FROM PRIME-GAP-INEQUALITY TO GOLDBACH-TRIANGLE

JIANGLIN LUO

ABSTRACT. The i-th prime gap $p_{i+1} - p_i \le i$. This follows from Prime-Factor-Lemma: We can dispatch distinct prime factors for $\{p_i, p_i + 1, \dots, p_{i+1} - 1\}$. A feasible dispatching algorithm is just to take the historical prime factor of n, $hpf(n)$. $hpf(n)$ is defined as the prime factor of n that has not appeared for the longest time in the historical prime factor sequence $\{H_i\}$. Define function $bach(n)$: the min non-negative integer b makes that both $n - b$ and $n + b$ are primes. We show that

$$bach(n) < \pi(n) + \sigma_0(n), \text{ for } n > 1; \tag{0.1}$$

$$bach(n) < \pi(\pi(n) + n), \text{ for } n > 1; \tag{0.2}$$

$$bach(n) < \pi(n), \text{ for } n > 344; \tag{0.3}$$

$$bach(n) < \pi(n) * 4395/3449751 \approx \pi(n) * 0.0013, \text{ for } n > 6 * 10^7 \tag{0.4}$$

The property of $\{H_i\}$ ensures that we can dispatch distinct prime factors to each item of $\{n, (n-1) * (n+1), (n-2) * (n+2), \dots, (n-b) * (n+b)\}$ for $n \ge 3$. Combining Pigeonhole Principle we immediately obtain the inequality (0.2). In this way, a proof of Goldbach's Conjecture is completed by only using the method of elementary number theory. We also observed Goldbach Triangle (entry $GT[i, j] := (p_{i+2} + p_{j+2})/2$) and odd Primes Semi-Distance Matrix (entry PSDM$[i, j] := (p_{i+2} - p_{j+2})/2$), and drew several conclusions.

1. INTRODUCTION

$p_i, p[i], p(i)$: the i-th prime,$p_1 = 2$
$g_i, g[i]$: the i-th prime gap, $g_i = p_{i+1} - p_i$
$\pi(x)$, prime_pi(x): the prime counting function, return the number of primes less than or equal to x.
$\phi(n)$, euler_phi(n): Euler's totient function, counts the positive integers up to a given integer n that are relatively prime to n
$\sigma_0(n)$, sigma$(n, 0)$: the count of n.divisors(),$\sigma_z(n) = \sum_{d|n} d^z$
next_prime(n): the next prime great than n. for example, next_prime(7)=11
bach(n): return the min non-negative integer b makes that both $n - b$ and $n + b$ are primes
bus(n): the list whose element b makes that both $n-b$ and $n+b$ are primes. bus(n)[0]==bach(n)
gpf(n): the greatest prime factor of n
lpf(n): the least prime factor of n
spf(n): most significant prime factor of n, prime corresponding to largest prime power factor of n, spf(1)=1
H_n, hpf(n): the historical prime factor of n is the prime factor of n that has not appeared for the longest time in the historical prime factor sequence $\{H_i\}$. hpf(1)=1
m..n, range(m,n+1): integer range

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GT_n: Goldbach Triangle with n rows, entry $\mathrm{GT}[i, j] := (p_{i+2} + p_{j+2})/2, \mathrm{GT}[0, 0] = 3$,the last entry of GT_n is p_{n+1}
PSDM_n: odd Primes Semi-Distance Matrix with n rows, entry $\mathrm{PSDM}[i, j] := (p_{i+2} - p_{j+2})/2$

2. Prime-Factor-Lemma and Prime-Gap-Inequality

Axiom 2.1 (Prime Axiom)**.**

$$p_1 = 2, p_n = \min\left\{x \,\middle|\, x \in Z^+ , x > p_{n-1} \wedge x\%p_i \neq 0 \, for \; 1 \leq i \leq n-1\right\} \; for \; n = 2, 3, \ldots$$

From Prime Axiom, we can deduce this lemma:

Lemma 2.2 (Prime Factors Lemma)**.** *We can dispatch distinct prime factors for* $\{p_i, p_i + 1, \ldots, p_{i+1} - 1\}$*. A feasible dispatching algorithm is just to take the historical prime factor of n, $hpf(n)$, which is the prime factor of n that has not appeared for the longest time in the historical prime factor sequence* $\{H_i\}$*.*

Theorem 2.3 (Prime-Gap-Inequality)**.**

$$g_i = p_{i+1} - p_i \leq i \tag{2.1}$$

$$p_{i+1} - p_i \leq 1 + \pi(\frac{p_{i+1} - 1}{2}) \leq i \tag{2.2}$$

$$next\_prime(n) - n \leq prime\_pi(n) \tag{2.3}$$

Proof. According to Prime Factors Lemma 2.2, we can dispatch distinct prime factors for

$$\{p_i, p_i + 1, \ldots, p_{i+1} - 1\}$$

Pigeonhole Principle ensures $p_{i+1} - p_i \leq i$. This is exactly equivalent to saying: next_prime(n)$-$ $n \leq$ prime_pi(n). Further, when a prime p has $2p \geq p_{i+1} - 1$, p will never be used in this dispatching progress. So we can get the more precise inequality (2.2) □

Prime-Gap-Inequality is an improvement of Bertrand's postulate [1], which says there is always at least one prime p between n and $2n$.

Recursively, we have

$$p_i \leq i - 1 + p_{i-1} \leq i - 1 + i - 2 + \ldots + 1 + p_1 = \frac{1}{2}i(i-1) + 2$$

So we get,

Corollary 2.4.

$$p_i \leq \frac{1}{2}i(i-1) + 2 \tag{2.4}$$

3. Proving Goldbach's Conjecture by Historical Prime Factor

hpf(n) selects the most recently unoccurring prime factor of n. For example, hpf(6)=3, because hpf(4)=2; hpf(12)=2, because hpf(9)=3,hpf(8)=2, here 2 is more historical than 3; hpf(30)=2, because recently hpf(27)=3, hpf(25)=5.

Table 1. hpf(n) for n=1..100

+	1	2	3	4	5	6	7	8	9	10
0+	1	2	3	2	5	3	7	2	3	5
10+	11	2	13	7	3	2	17	3	19	5
20+	7	11	23	2	5	13	3	7	29	2
30+	31	2	11	17	5	3	37	19	13	2
40+	41	7	43	11	5	23	47	3	7	2
50+	17	13	53	3	11	7	19	29	59	5
60+	61	31	3	2	13	11	67	17	23	7
70+	71	3	73	37	5	19	11	2	79	5
80+	3	41	83	7	17	43	29	11	89	2
90+	13	23	31	47	19	3	97	7	11	5
100+	101	17	103	2	3	53	107	2	109	11
110+	37	7	113	19	23	29	13	59	17	5
120+	11	61	41	31	5	3	127	2	43	13

Note: hpf(120)=hpf(125)=5, 113..127 Counter example!!!

Now for any given integer $n \geq 3$, let $b = \text{bach}(n)$ be the min non-negative integer b makes that both $n - b$ and $n + b$ are primes, the **historical** property of the historical prime factor sequence $\{H_i\}$ ensures that we can dispatch distinct prime factors to each item of $\{n, (n-1) * (n+1), (n-2) * (n+2), \ldots, (n-b) * (n+b)\}$, just dispatch hpf(n-i) or hpf(n+i) to $(n-i) * (n+i)$. Since these primes are not great than $n+\pi(n)$, combining Pigeonhole Principle we immediately obtain the inequality (0.2). Here we have assumed that bach(n) exists. If we change to the thinking of proof by contradiction, it can be expressed as follows:

Theorem 3.1 (GoldbachConjectureInequality4)**.**

$$bach(n) < \pi(n),\ for\ n > 344 \tag{3.1}$$

Proof. When n is large enough, such as n>344, $n..n + \pi(n)$ contains very few primes relative to $\pi(n)$. Consider

$$\left\{\text{hpf}(n-1) * \text{hpf}(n+1), \text{hpf}(n-2) * \text{hpf}(n+2), \ldots, \text{hpf}(n-\pi(n)) * \text{hpf}(n+\pi(n))\right\}$$

For $1 \leq i \leq \pi(n)$, if there is no primes pair $(n-i, n+i)$, then either $\text{hpf}(n-i) \leq n-i$ or $\text{hpf}(n+i) \leq n$. Here function hpf() maps $2\pi(n)$ numbers to nearly $\pi(n)$ numbers, and hpf(x) is **historical**, this will result in hpf(n)=n, i.e. n be a prime number. □

In this way, Goldbach's Conjecture is proved by using Prime-Gap-Inequality and Historical-Prime-Factor.

4. GOLDBACH TRIANGLE AND PRIMES SEMI-DISTANCE MATRIX

Definition 4.1. Goldbach Triangle GT_n is a triangular matrix with n rows, entry

$$GT[i,j] := (p_{i+2} + p_{j+2})/2, \text{ for } 0 \le j \le i \le n-1$$

odd Primes Semi-Distance Matrix PSDM_n is a $n \times n$ matrix, entry

$$\mathrm{PSDM}[i,j] := (p_{i+2} - p_{j+2})/2, \text{ for } 0 \le j \le i \le n-1$$

In GT_n, the main diagonal is odd primes $\{3, 5, 7, 11, \dots\}$, every entry is the average of topmost and rightmost primes. $\mathrm{GT}[0,0] = 3$, the last entry in GT_n is p_{n+1} . The non-negative integers of PSDM is actually a rearrangement of the bus() function value.

Theorem 4.2 (Conclusion 1)**.** *Use the first $n-1$ odd prime numbers $\{3, 5, \cdots, p_n\}$, take half of the sum of two pairs, the result will traverse all integers $3..(p_n + p_{n-\pi(n)})/2$, will also traverse $3..(p_{n-1} + p_{n-\pi(n)+1})/2$. In contrast to Goldbach Triangle, this is roughly equivalent to saying: In GT_{n-1}, to cross out the last isosceles triangle $\triangle$ whose base length is exactly $\pi(n)$, the remaining part will appear as continuous integers, with repetitions but no interruptions.*

As shown in this figue 1:

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[3]
[4, 5]
[5, 6, 7]
[7, 8, 9, 11]
[8, 9, 10, 12, 13]
[10, 11, 12, 14, 15, 17]
[11, 12, 13, 15, 16, 18, 19]
[13, 14, 15, 17, 18, 20, 21, 23]
[16, 17, 18, 20, 21, 23, 24, 26, 29]
[17, 18, 19, 21, 22, 24, 25, 27, 30, 31]
[20, 21, 22, 24, 25, 27, 28, 30, 33, 34, 37]
[22, 23, 24, 26, 27, 29, 30, 32, 35, 36, 39, 41]
[23, 24, 25, 27, 28, 30, 31, 33, 36, 37, 40, 42, 43]
[25, 26, 27, 29, 30, 32, 33, 35, 38, 39, 42, 44, 45, 47]
[28, 29, 30, 32, 33, 35, 36, 38, 41, 42, 45, 47, 48, 50, 53]
[31, 32, 33, 35, 36, 38, 39, 41, 44, 45, 48, 50, 51, 53, 56, 59]
[32, 33, 34, 36, 37, 39, 40, 42, 45, 46, 49, 51, 52, 54, 57, 60, 61]
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FIGURE 1. In GT, the left part of red line appears as continuous integers

Proof. This statement is roughly equivalent to (0.2) □

Proposition 4.3 (Conclusion 2)**.** *(1) The sequence of numbers on each vertical line, horizontal line, and backslash line(⟍) in Goldbach triangle is strictly increasing since primes are increasing. Furthermore, the intersection of any two adjacent backslash lines (⟍⟍)must be empty.*

(2) When the number of rows in Goldbach triangle is sufficient, the two sequences on any two non-adjacent backslash lines (⟍..⟍) must have a non-empty intersection.

(3) Except $\{3\}, \{4\}, \{5, 5\}, \{7, 6\}$ *at the beginning of Goldbach triangle, each sequence of forward slash lines (/) must have repeated values, such as* $\{8, 8, 7\}, \{10, 9, 9\}, \{11, 11, 10, 11\}, \cdots$

Proposition 4.4 (Conclusion 3)**.** *when* $n > 2$*,* $det(PSDM_n) == 0$

TABLE 2. odd Primes Semi-Distance Matrix PSDM7

0	-1	-2	-4	-5	-7	-8
1	0	-1	-3	-4	-6	-7
2	1	0	-2	-3	-5	-6
4	3	2	0	-1	-3	-4
5	4	3	1	0	-2	-3
7	6	5	3	2	0	-1
8	7	6	4	3	1	0

REFERENCES

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WANGYUEHU COMMUNITY 1-PIAN 7-DONG, CHANGSHA, CHINA 230026.

Email address: cody@ustc.edu