

# Low-Rank Matrix Completion Project

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## Task

Given the amount of observed matrix entries to reconstruct low-rank matrix approximation.

## Applications:

- recommender systems;
- image-processing;
- imputation of NAs for genomic data;
- rank estimation for SVD.

# Problem Statement

## Notations:

- $M$  —  $n \times m$  unknown matrix;
- $\Omega \in \{1, \dots, n\} \times \{1, \dots, m\}$  indices of observed elements;
- 

$$P_{\Omega}(M) = \begin{cases} M_{ij}, & \text{if } (i, j) \in \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

## Optimization Task (*NP*-hard)

$$\begin{array}{ll} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} & \text{rank}(X) \\ \text{subject to} & P_{\Omega}(X) = P_{\Omega}(M) \end{array}$$

# Relaxations

## Original Task (*NP-hard*)

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \text{rank}(X), \quad \text{subject to} \quad P_{\Omega}(X) = P_{\Omega}(M)$$

## SVP

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \|P_{\Omega}(X) - P_{\Omega}(M)\|_F^2$$
$$\text{subject to} \quad \text{rank}(X) \leq k$$

## SVT

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \tau \|X\|_* + \|X\|_F^2$$
$$\text{subject to} \quad P_{\Omega}(X) = P_{\Omega}(M)$$

## RISMF

$$\underset{\substack{U \in \mathbb{R}^{n \times k} \\ V \in \mathbb{R}^{k \times m}}}{\text{minimize}} \quad \|U\|_F^2 + \|V\|_F^2$$
$$\text{subject to} \quad P_{\Omega}(UV) = P_{\Omega}(M)$$

## SoftImpute

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \|X\|_*$$
$$\text{subject to} \quad \|P_{\Omega}(X) - P_{\Omega}(M)\|_F \leq \delta$$

- ① Candes E. J., Recht B. Exact matrix completion via convex optimization. 2009.
- ② Cai J. F., Candes E. J., Shen Z. A singular value thresholding algorithm for matrix completion. 2010.
- ③ Mazumder R., Hastie T., Tibshirani R. Spectral regularization algorithms for learning large incomplete matrices. 2010.
- ④ Jain P., Meka R., Dhillon I. S. Guaranteed rank minimization via singular value projection. 2010.
- ⑤ Takacs G. et al. Scalable collaborative filtering approaches for large recommender systems. 2009.
- ⑥ Vandereycken B. Low-rank matrix completion by Riemannian optimization. 2013.

## Relaxation

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \|P_{\Omega}(X) - P_{\Omega}(M)\|_F^2, \quad \text{subject to} \quad \text{rank}(X) \leq k$$

**Input:**  $\Omega, P_{\Omega}(M), k, \eta$ ;

**Output:**  $X$ ;

1:  $X := 0$ ;

2: **repeat**

3:    $X := X - \eta P_{\Omega}^{-1}(P_{\Omega}(X) - P_{\Omega}(M))$ ;

4:    $U, \Sigma, V := \text{SVD}(X)$ ;

5:    $X := U_k \Sigma_k V_k^T$ ;

6: **until**  $\|P_{\Omega}(M) - P_{\Omega}(X)\|_F / \|P_{\Omega}(X)\|_F > \varepsilon$

## Relaxation

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \lambda \|X\|_* + \|X\|_F^2, \quad \text{subject to} \quad P_\Omega(X) = P_\Omega(M)$$

**Input:**  $\Omega, M, \varepsilon, \lambda, \eta$ ;

**Output:**  $X$ ;

- 1:  $Y := \eta P_\Omega(M)$ ;
- 2: **repeat**
- 3:    $X := S_\lambda(Y)$ ;
- 4:    $P_\Omega(Y) := P_\Omega(Y) + \eta P_\Omega(M - X)$ ;
- 5: **until**  $\|P_\Omega(X - M)\|_F / \|P_\Omega(M)\|_F > \varepsilon$

$$S_\lambda(W) = U[D - \lambda I]_+ V^*; \quad U, D, V := \text{SVD}(W)$$

## Lagrangian Relaxation

$$\underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad \lambda \|X\|_* + \|P_\Omega(X) - P_\Omega(M)\|_F^2$$

**Input:**  $M, \Omega, \varepsilon, \lambda_1 > \dots > \lambda_k$ ;

**Output:** sequence  $X_{\lambda_1}, \dots, X_{\lambda_k}$ ;

1:  $X := 0$ ;

2: **for**  $i = 1, \dots, k$  **do**

3:   **repeat**

4:      $X := S_{\lambda_i}(P_\Omega(M) + P_\Omega^\perp(X))$ ;

5:   **until**  $\|P_\Omega(M) - P_\Omega(X)\|_F / \|P_\Omega(M)\|_F > \varepsilon$

6:    $X_{\lambda_i} := X$ ;

$$P_\Omega^\perp(M) = M - P_\Omega(M).$$



## Relaxation

$$\begin{array}{l} \text{minimize} \\ U \in \mathbb{R}^{n \times k} \\ V \in \mathbb{R}^{k \times m} \end{array} \quad \|P_{\Omega}(UV) - P_{\Omega}(M)\|_F^2 + \lambda (\|U\|_F^2 + \|V\|_F^2)$$

**Input:**  $M, \Omega, \varepsilon, \lambda, \eta$ ;

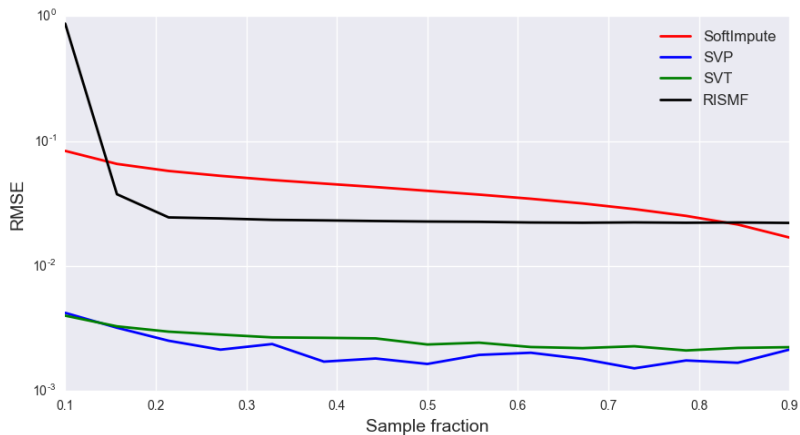
**Output:**  $U, V$ ;

- 1: Initialization of  $U, V$  with small random numbers;
- 2: **repeat**
- 3:   **for**  $(i, j) \in \Omega$  **do**
- 4:     Gradient step for  $U[i, :]$  and  $V[:, j]$  by regularized error function;
- 5: **until**  $\|P_{\Omega}(M) - P_{\Omega}(UV)\|_F / \|P_{\Omega}(M)\|_F > \varepsilon$

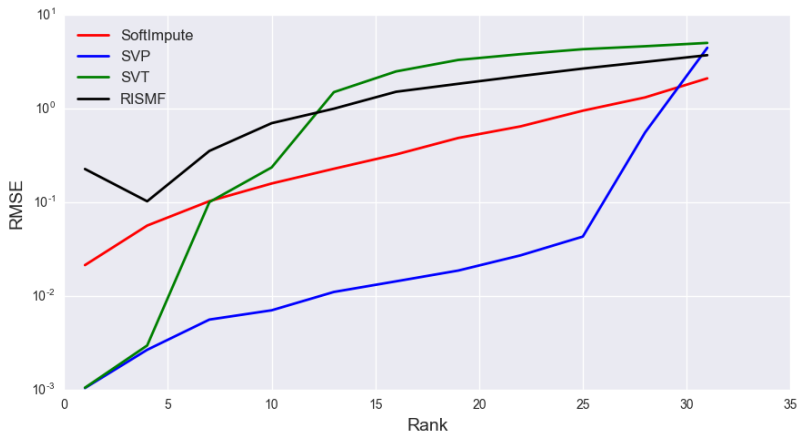
# Numerical Experiments

- Synthetic data  
Low-rank matrices with random noise.
- Low-rank images  
Visual demonstration of algorithms.
- Assessment dataset  
Real dataset for collaborating filtering.

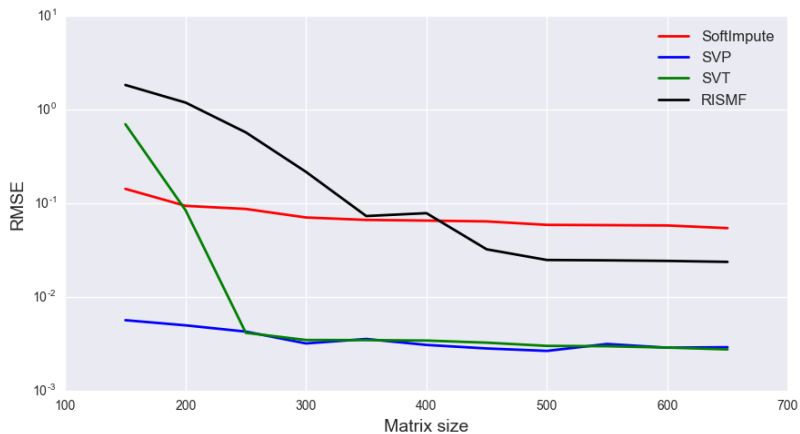
# Synthetic Data



# Synthetic Data



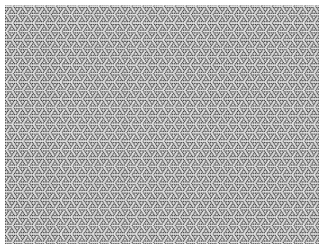
# Synthetic Data



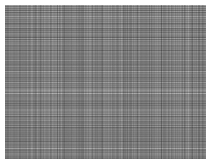
Sampled Image



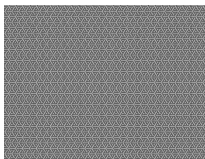
SoftImpute



RISMF



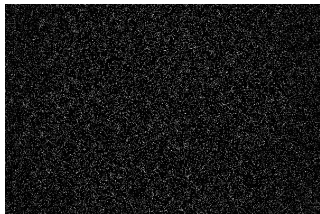
SVT



SVP



Sampled Image



SVT



RISMF



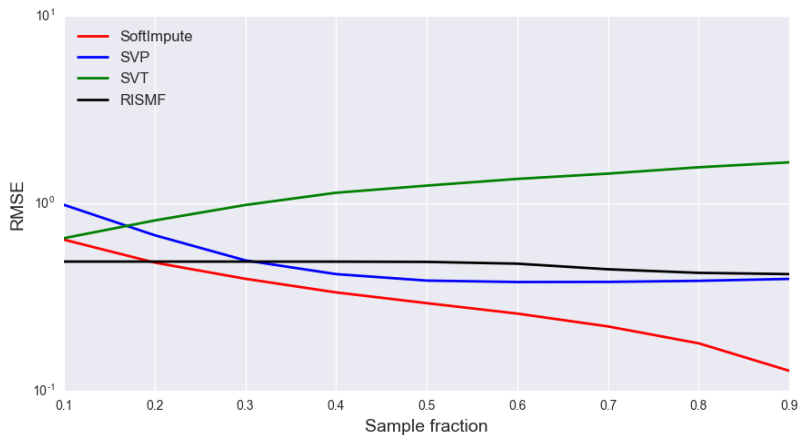
SVT



SVP

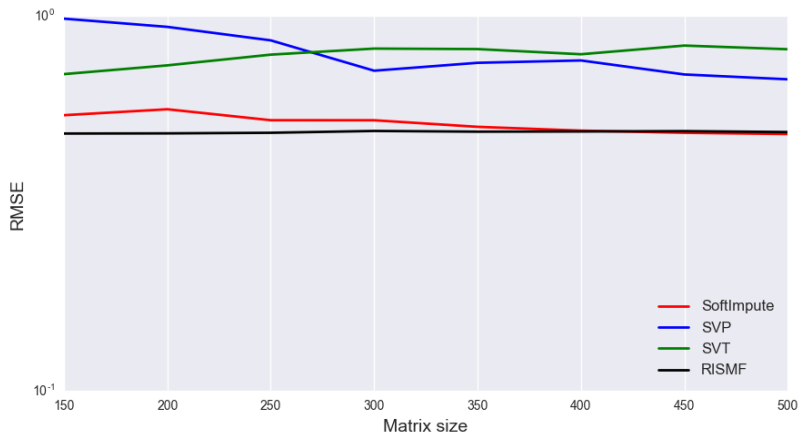


# Assessment data





# Assessment data



# Conclusions

- Different approaches to low-rank matrix completion problem were studied.
- The software for solution was developed.
- Performance of these methods was evaluated on synthetic and real datasets.