Formula to Express Ramsey Numbers $R(m,n)$ using Partition Numbers

JiangLin Luo

WangYueHu Community 1 pian 7 dong
Changsha, China 230026

cody@ustc.edu

November 26, 2023

Mathematics Subject Classifications: 05C15, 05C55, 05C85

Abstract

This paper proposes Fence Conjecture: Ramsey Number

$$R(m,n) = (2m-1) * p(2m-6+n,m) + \{1,m,m+1\}, \text{for } 3 \leq m \leq n$$

. Here $p(n,k)$ denotes the number of partitions of n into exactly k parts. The last summand chooses one from three values $\{1,m,m+1\}$. When $m=3$ or 4, it is

$$R(3,n) = 5 * p(n,3) + \{1,3,4\}, \text{for } 3 <= n$$

$$R(4,n) = 7 * p(n+2,4) + \{1,4,5\}, \text{for } 4 <= n$$

.

1 Introduction

The classical two color Ramsey numbers $R(r,s)$ in Ramsey's theorem is the minimum number of vertices, $v = R(m,n)$, such that all undirected simple graphs of order v, contain a clique of order m, or an independent set of order n. Or to say in any 2-coloring of the edges of the complete graph K_v, there is a monochromatic copy of K_m in color 1 or of K_n in color 2. By symmetry, $R(m,n) = R(n,m)$. Radziszowski's survey on Small Ramsey Numbers [1] gives the basic terminology and the current status of a host of problems related to Ramsey numbers.

Brendan McKay listed some small Ramsey Graphs [4]. Around 2010, I observed that $R(3,n) - 1 \equiv 0, 2, 3 (\text{mod } 5)$ for $n = 1..9$ those all known $R(3,n)$. This observation is the basis of the following conjecture.

Table 1: divmod(R(3,n)-1,5)

n	1	2	3	4	5	6	7	8	9	10
(R(3,n)-1)/5	0	0	1	1	2	3	4	5	7	?8
(R(3,n)-1)%5	0	2	0	3	3	2	2	2	0	?0

Table 2: layout Ramsey critical graphs for $R(3, n) - 1$ by interlacing with 2 or 3 vertices

o	o	o	o	o	o	o
	o		o		o	
o	o	o	o	o	o	o

2 R(m,n) Formula Conjecture

Conjecture 2.1 (Fence Conjecture part1) *When constructing critical Ramsey graph for $R(3, n)$, the $R(3, n) - 1$ vertices can be laid out interlacing with 2 vertices and 3 vertices. This means*

$$R(3, n) - 1 \equiv 0, 2, 3(mod\ 5)$$

.

In 2023-11-22, When I searched $0, 1, 1, 2, 3, 4, 5, 7, 8$ in OEIS, I got sequence A001399 [3] partitions numbers $p(n, 3)$. So I proposed,

Conjecture 2.2 (Fence Conjecture part2)

$$R(3, n) = 5 * p(n, 3) + \{1, 3, 4\}, for\ n \geq 3$$

.

Also, the other two known Ramsey numbers have $R(4, 4) = 18 = 7 * 2 + 4; R(4, 5) = 25 = 7*3+4$. It appears that $R(m, n)$ formula has modular form and relates to partitions function closely.

Conjecture 2.3 (Fence Conjecture) *Ramsey numbers*

$$R(m, n) = (2m - 1) * p(2m - 6 + n, m) + \{1, m, m + 1\}, for\ 3 \leq m \leq n$$

.

Here partitions function $p(n, k)$ [2] or denotes as $p_k(n)$, is both the number of partitions of n into exactly k parts, and the number of partitions of n into parts of maximum size exactly k. These two types of partition conjugate in their Young diagrams. The last summand chooses one value among three integers $\{1, m, m + 1\}$.

Table 3: (2m-1)*p(2m-6+n,m)

	n=3	4	5	6	7	8	9	10	11	12	13	14	15
m=3	5	5	10	15	20	25	35	40	50	60	70	80	95
4		14	21	35	42	63	77	105	126	161	189	238	273
5			45	63	90	117	162	207	270	333	423	513	630
6				121	154	220	286	385	484	638	781	990	1210
7					273	364	494	637	845	1066	1365	1703	2132
8						600	780	1050	1335	1740	2190	2790	3450
9							1241	1598	2091	2669	3417	4284	5406
10								2432	3116	4028	5073	6460	8037
11									4599	5838	7455	9345	11760
12										8418	10580	13386	16675
13											14925	18675	23375
14												25839	32076
15													43732

3 Experiments and Data

```
def p(n,k):
  return Partitions(n,length=k).cardinality()

P=matrix(ZZ,30,30,lambda n,k:p(n,k))

top=15
data=[[(2*m - 1)*p(2*m - 6 + n, m) for n in (m..top)] for m in (3..top)]
print(data)
```

Ramsey numbers have basic inequality relation $R(r,s) \leq R(r-1,s)+R(r,s-1)$, the conjectured formula satisfies this inequality completely. Fence conjecture implies that $R(3,10)=41, R(5,5)=46, R(6,6)=122$ or 127 or 128. The last summand has not be decided from $1, m, m+1$ yet. The conjecture needs to be supplemented or even corrected.

References

[1] S. P. Radziszowski, Small Ramsey Numbers, *The Electronic Journal of Combinatorics*, DS1, 2014

[2] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A008284`.

[3] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A001399`.

[4] Brendan McKay, Ramsey Graphs, http://users.cecs.anu.edu.au/~bdm/data/ramsey.html.