Low-Rank Matrix Completion project

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Problem Statement

Task

Given the amount of observed matrix entries to reconstruct low-rank matrix approximation.

Applications:

- recommender systems;
- image-processing;
- imputation of NAs for genomic data;
- rank estimation for SVD.

Problem Statement

Notations:

- M — $n \times m$ unknown matrix;
- $\Omega \in \{1, \dots, n\} \times \{1, \dots, m\}$ indices of observed elements;
- $$P_\Omega(M) = \begin{cases} M_{ij}, & \text{if } (i,j) \in \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

Optimization Task

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \text{rank}(X) \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M). \end{aligned}$$

Related Works

1. Candes E. J., Recht B. Exact matrix completion via convex optimization. 2009.
2. Cai J. F., Candes E. J., Shen Z. A singular value thresholding algorithm for matrix completion. 2010.
3. Mazumder R., Hastie T., Tibshirani R. Spectral regularization algorithms for learning large incomplete matrices. 2010.
4. Jain P., Meka R., Dhillon I. S. Guaranteed rank minimization via singular value projection. 2010.
5. Takacs G. et al. Scalable collaborative filtering approaches for large recommender systems. 2009.
6. Vandereycken B. Low-rank matrix completion by Riemannian optimization. 2013.

Convex Relaxation (Candes E. J., Recht B., 2009)

Original Task

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M). \end{aligned}$$

Relaxation

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \|X\|_* \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M). \end{aligned}$$

Motivation:

$$\operatorname{rank}(X) = |\{i : \sigma_i(X) \neq 0\}|; \quad \|X\|_* = \sum_{i=1}^{k} \sigma_i(X).$$

SVP

Original Task

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M). \end{aligned}$$

Relaxation

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \|P_\Omega(X) - P_\Omega(M)\|_F \\ \text{subject to} \quad & \operatorname{rank}(X) \leq k. \end{aligned}$$

SVT

SoftImpute (Hastie T., Tibshirani R., 2010)

Original Task

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M). \end{aligned}$$

Relaxation

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times m}}{\text{minimize}} \quad & \|X\|_* \\ \text{subject to} \quad & \|P_\Omega(X) - P_\Omega(M)\|_F \le \delta. \end{aligned}$$

Motivation:

The method is the same as *SVT* with $\delta = 0$. When $\delta > 0$ the overfitting is less possible.

RISMF

Riemannian Optimization