SUBMODULAR KNAPSACKS : A DISCUSSION

AVINASH BHARDWAJ

ABSTRACT. In the following discussion we study the facial structure of the convex hull of the level sets of a given submodular set function $f : \{0,1\}^n \mapsto \mathbb{R}$. In particular we derive valid inequalities and their extensions for the general lower level sets of submodular set functions, and propose the facet defining conditions for the same. The following discussion is novel in sense that it relaxes the monotonicity assumptions made in earlier studies and thus offers a generalization to earlier studies on this subject matter. We will follow a bottom up hierarchical structure to derive the appropriate expressions for the valid inequalities and their extensions from the linear knapsack inequalities corresponding to the extended polymatroid of the set function in context.

1. INTRODUCTION

Given a finite ground set N, a set function $f : 2^N \to \mathbb{R}$ is supermodular on N if

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T) \ \ \forall \, S, T \subseteq N.$$

Throughout, by an abuse of notation we, refer to a set function also as $f(\mathcal{X}_S)$, where $\mathcal{X}_S$ denotes the binary characteristic vector of $S \subseteq N$.

For notational simplicity, we denote a singleton set $\{i\}$ with its unique element i. For a vector $\mathbf{v} \in \mathbb{R}^N$, we use $\mathbf{v}(S)$ to denote $\sum_{i \in S} v_i$ for $S \subseteq N$. Given a set function f on N and $i \in N$, denote the difference function with respect to the set function f as,

$$\rho_i(S) := f(S \cup i) - f(S) \ \text{ for } S \subseteq N \setminus i$$

Observe that f is supermodular if and only if $\rho_i(S) \leq \rho_i(T) \, \forall \, S \subseteq T \subseteq N \setminus i$ and $i \in N$; that is the difference function ρ_i is non-decreasing on $N \setminus i$ (Schrijver, [38]).

1.1. **Motivation.**

1.1.1. *Chance Constraints with Uncertain Coefficients.*

Our prime motivation to study knapsack sets with submodular set functions stems from the need to analyze linear $0-1$ covering constraints with uncertain coefficients. For example, consider the following covering constraint,

$$\boldsymbol{\xi}'\mathbf{x} \geq d \tag{1}$$

where ξ_i, $i \in \{1, 2, \ldots, N\}$ are random with a support Ξ. Constraint (1) does not define a deterministic feasible set. In a deterministic realm we can approximate the above covering constraint by the following probabilistic constraint,

$$\mathbf{Pr}(\boldsymbol{\xi}'\mathbf{x} \geq d) \geq 1 - \epsilon, \ \ \epsilon > 0. \tag{2}$$

For $x \in \{0,1\}^N$, Charnes and Cooper [16] suggest under *nominal* conditions that Constraints of type (2) (or approximations thereof) can be modeled as conic-quadratic $0-1$ constraints,

$$(3) \qquad X_{CQ} := \left\{ \mathbf{x} \in \{0,1\}^m \ : \ \mathbf{u}^T \mathbf{x} - \Omega(\epsilon) \parallel \mathbf{\Sigma}^{\frac{1}{2}} \mathbf{x} \parallel \geq d \right\}$$

where u_e is the nominal value, $\mathbf{\Sigma}$ is the covariance matrix, and $\Omega(\epsilon) > 0$ is the scaling factor. The term $\Omega(\epsilon) \parallel \mathbf{\Sigma} \mathbf{x} \parallel$ is used to build sufficient slack into the constraint to accommodate the variability of ξ_i around the nominal value u_i. If ξ_i's are normally distributed independent random variables, then letting u_i and σ_{ii} be the mean and standard deviation of ξ_i, $0 \leq i \leq m$, and $\Omega(\epsilon) = -\phi^{-1}(\epsilon)$ with $0 < \epsilon < 0.5$, where ϕ is the standard normal cumulative distribution function, the set of $0-1$ solutions for the probabilistic covering constraint (2) is exactly X_{CQ}. On the other hand, if ξ_i's are known only through their first two moments u_i and σ_{ii}^2, then any point in X_{CQ} with $\Omega(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$ satisfies the probabilistic constraint (2) [13, 19]. Alternatively, if ξ_i's are only known to be symmetric with support $[u_i - \sigma_{ii}, u_i + \sigma_{ii}]$, then points in X_{CQ} with $\Omega(\epsilon) = \sqrt{\ln(1/\epsilon)}$ satisfy constraint (2) [11, 12]. Hence, under different assumptions of uncertainty on $\mathbf{\xi}$, one arrives at different instances of the conic quadratic knapsack set X_{CQ}.

Consider now a set function $f : 2^N \rightarrow R$ defined as,

$$(4) \qquad f(S) = \mathbf{u}(S) + g(\mathbf{c}(S)),$$

where g is a concave function and $\mathbf{u}(S)$ and $\mathbf{c}(S)$ denote the sums of the components of $\mathbf{u}, \mathbf{c} \in \mathbb{R}^N$ on $S \subseteq N$. Ahmed and Atamtürk [1] show that if $\mathbf{c} > 0$, then f is submodular on N. Observe that, in case ξ_i's are uncorrelated, X_{CQ} is a special case of X, observe that the conic quadratic constraint defining X_{CQ} can be written as

$$(5) \qquad -\mathbf{u}^T \mathbf{x} + \Omega \parallel \mathbf{\Sigma}^{\frac{1}{2}} \mathbf{x} \parallel \leq -d$$

Because $\mathbf{x}$ is a binary vector, letting $c_i = \Omega^2 \sigma_i^2$ for $i \in N$, we see that $\mathcal{X}_S$, $S \subseteq N$ is feasible for (5) if and only if S satisfies,

$$(6) \qquad f(S) = -\mathbf{u}(S) + \sqrt{\mathbf{c}(S)} \leq -d$$

The majority of the research on the nonlinear knapsack problem is devoted to the case with separable nonlinear functions [33]. Hochbaum [27] maximize a separable concave objective function, subject to a single "packing" constraint. They prove that the although seemingly more difficult in comparison with the linear knapsack, the complexity of the non-linear knapsack is similar. There are few studies on the nonseparable knapsack problem, most notably on the knapsack problem with quadratic objective and linear constraint. Helmberg et al. [26] give semidefinite programming relaxations of knapsack problems with quadratic objective. Ahmed and Atamtürk [1] consider maximizing a submodular function over a linear knapsack constraint. Atamtürk and Narayanan [7] describe a general mixed-integer rounding for conic quadratic mixed integer sets. Çezik and Iyengar [15] study techniques for generating valid convex constraints for mixed 0-1 conic programs and show that many of the techniques developed for generating linear cuts for mixed 0-1 linear programs, such as the Gomory cuts, the lift-and-project cuts, and cuts from other hierarchies of tighter relaxations, extend in a straightforward manner to mixed 0-1 conic programs. These approaches although efficient, are generic, and do not exploit any special structure(s) associated with the problem. We refer the reader to [14, 29] for a survey of nonlinear knapsack problems. The literature is scarce in case of knapsack sets associated with set functions. Assume f to be a given submodular set function on a finite set N. The lower level set of f, $K_f := \{\mathbf{x} \in \{0,1\}^n \ : \ f(\mathbf{x}) \leq b\}$, $b \in \mathbb{R}^n$ in case when f is non-decreasing on N was recently studied by Atamtürk and Narayanan [6]. Alternatively, when f is non-increasing as discussed above, the facial structure has been studied by Atamtürk and Bhardwaj in form of pack inequalities and their extensions. The most recent study on quadratically constrained knapsacks is by Atamtürk et al. [5] where they study the extensions and separation of cover inequalities for conic quadratic knapsack constraints with generalized upper bounds.

It is of interest to study the polyhedral structure of this level set when the monotonicity assumptions are relaxed. An additional application lies in form of the following second order cone set,

$$S_{\text{SOC}} := \left\{ \mathbf{x} \in \{0,1\}^n \ : \ \mathbf{a}'\mathbf{x} - \Omega\sqrt{\mathbf{x}'\Sigma\mathbf{x}} \geq b \right\},$$

$\Sigma = [\sigma_{ij}]_{1 \leq i,j \leq n} \geq 0$, $a_i \geq \Omega \cdot \sigma_i \ \forall \ 1 \leq i \leq n$. This is the inner approximation of the chance constraint (2) when $\xi_i, \ i \in N$ are correlated.

It can be seen $S_{\text{SOC}} = S_{\text{QP}} \cap \{x \in \{0,1\} \ : \ \mathbf{a}'\mathbf{x} \geq b\}$, where,

$$S_{\text{QP}} := \left\{ \mathbf{x} \in \{0,1\}^n \ : \ \sum_i \alpha_i x_i + 2\sum_{j>i} \beta_{ij} x_i x_j \leq \gamma \right\}$$

and $\alpha_i = (2a_i b + \Omega^2 \sigma_{ii}^2 - a_i^2)$, $\beta_{ij} = -(a_i a_j - \Omega^2 \sigma_{ij}) \leq (\Omega^2 \sigma_i \sigma_j - a_i a_j) \leq 0$ and $\gamma = b^2$.

Proposition 1. *Consider a set function* $f : \{0,1\}^n \mapsto \mathbb{R}$, *defined as a binary quadratic mapping,*

(7)
$$f(\mathbf{x}) := \mathbf{x}' Q\, x$$

where Q *is a* $n \times n$ *off-diagonal matrix.* $f(\cdot)$ *is submodular if and only if*

$$Q_{ij} \leq 0, \ 1 \leq i, j \ \leq n.$$

Proof. Consider, $S \subseteq T \subseteq N \setminus i \quad$ for some $i \in N$ Now,

$$\begin{aligned}
\rho_i(T) - \rho_i(S) &= f(T \cup i) - f(S \cup i) - f(T) + f(S) \\
&= Q_{i(j \in T)} - Q_{i(j \in S)} \\
&= Q_{i(j \in T \setminus S)} \\
&\leq 0
\end{aligned}$$

where the last inequality follows from non-positivity of $Q_{ij} \leq 0$, $1 \leq i, j \ \leq n$. Thus, we have $\rho_i(S) \leq \rho_i(T)$ $\forall \ S \subseteq T \subseteq N \setminus i \quad i \in N$ proving submodularity of $f(\cdot)$.

Alternatively, assume $f(\cdot)$ is submodular, thus $\forall \ S \subseteq T \subseteq N \setminus i \quad i \in N$,

$$\begin{aligned}
0 &\leq \rho_i(S) - \rho_i(T) \\
0 &\leq f(S \cup i) - f(T \cup i) - f(S) + f(T) \\
0 &\leq Q_{i(j \in S)} - Q_{i(j \in T)} \\
0 &\geq Q_{i(j \in T \setminus S)}
\end{aligned}$$

Thus, $Q_{i(j \in T \setminus S)} \leq 0$, $\forall \ S \subseteq T \subseteq N \setminus i \quad i \in N$, which yields $Q_{ij} \leq 0$, $1 \leq i, j \ \leq n$. $\qquad\square$

Corollary 2. *Consider a set function* $f : \{0,1\}^n \mapsto \mathbb{R}$, *defined as a binary quadratic mapping,*

(8)
$$f(\mathbf{x}) := \mathbf{a}'x + \mathbf{x}' Q\, x$$

where Q *is a* $n \times n$ *matrix and* $\mathbf{a} \in \mathbb{R}$. $f(\cdot)$ *is submodular if and only if*

$$Q_{ij} \leq 0, \ 1 \leq i, j \ \leq n, i \neq j.$$

Proposition 1 and Corollary 2 in effect suggest that S_{QP} is the lower level set of a submodular set function. The associated set however is not necessarily monotone. A requirement to study the polyhedral structure of the level sets associated with the generic submodular set functions is thus apparent. Albeit the motivation stems from the submodular quadratic maps, the following discussion is valid for knapsack sets associated with submodular functions in general.

2. Linear $0-1$ Knapsack Set

The knapsack problem is one of the most celebrated problems in optimization literature. Most of the literature available on the knapsack problem is for the linear knapsack set (binary, integer, mixed-integer). The polyhedral analysis of the linear knapsack set was initiated by Balas [8, 9], Hammer et al. [25], and Wolsey [42]. For an extensive review of theoretical and computational results on linear knapsacks we refer the reader to [2, 4, 22, 32]. In this section we'll review the results and literature for the case of linear $0-1$ knapsack set $K := \{\mathbf{x} \in \{0,1\}^n \ : \ \mathbf{ax} \le b\}, \mathbf{a} \ge 0$.

In the following, we review some of the results on the valid inequalities for $\mathrm{conv}(K)$

Proposition 3. *Inequality $x_i \ge 0$, $i \in N$ is facet defining for* $\mathrm{conv}(K)$.

Proposition 4. *Inequality $x_i \le 1$, $i \in N$ is facet defining for* $\mathrm{conv}(K)$ *if and only if $a_i + a_j \le b$, $\forall\ j \in N \setminus \{i\}$.*

Proposition 5. *If the inequality $\boldsymbol{\alpha}\mathbf{x} \le \beta$ defines a facet for* $\mathrm{conv}(K)$ *not including* $\mathbf{0}$ *then $\beta > 0$ and $\alpha_j \ge 0$, $\forall\ j \in N$.*

2.1. **Cover Inequalities.**

Definition 1. A subset C of the index set N is called a *cover* if $\lambda := a(C) - b > 0$.

For a cover C, let us consider the restriction K_C obtained by fixing all x_i, $i \in N \setminus C$ to zero. Since the sum of the coefficients a_i, $i \in C$ exceeds the knapsack capacity by $\lambda > 0$, all variables x_i, $i \in C$ cannot take a value of one simultaneously for any $\mathbf{x} \in K$. Therefore, the *cover inequality* [8, 25, 42]

$$(9) \qquad\qquad\qquad x(C) \le |C| - 1$$

is valid for $\mathrm{conv}(K)$. Cover inequality defines a facet for $\mathrm{conv}(K_C)$ if and only if C is a minimal cover, that is, $\mathbf{a}(C \setminus \{i\}) \le b$, $\forall\ i \in C$.

2.1.1. *Extensions of Cover Inequalities.*

Extensions are a means to strengthen the inequalities based on restrictions of feasible sets, in this case the cover inequalities. For a cover $C \subseteq N$, consider the cover inequality (9).

$$\mathbf{x}(C) \le |C| - 1$$

An important and very useful concept in deriving strong valid inequalities is lifting of inequalities. Lifting refers to extending valid inequalities for low dimensional restrictions of polyhedra to ones that are valid in high dimensions. The concept of lifting has been introduced by Gomory [20] in the context of the group problem. Padberg [35] described the sequential lifting procedure for $0-1$ programming. Since then lifting has been studied and used extensively (Atamtürk [3]; Balas and Zemel, [9]; Escudero, Garín, and Péres, [18]; Gu, Nemhauser, and Savelsbergh,[22, 23, 21, 24]; Johnson and Padberg, [28]; Louveaux and Wolsey, [30]; Marchand and Wolsey, [31]; Nemhauser and Vance, [34]; Padberg, [36]; Richard, de Farias, and Nemhauser, [37]; Sherali and Lee, [39]; Wolsey, [42, 41]; Zemel, [43, 44]), particularly, for $0-1$ and mixed $0-1$ programming problems. We can strengthen the cover inequality by introducing the variables not present in the inequality (x_i, $\forall\ i \in N \setminus C$) multiplied with appropriate coefficients. The lifted cover inequality in this case can be represented as,

$$(10) \qquad\qquad\qquad \mathbf{x}(C) + \sum_{i \in N \setminus C} \alpha_i x_i \le |C| - 1$$

where $\alpha_i \ge 0$, $i \in N \setminus C$ are the corresponding lifting coefficients. Obtaining exact lifting coefficients entails computing the lifting function $\Theta(\cdot)$ of cover inequality (9),

$$(11) \qquad\qquad\qquad \Theta(a) := |C| - 1 - \max\left\{\sum_{i \in C} x_i \ : \ \sum_{i \in C} a_i x_i \le b - a\right\}$$

for $a \geq 0$. Obtaining a closed form expression for the lifting function is not always trivial. Extensions are computationally inexpensive special cases of lifting, wherein instead of deriving the exact lifting coefficients, we identify whether a variable x_i, $i \in N \setminus C$ can be introduced into the cover inequality with a coefficient $\alpha_i \geq 1$. Let E be a set of all such indices, i.e. $E := \{i \in N \setminus C : \alpha_i \geq 1\}$. The extended cover inequality thus is,

$$\text{(12)} \qquad \mathbf{x}(C \cup E) \leq |C| - 1$$

Consider a subset of indices $\bar{E} \subseteq N \setminus C$. Assume that the corresponding variables x_i, $i \in \bar{E}$ have been included in the extended cover inequality. Consider the variable $x_k, k \in N \setminus (C \cup \bar{E})$ to be included next in the extended cover.

Proposition 6. $\alpha_k \geq 1$, if $a_k > \max\limits_{j \in C \cup \bar{E}} a_j - \lambda$

Atamtürk [4], Balas and Zemmel [9], Gu et al. [24] provide a characterization of the lifting function $\Theta(a)$ for the cover inequalities for $0-1$ integer knapsack set. This characterization yields a sequence independent lifting procedure, and as a special case, a sequence independent extension for the cover inequalities. Attributed to this, we can further simplify the extension procedure to a sequence independent extension.

Corollary 7. *For $k \in N \setminus C$, $\alpha_k \geq 1$, if $a_k > \max\limits_{j \in C} a_j - \lambda$.*

The corresponding extension set $E \subseteq N \setminus C$ can now be defined as,

$$\text{(13)} \qquad E := \left\{ i \in N \setminus C : a_i > \max_{j \in C} a_j - \lambda \right\}$$

Proposition 8. *The extended cover inequality (12) is valid for $\text{conv}(K)$. In addition, inequality (12) defines a facet of $\text{conv}(K_{C \cup E})$ if and only if C is minimal and $\forall i \in E \exists$ distinct $\{j, k\} \in C$ such that $\mathbf{a}(C \cup \{i\} \setminus \{j, k\}) \leq b$.*

2.1.2. *Sequence Independent Bounds for Lifting Coefficients.*

As a follow up to extensions, it is of interest to understand whether the lifting coefficients α_i, $i \in N \setminus C$ in the lifted cover inequality (10) can be computed efficiently in general. Sequential lifting and simultaneous lifting of cover inequalities have been studied extensively (Balas, [8]; Balas and Zemel, [9, 10]; Gu, Nemhauser, and Savelsbergh, [22, 23]; Hammer, Johnson, and Peled, [25]; Wolsey, [42]; Zemel, [44]) to extend them to strong inequalities for $\text{conv}(K)$. Wolsey [41] introduced the idea of superadditivity to strengthen valid inequalities for $0-1$ integer programs. Gu et al. [24] and Atamtürk [3] have explored the idea of superadditive lifting further in relation to mixed integer programming.

Atamtürk [4] present a unified review of the inequalities, whilst focusing on the use of superadditive functions for the analysis of knapsack polyhedra. They present a superadditive lower bound on the lifting function for cover inequalities. Following is a direct consequence of the subadditive lower bound $\varphi(\cdot)$ on the lifting function $\Theta(\cdot)$ which can be used to obtain lower bounds on the lifting coefficients α_i, $i \in N \setminus C$.

Proposition 9. *For $k \in N \setminus C$, $\alpha_k \geq h$, if $a_k \geq A_h$, where $A_h = \max\limits_{\substack{T \subseteq C \\ |T| = h}} \mathbf{a}(T)$.*

Observe that A_h, $0 \leq h \leq |N|$ can be efficiently computed using greedy algorithm since $a_i \geq 0$, $i \in N$. Conversely, to obtain an upper bound on the lifting coefficients, consider the lifting function $\Theta(a)$ (11). It is easy to observe that any feasible solution to the optimization problem,

$$\text{(14)} \qquad \max \left\{ \sum_{i \in C} x_i : \sum_{i \in C} a_i x_i \leq b - a \right\}$$

provides an upper bound on $\Theta(a)$. A greedy algorithm yields a feasible solution to (14). Consider $T \subseteq S$ such that $|T| = h$. It can be seen that $\alpha_k \leq |C| - 1 - h$, if $\mathbf{a}(T) \leq b - a_k$ for $k \in N \setminus C$ for all possible lifting sequences. This yields the following result,

Proposition 10. *For $k \in N \setminus C$, $\alpha_k \leq h$, if $a_k \leq b - B_{|C|-1-h}$, where $B_h = \min\limits_{\substack{T \subseteq C \\ |T|=h}} \mathbf{a}(T)$.*

As earlier, B_h can be computed efficiently using a greedy approach.

The motivation for deriving these bounds will become more apparent in Section 3.5 where we will derive sequence independent lifting coefficients for submodular knapsack polytope, since deriving exact lifting coefficients for the submodular knapsack polytope can be computationally intractable.

2.2. Pack Inequalities.

As seen in previous section, covers in effect are infeasible sets, and cover inequalities (9) are valid inequalities obtained from these infeasible sets. Contrary to that, we can also derive valid inequalities related to the feasible sets termed as *packs*. These inequalities have been studied in different forms and by different names in the literature *viz.* weight inequalities [40], reverse-cover inequalities [31], pack inequalities [4]. We refer to these feasible sets as packs through out the following discussion, starting with the preliminaries.

Definition 2. *A subset P of the index set N is called a* pack *if $\delta := b - \mathbf{a}(P) > 0$.*

For a pack P, consider the restriction K_P, obtained by xing all x_i, $i \in P$ to one. Since any x_i, $i \in N \setminus P$ with coefficient a_i greater than the residual capacity δ cannot take the value one in a feasible solution to K_P, the *weight inequality [40]/pack inequality [4]*,

$$(15) \qquad \sum_{i \in N \setminus P} (a_i - \delta)^+ x_i \leq 0$$

is valid for $\text{conv}(K_P)$. Weismantel [40] extended these inequalities to derive valid inequalities for $\text{conv}(K)$, their main result on reduction of weight inequalities,

Proposition 11 (Weismantel [40]). *Let $P \subseteq N$ satisfy $\mathbf{a}(P) < b$ and the index set $M := \{i \in N \setminus P : a_i > \delta\} \neq \varnothing$ where $\delta = b - \mathbf{a}(P)$. For an item $k \in P$ such that $a_k \geq a_i$ for all $i \in P$ and $\psi \in [0, \delta]$ with $a_k - \psi > 0$ the weight-reduction inequality with respect to T (and k) and ψ is defined as,*

$$(16) \qquad \sum_{i \in P \setminus \{k\}} a_i x_i + (a_k - \psi) x_k + \sum_{j \in M} c_j x_j \leq \mathbf{a}(P) - \psi.$$

where,

$$c_j = \begin{cases} (a_j - \delta), & \text{if } \delta < a_j \leq a_k + \delta - \psi, \\ (a_j - \psi), & \text{if } a_k + \delta - \psi < a_j \leq a_k + \delta, \\ (a_j - \delta - \psi), & \text{if } a_j > a_k + \delta. \end{cases}$$

The weight reduction inequality with respect to T and ψ is valid for $\text{conv}(K)$.

Atamtürk [4] strengthened the result of Weismantel [40] *via* superadditive lifting of the inequality (15).

Proposition 12 (Atamtürk [4]). *Suppose $\{i \in N \setminus P : a_i > \delta\} = \{1, 2, \ldots, q\}$ and $a_1 \geq a_2 \geq \ldots \geq a_q$. Then let $A_i = -\sum_{k=1}^{i} a_k$ for $i \in \{1, 2, \ldots, q\}$ and $A_0 = 0$. The inequality*

$$\sum_{i \in P} \psi(-a_i)(1 - x_i) + \sum_{i \in N \setminus P} (a_i - \delta)^+ x_i \leq 0$$

is valid for $\text{conv}(K)$ where,

$$\psi(a) = \begin{cases} i\delta + a, & \text{if } A_{i+1} + \delta \leq a \leq A_i, \\ i\delta + A_i, & \text{if } A_i \leq a \leq A_i + \delta, \\ q\delta + A_q, & \text{if } a \leq A_q + \delta. \end{cases}$$

is the superadditive lower bound on the lifting function of weight inequality (15).

In the following, we will discuss the inequalities obtained from aforementioned feasible sets, and derive extensions and sequence independent bounds on lifting coefficients as in the case of covers. To begin with, consider the index set $M := \{i \in N \setminus P \ : \ a_i > \delta\}$

$$\sum_{i \in M} x_i \leq 0 \tag{17}$$

is an alternate version of the weight inequality (15) and is thus valid for K_P. We term the class of inequalities (17) derived from feasible sets (packs) as *pack inequalities*. Furthermore, pack inequality defines a facet of K_P if and only if P is a maximal pack, that is, $\mathbf{a}(P \cup \{i\}) > b, \ \forall \, i \in N \setminus P$.

2.2.1. *Extensions of Pack Inequalities.*

Pack inequalities obtained from the feasible lower dimensional restrictions of the knapsack set K are valid for the respective restrictions. Lifting of pack inequalities is thus necessary to obtain valid inequalities for $conv(K)$. Atamtürk [4] explore superadditive lower bounds for the lifting functions of the weight inequality (15) to derive the valid inequalities for conv(K).

For a pack $P \subseteq N$ and the index set M as defined in Section 2.2, consider the pack inequality (17).

$$\mathbf{x}(M) \leq 0$$

As in the case of cover inequalities, we can strengthen the above inequality by introducing the variables not present in the inequality $(x_i, \ \forall \, i \in N \setminus M)$ multiplied with appropriate coefficients. A trivial procedure to extend the pack inequality (17) is by extending the pack P to obtain a maximal pack $\bar{P}$ (if $\exists \, a_j \leq \delta, \, j \in N \setminus P$ define $\bar{P} = P \cup \{j\}$) and $\bar{\delta} = b - \mathbf{a}(\bar{P})$. The corresponding *maximal pack inequality*,

$$\mathbf{x}(N \setminus \bar{P}) \leq 0 \tag{18}$$

The maximal pack inequality can further be lifted to the lifted pack inequality,

$$\mathbf{x}(N \setminus \bar{P}) - \sum_{i \in \bar{P}} \alpha_i (1 - x_i) \leq 0$$

where $\alpha_i, \, i \in \bar{P}$ are the corresponding lifting coefficients. The lifting function of (18) can be expressed as,

$$\Lambda(a) = \max \left\{ \mathbf{x}(N \setminus \bar{P}) \ : \ \sum_{i \in N \setminus \bar{P}} a_i x_i \leq \bar{\delta} + a \right\} \tag{19}$$

for $a \geq 0$.

Define,

(i) $\bar{A}_i = \min \left\{ \mathbf{a}(T) \ : \ T \subseteq N \setminus \bar{P}, |T| = i \right\}$

(ii) $\bar{B}_i = \max \left\{ \mathbf{a}(T) \ : \ T \subseteq N \setminus \bar{P}, |T| = i \right\}$

Observe that $\bar{A}_i$ and $\bar{B}_i$ can both be computed efficiently by a greedy approach. Suppose (without loss of generality) $N \setminus \bar{P} = \{1, 2, 3, \ldots, m\}$ and $a_1 \geq a_2 \geq \ldots \geq a_m$. Then $\bar{A}_i = \sum_{j=m-i+1}^{m} a_j$ and $\bar{B}_i = \sum_{j=1}^{i} a_i$ with $\bar{A}_0 = \bar{B}_0 = 0$. The lifting function $\Lambda(a), \, a \geq 0$, can be expressed in closed form as,

$$\Lambda(a) = \begin{cases} 0, & \text{if } \bar{A}_0 \leq a < \bar{A}_1 - \bar{\delta}, \\ i, & \text{if } \bar{A}_i - \bar{\delta} \leq a < \bar{A}_{i+1} - \bar{\delta}, \\ m, & \text{if } \bar{A}_m - \bar{\delta} \leq a. \end{cases} \tag{20}$$

$\Lambda(\cdot)$ is a step function, however $\psi(\cdot)$,

$$(21) \qquad \psi(a) = \begin{cases} i + \left(\frac{a - \bar{A}_i}{\bar{A}_{i+1} - \bar{A}_i - \bar{\delta}} \right), & \text{if } \bar{A}_i \le a \le \bar{A}_{i+1} - \bar{\delta}, \\ i, & \text{if } \bar{A}_i - \bar{\delta} \le a \le \bar{A}_i, \\ m, & \text{if } \bar{A}_m - \bar{\delta} \le a. \end{cases}$$

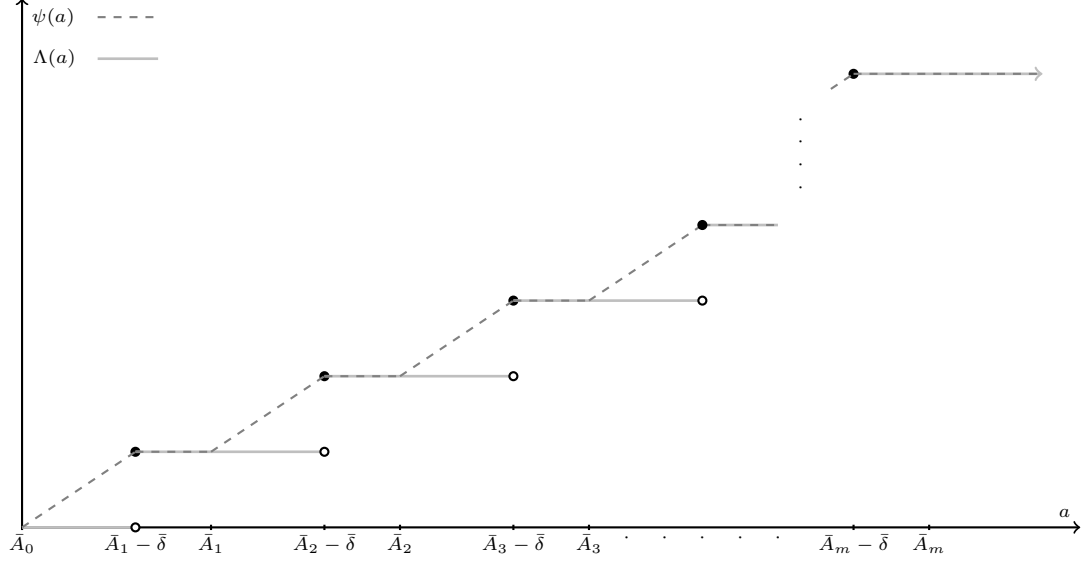for $a \ge 0$, and $i \in \{0, 1, 2 \ldots, m-1\}$ represents a subadditive upper bound on $\Lambda(a)$.

FIGURE 1: Lifting function $\Lambda(a)$ and subadditive upper bound $\psi(a)$, $a \ge 0$

Proposition 13. *The lifted pack inequality,*

$$(22) \qquad \mathbf{x}(N \setminus \bar{P}) - \sum_{i \in \bar{P}} \psi(a_i)(1 - x_i) \le 0$$

is valid for $\mathrm{conv}(K)$. *In addition,* (22) *defines a* $n - m - 1$ *dimensional face of* $\mathrm{conv}\, K$ *if* $A_{i_k} - \bar{\delta} \le a_k \le A_{i_k}$, *for some* $i_k \in \{0, 1, 2, \ldots, m-1\}$ *for all* $k \in \bar{P}$.

As in the case of cover inequalities, a computationally inexpensive procedure to lift maximal pack inequalities is to identify whether a variable x_i, $i \in \bar{P}$ can be introduced into the maximal pack inequality with a coefficient $\alpha_i \ge 1$. Let E be a set of all such indices, i.e. $E := \{i \in \bar{P} : \alpha_i \ge 1\}$. The extended pack inequality thus is,

$$(23) \qquad \mathbf{x}(E \cup (N \setminus \bar{P})) \le |E|$$

Consider a subset of indices $\bar{E} \subseteq \bar{P}$. Assume that the corresponding variables x_i, $i \in \bar{E}$ have been included in the extended cover inequality. Consider the variable $x_k, k \in \bar{P} \setminus \bar{E})$ to be included next in the extended cover.

Proposition 14. $\alpha_k \ge 1$, *if* $a_k + \bar{\delta} \ge \max\limits_{j \in N \setminus \bar{P}} a_j$.

where $\bar{\delta} := b - a(\bar{P})$ is the residual capacity of the maximal pack. The corresponding extension set $E \subseteq \bar{P}$ can now be defined as,

$$(24) \qquad E := \left\{ i \in \bar{P} : a_i + \bar{\delta} \ge \max\limits_{j \in N \setminus \bar{P}} a_j \right\}$$

Proposition 15. *The extended pack inequality* (23) *is valid for* $\mathrm{conv}(K_{\bar{P} \setminus E})$. *In addition, inequality* (23) *defines a facet of* $\mathrm{conv}(K_{\bar{P} \setminus E})$ *if and only if* $\bar{P}$ *is maximal and* $\forall\, i \in E \; \exists\, distinct \; \{j, k\} \in N \setminus \bar{P}$ *such that* $\mathbf{a}(\bar{P} \cup \{j, k\} \setminus \{i\}) > b$.

2.2.2. *Sequence Independent Bounds on the Lifting Coefficients.*

As in the case of covers, it is of interest to determine sequence independent bounds on the lifting coefficients of the lifted pack inequality (19). The particular bounds can be obtained using the lifting function $\Lambda(\cdot)$ and the subadditive upper bound $\psi(\cdot)$. Observe that any feasible solution to $\max\left\{\mathbf{x}(N\setminus\bar{P}) \; : \; \sum_{i\in N\setminus\bar{P}} a_i x_i \leq \bar{\delta} + a_j\right\}$ provides a lower bound on the lifting coefficient α_j, $j \in \bar{P}$. Alternatively, the subadditive upper bound $\psi(a_j)$ yields an upper bound on α_j, $j \in \bar{P}$. Consider the following,

Proposition 16. *Suppose that $\bar{P} \subseteq N$ is a maximal pack, and the lifted pack inequality (19) defines a $n-m-1$ dimensional face of $\mathrm{conv}(K)$, then the following statements hold for all α_j, $j \in \bar{P}$,*

    (i) $\alpha_j \leq h$, if $a_j \leq \bar{A}_h$.
    (ii) $\alpha_j \geq h$, if $a_j \geq \bar{B}_h - \bar{\delta}$.

Proof. (i) Follows from the definition of subadditive upper bound $\psi(a_j)$, $j \in \bar{P}$.
(ii) Consider $T \subseteq S$ such that $|T| = h$. Observe that, for $j \in \bar{P}$, $\alpha_j \geq h$, if $a(T) \leq \bar{\delta} + a_j$ for all possible lifting sequences. In other words,

$$\alpha_j \geq h \quad \text{if} \max_{\substack{T\subseteq N\setminus\bar{P} \\ |T|=h}} \mathbf{a}(T) \leq a_j + \bar{\delta}$$

$$\alpha_j \geq h \quad \text{if } \bar{B}_h \leq a_j + \bar{\delta}.$$

The result follows. □

Corollary 17. *The extended pack inequality,*

$$(25) \qquad \mathbf{x}(N\setminus\bar{P}) + \sum_{j\in\bar{P}} c_j x_j \leq \mathbf{c}(\bar{P})$$

is valid for $\mathrm{conv}(K)$, where $c_j = h$, if $\bar{A}_h \leq a_j \leq \bar{B}_h - \bar{\delta}$, $j \in \bar{P}$.

2.3. **Generalizing the linear 0-1 knapsack.** In this section, we present a more general expression for the cover inequalities whilst relaxing the earlier assumption of $\mathbf{a} > 0$. While this is a trivial exercise for the reader, and these expressions can be readily obtained via complementing variables x_i where corresponding $a_i < 0$, we present the following results to provide a sense of completeness to this analysis.

Reconsider the knapsack set K, with $\mathbf{a} \in \mathbb{R}$.

Define the indexed sets, I^+ and I^- as following,

$$I^+ := \{i \in N \; : \; a_i > 0\} \qquad I^- = N\setminus I^+ := \{i \in N \; : \; a_i < 0\}$$

The knapsack set K can be redefined for the aforementioned indexed sets as following,

$$(26) \qquad K = \left\{\mathbf{x} \in \{0,1\}^n \; : \; \sum_{i\in I^+} a_i x_i + \sum_{j\in I^-} |a_j|\bar{x}_j \leq b + \sum_{j\in I^-} |a_j|\right\}$$

where $\bar{x}_j = 1 - x_j$, $\forall\, j \in I^-$.

Proposition 18. *Inequality $x_i \geq 0$, $i \in I^+$ is facet defining for $\mathrm{conv}(K)$.*

Proposition 19. *Inequality $x_i \leq 1$, $i \in I^-$ is facet defining for $\mathrm{conv}(K)$.*

Proposition 20. *Inequality $x_i \leq 1$, $i \in I^+$ is facet defining for $\mathrm{conv}(K)$ if and only if $a_i + |a_k| \leq b + |\mathbf{a}|(I^-)$, $\forall\, k \in N\setminus\{i\}$.*

Proposition 21. *Inequality $x_i \geq 0$, $i \in I^-$ is facet defining for $\mathrm{conv}(K)$ if and only if $|a_i| + |a_k| \leq b + |\mathbf{a}|(I^-)$, $\forall\, k \in N\setminus\{i\}$.*

Proposition 22. *If the inequality $\boldsymbol{\alpha}\mathbf{x} \leq \beta$ defines a facet for $\mathrm{conv}(K)$ not including $\mathbf{0}$ then $\beta > \boldsymbol{\alpha}(I^-)$, $\alpha_i \geq 0$, $\forall\, i \in I^+$ and $\alpha_j \leq 0$, $\forall\, j \in I^-$.*

2.3.1. *Cover Inequalities,* $\mathbf{a} \in \mathbb{R}$.

Consider sets, $C^+ \subseteq I^+$ and $C^- \subseteq I^-$, such that $C^+ \cup C^-$ constitutes a cover, i.e.

$$\sum_{i \in C^+} a_i + \sum_{j \in C^-} |a_j| \qquad > b + \sum_{j \in I^-} |a_j|$$

$$\sum_{i \in C^+} a_i - \sum_{j \in I^- \setminus C^-} |a_j| \qquad > b$$

$$\sum_{i \in C^+} a_i + \sum_{j \in I^- \setminus C^-} a_j \qquad > b$$

$$\sum_{i \in C^+ \cup I^- \setminus C^-} a_i \qquad > b.$$

then, the corresponding cover inequality can be expressed as,

$$\sum_{i \in C^+} x_i + \sum_{j \in C^-} \bar{x}_j \le |C^+ \cup C^-| - 1$$

$$\sum_{i \in C^+} x_i + \sum_{j \in C^-} (1 - x_j) \le |C^+| + |C^-| - 1$$

$$\sum_{i \in C^+} x_i - \sum_{j \in C^-} x_j + |C^-| \le |C^+| + |C^-| - 1$$

$$\sum_{i \in C^+} x_i - \sum_{j \in C^-} x_j \le |C^+| - 1$$

The following result follows,

Proposition 23. *The* generalized cover inequality,

(27) $$\mathbf{x}(C^+) - \mathbf{x}(C^-) \le |C^+| - 1$$

is valid for $\mathrm{conv}(K)$. *Furthermore, inequality* (27) *defines a facet for* $\mathrm{conv}\, K_{C^+ \cup C^-}$ *if and only if* $C^+ \cup C^-$ *is of minimal cardinality, i.e.*

$$|a_i| \ge \lambda, \quad \forall \ i \in C^+ \cup C^-$$

It can also be seen that the corresponding the generalized lifted cover inequality in this case can be represented as,

$$\mathbf{x}(C^+) + \sum_{i \in I^+ \setminus C^+} \alpha_i x_i + \sum_{j \in I^- \setminus C^-} \alpha_j \bar{x}_j - \mathbf{x}(C^-) \le |C^+| - 1$$

$$\mathbf{x}(C^+) + \sum_{i \in I^+ \setminus C^+} \alpha_i x_i + \sum_{j \in I^- \setminus C^-} \alpha_j (1 - x_j) - \mathbf{x}(C^-) \le |C^+| - 1$$

$$\mathbf{x}(C^+) + \sum_{i \in I^+ \setminus C^+} \alpha_i x_i - \sum_{j \in I^- \setminus C^-} \alpha_j x_j - \mathbf{x}(C^-) \le |C^+| - \boldsymbol{\alpha}(I^- \setminus C^-) - 1$$

which yields the generalized lifted cover inequality,

(28) $$\mathbf{x}(C^+) + \sum_{i \in I^+ \setminus C^+} \alpha_i x_i - \sum_{j \in I^- \setminus C^-} \alpha_j x_j - \mathbf{x}(C^-) \le |C^+| - (\boldsymbol{\alpha}(I^- \setminus C^-) + 1)$$

where $\alpha_i \ge 0$, $i \in I^+ \setminus C^+$ and $\alpha_j \ge 0$, $j \in I^- \setminus C^-$ are the corresponding lifting coefficients. Let E be a set of all such indices, i.e. $E := \{i \in N \setminus (C^+ \cup C^-) \ : \ \alpha_i \ge 1\}$.

Sequence independent extensions for this generalization can also be obtained in a manner similar to Section 2.1.1. We characterize the extension set E with respect to $(C^+ \cup C^-)$ as,

(29) $$E := \left\{ i \in N \setminus (C^+ \cup C^-) \ : \ |a_i| > \max_{j \in C^+ \cup C^-} |a_j| - \lambda \right\}$$

where $\lambda = \mathbf{a}(C^+ \cup (I^- \setminus C^-)) - b$.

Further define subsets E^+ and E^- as,

$$E^+ := I^+ \cap E \qquad E^- := I^- \cap E$$

The *generalized extended cover inequality* can now be expressed as,

$$\mathbf{x}(C^+ \cup E^+) + \bar{\mathbf{x}}(C^- \cup E^-) \leq |C^+| + |C^-| - 1$$
$$\mathbf{x}(C^+ \cup E^+) + |C^- \cup E^-| - \mathbf{x}(C^- \cup E^-) \leq |C^+| + |C^-| - 1$$
$$\mathbf{x}(C^+ \cup E^+) + |C^-| + |E^-| - \mathbf{x}(C^- \cup E^-) \leq |C^+| + |C^-| - 1$$
$$\mathbf{x}(C^+ \cup E^+) - \mathbf{x}(C^- \cup E^-) \leq |C^+| - (|E^-| + 1)$$

The result follows,

Proposition 24. *The generalized extended cover inequality,*

(30) $$\mathbf{x}(C^+ \cup E^+) - \mathbf{x}(C^- \cup E^-) \leq |C^+| - (|E^-| + 1)$$

is valid for $\mathrm{conv}(K)$. *In addition, inequality* (30) *defines a facet of* $\mathrm{conv}(K_{C^+ \cup C^- \cup E^+ \cup E^-})$ *if and only if* $C^+ \cup C^-$ *is of minimal cardinality and* $\forall\, i \in E^+ \cup E^-\ \exists\, distinct\ \{j, k\} \in C^+ \cup C^-$ *such that* $|\mathbf{a}|(C^+ \cup C^- \cup \{i\} \setminus \{j, k\}) \leq b + |\mathbf{a}|(I^-)$

2.3.2. *Sequence Independent Bounds on Lifting Coefficients.*

As a direct generalization of the results presented in Section 2.1.2 we present the following conditions to provide sequence independent bounds on the lifting coefficients of the lifted cover inequality (28),

$$\mathbf{x}(C^+) + \sum_{i \in I^+ \setminus C^+} \alpha_i x_i - \sum_{j \in I^- \setminus C^-} \alpha_j x_j - \mathbf{x}(C^-) \leq |C^+| - (\boldsymbol{\alpha}(I^- \setminus C^-) + 1)$$

where $\alpha_i \geq 0$, $i \in I^+ \setminus C^+$ and $\alpha_j \geq 0$, $j \in I^- \setminus C^-$ are the corresponding lifting coefficients.

To obtain the lower bounds on the aforementioned lifted coefficients, we use the alternate representation of the linear $0-1$ knapsack (26). Proposition 9 and Proposition 10 can be readily extended to this case to yield lower and upper bounds on the lifting coefficients, α_i, $i \in N \setminus (C^+ \cup C^-)$.

Definition 3. Let $(C^+ \cup C^-) \subset N$ be of minimal cardinality such that

$$\mathbf{a}(C^+ \cup (I^- \setminus C^-)) > b.$$

For $h = 0, 1, 2, 3, \ldots, |(C^+ \cup C^-)|$, define,

(31) $$A_h := \max\, \{|\mathbf{a}|(T)\ :\ T \in \mathcal{T}_h\}$$
(32) $$B_h := \min\, \{|\mathbf{a}|(T)\ :\ T \in \mathcal{T}_h\}$$

where $\mathcal{T}_h := \{T\ :\ |T| = h,\ T \subseteq (C^+ \cup C^-)\}$.

Proposition 25. *Let* $(C^+ \cup C^-) \subset N$ *be a cover with* $\lambda := \mathbf{a}(C^+ \cup (I^- \setminus C^-)) - b$ *and* A_h *and* B_h, $h = 0, 1, 2, 3, \ldots, |C^+ \cup C^-|$ *be defined as in* (31) *and* (32). *Suppose that the lifted pack inequality* (28) *defines a facet of* $\mathrm{conv}(K)$. *Then, for* $k \in N \setminus (C^+ \cup C^-)$, *the following statements hold,*

    (i) If $|a_k| \geq A_h$, *then* $\alpha_k \geq h$.
    (ii) If $|a_k| \leq b + |\mathbf{a}|(I^-) - B_{|C^+ \cup C^-| - 1 - h}$, *then* $\alpha_k \leq h$

2.3.3. *Pack Inequalities,* $\mathbf{a} \in \mathbb{R}$.

Akin to the cover inequalities presented for K, $\mathbf{a} \in \mathbb{R}$ above, we can rewrite the pack inequalities similarly, i.e. *(via. complementation)*

Reconsider the knapsack set K, with $\mathbf{a} \in \mathbb{R}$. Apropos of the index sets I^+ and I^- and corresponding feasible set K (26) defined earlier in Section 2.3.1

Consider now, $P^+ \subseteq I^+$ and $P^- \subseteq I^-$, such that $P^+ \cup P^-$ defines a pack, i.e.,

$$\sum_{i \in P^+} a_i + \sum_{j \in P^-} |a_j| \quad < b + \sum_{j \in I^-} |a_j|$$

$$\sum_{i \in P^+} a_i - \sum_{j \in I^- \setminus P^-} |a_j| \quad < b$$

$$\sum_{i \in P^+} a_i + \sum_{j \in I^- \setminus P^-} a_j \quad < b$$

$$\sum_{i \in P^+ \cup I^- \setminus P^-} a_i \quad < b.$$

For $\delta := b - \mathbf{a}(P^+ \cup (I^- \setminus P^-))$ and $M := \{i \in N \setminus (P^+ \cup P^-) \ : \ |a_i| > \delta\}$, the corresponding pack inequality can be expressed as,

$$\sum_{i \in M \cap I^+} x_i + \sum_{j \in M \cap I^-} \bar{x}_j \leq 0$$

$$\sum_{i \in M \cap I^+} x_i + \sum_{j \in M \cap I^-} (1 - x_j) \leq 0$$

$$\sum_{i \in M \cap I^+} x_i + |M \cap I^-| \leq \sum_{j \in M \cap I^-} x_j$$

$$\sum_{i \in M \cap I^-} x_i - \sum_{j \in M \cap I^+} x_j \geq |M \cap I^-|$$

The following result follows,

Proposition 26. *The* generalized pack inequality,

$$(33) \qquad\qquad \mathbf{x}(M \cap I^-) - \mathbf{x}(M \cap I^+) \geq |M \cap I^-|$$

is valid for $\mathrm{conv}(K_{P^+ \cup P^-})$. *Furthermore, inequality* (33) *defines a facet for* $\mathrm{conv}(K_{P^+ \cup P^-})$ *if and only if* $P^+ \cup P^-$ *is of maximal cardinality, i.e.*

$$|a_i| > \delta, \quad \forall \ i \in N \setminus (P^+ \cup P^-)$$

or $M = N \setminus (P^+ \cup P^-)$

In addition, attributing to the above preliminaries we can rewrite the weight inequalities for the generalized $0 - 1$ knapsack,

Proposition 27. *The generalized weight inequality,*

$$(34) \qquad \sum_{i \in P^-} a_i x_i - \sum_{i \in P^+} a_i(1 - x_i) + \sum_{j \in M \cap I^+} (a_j - \delta)^+ x_j + \sum_{j \in M \cap I^-} (-a_j - \delta)^+ (1 - x_j) \leq 0$$

is valid for $\mathrm{conv}(K)$.

For $\bar{P} = \bar{P}^+ \cup \bar{P}^-$ maximal and corresponding residue $\bar{\delta} = b - \mathbf{a}(\bar{P}^+ \cup (I^- \setminus \bar{P}^-))$, we have the maximal pack inequality,

$$(35) \qquad\qquad \mathbf{x}(I^- \setminus \bar{P}) - \mathbf{x}(I^+ \setminus \bar{P}) \geq |I^- \setminus \bar{P}|$$

and the maximal weight inequality,

$$(36) \qquad \sum_{i \in P^-} a_i x_i - \sum_{i \in P^+} a_i(1 - x_i) + \sum_{j \in I^+ \setminus P^+} (a_j - \bar{\delta})^+ x_j + \sum_{j \in I^- \setminus P^-} (-a_j - \bar{\delta})^+ (1 - x_j) \leq 0$$

It can also be seen that the corresponding generalized lifted pack inequality for $\bar{P}$ maximal, can be represented as,

$$\mathbf{x}(I^- \setminus \bar{P}) - \sum_{i \in \bar{P}^+} \alpha_i(1 - x_i) - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} \alpha_j(1 - \bar{x}_j) \geq |I^- \setminus \bar{P}|$$

$$\mathbf{x}(I^- \setminus \bar{P}) + \sum_{i \in \bar{P}^+} \alpha_i x_i - \boldsymbol{\alpha}(\bar{P}^+) - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} \alpha_j x_j \geq |I^- \setminus \bar{P}|$$

$$\mathbf{x}(I^- \setminus \bar{P}) + \sum_{i \in \bar{P}^+} \alpha_i x_i - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} \alpha_j x_j \geq |I^- \setminus \bar{P}| + \boldsymbol{\alpha}(\bar{P}^+)$$

which yields the generalized lifted pack inequality,

$$(37) \qquad \mathbf{x}(I^- \setminus \bar{P}) + \sum_{i \in \bar{P}^+} \alpha_i x_i - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} \alpha_j x_j \geq |I^- \setminus \bar{P}| + \boldsymbol{\alpha}(\bar{P}^+)$$

where α_i, $i \in \bar{P}^+ := \bar{P} \cap I^+$ and α_j, $j \in \bar{P}^- := \bar{P} \cap I^-$ are the corresponding lifting coefficients. Let E be a set of all such indices, i.e. $E := \{i \in \bar{P} : \alpha_i \geq 1\}$.

We characterize the extension set E with respect to $(\bar{P}^+ \cup \bar{P}^-)$ as,

$$(38) \qquad E := \left\{i \in \bar{P} : |a_i| + \bar{\delta} \geq \max_{j \in N \setminus \bar{P}} |a_j|\right\}$$

where $\bar{\delta} = b - \mathbf{a}(\bar{P}^+ \cup (I^- \setminus \bar{P}^-))$.

Further define subsets E^+ and E^- as,

$$E^+ := I^+ \cap E \qquad E^- := I^- \cap E$$

The *generalized pack inequality* can now be expressed as,

$$\mathbf{x}(E^+ \cup I^- \setminus \bar{P}) - \mathbf{x}(E^- \cup I^+ \setminus \bar{P}) \geq |I^- \setminus \bar{P}| + |E^+|$$

The result follows,

Proposition 28. *The generalized extended pack inequality,*

$$(39) \qquad \mathbf{x}(E^+ \cup I^- \setminus \bar{P}) - \mathbf{x}(E^- \cup I^+ \setminus \bar{P}) \geq |I^- \setminus \bar{P}| + |E^+|$$

is valid for $\mathrm{conv}(K_{\bar{P}^+ \cup \bar{P}^- \setminus (E^+ \cup E^-)})$. *In addition, inequality* (39) *defines a facet of* $\mathrm{conv}(K_{\bar{P}^+ \cup \bar{P}^- \setminus (E^+ \cup E^-)})$ *if and only if* $\bar{P}$ *is of maximal cardinality and* $\forall\, i \in E\; \exists\, distinct\; \{j,k\} \in N \setminus \bar{P}$ *such that* $|\mathbf{a}|(\bar{P} \cup \{j,k\} \setminus \{i\}) > b + |\mathbf{a}|(I^-)$

The extended pack inequalities are often valid for low dimensional extensions of the knapsack set and not the original set itself. Lifting the pack inequalities yields us strong inequalities that are valid for the convex hull of the original knapsack set, $\mathrm{conv}(K)$. The lifted pack inequality for the generalized linear $0-1$ knapsack can be expressed as (37),

$$\mathbf{x}(I^- \setminus \bar{P}) + \sum_{i \in \bar{P}^+} \alpha_i x_i - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} \alpha_j x_j \geq |I^- \setminus \bar{P}| + \boldsymbol{\alpha}(\bar{P}^+)$$

where $\bar{P}$ represents a maximal pack, and $\alpha_i \geq 0$, $i \in \bar{P}^+$ and $\alpha_j \geq 0$, $j \in \bar{P}^-$ are the corresponding lifting coefficients.

Sequence independent bounds on the aforementioned lifting coefficients can be obtained as a direct generalizations of Proposition 16, the results follows,

Proposition 29. *Suppose that* $\bar{P} \subseteq N$ *is a maximal pack. Let* $m = |N \setminus \bar{P}|$ *and suppose the lifted pack inequality* (19) *defines a* $n - m - 1$ *dimensional face of* $\mathrm{conv}(K)$, *then the following statements hold for all* α_j, $j \in \bar{P}$,

(i) $\alpha_j \leq h$, *if* $|a_j| \leq \bar{A}_h$.

(ii) $\alpha_j \geq h$, if $|a_j| \geq \bar{B}_h - \bar{\delta}$.

where $\bar{A}_h = \min \{ \mathbf{a}(T) \ : \ T \subseteq N \setminus \bar{P}, |T| = h \}$ and $\bar{B}_h = \max \{ \mathbf{a}(T) \ : \ T \subseteq N \setminus \bar{P}, |T| = h \}$ for $h = 0, 1, 2, \ldots, m$ and the excess $\bar{\delta} = b - \mathbf{a}(\bar{P}^+ \cup (I^+ \setminus \bar{P}^-))$

Corollary 30. *The extended generalized pack inequality,*

$$(40) \qquad \mathbf{x}(I^- \setminus \bar{P}) + \sum_{i \in \bar{P}^+} c_i x_i - \mathbf{x}(I^+ \setminus \bar{P}) - \sum_{j \in \bar{P}^-} c_j x_j \geq |I^- \setminus \bar{P}| + \mathbf{c}(\bar{P}^+)$$

is valid for $\mathrm{conv}(K)$, *where* $c_j = h$, *if* $\bar{A}_h \leq a_j \leq \bar{B}_h - \bar{\delta}$, $j \in \bar{P}$.

3. Generalizations to Submodular Knapsacks

In this section we will extend the results of the $0-1$ linear knapsack to submodular knapsacks. Consider a set function $f : \{0,1\}^n \mapsto \mathbb{R}$, $f(\varnothing) = 0$. For f submodular on a finite set N and $b \in \mathbb{R}$, we define the submodular knapsack set K^f as lower level set of f,

$$K^f = \{ \mathbf{x} \in \{0,1\}^n \ : \ f(\mathbf{x}) \leq b \}$$

A special case for the submodular knapsack polytope, namely when f is non-decreasing on N has been studied by Atamtürk and Narayanan [6] where they derive cover inequalities and their extensions for this special case. The generalization of cover inequalities to this special case, although interesting appears to lack the understanding of their *evolution*. It also appears that the general submodular knapsack has not been studied yet in the literature. In this particular section we will present the corresponding valid inequalities and their extensions for the generalized submodular polytope $\mathrm{conv}(K^f)$, whilst also providing an understanding *vis-à-vis* how these inequalities can be derived from the conjunction of two of the main understandings for knapsacks and submodular functions, namely the cover inequalities and the extended polymatroids. These inequalities per our understanding serve as the most generalized version of the cover and pack inequalities for the knapsack polytope.

3.1. Submodular functions and Extended Polymatroids.

Definition 4. Let $f : \{0,1\}^n \mapsto \mathbb{R}$ be a submodular set function on a finite set N. The set EP_f is called the extended polymatroid associated with f, where,

$$EP_f := \{ \mathbf{v} \in \mathbb{R}^n \ : \ \mathbf{v}(S) \leq f(S), \ S \subset N \}.$$

Theorem 31. *(Edmonds, 1970) [17] For any submodular function f on a finite set N ($f(\varnothing) = 0$), the vertices of the extended polymatroid EP_f are given by,*

$$v_j^{\boldsymbol{\pi}} = f(S_j^{\boldsymbol{\pi}}) - f(S_{j-1}^{\boldsymbol{\pi}}),$$

for a permutation $\boldsymbol{\pi}$ of $\{1, 2, \ldots, n\}$, and $S_j^{\boldsymbol{\pi}}$ denotes the set consisting of first j elements of the permutation $\boldsymbol{\pi}$.

Definition 5. Let $\mathbf{V}_f$ denote the set of all extreme points of EP_f.

Corollary 32. *Consider the following set,*

$$B := \{ (\mathbf{x}, z) \in \{0,1\}^n \times \mathbb{R} \ : \ f(x) \leq z \}$$

then

$$\mathbf{v}\mathbf{x} \leq z, \ \mathbf{v} \in EP_f$$

is valid for B. In particular, $\mathbf{v}\mathbf{x} \leq z$ defines a facet of B for all $\mathbf{v} \in \mathbf{V}_f$.

Corollary 33. *Define, for some $\gamma \in \mathbb{R}$, $K_{\mathbf{v}} := \{ \mathbf{x} \in \{0,1\}^n \ : \ \mathbf{v}\mathbf{x} \leq \gamma \}$ for some $\mathbf{v} \in \mathbf{V}_f$ and $K_f := \{ \mathbf{x} \in \{0,1\}^n \ : \ f(\mathbf{x}) \leq \gamma \}$.*

(i) $\bigcap_{\mathbf{v} \in \mathbf{V}_f} K_{\mathbf{v}} = K_f$

(ii) *If $\boldsymbol{\alpha}\mathbf{x} \leq \beta$, is valid for $K_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbf{V}_f$ then $\boldsymbol{\alpha}\mathbf{x} \leq \beta$ is valid for K_f.*

(iii) If $\boldsymbol{\alpha}\mathbf{x} \le \beta$ defines a facet of $K_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbf{V}_f$ and $\exists\ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k,\ 1 \le k \le n$ affinely independent points on the hyperplane $\boldsymbol{\alpha}\mathbf{x} = \beta$ satisfying $f(\mathbf{x}^i) \le b,\ 1 \le i \le k$ then $\boldsymbol{\alpha}\mathbf{x} \le \beta$ defines a $k-1$ dimensional face of K_f. In particular, when $k = n$, $\boldsymbol{\alpha}\mathbf{x} \le \beta$ defines a facet of K_f.

(iv) Let $\boldsymbol{\alpha}\mathbf{x} \le \beta_{\mathbf{v}}$ imply that the set of conditions Γ are satisfied for some $K_{\mathbf{v}}$, $\mathbf{v} \in \mathbf{V}_f$. If $\exists\ \beta < \beta_{\mathbf{v}}\ \forall\ \mathbf{v} \in \mathbf{V}_f$, then Γ are satisfied for K_f if $\boldsymbol{\alpha}\mathbf{x} \le \beta$.

3.2. **Valid Inequalities for K^f.**

As earlier, let $f : \{0,1\}^n \mapsto \mathbb{R}$ be a submodular set function on a finite set N. Also let $\mathbf{V}_f$ denote the set of all extreme points of the extended polymatroid of f, EP_f. Moreover, for N, define a permutation $\boldsymbol{\pi}$ for the elements of N,

$$\boldsymbol{\pi} := \{(1), (2), (3), \ldots, |N|\}$$

Also, define $S \subseteq N$ as,

(41)
$$S := \{(1), (2), \ldots, (|S|)\}$$

Observe that $\sum_{(i) \in S} v_i^{\boldsymbol{\pi}} = f(S)$. For the permutation $\boldsymbol{\pi}$, define the index sets $I_{\boldsymbol{\pi}}$ and $J_{\boldsymbol{\pi}}$ as following,

$$I_{\boldsymbol{\pi}} = \left\{(i) \in \boldsymbol{\pi}\ :\ v_{(i)}^{\boldsymbol{\pi}} > 0\right\}$$
$$J_{\boldsymbol{\pi}} = \left\{(j) \in \boldsymbol{\pi}\ :\ v_{(i)}^{\boldsymbol{\pi}} < 0\right\}$$

In addition define the sets, I^+ and I^- as,

$$I^+ = \{i \in N\ :\ \rho_i(N \setminus i) > 0\}$$
$$I^- = \{j \in N\ :\ \rho_i(\varnothing) < 0\}$$

One can easily observe that $I^+ \subseteq I_{\boldsymbol{\pi}}$ and $I^- \subseteq J_{\boldsymbol{\pi}}$. We begin our subsequent discussion on valid inequalities for K_f by presenting the results on the trivial facets of the K^f,

Proposition 34. *Inequality $x_i \ge 0$, $i \in I^+$ is facet defining for* $\mathrm{conv}(K^f)$.

Proposition 35. *Inequality $x_i \le 1$, $i \in I^-$ is facet defining for* $\mathrm{conv}(K^f)$.

Proposition 36. *Inequality $x_i \le 1$, $i \in I^+$ is facet defining for $\mathrm{conv}(K^f)$ if and only if $f(i) + |\rho_j(i)| \le b + |f(I^-)|,\ \forall\ k \in N \setminus \{i\}$.*

Proposition 37. *Inequality $x_i \ge 0$, $i \in I^-$ is facet defining for $\mathrm{conv}(X)$ if and only if $|f(i)| + |\rho_j(i)| \le b + |f(I^-)|,\ \forall\ k \in N \setminus \{i\}$.*

Proposition 38. *If the inequality $\boldsymbol{\alpha}\mathbf{x} \le \beta$ defines a facet for $\mathrm{conv}(X)$ not including $\mathbf{0}$ then $\beta > \boldsymbol{\alpha}(I^-)$, $\alpha_i \ge 0,\ \forall\ i \in I^+$ and and $\alpha_j \le 0,\ \forall\ j \in I^-$.*

3.2.1. *Cover Inequalities.*

Corollary 32 suggests that for $\mathbf{v} \in \mathbf{V}$ of elements of N, $\mathbf{v}\mathbf{x} \le b$ is valid for K^f. Analogous to our discussion of the general $0-1$ linear knapsack, define the sets $C_{\boldsymbol{\pi}}^+ \subseteq I_{\boldsymbol{\pi}}$ and $C_{\boldsymbol{\pi}}^- \subseteq J_{\boldsymbol{\pi}}$,

$$C_{\boldsymbol{\pi}}^+ := I_{\boldsymbol{\pi}} \cap S$$
$$C_{\boldsymbol{\pi}}^- := J_{\boldsymbol{\pi}} \cap S$$

Definition 6. We define $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ for some permutation $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$ as a *submodular cover* if $f(S) > b$. The corresponding excess is $\lambda := f(S) - b > 0$.

Proposition 39. *Consider $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ for some permutation of $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$, then S is a submodular cover if and only if S is an extended polymatroid cover.*

Proof. The proof follows from the definitions. Let S be an extended polymatroid cover, thus,

$$b < \mathbf{v}(S) = \sum_{i \in S} v_i^{\boldsymbol{\pi}}$$

$$= \sum_{(i) \in S} f(\{S_i^{\boldsymbol{\pi}}\}) - f(\{S_{i-1}^{\boldsymbol{\pi}}\})$$

$$= f(S) - f(\varnothing)$$

$$= f(S)$$

Conversely, let S be a submodular cover, i.e.

$$b < f(S) = f(S) - f(\varnothing)$$

$$= f(S_{|S|}^{\boldsymbol{\pi}}) - \sum_{j=1}^{|S|-1} (f(S_j^{\boldsymbol{\pi}}) - f(S_j^{\boldsymbol{\pi}})) - f(\varnothing)$$

$$= \sum_{j=1}^{|S|} (f(S_j^{\boldsymbol{\pi}}) - f(S_{j-1}^{\boldsymbol{\pi}}))$$

$$= \sum_{j=1}^{|S|} v_j^{\boldsymbol{\pi}} = \mathbf{v}(S)$$

The result follows. $\square$

Definition 7. We define $S \subseteq N$, $S \coloneqq \{(1), (2), \ldots, (|S|)\}$ for some permutation $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$ as a *minimal submodular cover* if, minimal submodular cover with respect to $\boldsymbol{\pi}$ if,

$$f(S \cup (j)) \le b, \ \forall \, (j) \in J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^{-} \text{ and },$$

$$f(S \setminus (i)) \le b, \ \forall \, (i) \in C_{\boldsymbol{\pi}}^{+}$$

Proposition 40. *Consider $S \subseteq N$, $S \coloneqq \{(1), (2), \ldots, (|S|)\}$ for some permutation of $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$, then S is a minimal extended polymatroid cover if S is a minimal submodular cover with respect to $\boldsymbol{\pi}$.*

Proof. Notice that, for S to be a minimal extended polymatroid cover,

$$\sum_{i \in S} v_i + v_j \le b, \ \forall \, j \in C_{\boldsymbol{\pi}}^{-} \text{ and },$$

$$\sum_{i \in S} v_i - v_j \le b, \ \forall \, j \in C_{\boldsymbol{\pi}}^{+}$$

In other words,

$$-v_j \ge \lambda, \ \forall \, j \in C_{\boldsymbol{\pi}}^{-} \text{ and },$$

$$v_j \ge \lambda, \ \forall \, j \in C_{\boldsymbol{\pi}}^{+}$$

If S is a minimal submodular cover then,

$$-\rho_{(j)}(S) \ge \lambda, \ \forall \, (j) \in J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^{-} \ \text{ and } \rho_{(j)}(S \setminus (j)) \ge \lambda, \ \forall \, (j) \in C_{\boldsymbol{\pi}}^{+}$$

Since, f is submodular, the difference function $\rho(\cdot)$ is a non-increasing function, which yields,

$$v_{(j)}^{\boldsymbol{\pi}} \le \rho_{(j)}(S), \ \forall \, (j) \in J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^{-} \ \text{ and } v_{(j)}^{\boldsymbol{\pi}} \ge \rho_{(j)}(S \setminus (j)), \ \forall \, (j) \in C_{\boldsymbol{\pi}}^{+}$$

thus we have,

$$-v_{(j)}^{\boldsymbol{\pi}} \ge \lambda, \ \forall \, (j) \in J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^{-} \ \text{ and } v_{(j)}^{\boldsymbol{\pi}} \ge \lambda, \ \forall \, (j) \in C_{\boldsymbol{\pi}}^{+}$$

The result follows. $\square$

The following result follows directly from Proposition 27.

Proposition 41. *Let* $S \subseteq N$, $\cancel{S := \{(1),(2),\ldots,(|S|)\}\ \textit{for some permutation of}\ \pi = \{(1),(2),(3),\ldots,|N|\}}$ *be a submodular cover* with respect to a given permutation $\pi = \{(1),(2),(3),\ldots,|N|\}$, *then,*

$$(42) \qquad \sum_{i \in C_\pi^+} x_{(i)} - \sum_{j \in J_\pi \setminus C_\pi^-} x_{(j)} \leq |C_\pi^+| - 1$$

is valid for $\mathrm{conv}(K^f)$. *In addition, if S is a minimal submodular cover, then (42) defines a facet of* $\mathrm{conv}(K^f_{C_\pi^+ \cup J_\pi \setminus C_\pi^-})$.

Remark 1. Every permutation of S, $\pi^S := \{(1),(2),\ldots,(|S|)\}$ (observe $f(\pi^S) > b$, $\forall\, \pi^S$) and of the complement $\pi^{(N \setminus S)} := \{(|S|+1),(|S|+2)\ldots,(|N|)\}$ yield different index sets C_π^+ and C_π^- and hence a new valid inequality.

Remark 2. In the special case when $J_\pi = \varnothing$, $\forall\, \pi$ i.e. f is non decreasing on N, inequality (42) yields the result of Proposition 4 [6].

Example. Consider the set K as defined below,

$$K = \left\{ \mathbf{x} \in \{0,1\}^3 \ : \ 2x_1^2 + x_2^2 + 2x_3^2 - x_1 x_2 - 2x_1 x_3 - 4x_2 x_3 \leq 1 \right\}$$

The set K can be enumerated as,

$$K = \{(0,0,0),(0,1,0),(0,1,1),(1,1,1)\}$$

We have $3! = 6$ permutations and corresponding extended polymatroid inequalities. We tabulate the
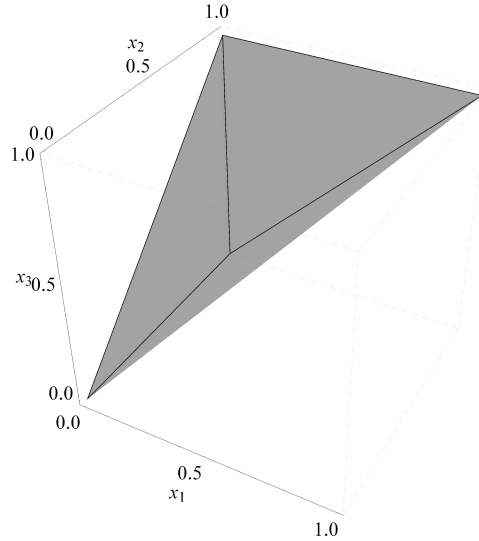
FIGURE 1. Convex hull $conv(K)$

coefficient pertaining to these inequalities in Table 1 In the first case, the natural permutation $\{1, 2, 3\}$

TABLE 1. Tabulating polymatroid coefficients for different permutations

$\pi = \{(1),(2),(3)\}$	$f(\{(1)\})$	$f(\{(1),(2)\})$	$f(N)$	v_1	v_2	v_3
$\{1,2,3\}$	2	2	-2	2	0	-4
$\{1,3,2\}$	2	2	-2	2	0	-4
$\{2,1,3\}$	1	2	-2	1	1	-4
$\{2,3,1\}$	1	-1	-2	1	-2	-1
$\{3,1,2\}$	2	2	-2	2	0	-4
$\{3,2,1\}$	2	-1	-2	2	-3	-1

yields $S = \{1, 2\}$ and $C_{\boldsymbol{\pi}}^+ = \{1\}$, $J_{\boldsymbol{\pi}} = \{3\}$, $C_{\boldsymbol{\pi}}^- = \varnothing$. Proposition 41 yields the corresponding inequality,

$$(43) \qquad\qquad\qquad\qquad x_1 \le x_3$$

The remaining aforementioned permutations yield the following submodular cover inequalities,

$$(44) \qquad\qquad\qquad\qquad x_1 \le x_2$$

$$(45) \qquad\qquad\qquad\qquad x_1 + x_2 \le x_3 + 1$$

$$(46) \qquad\qquad\qquad\qquad x_3 \le x_2$$

$$(47) \qquad\qquad\qquad\qquad x_3 \le x_1 + x_2$$

All of the aforementioned inequalities (43)-(47) are valid for conv K. Furthermore, inequalities (43) and (46) define the facets of conv K.

3.2.2. *Pack Inequalities.*

As per the preliminaries defined in Section 3.2.1, we define the notion of *submodular packs* and characterize *maximal submodular packs* as follows,

Definition 8. We define $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ for some permutation $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$ as a *submodular pack* if $f(S) < b$. The corresponding residual capacity is $\delta := b - f(S) > 0$.

Proposition 42. *Consider* $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ *for some permutation of* $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$, *then S is a submodular pack if and only if S is an extended polymatroid pack.*

Proof. The proof follows from the definitions. Let S be an extended polymatroid pack, thus,

$$\begin{aligned} b > \mathbf{v}(S) &= \sum_{i \in S} v_i^{\boldsymbol{\pi}} \\ &= \sum_{(i) \in S} f(\{S_i^{\boldsymbol{\pi}}\}) - f(\{S_{i-1}^{\boldsymbol{\pi}}\}) \\ &= f(S) - f(\varnothing) \\ &= f(S) \end{aligned}$$

Conversely, let S be a submodular cover, i.e.

$$\begin{aligned} b > f(S) &= f(S) - f(\varnothing) \\ &= f(S_{|S|}^{\boldsymbol{\pi}}) - \sum_{j=1}^{|S|-1} \left(f(S_j^{\boldsymbol{\pi}}) - f(S_j^{\boldsymbol{\pi}}) \right) - f(\varnothing) \\ &= \sum_{j=1}^{|S|} (f(S_j^{\boldsymbol{\pi}}) - f(S_{j-1}^{\boldsymbol{\pi}})) \\ &= \sum_{j=1}^{|S|} v_j^{\boldsymbol{\pi}} = \mathbf{v}(S) \end{aligned}$$

The result follows. $\square$

Definition 9. We define $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ for some permutation $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$ as a *maximal submodular pack with respect to $\boldsymbol{\pi}$* if,

$$f(S \cup (j)) > b, \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \text{ and },$$
$$f(S \setminus (i)) > b, \ \forall \, (i) \in J_{\boldsymbol{\pi}} \cap S$$

Proposition 43. *Consider* $S \subseteq N$, $S := \{(1), (2), \ldots, (|S|)\}$ *for some permutation of* $\boldsymbol{\pi} = \{(1), (2), (3), \ldots, |N|\}$, *then S is a maximal submodular pack if S is a maximal extended polymatroid pack* *with respect to $\boldsymbol{\pi}$*.

Proof. Notice that, for S to be a maximal submodular pack with respect to $\boldsymbol{\pi}$,

$$f(S \cup (j)) > b, \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \text{ and },$$
$$f(S \setminus (i)) > b, \ \forall \, (i) \in J_{\boldsymbol{\pi}} \cap S$$

In other words,

$$\rho_j(S) > \delta, \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \text{ and },$$
$$-\rho_j(S \setminus j) > \delta, \ \forall \, (i) \in J_{\boldsymbol{\pi}} \cap S$$

If S is a maximal extended polymatroid pack with respect to $\boldsymbol{\pi}$ then,

$$v_j^{\boldsymbol{\pi}} > \delta, \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \ \text{ and } - v_j^{\boldsymbol{\pi}} > \delta, \ \forall \, (j) \in J_{\boldsymbol{\pi}} \cap S$$

Since, f is submodular, the difference function $\rho(\cdot)$ is a non-increasing function, which yields,

$$v_{(j)}^{\boldsymbol{\pi}} \leq \rho_{(j)}(S), \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \ \text{ and } v_{(j)}^{\boldsymbol{\pi}} \geq \rho_{(j)}(S \setminus (j)), \ \forall \, (j) \in J_{\boldsymbol{\pi}} \cap S$$

thus we have,

$$\rho_j(S) > \delta, \ \forall \, (j) \in I_{\boldsymbol{\pi}} \setminus S \text{ and } - \rho_j(S \setminus j) > \delta, \ \forall \, (i) \in J_{\boldsymbol{\pi}} \cap S.$$

The result follows. $\qquad\square$

Define the sets $M_{\boldsymbol{\pi}}^+$ and $M_{\boldsymbol{\pi}}^-$ as follows,

(48) $$M_{\boldsymbol{\pi}}^+ := \left\{ (i) \in I_{\boldsymbol{\pi}} \setminus S \ : \ \rho_{(i)}(N \setminus (i)) > \delta \right\}$$

(49) $$M_{\boldsymbol{\pi}}^- := \left\{ (j) \in J_{\boldsymbol{\pi}} \cap S \ : \ -\rho_{(j)}(\varnothing) > \delta \right\}$$

Proposition 44. *Consider $S \subseteq N$, $S := \{(1), (2), \dots, (|S|)\}$ for some permutation of $\boldsymbol{\pi} = \{(1), (2), (3), \dots, |N|\}$. If S is a submodular pack then the* submodular pack inequality *with respect to $\boldsymbol{\pi}$,*

(50) $$\sum_{(j) \in M_{\boldsymbol{\pi}}^-} x_{(j)} - \sum_{(i) \in M_{\boldsymbol{\pi}}^+} x_{(i)} \geq |M_{\boldsymbol{\pi}}^-|$$

is valid for K_S^f.

Submodular pack inequalities (50) are derived from the restrictions of the submodular knapsack polytope and are valid for the corresponding projections and not necessarily for K^f. Weight inequalities [40] as seen in Sections 2.2 and 2.3.3 that are valid for the knapsack set, can be directly generalized to the case of submodular knapsack *via* extended polymatroid associated with the corresponding set function f. We formalize this in the following result.

Proposition 45. *The submodular weight inequality,*

(51) $$\sum_{j \in J_{\boldsymbol{\pi}} \setminus S} \rho_j(N \setminus j) x_j - \sum_{i \in S \cap I_{\boldsymbol{\pi}}} f(i)(1 - x_i) + \sum_{i \in I_{\boldsymbol{\pi}} \setminus S} (\rho_i(N \setminus i) - \delta)^+ x_i + \sum_{j \in S \cap J_{\boldsymbol{\pi}}} (-f(j) - \delta)^+ (1 - x_j) \leq 0$$

is valid for K_f.

Proof. Consider a permutation $\boldsymbol{\pi} = \{1, 2, \dots, n\}$ of the elements of the index set N. Define the corresponding index sets $I_{\boldsymbol{\pi}}$ and . Proposition 27 suggests that the following weight inequality,

$$\sum_{j \in J_{\boldsymbol{\pi}} \setminus S} v_j^{\boldsymbol{\pi}} x_j - \sum_{i \in S \cap I_{\boldsymbol{\pi}}} v_i^{\boldsymbol{\pi}} (1 - x_i) + \sum_{i \in I_{\boldsymbol{\pi}} \setminus S} (v_i^{\boldsymbol{\pi}} - \bar{\delta})^+ x_i + \sum_{j \in S \cap J_{\boldsymbol{\pi}}} (-v_j^{\boldsymbol{\pi}} - \bar{\delta})^+ (1 - x_j) \leq 0$$

is valid for K_f. Furthemore, submodularity of the set function f suggests,

$$\rho_j(N \setminus j) \leq v_j^{\boldsymbol{\pi}} \leq f(j), \forall \, j \in N, \ \forall \ \boldsymbol{\pi}$$

The result follows. $\qquad\square$

Example. *To be added...*

3.3. **Strengthening the valid inequalities *via* Extensions.**

As in the case of linear $0-1$ knapsack, we can strengthen the inequalities corresponding to *submodular covers* (42) and *submodular packs* (50) via computationally efficient extensions. In the following discussion we provide an analogous procedure to extend these inequalities as in Section 2.1.1.

Consider the subset $S \subseteq N$ as earlier in Section 42. Before we generalize the extensions for submodular set functions, we would like to define a few preliminaries.

Definition 10. Reconsider the permutation $\boldsymbol{\pi}$, and the corresponding complement set $\boldsymbol{\pi}^{(N \setminus S)} = \{(|S| + 1), (|S| + 2) \dots, (|N|)\}$ and let $S_k = S \cup \{(|S| + 1), (|S| + 2), \dots, (|S| + k)\}$, $\forall\, k = 1, 2, \dots, |N \setminus S|$. We define the extension of $C_{\boldsymbol{\pi}}^+ \subseteq S$ and reduction of $C_{\boldsymbol{\pi}}^- \subseteq S$ with respect to $\boldsymbol{\pi}^{(N \setminus S)}$ as $E_{\boldsymbol{\pi}}^+ := (I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+) \cap E$ and $R_{\boldsymbol{\pi}}^- := C_{\boldsymbol{\pi}}^- \cap E$ respectively, where,

$$E := \left\{ (i) \in (I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+) \cup C_{\boldsymbol{\pi}}^- \ : \ |v_{(i)}^{\boldsymbol{\pi}}| > \max_{(j) \in (C_{\boldsymbol{\pi}}^+ \cup J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-)} \{f(j), f(N \setminus (j)) - f(N)\} - \lambda \right\}$$

Alternatively,

$$E := \left\{ (k) \in (I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+) \cup C_{\boldsymbol{\pi}}^- \ : \ |\rho_{(k)}(S_{k-1})| > \max_{(j) \in (C_{\boldsymbol{\pi}}^+ \cup J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-)} \{f(j), |\rho_{(j)}(N \setminus (j))|\} - \lambda \right\}$$

where $\lambda = f(S) - b$

Owing to the above preliminaries, we now present the following result leading to the extensions of submodular cover inequalities (42) which is a direct generalization of Proposition 24.

Proposition 46. *If $f(S) > b$, the extended submodular cover inequality,*

(52) $$\mathbf{x}(\bar{C}_{\boldsymbol{\pi}}^+) - \mathbf{x}(J_{\boldsymbol{\pi}} \setminus \bar{C}_{\boldsymbol{\pi}}^-) \le |C_{\boldsymbol{\pi}}^+| - (|C_{\boldsymbol{\pi}}^- \setminus \bar{C}_{\boldsymbol{\pi}}^-| + 1)$$

where $\bar{C}_{\boldsymbol{\pi}}^+ = C_{\boldsymbol{\pi}}^+ \cup E_{\boldsymbol{\pi}}^+$ and $\bar{C}_{\boldsymbol{\pi}}^- = C_{\boldsymbol{\pi}}^- \setminus R_{\boldsymbol{\pi}}^-$, is valid for $\operatorname{conv}(K^f)$. Additionally, inequality (52) defines a facet of $\operatorname{conv}(K^f_{\bar{C}_{\boldsymbol{\pi}}^+ \cup J_{\boldsymbol{\pi}} \setminus \bar{C}_{\boldsymbol{\pi}}^-})$ if and only if S is maximal with $C_{\boldsymbol{\pi}}^+$ minimal and if $\forall\, (i) \in E_{\boldsymbol{\pi}}^+ \cup R_{\boldsymbol{\pi}}^- \ \exists$ distinct $\{(j), (k)\} \in S$ such that

$$f(S \cup (i) \setminus \{(j), (k)\}) \le b$$

Remark 3. It should be observed that as in the case of $0-1$ linear knapsack, and stated in Proposition 24, the extensions for submodular knapsack are *not* sequence independent.

Remark 4. In the special case when $J_{\boldsymbol{\pi}} = \varnothing, \forall\, \boldsymbol{\pi}$ i.e. f is non decreasing on N, inequality (52) yields an extension for the case considered in [6]. Though their result of Proposition 5 [6] is derived directly from the subadditive lower bound $\varphi(a)$ for lifting function $\Theta(a)$ [4].

Definition 11. Reconsider the permutation $\boldsymbol{\pi}$, and the corresponding complement set $\boldsymbol{\pi}^{(N \setminus S)} = \{(|S| + 1), (|S| + 2) \dots, (|N|)\}$ and let $S_k = S \cup \{(|S| + 1), (|S| + 2), \dots, (|S| + k)\}$, $\forall\, k = 1, 2, \dots, |N \setminus S|$. We define the extension of $M_{\boldsymbol{\pi}}^+ \subseteq N \setminus S$ and $M_{\boldsymbol{\pi}}^- \subseteq N \setminus S$ with respect to $\boldsymbol{\pi}^{(N \setminus S)}$ as $E_{\boldsymbol{\pi}}^+ := (I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+) \cap E$ and $R_{\boldsymbol{\pi}}^- := C_{\boldsymbol{\pi}}^- \cap E$ respectively, where,

$$E := \left\{ (i) \in (N \setminus (M_{\boldsymbol{\pi}}^+ \cup M_{\boldsymbol{\pi}}^-)) \ : \ |v_{(i)}^{\boldsymbol{\pi}}| + \delta > \max_{(j) \in (M_{\boldsymbol{\pi}}^+ \cup M_{\boldsymbol{\pi}}^-)} \{f(j), f(N \setminus (i)) - f(N)\} \right\}$$

Alternatively,

$$E := \left\{ (k) \in (I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+) \cup C_{\boldsymbol{\pi}}^- \ : \ |\rho_{(k)}(S_{k-1})| > \max_{(j) \in (C_{\boldsymbol{\pi}}^+ \cup J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-)} \{f(j), |\rho_{(i)}(N \setminus (i))|\} - \lambda \right\}$$

where $\lambda = f(S) - b$

3.4. **Lifted Submodular Cover Inequalities.**

In this section we study the lifting problem of the valid inequalities discussed earlier in order to strengthen them. The lifting procedure has been very effective in strengthening inequalities for the linear $0-1$ min-knapsack set [8, 9, 10, 22, 25, 42] as described in Section 2.1.1.

3.4.1. *Cover Inequalities.*

The lifting problem for the submodular cover inequalities for K_f is itself an optimization problem over the supermodular knapsack set.

In particular, we lift the submodular cover inequality (42) to a valid inequality of the form,

$$(53) \qquad \mathbf{x}(C_{\boldsymbol{\pi}}^+) - \mathbf{x}(J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-) + \sum_{i \in I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+} \alpha_i x_i - \sum_{j \in C_{\boldsymbol{\pi}}^-} \alpha_j x_j \le |C_{\boldsymbol{\pi}}^+| - (\boldsymbol{\alpha}(C_{\boldsymbol{\pi}}^-) + 1)$$

where $\alpha_i \in \mathbb{R}$, $i \in (C_{\boldsymbol{\pi}}^- \cup I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+)$ are the corresponding lifting coefficients. Observe that, $\alpha_i \ge 0$, $i \in I^+ \setminus C_{\boldsymbol{\pi}}^+$ and $\alpha_j \ge 0$, $j \in I^- \cap C_{\boldsymbol{\pi}}^-$

3.5. **Sequence Independent Bounds on Lifting Coefficients.**

Akin to the linear $0 - 1$ knapsacks, we can derive the sequence independent lifting coefficients for the submodular knapsack K_f. It is now that we will utilize the results from Section 2.3.1 to derive these bounds. In addition we will also demonstrate that the earlier results by Atamtürk and Narayanan [6] are a special case of these generalizations. Before proceeding, we define the following preliminaries,

Definition 12. For an index set T, and a permutation $\boldsymbol{\pi} = \{(1), (2), \ldots, (|T|)\} \in \Pi_T$ the set of all permutations of elements of T, we define the set function $\bar{f} : \{0, 1\}^{|T|} \mapsto \mathbb{R}$ as,

$$(54) \qquad \bar{f}(\boldsymbol{\pi}) = \sum_{i \in 1}^{|T|} |\rho_{(i)}(T_{i-1})|$$

where $T_k = \{(1), (2), \ldots, (k)\}$, $\forall k = 1, 2, \ldots, |T|$ and $T_0 = \varnothing$.

Observe that when f is monotone non-decreasing (non-increasing) then $\bar{f}(\boldsymbol{\pi})$ is independent of the permutation $\boldsymbol{\pi} \in \Pi_T$ and evaluates to $f(T)$ $(-f(T))$.

Definition 13. For an index set T, and a permutation $\boldsymbol{\pi} = \{(1), (2), \ldots, (|T|)\} \in \Pi_T$ the set of all permutations of elements of T, we define

$$(55) \qquad A_h := \max \left\{ \bar{f}(\boldsymbol{\pi}) \ : \ T \subseteq C_{\boldsymbol{\pi}}^+ \cup (J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-), \ |T| = h, \ \boldsymbol{\pi} \in \Pi_T \right\}.$$

$$(56) \qquad B_h := \min \left\{ \bar{f}(\boldsymbol{\pi}) \ : \ T \subseteq C_{\boldsymbol{\pi}}^+ \cup (J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-), \ |T| = h, \ \boldsymbol{\pi} \in \Pi_T \right\}.$$

Proposition 47. *For* $k \in (C_{\boldsymbol{\pi}}^- \cup I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+)$, $\alpha_k \ge h$, *if*

$$(57) \qquad \min\{|f(k)|, |\rho_i(N \setminus i)|\} \ge A_h$$

Proposition 48. *For* $k \in (C_{\boldsymbol{\pi}}^- \cup I_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^+)$, $\alpha_k \le h$, *if*

$$(58) \qquad \max\{|f(k)|, |\rho_i(N \setminus i)|\} \le b + \min_{\boldsymbol{\pi} \in \Pi_{J_{\boldsymbol{\pi}}}} \bar{f}(\boldsymbol{\pi}) - B_{|C_{\boldsymbol{\pi}}^+ \cup (J_{\boldsymbol{\pi}} \setminus C_{\boldsymbol{\pi}}^-)| - 1 - h}$$

Remark 5. In the special case when $J_{\boldsymbol{\pi}} = \varnothing$, $\forall \boldsymbol{\pi}$ i.e. f is non decreasing on N, inequality (52) yields the result of Proposition 7 on sequence independent lifting coefficients [6].

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