Intro to Quantum Computing

Lecture 2 bonus slides: noise metrics in quantum computing

Olivia Di Matteo

Quantum Information Science Associate, TRIUMF

In these ‘bonus’ slides, I will give a brief physical description of the noise metrics we discussed, as well as introduce to you the idea of density matrices.

Mixed states

A quantum state $|\psi\rangle$ is *pure* if it can be written as a single vector.

The most general representation of a quantum system is using a **density matrix** ρ. For the pure state $|\psi\rangle$,

$$\rho = |\psi\rangle\langle\psi| \tag{1}$$

We can also have a linear combination of pure states:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad \sum_k p_k = 1 \tag{2}$$

Properties of ρ:

- $2^n \times 2^n$ matrix
- Trace 1
- Positive semidefinite (all eigenvalues ≥ 0)

Mixed states and the Bloch sphere

Recall that single-qubit pure states live on the *surface* of the Bloch sphere.

Single-qubit mixed-states live *inside* the Bloch sphere.

The centre of the Bloch sphere is called the *maximally mixed state*:

$$\rho_{MM} = \frac{1}{2}|0\rangle \langle 0| + \frac{1}{2}|1\rangle \langle 1| \tag{3}$$

Noise and the maximally mixed state

The presence of noise in a quantum system can turn pure states to mixed states!

Suppose $\mathcal{N}$ is some noisy process that sends

$$\mathcal{N}(\rho) \to \rho' \tag{4}$$

$\mathcal{N}$ is an example of a quantum channel - it eats a state and spits out a (potentially) different one.

We can mathematically quantify noise by comparing ρ' and ρ.

Fidelity

Fidelity defines a notion of *distance* between quantum states that we can use to quantify the quality of our operations.

$$\mathcal{F}(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \tag{5}$$

This expression is more illuminating if we consider the fidelity of a pure state $|\psi\rangle$ with a density operator ρ:

$$\mathcal{F}(|\psi\rangle, \rho') = \sqrt{\langle\psi|\rho'|\psi\rangle} \tag{6}$$

Fidelity (Example)

Suppose we start with state $|\psi\rangle = |0\rangle$ and apply a Hadamard to it. The output state should be

$$|\psi'\rangle = H|0\rangle = |+\rangle \tag{7}$$

Unitaries act on density matrices using conjugation:

$$\rho = |0\rangle\langle 0| \to \rho' = (H|0\rangle)(\langle 0| H^\dagger) = |+\rangle\langle +| \tag{8}$$

If the H operation was perfect, we should compute the fidelity of the outcome with the expected outcome to be 1:

$$\begin{aligned} \mathcal{F}(|+\rangle, \rho') &= \sqrt{\langle +|\rho'|+\rangle} \\ &= \sqrt{\langle +|+\rangle\langle +|+\rangle} \\ &= 1 \end{aligned}$$

But what if H was noisy?

Depolarizing noise

Depolarizing noise acts like:

$$\mathcal{N}(\rho) = \lambda\rho + (1-\lambda)\frac{\mathbb{1}}{d} \tag{9}$$

Let's deconstruct this. We transform the system from the original ρ to a mixture of ρ and the maximally mixed state $\frac{\mathbb{1}}{d}$. λ controls the strength of the depolarization - lower λ means more mixing.

Suppose there is depolarizing noise on the output of our Hadamard operation.

$$\mathcal{N}(|+\rangle\langle+|) = \lambda|+\rangle\langle+| + (1-\lambda)\frac{\mathbb{1}}{d} \tag{10}$$

Fidelity (Example)

Then the fidelity of $\mathcal{N}(|+\rangle\langle+|)$ and the expected output $|+\rangle\langle+|$ is:

$$
\begin{aligned}
\mathcal{F}(|+\rangle, \mathcal{N}(|+\rangle\langle+|)) &= \sqrt{\langle+|\mathcal{N}(|+\rangle\langle+|)|+\rangle} \\
&= \sqrt{\langle+|\left(\lambda|+\rangle\langle+| + (1-\lambda)\frac{\mathbb{1}}{d}\right)|+\rangle} \\
&= \sqrt{\lambda\langle+|+\rangle\langle+|+\rangle + (1-\lambda)\frac{\langle+|\mathbb{1}|+\rangle}{d}} \\
&= \sqrt{\lambda + \frac{1-\lambda}{d}} \\
&= \sqrt{\frac{1}{d} + \lambda\left(1 - \frac{1}{d}\right)}
\end{aligned}
$$

Smaller λ (more depolarizing noise) decreases the fidelity!

Coherence times

How can we quantify the degradation of qubit states due to interaction with the environment?

Two important quantities:

- T_1, spin relaxation time
- T_2, spin decoherence time
- T_1^*, ensemble spin relaxation time
- T_2^*, ensemble spin decoherence time

How long your qubit retains its state governs how many operations you can perform on it.

Coherence times

T_1, spin relaxation time

Suppose states $|0\rangle$ and $|1\rangle$ are implemented using the ground and excited states of a system. The *spin relaxation time*, T_1, is a measure of the average time it takes for the system to transition from $|1\rangle$ to $|0\rangle$ due to the influence of the environment.

T_2, spin decoherence time

Consider a qubit in superposition $|0\rangle + |1\rangle$. As the system evolves, it will begin to *precess* and pick up a relative phase:

$$|0\rangle + |1\rangle \to |0\rangle + e^{i\alpha}|1\rangle \tag{11}$$

We say the system has *decohered* when this phase is essentially random, and T_2 is the average time it takes for this to occur. The T_2^* time is like the T_2 time but also considers that there is additional decoherence *between* spins as well as within each spin.