrained knapsack polytopes in the context of coefficient reductions for 0-1 integer programming problems. In the same spirit, model GKP can also be used to generate classes of valid inequalities for certain scheduling polytopes (see Lee and Sherali [12] and Wolsey [14]) to tighten their underlying linear programming relaxations. In this regard, Wolsey [14] defines a "GUB cover" inequality for problem GKP and presents some implementations of GUB cover inequalities for solving machine sequencing problems, generalized assignment problems, and variable-upper-bounded flow problems with GUB constraints. He also shows that for a special case of a GUB-constrained knapsack problem, this class of inequalities is sufficient to describe the entire convex hull. However, we will be concerned in this paper with the polyhedral properties of the convex hull of feasible solutions to problem GKP through an extension of the well-known minimal cover inequalities for the ordinary knapsack polytope. Johnson and Padberg [7] also briefly treat a partial characterization of the facets defining the convex hull of feasible solutions to GKP. In particular, they identify conditions under which facet coefficients would be zero, ordered in magnitude within sets, or have the same magnitude within sets. We will, however, be concerned with the actual generation of facets via a lifting process, and we will provide a complete characterization of facets obtainable in this fashion.

In §2 below, we present a class of valid inequalities for problem GKP obtained by a generalization of the minimal cover inequalities for the ordinary knapsack polytope. We also develop a necessary and sufficient condition for such an inequality to define a facet of a lower dimensional polytope. Subsequently, in §3, we develop a sequential lifting procedure to obtain a family of facets. The sequential lifting procedure developed herein computes lifted coefficients of the variables in each GUB set simultaneously, in contrast with the usual sequential lifting procedure that lifts only one variable at a time. Moreover, we show that this sequential lifting procedure can be implemented in polynomial time of complexity $O(nm)$. In §4, we use the reformulation–linearization technique of Sherali and Adams [11] to easily characterize facets obtainable through a simultaneous lifting procedure. This characterization enables us to derive lower and upper bounds on the lifted coefficients. Finally, in §5, for the special case of the ordinary knapsack polytope, we use this analysis to further tighten a known lower bound on the coefficients of lifted facets derived from minimal covers.

2. Valid inequalities from minimal GUB covers. Denote the constraint set of model GKP as

$$X \equiv \left\{ x \in (0,1)^n : \sum_{i \in M} \sum_{j \in N_i} a_j x_j \geq b, \sum_{j \in N_i} x_j \leq 1 \ \forall \ i \in M \right\}.$$

We start by introducing some notation. For $K \subseteq N$ let $M_K = \{i \in M : j \in N_i$ for some $j \in K\}$. Also, for $k \in N$, we denote $M_{\{k\}}$ simply as M_k. For each $i \in M$, define a *key index* $j(i)$ such that $j(i) \in \arg\max_{j \in N_i}(a_j)$. Similarly, given any $B \subseteq N$, for each $i \in M_B$, define a *key index* $j_B(i)$ such that $j_B(i) \in \arg\max_{j \in N_i \cap B}(a_j)$. For $A \subseteq M$, denote $A_+ = \{j(i) : i \in A\}$. Similarly, for $B \subseteq N$, denote $B_+ = \{j_B(i) : i \in M_B\}$, and let $B_- = B - B_+$.

Let us suppose that for each $k \in N$

$$a_k + \sum_{i \in (M-M_k)} a_{j(i)} \geq b. \tag{1}$$

Otherwise, $x_k = 0$ in every feasible solution to X. Denoting the convex hull operation by conv $(\cdot)$, let GUBKP $\equiv$ conv(X), and let dim(GUBKP) be the dimension of GUBKP, which is the maximum number of affinely independent points in GUBKP minus one.

PROPOSITION 2.1. *dim (GUBKP) $= n - |M_0|$, where $M_0 = \{i \in M : \sum_{p \in (M-i)} a_{j(p)} < b\}$.*

Proof. By the definition of M_0, we must have $\sum_{j \in N_i} x_j = 1$ for each $i \in M_0$. Hence, it follows that dim(GUBKP) $\leq n - |M_0|$. To prove that dim(GUBKP) $= n - |M_0|$, it suffices to show that there exist $n - |M_0| + 1$ affinely independent points in GUBKP.

For each $i \in (M - M_0)$, we construct a set of feasible points in GUBKP as follows. For each $k \in N_i$, construct $x^{(k,i)} \equiv \{x_k = 1,\ x_j = 1 \text{ for } j = j(p),\ \forall\ p \in (M-i),\ x_j = 0 \text{ otherwise}\}$, and let $x^{(0,i)} \equiv \{x_j = 1 \text{ for } j = j(p),\ \forall\, p \in (M-i),\ x_j = 0 \text{ otherwise}\}$. Similarly, for each $i \in M_0$, we construct a set of feasible points in GUBKP as follows. For each $k \in N_i$, construct $x^{(k,i)} \equiv \{x_k = 1,\ x_j = 1 \text{ for } j = j(p),\ \forall\ p \in M - i,\ x_j = 0 \text{ otherwise}\}$. Then, the total number of distinct feasible points thus constructed is $n - |M_0| + 1$. Let $\widetilde{X}$ be the set of these distinct points x^j, indexed by $j = 1, \ldots, n - |M_0| + 1$. Without loss of generality, let $x^{n-|M_0|+1} \equiv \{x_j = 1 \text{ for } j \in N_+,\ x_j = 0 \text{ otherwise}\}$. Construct a matrix D whose row vectors are $x^j - x^{n-|M_0|+1}$, $j = 1, \ldots, n - |M_0|$. Then the matrix D can be readily seen to possess a block-diagonal structure, with the rows corresponding to each block being linearly independent. Hence, x^j, $j = 1, \ldots, n - |M_0| + 1$, are affinely independent. This completes the proof. $\square$

Now, if dim(GUBKP) is less than n, then by writing each inequality constraint $i \in M_0$ as an equality constraint and using this equation to eliminate the variable that has the smallest a_j coefficient, we get a full-dimensional subpolytope of dimension of $n - |M_0|$. Hence, without loss of generality, we can assume henceforth that GUBKP is a full-dimensional polytope. The following results are readily evident.

COROLLARY 2.2. *For a given $H \subseteq N$, define $X(H) = X \cap \{x \in (0,1)^n : x_{j(i)} = 1,\ \forall\ i \in M_H\}$. Then, for any $A \subseteq M$, dim(conv$(X(\cup_{i \in A} N_i))) = |\cup_{i \in (M-A)} N_i|$.*

PROPOSITION 2.3. *For each $j \in N_-$, the inequality $x_j \geq 0$ is a facet of GUBKP.*

PROPOSITION 2.4. *The GUB constraints $\sum_{j \in N_i} x_j \leq 1$, $i \in M$, are facets of GUBKP.*

Balas [1] discusses the well-known classes of minimal cover inequalities for the ordinary knapsack polytope, along with its tightened variants, namely the extended and the strong minimal cover inequalities. We generalize these definitions and derive related results for the GUB-constrained knapsack polytope below. In addition, an alternate tightened variant that is peculiar to the GUB-constrained situation is derived. Note that these inequalities are different from the valid inequalities treated by Wolsey [14].

We will say that a set $K = \cup_{i \in Q} N_i$, for some $Q \subseteq M$, is called a *GUB cover* of X, if $\sum_{i \in M_{\overline{K}}} a_{j(i)} \leq b - 1$, where $\overline{K} \equiv N - K$. A GUB cover K is called a *minimal GUB cover* of X if $\sum_{i \in M_{\overline{K}}} a_{j(i)} + \min_{i \in M_K}(a_{j(i)}) \geq b$. Accordingly, we define a *minimal*

GUB cover inequality as the valid inequality

$$\sum_{j \in K} x_j \geq 1. \tag{2}$$

For a minimal GUB cover K, we define $R = \{j \in \overline{K} : a_j \geq \max_{j \in K}(a_j)\}$. An *extension of the minimal GUB cover* K of X, denoted by $E(K)$, is defined as $E(K) = K \cup S$, where $S = \cup_{i \in M_R} N_i$.

PROPOSITION 2.5. *If K is a minimal GUB cover of X, then the inequality defined as*

$$\sum_{j \in E(K)} x_j \geq 1 + |M_R| \tag{3}$$

is a valid inequality for GUBKP. Moreover, this inequality dominates the minimal GUB cover inequality if $R \neq \phi$.

Proof. The validity of (3) follows by verifying that if a binary $\overline{x}$ satisfies the GUB constraints and is such that $\sum_{j \in E(K)} \overline{x}_j \leq |M_R|$, then $\overline{x} \notin X$. The dominance statement is evident by rewriting (3) as $\sum_{j \in K} x_j \geq 1 + \sum_{i \in M_R}(1 - \sum_{j \in N_i} x_j)$. This completes the proof. □

The idea of using strong minimal covers to generate nondominated extensions as for the case of the ordinary knapsack problem (see Balas [1]) can be readily extended to the GUB-constrained situation as follows. We will call a minimal GUB cover K *strong* if either $E(K) = N$, or else no set of the form $S = \cup_{i \in (M_K - M_{j_1}) \cup \{p\}} N_i$ for any $p \in M_{\overline{E(K)}}$ is a cover, where $j_1 \in \arg\max_{j \in K}(a_j)$. That is, a set $K \subseteq N$ is a *strong minimal GUB cover* of X if K is a minimal GUB cover for which, if $E(K) \neq N$, then $a_{j_1} + \sum_{i \in (M_{\overline{E(K)}} - p)} a_{j(i)} \geq b \;\; \forall\, p \in M_{\overline{E(K)}}$. Hence, a minimal GUB cover K is strong if there exists no minimal GUB cover of the same size as K whose extension strictly contains that of K.

We now consider another strengthening procedure for the minimal GUB cover inequality.

PROPOSITION 2.6. *If K is a minimal GUB cover and (K_1, K_2) is a partition of K with $K_2 \neq \phi$ such that*

$$\sum_{i \in M_{K_2}} \max_{j \in N_i \cap K_2}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} < b,$$

then the inequality

$$\sum_{j \in K_1} x_j \geq 1 \tag{4}$$

is valid for GUBKP and dominates the minimal GUB cover inequality $\sum_{j \in K} x_j \geq 1$. Moreover, if $\min_{j \in K_1}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} \geq b$, then the inequality (4) is a facet of $\operatorname{conv}(X(K_2, \overline{K}))$ *where $X(K_2, \overline{K}) = X \cap \{x \in (0,1)^n : x_j = 0 \;\; \forall\, j \in K_2,\; x_{j(i)} = 1$ $\forall\, i \in M_{\overline{K}}\}$.*

Proof. It is readily shown that the inequality (4) is valid for GUBKP. Since $\min_{j \in K_1}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} \geq b$, the unit vectors e_j, for $j \in K_1$, are feasible to

$\text{conv}(X(K_2, \overline{K}))$, and moreover, they satisfy (4) as an equality and are linearly independent. Hence, the inequality (4) is a facet of $\text{conv}(X(K_2, \overline{K}))$. This completes the proof. □

Furthermore, if no such partition with $K_2 \neq \phi$ exists, i.e., if $\min_{j \in K}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} \geq b$, then we have the following result.

PROPOSITION 2.7. *For a minimal GUB cover K, the minimal GUB cover inequality is a facet of* $\text{conv}(X(\overline{K}))$ *if and only if* $\min_{j \in K}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} \geq b$.

Proof. The proof follows easily by examining the $|K|$ linearly independent unit vectors e_j, one for each $j \in K$, which belong to $\text{conv}(X(\overline{K}))$, and which satisfy the minimal GUB cover inequality as an equality. □

Moreover, canonical facets of GUBKP are related to inequalities (4) as follows.

PROPOSITION 2.8. *If for some $H \subseteq N$, the inequality $\sum_{j \in H} x_j \geq 1$ is a facet of GUBKP, then $K = \cup_{i \in M_H} N_i \supseteq H$ is a GUB cover such that within K, H is a minimal set satisfying $\sum_{i \in M_H} \max_{j \in (N_i - H)}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} < b$.*

Proof. Since $\sum_{j \in H} x_j \geq 1$ is valid for GUBKP, we have that $\sum_{j \in \overline{H}_+} a_j \leq b - 1$, which can be restated as $\sum_{i \in (M - M_H)} a_{j(i)} + \sum_{i \in M_H} \max_{j \in (N_i - H)}(a_j) \leq b - 1$. Since $\sum_{i \in (M - M_H)} a_{j(i)} \leq b - 1$, $K = \cup_{i \in M_H} N_i$ is a GUB cover set that contains H and satisfies the condition of the proposition. If H is not minimal, then there exists an $s \in H$ such that $H' = H - s$ satisfies the condition of the proposition. Then, by Proposition 2.6, $\sum_{j \in H'} x_j \geq 1$ is valid for GUBKP. Since $\sum_{j \in H'} x_j \geq 1$ strictly dominates $\sum_{j \in H} x_j \geq 1$, we have a contradiction, and this completes the proof. □

Example 2.1. Consider the following example, where $X \equiv \{x \in (0,1)^8 : x_1 + 5x_2 + x_3 + 5x_4 + x_5 + 3x_6 + x_7 + 3x_8 \geq 9,\ x_1 + x_2 \leq 1,\ x_3 + x_4 \leq 1,\ x_5 + x_6 \leq 1,\ x_7 + x_8 \leq 1\}$.

Since for all $i \in M$, $\sum_{p \in (M-i)} a_{j(p)} \geq 9$, the convex hull of X is a full-dimensional polytope by Proposition 2.1. A GUB cover is given by $K = \{1,2,3,4,5,6\}$, where $K_+ = \{2,4,6\}$ and $K_- = \{1,3,5\}$. A minimal GUB cover is given by $K = \{1,2,3,4\}$, and a minimal GUB cover inequality is $x_1 + x_2 + x_3 + x_4 \geq 1$. This minimal GUB cover does not admit any extension. However, consider a partition of K such that $K_1 = \{2,4\}$ and $K_2 = \{1,3\}$. Then a strengthened minimal GUB cover inequality of the form (4) is $x_2 + x_4 \geq 1$, which is a facet of $\text{conv}(X(K_2, \overline{K}))$, where $X(K_2, \overline{K}) \equiv X \cap \{x \in (0,1)^8 : x_1 = x_3 = 0,\ x_6 = x_8 = 1\}$. Note that the foregoing minimal cover is not strong, since an extension of the minimal GUB cover $K = \{3,4,5,6\}$ is $E(K) = \{1,2,3,4,5,6\} \supset \{1,2,3,4\}$, and the extended inequality of the type (3) is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 2$. This minimal GUB cover $K = \{3,4,5,6\}$ can be verified to be strong. Moreover, since $\min_{j \in K}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} = 9 = b$, by Proposition 2.7, the corresponding minimal GUB cover inequality $x_3 + x_4 + x_5 + x_6 \geq 1$ is a facet of $\text{conv}(X(\overline{K})) \equiv X \cap \{x \in (0,1)^8 : x_2 = 1,\ x_8 = 1\}$.

Remark 2.1. In concluding this section, we remark that within the context of 0-1 programming problems that contain GUB constraints, given a fractional solution to the continuous relaxation, one can set up a separation problem using individual problem constraints along with (a subset of) the GUB constraints to possibly generate a minimal GUB cover that deletes this fractional solution. Such an approach would be similar to that used by Crowder, Johnson, and Padberg [4] (also, see Hoffman and Padberg [6]), except that this separation problem would be a GUB-constrained knapsack problem. Note that the transformation suggested by Johnson and Padberg [7] can be used to put this GUB-constrained knapsack problem in the standard form considered herein. Having generated such a minimal cover, this can either be tightened using the foregoing discussion, or it can be possibly lifted into a facet of GUBKP as discussed in §3 below.

3. Sequentially lifted facets from minimal GUB covers. We now consider a polynomial-time strengthening procedure that sequentially lifts a given minimal GUB cover inequality. For the case of the ordinary knapsack polytope, Balas and Zemel [15] exhibit that the sequential lifting procedure of Padberg [10], which lifts one variable at a time, obtains a facet when applied to a minimal cover inequality that is a facet of a certain lower-dimensional polytope. However, as we shall show, in the presence of GUB constraints, we need to lift all the variables in N_p for each GUB constraint $p \in M_{\overline{K}}$ *simultaneously*, where K is a minimal GUB cover of X, to obtain a lifted inequality. In particular, if the condition of Proposition 2.7 holds, then we show that the resulting inequality is a facet of GUBKP. Moreover, in the spirit of the procedure proposed by Zemel [16], we show that this type of a sequential-simultaneous lifting can also be conducted in polynomial time.

Let us begin our analysis by defining some notation. For a given $p \in M_{\overline{K}}$, define η^t, for $t \equiv j(p)$, as

$$\begin{aligned}\eta^t &\equiv \min\left\{\sum_{j\in K} x_j \;:\; x \in X,\; x_j = 0 \;\;\forall\, j \in N_p\right\}\\ &\equiv \min\left\{\sum_{i\in M_K} x_{j(i)} \;:\; \sum_{i\in M_K} a_{j(i)}x_{j(i)} \ge b - \sum_{i\in(M_{\overline{K}}\,-\,p)} a_{j(i)},\right.\\ &\qquad\qquad \left. x_{j(i)} \in (0,1)\;\forall\, i \in M_K\right\},\end{aligned} \tag{5}$$

and define ζ^s, for each $s \in (N_p - j(p)) \equiv (N_p - t)$, as

$$\begin{aligned}\zeta^s &\equiv \min\left\{\sum_{j\in K} x_j : x \in X,\; x_s = 1\right\}\\ &\equiv \min\left\{\sum_{i\in M_K} x_{j(i)} : \sum_{i\in M_K} a_{j(i)}x_{j(i)} \ge b - a_s - \sum_{i\in(M_{\overline{K}}\,-\,p)} a_{j(i)},\right.\\ &\qquad\qquad \left. x_{j(i)} \in (0,1)\;\forall\, i \in M_K\right\}.\end{aligned} \tag{6}$$

Since K is a minimal GUB cover of X and GUBKP is full-dimensional and by condition (1), note that $1 \le \eta^t \le |M_K|$ and $1 \le \zeta^s \le \eta^t \;\;\forall\, s \in (N_p - t)$.

PROPOSITION 3.1. (i) *For a given minimal GUB cover K of X, the following inequality, defined for any particular $p \in M_{\overline{K}}$ as*

$$\sum_{j\in K} x_j + \sum_{s\in(N_p\,-\,t)} \alpha_s x_s + \alpha_t x_t \ge 1 + \alpha_t, \tag{7}$$

where $t \equiv j(p)$, is a valid inequality for GUBKP for any $\alpha_t \le \eta^t - 1$ and for any $\alpha_s \ge -\zeta^s + \alpha_t + 1 \;\forall\, s \in (N_p - t)$.

(ii) *Moreover, if* $\alpha_t = \eta^t - 1$, $\alpha_s = \eta^t - \zeta^s \; \forall \; s \in (N_p - t)$ *and* $\min_{j\in K}(a_j) + \sum_{i\in M_{\overline{K}}} a_{j(i)} \geq b$, *then the inequality* (7) *is a facet of* $\mathrm{conv}(X(\overline{K} - N_p))$.

Proof. (i) Since the minimal GUB cover inequality is valid for GUBKP and (7) coincides with the minimal GUB inequality when $x_t = 1$ and $x_s = 0 \, \forall \, s \in (N_p - t)$, it is sufficient to show for establishing the validity of (7) that if $\bar{x} \in X$ and has $\bar{x}_s = 1$ for any $s \in (N_p - t)$, or if $\tilde{x} \in X$ and has $\tilde{x}_j = 0 \;\; \forall \; j \in N_p$, then such $\bar{x}$ and $\tilde{x}$ satisfy (7). That is, $\sum_{j\in K} \bar{x}_j + \alpha_s \geq 1 + \alpha_t$, and $\sum_{j\in K} \tilde{x}_j \geq 1 + \alpha_t$. By the definition of the quantities α_t and α_s, in the first case, we have, using (6), that $\sum_{j\in K} \bar{x}_j + \alpha_s \geq \zeta^s - \zeta^s + \alpha_t + 1 = 1 + \alpha_t$, and in the second case, we have, using (5), that $\sum_{j\in K} \tilde{x}_j \geq \eta^t \geq 1 + \alpha_t$. Hence, the inequality (7) is valid for GUBKP.

(ii) Since the minimal GUB cover inequality is a facet of $\mathrm{conv}(X(\overline{K}))$ by Proposition 2.7, and since $\mathrm{conv}(X(\overline{K}))$ is of full dimension $|K|$ by Corollary 2.2, there exist $|K|$ linearly independent vertices of $\mathrm{conv}(X(\overline{K}))$, indexed by x^j, $j = 1, \ldots, |K|$, that satisfy the minimal GUB cover inequality as an equality. Since $x^j_{j(i)} = 1$, for $i \in M_{\overline{K}}$, $j = 1, \ldots, |K|$, these vertices also satisfy the inequality (7) as an equality and moreover, these vertices belong to $X(\overline{K} - N_p)$. Now, for each $s \in (N_p \; - \; t)$, let $\hat{x}^s$ be a solution of (6) such that $\zeta^s = \sum_{j\in K} \hat{x}^s_j$. Note that $\hat{x}^s \in X(\overline{K} - N_p)$. Also, for $t = j(p)$, let $\hat{x}^t$ be a solution of (5) such that $\eta^t = \sum_{j\in K} \hat{x}^t_j$, and note that $\hat{x}^t \in X(\overline{K} - N_p)$. Moreover, $\hat{x}^s$ and $\hat{x}^t$ satisfy the inequality (7) as an equality when $\alpha_s = \eta^t - \zeta^s$, and $\alpha_t = \eta^t - 1$.

By Corollary 2.2, we have that $\dim(\mathrm{conv}(X(\overline{K} - N_p))) = |K \cup N_p|$. Let $\widetilde{X} \equiv \{(x^j, 1) \text{ for } j = 1, \ldots, |K|, (\hat{x}^t, 1), (\hat{x}^s, 1) \text{ for } s \in (N_p - t)\}$ be the set of vectors obtained by adding a new component having value 1 to each vector $x^j, \hat{x}^t$, and $\hat{x}^s$ in the collection as shown. Let us show that the vectors in $\widetilde{X}$ are linearly independent. On the contrary, suppose that these vectors are linearly dependent. Then there exists a set of multipliers $\{(\lambda_j \text{ for } j = 1, ..., |K|), \delta_t, (\mu_s \text{ for } s \in (N_p - t))\} \neq 0$ such that

$$\text{(8)} \qquad \sum_{j=1}^{|K|} \lambda_j x^j + \delta_t \hat{x}^t + \sum_{s\in(N_p-t)} \mu_s \hat{x}^s = 0 \text{ and } \sum_{j=1}^{|K|} \lambda_j + \delta_t + \sum_{s\in(N_p-t)} \mu_s = 0.$$

Since for each $s \in (N_p - t)$, $\hat{x}^t_s = 0$, $x^j_s = 0$ for $j = 1, ..., |K|$, and $\hat{x}^s_s = 1$, it follows that $\mu_s = 0 \;\; \forall \; s \in (N_p - t)$. Now, if $\delta_t = 0$, then by the linear independence of x^j, $j = 1, \ldots, |K|$, we would have $\lambda_j = 0, \;\; \forall \;\; j = 1, \ldots, |K|$, a contradiction. Hence, without loss of generality, suppose that $\delta_t = -1$, so that (8) becomes $\hat{x}^t = \sum_{j=1}^{|K|} \lambda_j x^j$ and $\sum_{j=1}^{|K|} \lambda_j = 1$. Now, since $\hat{x}^t_t = 0$, $x^j_t = 1$ for $j = 1, \ldots, |K|$, it follows that $\sum_{j=1}^{|K|} \lambda_j = 0$, which is a contradiction. Hence, the vectors $\{(x^j, \text{ for } j = 1, \ldots, |K|), \hat{x}^t, (\hat{x}^s \text{ for } s \in (N_p - t))\}$ are affinely independent. This completes the proof. $\square$

PROPOSITION 3.2. *Let* K *be a minimal GUB cover such that the corresponding minimal GUB cover inequality gives a facet of* $\mathrm{conv}(X(\overline{K}))$. *Let* $M_{\overline{K}} \equiv \{i_1, \ldots, i_k\}$ *be arbitrarily ordered, where* $k \equiv |M_{\overline{K}}|$. *Let* $M(q) = \{i_1, \ldots, i_q\} \subseteq M_{\overline{K}}$, *and let* $N(q) = \cup_{i\in M(q)} N_i$ *for* $q = 1, \ldots, k$. *For* $q = 0$, *let* $M(q) = N(q) = \phi$. *Let* $q \in \{0, \ldots, k-1\}$ *and suppose that* $\sum_{j\in K\cup N(q)} \alpha_j x_j \geq \alpha_0$ *is valid for GUBKP and is a facet of* $\mathrm{conv}(X(\overline{K} - N(q)))$. *Consider* i_{q+1}. *Denote* $t = j(i_{q+1})$, *and compute*

$$\text{(9)} \qquad \eta^t(q) = \min \left\{ \sum_{j\in K\cup N(q)} \alpha_j x_j : x \in X, \; x_j = 0 \; \forall \; j \in N_{i_{q+1}} \right\}.$$

Also, for each $s \in N(i_{q+1}) - t$, compute

$$\zeta^s(q) = \min\left\{ \sum_{j \in K \cup N(q)} \alpha_j x_j : x \in X,\ x_s = 1 \right\}. \tag{10}$$

Then

$$\sum_{j \in K \cup N(q)} \alpha_j x_j + \sum_{s \in (N_{i_{q+1}} - t)} (\eta^t(q) - \zeta^s(q)) x_s + (\eta^t(q) - \alpha_0) x_t \geq \eta^t(q) \tag{11}$$

is (i) *valid for GUBKP, and* (ii) *is a facet of* $\operatorname{conv}(X(\overline{K} - N(q+1)))$.

The proof of Proposition 3.2 follows the same argument used in the proof of Proposition 3.1. Note that if $q = 0$, then $\eta^t(q) = \eta^t$ and $\zeta^s(q) = \zeta^s$, as given by (5) and (6), respectively. This proposition establishes an inductive sequential procedure for generating facets for GUBKP from minimal GUB cover inequalities in the spirit of Balas and Zemel [2] for the ordinary knapsack polytope, except that a simultaneous lifting of the variables within each GUB constraint needs to be conducted in this case, as mentioned earlier.

Now, examining (7) and (11), note that a sequentially lifted inequality obtained from a minimal GUB cover inequality is of the form

$$\sum_{j \in K} x_j + \sum_{j \in \overline{K}_-} \alpha_j x_j - \sum_{j \in \overline{K}_+} \alpha_j (1 - x_j) \geq 1,$$

or

$$\sum_{j \in K} x_j + \sum_{j \in \overline{K}_-} \alpha_j x_j + \sum_{j \in \overline{K}_+} \alpha_j x_j \geq 1 + \sum_{j \in \overline{K}_+} \alpha_j. \tag{12}$$

The derivation of the coefficients α_j of this lifted inequality requires the solution of a sequence of GUB-constrained knapsack problems. Furthermore, the values of the coefficients depend on the sequence in which the indices $i \in M_{\overline{K}}$ are considered. For each $i \in M_{\overline{K}}$, let $\alpha_j{}'$ denote the value of α_j for $j \in N_i$ when $i = i_1$, i.e., when i is taken to be the first index in $M_{\overline{K}}$. In other words, $\alpha_t{}' = \eta^t - 1$ for $t = j(i)$, and $\alpha_s{}' = \eta^t - \zeta^s \ \ \forall\ s \in (N_i - t)$. The subproblem that determines these initial coefficients has a simpler structure than the subsequent GUB-constrained knapsack problems that have to be solved to find the other coefficients of the sequence, and because of this structure, the values of η^t and ζ^s can be easily obtained, as shown in the following propositions. (Some of the proofs are obvious, and are hence omitted.)

PROPOSITION 3.3. *Let K_h be the index set of the h largest $a_{j(i)}$ for $i \in M_K$. For a minimal GUB cover K and for all $t \in \overline{K}_+$, we have by* (5) *that $\eta^t = h$, where h is defined by*

$$\sum_{k \in K_h} a_k \geq b - \sum_{i \in (M_{\overline{K}} - M_t)} a_{j(i)} > \sum_{k \in K_{h-1}} a_k.$$

PROPOSITION 3.4. *Let K_h be the index set of the h largest $a_{j(i)}$, for $i \in M_K$. For a minimal GUB cover K and for any $s \in \overline{K}_-$, we have $\zeta^s = h$ in* (6), *where h is defined by*

$$\sum_{k \in K_h} a_k \geq b - a_s - \sum_{i \in (M_{\overline{K}} - M_s)} a_{j(i)} > \sum_{k \in K_{h-1}} a_k.$$

Note that for a given $p \in M_{\overline{K}}$, $\zeta^{j1} \geq \zeta^{j2}$ whenever $a_{j1} \leq a_{j2}$, for $j1,\ j2 \in N_p \cap \overline{K}_-$.

COROLLARY 3.5. *For a given* $p \in M_{\overline{K}}$, *if* $\eta^t = 1$, *then* $\zeta^s = 1 \quad \forall\ s \in (N_p - t)$, *where* $t \equiv j(p)$.

Proof. Follows from the fact that $1 \leq \zeta^s \leq \eta^t \quad \forall\ s \in N_p - t$. □

COROLLARY 3.6. *Let* $t_1 \in \arg\max_{i \in M_{\overline{K}}}(a_{j(i)})$. *If* $\eta^{t_1} = 1$, *then* $\zeta^s = 1 \quad \forall\ s \in (N_i - j(i))$, $i \in M_{\overline{K}}$, *and* $\eta^{j(i)} = 1 \quad \forall\ i \in M_{\overline{K}}$.

Proof. From (5), it follows that $1 \leq \eta^{j(i)} \leq \eta^{t_1} \quad \forall\ i \in M_{\overline{K}}$. Using this fact along with Corollary 3.5 establishes the required result. □

The above results suggest a sufficient condition under which a minimal GUB cover inequality would be a facet of GUBKP.

PROPOSITION 3.7. *If the minimal GUB cover inequality is a facet of* $\operatorname{conv}(X(\overline{K}))$ *(see Proposition* 2.7), *and if* $\max_{j \in K}(a_j) + \sum_{i \in M_{\overline{K}} - M_{t_1}} a_{j(i)} \geq b$, *where* $t_1 \in \arg\max_{i \in M_{\overline{K}}}(a_{j(i)})$, *then the minimal GUB cover inequality is a facet of GUBKP.*

Proof. Since $\max_{j \in K}(a_j) + \sum_{i \in (M_{\overline{K}} - M_{t_1})} a_{j(i)} \geq b$, we have that $\eta^{t_1} = 1$. By Corollary 3.6, all the coefficients in the lifted inequality of the form (7) are zeros. Examining (9), (10), and (11), we continue to obtain zeros for the lifted coefficients in Proposition 3.1, and so the minimal GUB cover inequality is a facet of GUBKP. □

We now show that the readily obtained coefficients α'_j, $j \in \overline{K}$, provide bounds on the coefficients β_j, $j \in \overline{K}$, of arbitrary valid inequalities (not necessarily sequentially lifted) that have unit coefficients for all $j \in K$.

PROPOSITION 3.8. *If* $\sum_{j \in K} x_j + \sum_{j \in \overline{K}_-} \beta_j x_j - \sum_{j \in \overline{K}_+} \beta_j (1 - x_j) \geq 1$ *is valid for GUBKP, then* $\beta_j \leq \alpha_j' \quad \forall\ j \in \overline{K}_+$, *and* $\beta_j \geq \alpha'_j - (\alpha'_t - \beta_t)\ \forall\ j \in \overline{K}_-$, *where* $t = j(p)$, $p \in M_{\overline{K}}$, *such that* $j \in N_p$.

Proof. Assume that for some $t \in \overline{K}_+$, $\beta_t > \alpha_t'$. From Proposition 3.1, we have that $\alpha_t' = \eta^t - 1$. Let $\bar{x}$ be an optimal solution for (5) such that $\eta^t = \sum_{j \in K} \bar{x}_j$. Then $\overline{x} \in X$, but

$$\sum_{j \in K} \bar{x}_j + \sum_{j \in \overline{K}_-} \beta_j \bar{x}_j - \sum_{j \in \overline{K}_+} \beta_j (1 - \bar{x}_j) = \eta^t - \beta_t = \alpha_t' + 1 - \beta_t < 1,$$

and so, the inequality in the proposition is violated. Hence, $\beta_t \leq \alpha_t' \ \forall\ t \in \overline{K}_+$. For the case of β_j, $j \in \overline{K}_-$, suppose that for some $p \in M_{\overline{K}}$, and $s \in N_p$, we have $\beta_s < \alpha'_s - (\alpha'_t - \beta_t)$ where $t \equiv j(p) \neq s$. Consider problem (6), and let $\overline{x} \in X$ solve this problem. Then, since $\alpha'_s - \alpha'_t = 1 - \zeta^s$, we have that

$$\sum_{j \in K} \bar{x}_j + \sum_{j \in \overline{K}_-} \beta_j \bar{x}_j - \sum_{j \in \overline{K}_+} \beta_j (1 - \bar{x}_j) = \zeta^s + \beta_s - \beta_t < \zeta^s + \alpha'_s - \alpha'_t = 1,$$

and so the inequality is again violated. This completes the proof. □

We now consider the task of efficiently computing the coefficients α_j for $j \in \overline{K}$, of a sequentially lifted facet (12). As mentioned earlier, the task of computing α_j for $j \in \overline{K}$ involves the solution of a set of 0-1 GUB-constrained knapsack problems. However, because of the special structure of these problems, we can easily obtain the coefficients α_j for all $j \in \overline{K}$ within a time complexity of $O(n|M_K|)$ by adapting the procedure due to Zemel [16] that was proposed for the ordinary knapsack problem. Toward this end, consider the following propositions. (The proofs are straightforward and are hence omitted.)

PROPOSITION 3.9. *Suppose that $\overline{x}$ is an optimal solution to the following problem with $z = \overline{z}$, and with all data integer valued:*

$$w(z) = \max\left\{\sum_{j\in N} a_j x_j : \sum_{j\in N} c_j x_j \le z, \sum_{j\in N_i} x_j \le 1 \ \forall\ i \in M,\ x_j \in (0,1)\ \forall\ j \in N\right\}.$$

Then, $\overline{x}$ is an optimal solution to the following GUB-constrained knapsack problem

$$P(b) : v(b) = \min\left\{\sum_{j\in N} c_j x_j : \sum_{j\in N} a_j x_j \ge b, \sum_{j\in N_i} x_j \le 1 \ \forall\ i \in M,\ x_j \in (0,1)\ \forall\ j \in N\right\}$$

for all b satisfying $w(\overline{z}-1) < b \le w(\overline{z})$. □

PROPOSITION 3.10. *Let the function $w(z)$ and the problem $P(b)$ be as defined in Proposition* 3.9, *for any integers z and b. Then, $v(b) = \min\{z : w(z) \ge b\}$.*

Now, examining (12), suppose that we have a (partially) lifted inequality of the form

$$\sum_{j\in K} x_j + \sum_{j\in T_-} \alpha_j x_j - \sum_{j\in T_+} \alpha_j (1 - x_j) \ge 1. \tag{13}$$

We want to find a lifting of (13) with respect to the variables x_j, $\forall\ j \in N_p$, for some $p \in (M_{\overline{K}} - M_T)$. By Proposition 3.2 and inequality (12), we have that for $t = j(p)$ and $\forall\ s \in (N_p - t)$,

$$1 + \alpha_t = \min\left\{\sum_{j\in K} x_j + \sum_{j\in T_-} \alpha_j x_j - \sum_{j\in T_+} \alpha_j (1 - x_j) :\right.$$
$$\sum_{j\in K\cup T} a_j x_j \ge b - \sum_{i\in(M_{\overline{K}} - M_T - p)} a_{j(i)},$$
$$\left.\sum_{j\in N_i} x_j \le 1 \ \forall\ i \in M,\ x_j \in (0,1)\ \forall\ j \in N\right\},$$

and

$$1 + \alpha_t - \alpha_s = \min\left\{\sum_{j\in K} x_j + \sum_{j\in T_-} \alpha_j x_j - \sum_{j\in T_+} \alpha_j (1 - x_j) :\right.$$
$$\sum_{j\in K\cup T} a_j x_j \ge b - a_s - \sum_{i\in(M_{\overline{K}} - M_T - p)} a_{j(i)},$$
$$\left.\sum_{j\in N_i} x_j \le 1 \ \forall\ i \in M,\ x_j \in (0,1)\ \forall\ j \in N\right\}.$$

Let $L \equiv K \cup T$, $\overline{L} \equiv N - L$, and define

$$w_L(z) = \max\left\{\sum_{j\in L} a_j x_j : \sum_{j\in K} x_j + \sum_{j\in T_-} \alpha_j x_j - \sum_{j\in T_+} \alpha_j (1 - x_j) \le z,\right.$$
$$\left.\sum_{j\in N_i} x_j \le 1 \ \forall\ i \in M,\ x_j \in (0,1)\ \forall\ j \in N\right\}. \tag{14}$$

By Proposition 3.10, we have that

$$1+\alpha_t = \min\left\{ z : w_L(z) \geq b - \sum_{i\in(M_{\overline{L}}\ -\ p)} a_{j(i)} \right\},$$

and

$$1+\alpha_t-\alpha_s = \min\left\{ z : w_L(z) \geq b - a_s - \sum_{i\in(M_{\overline{L}}\ -\ p)} a_{j(i)} \right\}.$$

Hence, we can efficiently obtain the coefficients α_j for $j \in N_p$, by computing $w_L(z)$ efficiently for different pertinent values of z. Toward this end, consider the following recursive equation for computing the function $w_L(z)$. Note that

$$w_{L\cup N_p}(z) = \max\{ \max_{s\in(N_p-t)} [a_s + w_L(z+\alpha_t-\alpha_s)],\ [a_t + w_L(z)],\ [w_L(z+\alpha_t)]\}. \tag{15}$$

Hence, we can compute the coefficients α_j for $j \in \overline{K}$ by using the recursive equation (15). Consider the problem (14). Let $\hat{z}$ be the smallest among the alternative optimal solutions of $\max_z(w_L(z))$. Then, it follows that $w_L(z) = w_L(\hat{z})\ \ \forall\ z \geq \hat{z}$. For any minimal GUB cover K, to begin with, since $w_K(z) = w_K(|M_K|)\ \forall\ z \geq |M_K|$, we only need to compute $w_K(z)$ for $z = 1, \ldots, |M_K|$. Moreover, since $\alpha_t \geq 0$ and $\alpha_t - \alpha_s \geq 0$ $\forall\ s \in (N_p - t)$, we have recursively that the value in (15), for each L and N_p thereafter, also remains a constant for $z \geq |M_K|$. Therefore, each function $w_L(z)$ needs to be evaluated (recursively) via (15) only for $z = 1, \ldots, |M_K|$. Hence, the time complexity of computing the lifted coefficients α_j for $j \in \overline{K}$ is $O(n|M_K|)$.

4. Simultaneously lifted facets from minimal GUB covers. We now consider an implementation of the reformulation–linearization technique (RLT) (see Sherali and Adams [11]) to characterize a class of valid inequalities (facets) of GUBKP, obtainable via a simultaneous lifting of minimal GUB cover inequalities. Toward this end, define a set F corresponding to feasible solutions for GKP as

$$F \equiv \left\{ J \subseteq N : \sum_{j\in J} a_j \geq b,\ |J \cap N_i| \leq 1\ \forall\ i \in M \right\} \text{ and let } \overline{F} \equiv \{J \subseteq N : J \notin F\}.$$

Then, we can directly write conv(X) as a convex combination of all feasible solutions to X by associating, for each $J \subseteq N$, a convex combination weight y_J with a vector that has ones in positions $j \in J$ and zeros otherwise. Noting that feasibly requires that $y_J \equiv 0$ if $J \in \overline{F}$, we get

$$GUBKP \equiv \text{conv}(X) = \left\{ x : x_j = \sum_{J:\ j\in J} y_J\ \forall\ j \in N, \quad \sum_{J\subseteq N} y_J = 1, \quad y_J \geq 0,\ \ \forall\ J \in F,\ y_J \equiv 0\ \ \forall\ J \in \overline{F} \right\}. \tag{16}$$

Using the standard projection operation, the set of all x's for which there exist corresponding vectors y that yield a feasible solution to (16) is given by duality or Farkas's Lemma (see Nemhauser and Wolsey, [9]) as

$$GUBKP \equiv \left\{ x : \sum_{j\in N} \pi_j^k x_j \geq \pi_0^k \right. \tag{17}$$
$$\left. \text{where } (\pi_j^k, \pi_0^k),\ k = 1,\dots,K, \text{are the extreme directions of } \Pi \right\}$$

where $\Pi \equiv \{(\pi, \pi_0) : \sum_{j\in J} \pi_j - \pi_0 \geq 0\ \forall\ J \in F\}$.

We now consider the characterization of a family of valid inequalities of GUBKP obtainable via a simultaneous lifting of the minimal GUB cover inequality. Recall that the minimal GUB cover inequality $\sum_{j\in K} x_j \geq 1$ is a valid inequality for $\operatorname{conv}(X(\overline{K}))$, and a facet for $\operatorname{conv}(X(\overline{K}))$ if $\min_{j\in K}(a_j) + \sum_{i\in M_{\overline{K}}} a_{j(i)} \geq b$. We are interested in finding a lifted inequality, which is a facet of GUBKP, and is of the form $\sum_{j\in K} x_j + \sum_{j\in\overline{K}_-} \pi_j x_j - \sum_{j\in\overline{K}_+} \pi_j \bar{x}_j \geq 1$ where $\bar{x}_j \equiv (1 - x_j)\ \forall\ j \in N$, and $\pi_j\ \forall\ j \in \overline{K}$ are unrestricted in sign. This is of the form

$$\sum_{j\in K} x_j + \sum_{j\in\overline{K}_-} \pi_j x_j + \sum_{j\in\overline{K}_+} \pi_j x_j \geq 1 + \sum_{j\in\overline{K}_+} \pi_j. \tag{18}$$

Motivated by (17) and the form of (18), consider a polyhedral set $\Pi_{\overline{K}}$, where $\Pi_{\overline{K}} \equiv \{\pi_j,\ j \in \overline{K} : (\pi, \pi_0) \in \Pi,\ \pi_j = 1\ \forall\ j \in K,\ \pi_0 = 1 + \sum_{j\in\overline{K}_+} \pi_j\}$. Let us now proceed through a series of simplifications in characterizing $\Pi_{\overline{K}}$ more precisely, and then state our main result regarding this set and the validity of (18), in the same spirit as that of (17). To begin, observe that $\Pi_{\overline{K}}$ can be represented as follows, where $\pi_{\overline{K}} \equiv \{\pi_j : j \in \overline{K}\}$

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j\in J\cap\overline{K}} \pi_j \geq 1 + \sum_{j\in\overline{K}_+} \pi_j - |J \cap K|,\ \forall\ J \in F \right\}. \tag{19}$$

Equivalently, we have,

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j\in(\overline{K}_+\ -\ J)} \pi_j - \sum_{j\in J\cap\overline{K}_-} \pi_j \leq |J \cap K| - 1,\ \forall\ J \in F \right\}.$$

Note that we need to consider only those $J \in F$ above, for which $\overline{KJ} \equiv (\overline{K}_+\ -\ J) \cup (J \cap \overline{K}_-) \neq \phi$. Hence, we have that

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j\in(\overline{K}_+\ -\ J)} \pi_j - \sum_{j\in J\cap\overline{K}_-} \pi_j \leq |J \cap K| - 1, \right. \tag{20}$$
$$\left. \forall\ J \in F \text{ having } \overline{KJ} \neq \phi \right\}.$$

Furthermore, note that we need to examine only the most restrictive constraints in (20). Toward this end, define $GUB(\overline{K}) = \{T \subseteq \overline{K} : |T \cap N_i| \leq 1 \ \forall\, i \in M_{\overline{K}}\}$. Now for any $T \in GUB(\overline{K})$, define the feasible extension of T as $J(T) = \{J \in F : J = J_1 \cup T$ for some $J_1 \subseteq K\}$, and let $\overline{KT} = (\overline{K}_+ - T) \cup (T \cap \overline{K}_-)$. Accordingly, define

$$F_K = \{T \in GUB(\overline{K}) : J(T) \neq \phi, \text{ and } \overline{KT} \neq \phi\}.$$

Then, we can restate (20) as follows.

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in T \cap \overline{K}_-} \pi_j \leq \min\{|J \cap K| : J \in J(T)\} - 1 \quad \forall\, T \in F_K \right\}. \tag{21}$$

We now consider an explicit representation of the minimization problem in (21). Note that $T \in F_K$ if and only if (i) $|T \cap N_i| \leq 1$ for $i \in M_{\overline{K}}$, i.e., $T \in GUB(\overline{K})$, (ii) $\sum_{j \in K_+} a_j + \sum_{j \in T} a_j \geq b$, i.e., $J(T) \neq \phi$, and (iii) $\overline{KT} \neq \phi$. Hence, for each $T \in F_K$, we can represent the minimization problem in (21), denoted by $AGUBKP(T)$, as follows.

$$AGUBKP(T) : \text{minimize } \left\{ \sum_{j \in K_+} y_j : \sum_{j \in K_+} a_j y_j \geq b - \sum_{j \in T} a_j,\ y_j \in (0,1)\ \forall\, j \in K_+ \right\}.$$

Note that $1 \leq \nu(AGUBKP(T)) \leq |M_K|$, where $\nu(P)$ denotes the optimal objective value of the corresponding problem P. Of course, $AGUBKP$ is an easy problem in the sense that we can readily compute $\nu(AGUBKP(T))$ for each $T \in F_K$, using a greedy procedure. Let $\bar{b} = b - \sum_{j \in T} a_j$. If $\bar{b}$ is less than or equal to $\max_{j \in K_+}(a_j)$, then $\nu(AGUBKP(T)) = 1$. Otherwise, if $\bar{b}$ is less than or equal to the sum of the first two largest a_j's for $j \in K_+$, then $\nu(AGUBKP(T)) = 2$, and so on. Hence the time complexity of solving AGUBKP is $O(|M| \log |M|)$. Let $\mathbf{N}(T) = \nu(AGUBKP(T)) - 1$. Then, we have

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in T \cap \overline{K}_-} \pi_j \leq \mathbf{N}(T) \ \ \forall\, T \in F_K \right\}. \tag{22}$$

PROPOSITION 4.1. *For a minimal GUB cover K, the inequality* (18) *is a valid inequality for GUBKP if and only if $\pi_{\overline{K}} \in \Pi_{\overline{K}}$, where $\pi_{\overline{K}} = \{\pi_j : j \in \overline{K}\}$, and $\Pi_{\overline{K}}$ is given by* (22).

Proof. $\pi x \geq \pi_0$ is valid for GUBKP if and only if $\sum_{j \in J} \pi_j \geq \pi_0 \ \ \forall\, J \in F$, that is, from (17), if and only if $(\pi, \pi_0) \in \Pi$. Hence, noting the form of (18) and the derivation of (22), we have that (18) is valid for GUBKP if and only if $\pi_{\overline{K}} \in \Pi_{\overline{K}}$, where $\Pi_{\overline{K}}$ is given by (22). This completes the proof. $\square$

PROPOSITION 4.2. *Let K be a minimal GUB cover such that $\min_{j \in K}(a_j) + \sum_{i \in M_{\overline{K}}} a_{j(i)} \geq b$, and let $\Pi_{\overline{K}}$ be given by* (22). *Then, the inequality* (18) *having $1 + \sum_{j \in \overline{K}_+} \pi_j > 0$ is a facet of GUBKP if and only if $(\pi_j,\ j \in \overline{K})$ is a vertex of $\Pi_{\overline{K}}$ with $1 + \sum_{j \in \overline{K}_+} \pi_j > 0$.*

Proof. As shown similarly in Sherali and Adams [11], it is readily verified that $\pi x \geq 1$ is a facet of GUBKP if and only if π is an extreme point of Π_1, where

$$\Pi_1 = \left\{ \pi : \sum_{j \in J} \pi_j \geq 1 \; \forall \; J \in F \right\}. \tag{23}$$

Hence, (18) with $\pi_j = \overline{\pi}_j$ for $j \in \overline{K}$ is a facet of GUBKP if and only if the scaled partitioned vector $\hat{\pi}$, where

$$\hat{\pi} = \left\{ \left(\hat{\pi}_j = \frac{1}{(1 + \sum_{j \in \overline{K}_+} \overline{\pi}_j)},\ j \in K \right),\ \left(\hat{\pi}_j = \frac{\overline{\pi}_j}{(1 + \sum_{j \in \overline{K}_+} \overline{\pi}_j)},\ j \in \overline{K} \right) \right\} \tag{24}$$

is an extreme point of (23).

Now, for each $j \in K$, the set $J(j) \equiv \{j\} \cup \overline{K}_+ \in F$ by the hypothesis of the theorem, and the corresponding constraints of (23) are linearly independent and are binding at the solution (24). The latter $|K|$ linearly independent equality constraints appear as

$$\pi_j = 1 - \sum_{t \in \overline{K}_+} \pi_t \text{ for } j \in K, \tag{25}$$

and so determine $\pi_j,\ j \in K$, uniquely in terms of $\pi_j,\ j \in \overline{K}_+$. Now, (24) is a vertex of (23) if and only if it is feasible to (23) and there exist some $|\overline{K}|$ hyperplanes binding from (23) that are linearly independent in combination with (25). This happens if and only if $\{\hat{\pi}_j,\ j \in \overline{K}\}$ is an extreme point of the set obtained by imposing (25) on (23), i.e., the set

$$\left\{ \pi_{\overline{K}} : \sum_{j \in J \cap \overline{K}} \pi_j \geq 1 + |J \cap K| \left(\sum_{t \in \overline{K}_+} \pi_t - 1 \right) \; \forall \; J \in F \right\}. \tag{26}$$

This holds if and only if $(\hat{\pi}_j,\ j \in \overline{K})$ is feasible to (26), and there exist some $|\overline{K}|$ linearly independent hyperplanes that are binding at $(\hat{\pi}_j,\ j \in \overline{K})$. Feasibility of $\hat{\pi}$ to (26) requires from (24) that

$$\sum_{j \in J \cap \overline{K}} \frac{\overline{\pi}_j}{(1 + \sum_{t \in \overline{K}_+} \overline{\pi}_t)} \geq 1 + |J \cap K| \left\{ \frac{\sum_{t \in \overline{K}_+} \overline{\pi}_t}{(1 + \sum_{t \in \overline{K}_+} \overline{\pi}_t)} - 1 \right\} \; \forall \; J \in F.$$

That is, we must have

$$\sum_{j \in J \cap \overline{K}} \overline{\pi}_j \geq 1 + \sum_{t \in \overline{K}_+} \overline{\pi}_t - |J \cap K| \; \forall \; J \in F. \tag{27}$$

Note by (19) that (27) is equivalent to requiring that $\overline{\pi}_{\overline{K}}$ belongs to $\Pi_{\overline{K}}$. Moreover, an inequality in (26) is binding at $\hat{\pi}$ if and only if the corresponding inequality in (27) is binding. Also, a collection of $|\overline{K}|$ linearly independent equations from (26) give $\hat{\pi}$ as the unique solution if and only if the corresponding $|\overline{K}|$ equations from (27) give $\overline{\pi}$ as the unique solution, because from (24), there is a one-to-one correspondence between $\hat{\pi}$ and $\overline{\pi}$ according to

$$\left\{ \hat{\pi}_j = \frac{\overline{\pi}_j}{(1 + \sum_{j \in \overline{K}_+} \overline{\pi}_j)} \; \forall \; j \in \overline{K} \right\} \text{ and } \left\{ \overline{\pi}_j = \frac{\hat{\pi}_j}{(1 - \sum_{j \in \overline{K}_+} \hat{\pi}_j)} \; \forall \; j \in \overline{K} \right\}.$$

TABLE 1

T	$\overline{KT}$	$\sum_{j\in T} a_j$	$\nu(AGUBKP(T))$	Inequalities of $\Pi_{\overline{K}}$ in (22)
ϕ	9	0	2	$\pi_9 \leq 1$
7	7, 9	1	2	$\pi_9 - \pi_7 \leq 1$
8	8, 9	1	2	$\pi_9 - \pi_8 \leq 1$

Hence, $\hat{\pi}_{\overline{K}}$ is an extreme point of (26) if and only if $\overline{\pi}_{\overline{K}}$ is an extreme point of $\Pi_{\overline{K}}$ with $1 + \sum_{j\in \overline{K}_+} \overline{\pi}_j > 0$, and this completes the proof. □

Example 4.1. Consider the following example to illustrate the above simultaneous lifting procedure. Let $X \equiv \{x \in (0,1)^9 : x_1 + x_2 + 2x_3 + x_4 + x_5 + 2x_6 + x_7 + x_8 + 3x_9 \geq 4,\ x_1 + x_2 + x_3 \leq 1,\ x_4 + x_5 + x_6 \leq 1,\ x_7 + x_8 + x_9 \leq 1\}$. A minimal GUB cover inequality is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 1$, which is a facet of $\text{conv}(X(\overline{K}))$, where $\overline{K} = \{7,8,9\}$. For this minimal cover, we have that $F_K = \{\phi, \{7\}, \{8\}\}$, thereby leading to the computations shown in Table 1.

The point $(0,0,1)$ is the only vertex of $\Pi_{\overline{K}}$ with $1 + \sum_{j\in \overline{K}_+} \pi_j = 2 > 0$. Hence, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_9 \geq 2$ is the only facet obtainable from the minimal GUB cover inequality by the lifting procedure.

Remark 4.1. In a spirit similar to Remark 2.1, the foregoing proposition can be used in the context of a separation problem for generating a simultaneously lifted facet of GUBKP, based on a given minimal GUB cover, that deletes a given fractional solution to the continuous relaxation of a 0-1 problem. In this context, an additional linear program would need to be solved over the polytope (22) to generate the cut coefficients for $j \in \overline{K}$, where K is the minimal GUB cover being used. Note that if such a strategy is being used as in Crowder, Johnson, and Padberg [4] and Hoffman and Padberg [6] within an overall algorithm for solving a 0-1 integer program, then if this problem is sparse, we might expect $|\overline{K}|$ and $|M|$ to be manageably small when employing a GUB-constrained polytope based on a single problem constraint. Thus, the generation of the foregoing facetial cut would not be too computationally burdensome. In this type of an analysis, for further restricting a subset of the cut coefficients a priori before employing a reduced sized set (22) along with Propositions 4.1 and 4.2 to generate the remaining coefficients defining strong valid inequalities, it would be computationally useful to have knowledge of lower and upper bounds on the simultaneously lifted facet coefficients π_j for $j \in \overline{K}$ in (18). This topic is addressed next.

4.1 Lower and upper bounds on the lifted coefficient π_j for $j \in \overline{K}$ in lifted inequality (18) under Proposition 4.2. To begin with, let us consider the lifted coefficients π_t for $t \in \overline{K}_+$. Since GUBKP is a full-dimensional polytope, it follows that for any $t \in \overline{K}_+$, we have $T = (\overline{K}_+ - t) \in F_K$, and from (22), we directly have that $\pi_t \leq \mathbf{N}(\overline{K}_+ - t)$. Hence, from (5), an upper bound UB_t on π_t is given by

$$UB_t = \mathbf{N}(\overline{K}_+ - t) = \eta^t - 1 \text{ for each } t \in \overline{K}_+.$$

Now, for given lower bounds LB_s on $\pi_s\ \forall\ s \in (N_p - t)$, where $p \equiv M_t$, let us derive a lower bound LB_t on π_t. Toward this end, examine any $T \in F_K$ such that $t \notin T$. Then, from (22), we have that

$$\sum_{j\in(\overline{K}_+ - T)} \pi_j - \sum_{j\in T\cap\overline{K}_-} \pi_j \leq \mathbf{N}(T)\ \forall\ T \in F_k \text{ such that } t \notin T. \tag{28}$$

From (28), since $t \in (\overline{K}_+ - T)$, we have that

$$(29)\quad \pi_t \le \mathbf{N}(T) - \left[\sum_{j\in(\overline{K}_+-T-t)} \pi_j - \sum_{j\in T\cap\overline{K}_-} \pi_j \right] \ \forall\, T \in F_K \text{ such that } \ t \notin T.$$

Note that (29) is comprised of all the constraints of $\Pi_{\overline{K}}$ that contain π_t. Since at any vertex of $\Pi_{\overline{K}}$, at least one of (29) must be binding, we have that

$$(30)\ \pi_t = \min\left\{ \mathbf{N}(T) - \left[\sum_{j\in(\overline{K}_+-T-t)} \pi_j - \sum_{j\in T\cap\overline{K}_-} \pi_j \right] : T \in F_K \text{ with } t \notin T \right\}.$$

For all $T \in F_k$ such that $t \notin T$, define

$$\nu(T) \equiv \sum_{j\in(\overline{K}_+-T)} \pi_j - \sum_{j\in T\cap\overline{K}_-} \pi_j, \qquad \overline{\nu}(T) \equiv \sum_{j\in(\overline{K}_+-T-t)} \pi_j - \sum_{j\in T\cap\overline{K}_-} \pi_j.$$

Then equation (30) reads

$$(31)\qquad \pi_t = \min\{\mathbf{N}(T) - \overline{\nu}(T) : T \in F_K \text{ with } t \notin T\}.$$

For any $T \in F_K$ such that $t \notin T$, if $(T+t) \in F_K$, then we have that $\overline{\nu}(T) = \nu(T+t) \le \mathbf{N}(T+t)$ by (28). On the other hand, if $(T+t) \notin F_K$, then there exists some $s \in T \cap N_p$ and $\overline{\nu}(T) = \nu(T+t-s) - \pi_s \le \mathbf{N}(T+t-s) - \pi_s$. Hence, from (31), we have

$$\pi_t = \min\{\min_{T\in T_1}[\mathbf{N}(T) - \mathbf{N}(T+t)],\ \min_{T\in T_2}[\mathbf{N}(T) - \mathbf{N}(T+t-s) + \pi_s]\},$$

where $T_1 \equiv \{T \in F_K : t \notin T \text{ and } (T+t) \in F_K\}$, $T_2 \equiv \{T \in F_K : t \notin T$ and there exists some $s \in (N_p - t) \cap T$ where $p \equiv M_t\}$. Hence, for given lower bounds LB_s on π_s for $s \in (N_p - t)$, a lower bound LB_t on π_t, for $t \in \overline{K}_+$, is given by

$$(32)\ \ LB_t = \min\{\min_{T\in T_1}[\mathbf{N}(T) - \mathbf{N}(T+t)],\ \min_{T\in T_2}[\mathbf{N}(T) - \mathbf{N}(T+t-s) + LB_s]\}.$$

Next, let us derive lower and upper bounds on the lifted coefficients π_s, for any $s \in \overline{K}_-$. From (22), we have that when $T = \{s\} \cup \{\overline{K}_+ - t\}$, where $t \equiv j(M_s)$, $\pi_t - \pi_s \le \mathbf{N}(T)$. Hence, we have $\pi_s \ge \pi_t$ - $\mathbf{N}(T)$. Consequently, for a given lower bound LB_t on π_t, $t \in \overline{K}_+$, a lower bound LB_s for any $s \in \overline{K}_-$ is given by

$$(33)\qquad LB_s = LB_t - \mathbf{N}(\overline{K}_+ - t + s) \text{ where } t \equiv j(M_s).$$

Remark 4.2. Note that the lower bounds (32) and (33) are conditional bounds, each being determined based on lower bounds of the other. These conditional bounds are useful if we restrict the class of facets to have prescribed lower bounds on the π_t or the π_s coefficients. Otherwise, we need to derive unconditional lower bounds LB_j $\forall\, j \in \overline{K}$. Toward this end, we derive an unconditional lower bound of zero on π_t, $\forall\, t \in \overline{K}_+$, as follows. By the transformation in Johnson and Padberg ([7], Prop. 2.1), since the inequality (18) is a facet of GUBKP and $|M_K| > 1$ (otherwise, dim(GUBKP) $< n$), it can be readily shown that the inequality

$$(34)\qquad \sum_{j\in K_+} z_j + \sum_{j\in\overline{K}_+} \pi_j z_j + \sum_{i\in M_{\overline{K}}} \sum_{j\in\{N_i-j(i)\}} (\pi_{j(i)} - \pi_j) z_j \le |M_K| - 1$$

is a facet of the convex hull of the polytope

$$Z \equiv \text{conv}\left\{ z \in (0,1)^n : \sum_{i \in M} \sum_{j \in N_i} \overline{a}_j z_j \leq \overline{b},\ \sum_{j \in N_i} z_j \leq 1 \ \forall\ i \in M \right\},$$

where for all $i \in M$, $\overline{a}_{j(i)} = a_{j(i)}$, $\overline{a}_j = a_{j(i)} - a_j \ \forall\ j \in \{N_i - j(i)\}$, and $\overline{b} = b + \sum_{j \in N_+} a_j$. Since the constraints of the polytope Z have all nonnegative coefficients, we have that all the coefficients of the facet (34) are nonnegative [5]. Accordingly, for each $t \in \overline{K}_+$, we have

$$\pi_t \geq 0 \text{ and } \pi_t - \pi_s \geq 0 \ \forall\ s \in (N_p - t), \text{ where } p = M_t. \tag{35}$$

From (35), we have a valid lower bound of zero on π_t, $\forall\ t \in \overline{K}_+$. Consequently, from (33), we have that $LB_s = -\mathbf{N}(\overline{K}_+ - t + s) \ \forall\ s \in (N_p - t)$, where $p \equiv M_t$.

Finally, let us derive an upper bound UB_s for any given $s \in \overline{K}_-$. Note from (22) that the collection of constraints defining $\Pi_{\overline{K}}$ that contain the coefficient π_s is given by

$$\pi_s \geq \sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in (T - s) \cap \overline{K}_-} \pi_j - \mathbf{N}(T) \ \forall\ T \in F_K \text{ such that } s \in T. \tag{36}$$

Again, at any vertex of $\Pi_{\overline{K}}$, since at least one of (36) must be binding, we have that

$$\pi_s = \max\left\{ \sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in (T - s) \cap \overline{K}_-} \pi_j - \mathbf{N}(T) : T \in F_K \text{ with } s \in T \right\}. \tag{37}$$

Now, in (37), if $(T - s) \in F_K$, we have from (22) that

$$\sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in (T - s) \cap \overline{K}_-} \pi_j \leq \mathbf{N}(T - s).$$

On the other hand, if $(T - s) \notin F_K$, then $\mathbf{N}(T - s) \equiv \infty$, and so the foregoing inequality holds for all $T \in F_K$ such that $s \in T$. Furthermore, for any $T \in F_K$ such that $s \in T$, we also have $(T - s + t) \in F_K$, where $t \equiv j(M_s)$. Consequently, from (22), the corresponding constraint for $T' \equiv (T - s + t) \in F_K$ yields

$$\begin{aligned} \sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in (T - s) \cap \overline{K}_-} \pi_j &= \sum_{j \in (\overline{K}_+ - T')} \pi_j - \sum_{j \in T' \cap \overline{K}_-} \pi_j + \pi_t \\ &\leq \mathbf{N}(T') + \pi_t \leq \mathbf{N}(T') + UB_t. \end{aligned} \tag{38}$$

Note that if $\overline{KT'} \equiv (\overline{K}_+ - T') \cup (T' \cap \overline{K}_-) = \phi$, we simply have $\mathbf{N}(T') \equiv 0$ in that case. Combining the last two inequalities, we may write for any $T \in F_K$ such that $s \in T$, and $t \equiv j(M_s)$,

$$\sum_{j \in (\overline{K}_+ - T)} \pi_j - \sum_{j \in (T - s) \cap \overline{K}_-} \pi_j \leq \min\{\mathbf{N}(T - s),\ \mathbf{N}(T - s + t) + UB_t\}.$$

Substituting this in (37) above, we obtain the following upper bound UB_s on π_s for any $s \in \overline{K}_-$.

$$UB_s = \max_{T \in F_K:\ s \in T} \{\min\{\mathbf{N}(T - s),\ \mathbf{N}(T - s + t) + UB_t\} - \mathbf{N}(T)\}, \tag{39}$$

where $t \equiv j(M_s)$.

Example 4.2. Consider the following constraints of a GUB-constrained knapsack problem, where $X \equiv \{x \in (0,1)^{12} : 2x_1 + 5x_2 + 2x_3 + 3x_4 + x_5 + 3x_6 + x_7 + 3x_8 + 2x_9 + 2x_{10} + 2x_{11} + 2x_{12} \geq 16,\ x_1 + x_2 \leq 1,\ x_3 + x_4 \leq 1,\ x_5 + x_6 \leq 1,\ x_7 + x_8 \leq 1,\ x_9 \leq 1,\ x_{10} \leq 1,\ x_{11} \leq 1,\ x_{12} \leq 1\}$. For a minimal GUB cover K={9, 10, 11, 12}, the minimal GUB cover inequality is $x_9 + x_{10} + x_{11} + x_{12} \geq 1$, which is a facet of conv$(X(\overline{K}))$ by Proposition 2.7. Note that $\overline{K}_+ = \{2,4,6,8\}$ and $\overline{K}_- = \{1,3,5,7\}$. We now consider a facet of the form (18) with $\pi_j \geq 0\ \forall\ j \in \overline{K}_-$. Let us derive lower and upper bounds on the coefficient π_2, where $\{2\} \in \overline{K}_+$. Since $\eta^2 = 4$, we have that $UB_2 = \eta^2 - 1 = 3$. Furthermore, the set $T_1 \equiv \{T : T \in F_K$ with $\{2\} \notin T$ and $(T + \{2\}) \in F_K\}$, is given by $T_1 = \{(3,6,8),\ (4,6,8)\}$. Also the set $T_2 \equiv \{T : T \in F_K$ with $\{2\} \notin T$ and $\{1\} \in T\}$ is given by $T_2 = \{(1,4,6),\ (1,4,8),\ (1,6,8),\ (1,3,5,8),\ (1,3,6,7),\ (1,3,6,8),\ (1,4,5,8),\ (2,4,6,7),\ (1,4,6,8)\}$. Hence, by (32), conditioned on $LB_1 = 0$, a lower bound LB_2 can be computed as follows.

$$LB_2 = \min\{\min_{T \in T_1}[\mathbf{N}(T) - \mathbf{N}(T + \{2\})],\ \min_{T \in T_2}[\mathbf{N}(T) - \mathbf{N}(T + \{2\} - \{1\}) + 0]\} = 1.$$

Next, let us select $\{1\} \in \overline{K}_-$, and illustrate the computation of an upper bound on the coefficient π_1 in any lifted facet (18). The set $T' \equiv \{T : T \in F_K$ with $\{1\} \in T\}$ is given by $\{(1,4,6),\ (1,4,8),\ (1,6,8),\ (1,3,5,8),\ (1,3,6,7),\ (1,3,6,8),\ (1,4,5,8),\ (1,4,6,7),\ (1,4,6,8)\}$. Since $UB_2 = 3$, we have by (39) that

$$UB_1 = \max_{T \in T'}\{\min\{\mathbf{N}(T - \{1\}),\ \mathbf{N}(T - \{1\} + \{2\}) + 3\} - \mathbf{N}(T)\} = 2.$$

Hence, we have that $0 \leq \pi_1 \leq 2$ and $1 \leq \pi_2 \leq 3$ in any lifted facet (18) having $\pi_1 \geq 0$. Note that we can also compute an unconditional lower bound LB_1, taking $LB_2 = 0$. By (33), $LB_1 = 0 - \mathbf{N}(\overline{K}_+ - \{2\} + \{1\}) = -2$. Hence, a set of unconditional bounds on π_1 and π_2 are given by $-2 \leq \pi_1 \leq 2$ and $0 \leq \pi_2 \leq 3$.

5. A special case: The zero-one knapsack polytope. Consider a special case of GUBKP with $|N_i| = 1\ \forall\ i \in M$, which represents the ordinary knapsack polytope, denoted by KP. That is, $KP \equiv \text{conv}\{x \in (0,1)^n : \sum_{j \in N} a_j x_j \geq b\}$ where the data is all integer, $N = \{1, \ldots, n\}$, $b > 0$, $0 < a_j \leq b\ \forall\ j \in N$, and $\sum_{j \neq k} a_j \geq b$ for all $k \in N$. Recall that the minimal (GUB) cover inequality $\sum_{j \in K} x_j \geq 1$ is a facet of conv$(KP(\overline{K}))$, where $KP(\overline{K}) = KP \cap \{x \in (0,1)^n : x_j = 1,\ \forall\ j \in \overline{K}\}$. Our interest is in characterizing a (simultaneously) lifted facet, as in Balas and Zemel [2], of the form

$$\sum_{j \in K} x_j + \sum_{j \in \overline{K}} \pi_j x_j \geq 1 + \sum_{j \in \overline{K}} \pi_j, \tag{40}$$

where K is a minimal (GUB) cover of KP. Toward this end, in the spirit of (21) and problem $AGUBKP(T)$, we define

$$f(\theta) = \min\left\{\sum_{j \in K} y_j : \sum_{j \in K} a_j y_j \geq \overline{b} + \theta\right\} - 1,$$

where $\overline{b} = b - \sum_{j \in \overline{K}} a_j$.

By Proposition 4.2, we have that (40) is a facet of KP if and only if $(\pi_j,\ j \in \overline{K})$ is an extreme point of $\Pi_{\overline{K}}$ with $1 + \sum_{j \in \overline{K}} \pi_j > 0$, where

$$\Pi_{\overline{K}} \equiv \left\{ \pi_{\overline{K}} : \sum_{j \in (\overline{K}-T)} \pi_j \leq f\left(\sum_{j \in (\overline{K}-T)} a_j \right) \ \forall\, T \in F_K \right\} \tag{41}$$

and where $F_K = \{T \subset \overline{K} : \sum_{j \in K \cup T} a_j \geq b\}$.

This is precisely Balas and Zemel's characterization of simultaneously lifted facets obtainable from minimal cover inequalities. We now derive upper and lower bounds on $\pi_j,\ j \in \overline{K}$, for such facets of KP.

5.1. Upper bound UB_t on π_t, $t \in \overline{K}$. From (41), by examining $T = \overline{K} - \{t\} \in F_K$, we directly have that $\pi_t \leq f(a_t)$. Hence, an upper bound UB_t on π_t is given by

$$UB_t = f(a_t) \text{ for each } t \in \overline{K}.$$

Note that UB_t is the same as Balas and Zemel's upper bound on $\pi_t,\ t \in \overline{K}$.

5.2. Lower bound LB_t on π_t, $t \in \overline{K}$. Balas and Zemel [2] derive the following lower bound, denoted by $LBBZ_t$:

$$LBBZ_t = h \qquad \forall\, t \in S_h,$$

where letting $E(K)$ denote the extension of K as before, we have $S_0 = \overline{E(K)}$, and $S_h = \{t \in (E(K) - K) : \sum_{j \in K_h} a_j \leq a_t < \sum_{j \in K_{h+1}} a_j\}$, where K_h is the index set of the h largest a_j for $j \in K$. Note that $UB_t = LBBZ_t$ or $LBBZ_t + 1$.

We now construct a tighter lower bound on π_t. Consider any $t \in \overline{K}$, and let us examine any $T \in F_K$ such that $t \notin T$. From (41), we have that

$$\pi_t \leq f\left(\sum_{j \in (\overline{K}-T)} a_j \right) - \sum_{j \in (\overline{K}-T-\{t\})} \pi_j \ \forall\, T \in F_K \text{ such that } t \notin T. \tag{42}$$

But since at an extreme point of $\Pi_{\overline{K}}$, at least one of (42) must be binding, we have that

$$\pi_t = \min \left\{ f\left(\sum_{j \in (\overline{K}-T)} a_j \right) - \sum_{j \in (\overline{K}-T-\{t\})} \pi_j : T \in F_K \text{ with } t \notin T \right\},$$

given other π_j values. Consequently, we get

$$LB_t = \min \left\{ f\left(\sum_{j \in (\overline{K}-T)} a_j \right) - f\left(\sum_{j \in (\overline{K}-T-\{t\})} a_j \right) : T \in F_K \text{ with } t \notin T \right\}. \tag{43}$$

PROPOSITION 5.1. $LB_t \geq LBBZ_t \ \forall\, t \in \overline{K}$.

Proof. By the monotone increasing nature of the function f, it follows that $LB_t \geq 0 \ \forall\, t \in \overline{K}$. Since $LBBZ_t = 0$ for $t \in \overline{E(K)}$, the result holds trivially for this case. Hence, suppose that $t \in (E(K) - K)$. Consider any $T \in F_K$ with $t \notin T$, and examine two cases.

Case i. $a_t = \sum_{j \in K_h} a_j$. It follows that $LBBZ_t = h \equiv f(a_t - \bar{b}) + 1$. But we have, defining $\hat{b} = b - \sum_{j \in T} a_j - a_t > 0$, that

$$
\begin{aligned}
f\left(\sum_{j \in (\overline{K}-T)} a_j\right) - f\left(\sum_{j \in (\overline{K}-T-\{t\})} a_j\right) &= \min\left\{\sum_{j \in K} y_j : \sum_{j \in K} a_j y_j \geq \hat{b} + a_t\right\} \\
&\quad - \min\left\{\sum_{j \in K} y_j : \sum_{j \in K} a_j y_j \geq \hat{b}\right\} \\
&\geq \min\left\{\sum_{j \in K} y_j : \sum_{j \in K} a_j y_j \geq a_t\right\} \\
&= f(a_t - \bar{b}) + 1. \qquad (44)
\end{aligned}
$$

This implies from (43) that $LB_t \geq LBBZ_t$.

Case ii. $\sum_{j \in K_h} a_j < a_t < \sum_{j \in K_{h+1}} a_j$. In this case, we have, $LBBZ_t = h \equiv f(a_t - \bar{b})$. Let Δ be the amount that needs to be subtracted from a_t so that $a_t - \Delta = \sum_{j \in K_h} a_j$. Then, using the monotone increasing nature of the function f and following (44), we have that

$$
\begin{aligned}
&f\left(\sum_{j \in (\overline{K}-T)} a_j\right) - f\left(\sum_{j \in (\overline{K}-T-\{t\})} a_j\right) \geq f\left(\sum_{j \in (\overline{K}-T)} a_j - \Delta\right) f\left(\sum_{j \in (\overline{K}-T-\{t\})} a_j\right) \\
&\qquad \geq f(a_t - \Delta - \bar{b}) + 1 = f(a_t - \bar{b}). \qquad (45)
\end{aligned}
$$

Hence from (43) and (45), we have that $LB_t \geq LBBZ_t$. Therefore, the result holds for any $t \in (E(K) - K)$ as well, and this completes the proof. □

Example 5.1. Consider $KP = \text{conv}\{x \in (0,1)^4 : 3x_1 + 3x_2 + 3x_3 + 2x_4 \geq 7\}$. Let $K = \{1,2\}$, so that $E(K) = \{1,2,3\}$ and $\bar{b} = 2$. Consider $t = \{4\} \in \overline{E(K)}$. Note that $\{T \in F_K : t \notin T\} = \{3\}$. Hence, from (43), we have that $LB_4 = f(a_4) - f(0) = f(a_4) = UB_4 = 1 > LBBZ_4 = 0$. Note, however, that K is not a strong cover, as evidenced by the (strong) minimal cover $K' = \{3,4\}$. Otherwise, by the definition of a strong cover, we would have had for any $t \in \overline{E(K)}$, if it exists, that $UB_t = f(a_t) = 0$, and so $LB_t = 0 \equiv LBBZ_t$ as well.

Example 5.2. Consider $KP = \text{conv}\{x \in (0,1)^4 : 3x_1 + 3x_2 + 3x_3 + 4x_4 \geq 7\}$. The minimal cover $K = \{1,2,3\}$ is a strong cover for KP. Since $\bar{b} = 3$, $\overline{K} = \{4\}$, and for $t = \{4\} \in (E(K) - K)$, we have that $\{T \in F_K : t \notin T\} = \phi$; this gives $LB_4 = f(a_4) - f(0) = f(a_4) \equiv UB_4 = 2$. However, $LBBZ_t = 1 < LB_4$.

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