# Intro to Quantum Computing

**Lecture 1 'bonus slides'**

Olivia Di Matteo

Quantum Information Science Associate, TRIUMF

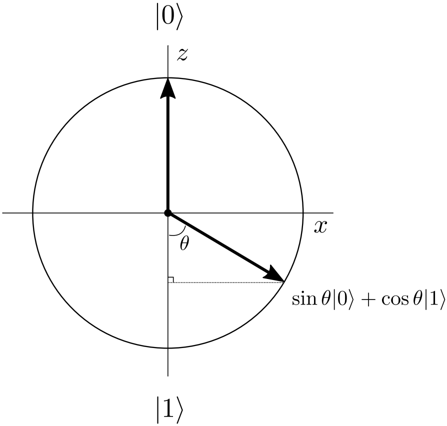
# Deriving the result of a projective measurement

Let $|\varphi\rangle$ be a single-qubit state and $\Pi_i = |\psi_i\rangle\langle\psi_i|$ be one component of a projective measurement. Then

$$
\begin{aligned}
\text{Pr(outcome i)} &= \text{Tr}(|\psi_i\rangle\langle\psi_i|\,|\varphi\rangle\langle\varphi|) && (1)\\
&= \sum_n \langle e_n|\psi_i\rangle\langle\psi_i|\varphi\rangle\langle\varphi|e_n\rangle && (2)\\
&= \sum_n \langle\psi_i|\varphi\rangle\langle\varphi|e_n\rangle\langle e_n|\psi_i\rangle && (3)\\
&= \langle\psi_i|\varphi\rangle\langle\varphi|\left(\sum_n |e_n\rangle\langle e_n|\right)|\psi_i\rangle && (4)\\
&= \langle\psi_i|\varphi\rangle\langle\varphi|\mathbb{1}|\psi_i\rangle && (5)\\
&= |\langle\psi_i|\varphi\rangle|^2 && (6)
\end{aligned}
$$

where $|e_n\rangle$ is vectors with all elements 0 except a 1 in element $n$.

# Measurements and the Bloch sphere



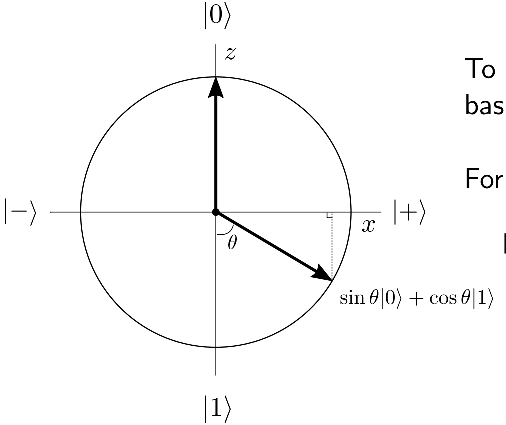

We can visualize measurements as projections onto the axes of the Bloch sphere:

For example,

$$\begin{aligned} \langle 1|\psi\rangle &= \sin\theta\langle 1|0\rangle + \cos\theta\langle 1|1\rangle \\ &= \cos\theta \end{aligned}$$

$$\begin{aligned} \Pr(1) &= |\langle 1|\psi\rangle|^2 \\ &= \cos^2\theta \end{aligned}$$

# Measurements and the Bloch sphere



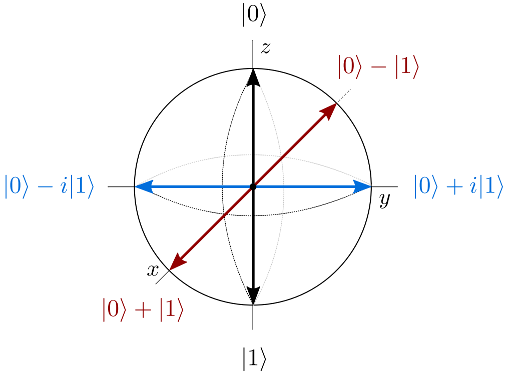

To measure in the Hadamard basis, we project onto the $x$ axis:

For example,

$$\begin{aligned}
\mathrm{Pr}(+) &= |\langle +|\psi\rangle|^2 \\
&= \ldots \\
&= \frac{1}{2}\left(1+\sin(2\theta)\right)
\end{aligned}$$

# Quantum tomography

This geometric interpretation makes the following fact very intuitive: to fully characterize the state of a single-qubit ket, we can make measurements in 3 different bases along $x, y$ and $z$.




We say that these 3 bases form a *tomographically complete* set of measurements.

We can relate the outcome probabilities to the Bloch sphere coordinates directly:

A Bloch vector in 3D space is given by

$$\vec{a} = (a_x, a_y, a_z) \tag{7}$$

where

$$\begin{aligned} a_x &= 2 \cdot \Pr(|+\rangle) - 1 && (8) \\ a_y &= 2 \cdot \Pr(|y_+\rangle) - 1 && (9) \\ a_z &= 2 \cdot \Pr(|0\rangle) - 1 && (10) \end{aligned}$$

Exercise: Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left( |00\rangle + |10\rangle - |11\rangle \right) \tag{11}$$

What is the probability of the first qubit being in state $|0\rangle$?
State $|1\rangle$?
What is the state of the second qubit after each measurement?

# Measuring multi-qubit systems

Solution:

$$\frac{1}{\sqrt{3}}\left(|00\rangle + |10\rangle - |11\rangle\right)$$

To get the probability the first qubit is in state $|0\rangle$, sum the probabilities for each measurement outcome where this is possible:

$$\begin{aligned}
\Pr(0 \text{ for qubit } 1) &= \Pr(00) + \Pr(01) \\
&= |\langle 00|\psi\rangle|^2 + |\langle 01|\psi\rangle|^2 \\
&= \frac{1}{3}
\end{aligned}$$

The second qubit will be in state $|0\rangle$

# Measuring multi-qubit systems

For the second part, let's factor the state:

$$\frac{1}{\sqrt{3}}\left(|00\rangle + |1\rangle\left(|0\rangle - |1\rangle\right)\right)$$

Then

$$\begin{aligned}
\Pr(1 \text{ for qubit } 1) &= |\langle 10|\psi\rangle|^2 + |\langle 11|\psi\rangle|^2 \\
&= \frac{1}{3} + \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}$$

The second qubit will be in state $|-\rangle$.